# A Systematic Approach to Bound Factor-Revealing LPs and its Application to the Metric and Squared Metric Facility Location Problems* 

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#### Abstract

A systematic technique to bound factor-revealing linear programs is presented. We show how to derive a family of upper bound factor-revealing programs (UPFRP), and show that each such program can be solved by a computer to bound the approximation factor of an associated algorithm. Obtaining an UPFRP is straightforward, and can be used as an alternative to analytical proofs, that are usually very long and tedious. We apply this technique to the Metric Facility Location Problem (MFLP) and to a generalization where the distance function is a squared metric. We call this generalization the Squared Metric Facility Location Problem (SMFLP) and prove that there is no approximation factor better than 2.04, assuming $\mathrm{P} \neq \mathrm{NP}$. Then, we analyze the best known algorithms for the MFLP based on primal-dual and LP-rounding techniques when they are applied to the SMFLP. We prove very tight bounds for these algorithms, and show that the LP-rounding algorithm achieves a ratio of 2.04 , and therefore has the best factor for the SMFLP. We use UPFRPs in the dual-fitting analysis of the primal-dual algorithms for both the SMFLP and the MFLP, improving some of the previous analysis for the MFLP.


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## 1 Introduction

Let $C$ and $F$ be finite disjoint sets. Call cities the elements of $C$ and facilities the elements of $F$. For each facility $i$ and city $j$, let $c_{i j}$ be a non-negative number representing the cost to connect $i$ to $j$. Additionally, let $f_{i}$ be a non-negative number representing the cost to open facility $i$. For each city $j$ and subset $F^{\prime}$ of $F$, let $c\left(F^{\prime}, j\right)=\min _{i \in F^{\prime}} c_{i j}$. The Facility Location Problem (FLP) consists of the following: given sets $C$ and $F$, and $c$ and $f$ as above, find a subset $F^{\prime}$ of $F$ such that $\sum_{i \in F^{\prime}} f_{i}+\sum_{j \in C} c\left(F^{\prime}, j\right)$ is minimum. Hochbaum [9] presented an $\mathrm{O}(\log n)$ approximation for the FLP.

A well-studied particular case of the FLP is its so called metric variant. We say that an instance ( $C, F, c, f$ ) of the FLP is metric if $c_{i j} \leq c_{i j^{\prime}}+c_{i^{\prime} j^{\prime}}+c_{i^{\prime} j}$, for all facilities $i$ and $i^{\prime}$, and cities $j$ and $j^{\prime}$. In the context of the FLP, this inequality is called the triangle inequality. The METRIC FLP, denoted by MFLP, is the particular case of the FLP that considers only metric instances. Several algorithms were proposed in the literature for the MFLP [2, 5, 7, 10, 12, [13, 15, 16]. In particular, the best known algorithm for the MFLP is a 1.488 -approximation proposed by Li [13]. Also, Guha and Khuller [8] proved an inapproximability result that states that there is no approximation algorithm for the MFLP with a ratio smaller than 1.463, unless $\mathrm{NP} \subseteq \mathrm{DTIME}\left[n^{\mathrm{O}(\log \log n)}\right]$. This result was strengthened by Sviridenko, who showed that the lower bound holds unless $\mathrm{P}=\mathrm{NP}$ (see [17]).

The Euclidean FLP is a particular case of the MFLP also considered in the literature. In the Euclidean FLP, one is given a position in an Euclidean space for each city and for each facility, and the cost $c_{i j}$ is the Euclidean distance between the position of facility $i$ and the position of city $j$. There is a PTAS for the Euclidean FLP in 2-dimensional space, by Arora, Raghavan, and Rao [1].

Yet another variant considered in the literature is the so called Squared Euclidean FLP, denoted here by $\mathrm{E}^{2}$ FLP. In this variant, as in the Euclidean case, one is given a position in an Euclidean space for each city and for each facility. Here, the cost $c_{i j}$ is the square of the Euclidean distance between the position of facility $i$ and the position of city $j$. This cost measure is known as $\ell_{2}^{2}$, and was, for instance, considered by Jain and Vazirani [12], pp. 292293] in the context of the FLP. Their approach implies a 9-approximation for the E ${ }^{2}$ FLP.

We consider instances ( $C, F, c, f$ ) of the FLP such that a relaxed version of the triangle inequality is satisfied. We say that a cost function $c$ is a squared metric, if, for all facilities $i$ and $i^{\prime}$, and cities $j$ and $j^{\prime}$, we have $\sqrt{c_{i j}} \leq \sqrt{c_{i j^{\prime}}}+\sqrt{c_{i^{\prime} j^{\prime}}}+\sqrt{c_{i^{\prime}} j}$. The particular case of the FLP that only considers instances with a squared metric is called SQuared Metric FLP, and is denoted by SMFLP. Notice that the SMFLP is a generalization of the $\mathrm{E}^{2}$ FLP and of the MFLP. Thus any approximation for the SMFLP is also an approximation for the $\mathrm{E}^{2} \mathrm{FLP}$ or the MFLP, and the inapproximability results for the MFLP are also valid for the SMFLP. The 9-approximation of Jain and Vazirani [12] applies also to the SMFLP and, to our knowledge, it has the best previously known approximation factor. The choice of squared metrics discourages excessive distances in the solution. This effect is important in several applications, such as $k$ means and classification problems.

Although there are several algorithms for the MFLP in the literature, there are very few works on the SMFLP. Nevertheless, one may try to solve an instance of the SMFLP using good algorithms designed for the MFLP. Since these algorithms and their analysis are based on the assumption of the triangle inequality, it is reasonable to expect that they generate good solutions also for the SMFLP. However, there is no trivial way to derive an approximation factor from the MFLP to the SMFLP, so each algorithm must be reanalyzed individually. In this paper, we analyze three primal-dual algorithms (the 1.861 and the 1.61 -approximation
algorithms of Jain et al. [10], and the 1.52-approximation of Mahdian, Ye, and Zhang [15]) and an LP-rounding algorithm (Chudak and Shmoys's algorithm [6] used in the 1.5-approximation of Byrka and Aardal [2]) when applied to SMFLP instances. We show that these algorithms achieve ratios of $2.87,2.43,2.17$, and 2.04 for the SMFLP, respectively. The last approximation factor is the best possible, as we show a 2.04 -inapproximability limit for the SMFLP. This was obtained by extending the metric case hardness results of Guha and Khuller [8].

The original analysis of the three primal-dual algorithms are based on the so called families of factor-revealing linear programs [10, 15]. The value of a computer calculated optimal solution for any program in this family gives a lower bound on the approximation factor. An upper bound, however, is obtained analytically by bounding the value of every program in this family, which requires long and tedious proofs. In this paper, we propose a way to obtain a new family of upper bound factor-revealing programs, as an alternative technique to achieve an upper bound. Now, the upper bound on the approximation factor is also obtained by a computer calculated solution of a single program. We note that, for the SMFLP, our factor-revealing programs are nonlinear, since the squared metric constraints contain square roots. We tackle this by replacing these constraints with an infinite set of linear constraints.

Recently, Mahdian and Yan [14] introduced the strongly factor-revealing linear programs. Our upper bound factor-revealing program is similar to a strongly factor-revealing program. The techniques involved in obtaining our program, however, are different. To obtain a strongly factor-revealing linear program, one projects a solution of an arbitrarily large linear program into a linear program with a constant number of variables, and guesses how to adjust the restrictions to obtain a feasible solution. In our approach, we define a candidate dual solution for a program with a fixed number of variables, and obtain an upper bound factor-revealing program directly in the form of a minimization program using only straightforward calculations. For the case of the SMFLP, we observed that calculating the dual upper bound program is easier than projecting the solutions on the primal. Also, we have considered the case of the MFLP, for which the obtained lower and upper bound factor-revealing programs converge.

Our contribution is two-folded. First, we make an important step towards generalizing the squared Euclidean distance and successfully analyze this generalization in the context of the FLP. Second, more importantly, we propose a new technique to systematically bound factor-revealing programs. This technique is used in the dual-fitting analysis of the primal-dual algorithms for both the SMFLP and the MFLP. We hope that this technique can also be used in the analysis of other dual-fitting algorithms analyzed through factor-revealing LPs.

The paper is organized as follows. In Section 2, we present the new technique analyzing the performance of the first algorithm of Jain et al. [10] for the SMFLP. Section 3] applies the technique in the analysis of the second algorithm of algorithm of Jain et al. [10]. Section 4 analyzes the performance of the algorithm of Mahdian, Ye, and Zhang [15] for the SMFLP. In Section 5], we present the analysis of the algorithm of Li [13], which happens to be optimal for SMFLP, according to the complexity result we present in Section 6. Finally we make some concluding remarks in Section 7 .

## 2 A new factor-revealing analysis

We analyze the algorithms of Jain et al. [10] using a new systematic factor-revealing technique. For each algorithm, Jain et al. [10] analysis uses a family of factor-revealing LPs parameterized by some $k$. The optimal value $z_{k}$ of the corresponding LP in the family is such that $\sup _{k \geq 1} z_{k}$ is the approximation factor of the algorithm. Thus each value $z_{k}$ is a lower bound on the approx-
imation factor and one has to analytically upper bound $\sup _{k \geq 1} z_{k}$ to obtain an approximation factor. This is a nontrivial analysis, since it is done by guessing a general suboptimal dual solution for the LP, usually inspired by numerically obtained dual LP solutions for small values of $k$. In this section, we show how to derive a family of upper bound factor-revealing programs (UPFRP) parameterized by some $t$, so that, for any given $t$, the optimal value $x_{t}$ of one such program is an upper bound on $\sup _{k \geq 1} z_{k}$. Obtaining a UPFRP and solving it using a computer is much simpler and more straightforward than using an analytical proof to obtain the approximation factor, since this does not include a guessing step and a manual verification of the feasibility of the solution. Additionally, as a property of the UPFRPs, we may tighten the obtained factor by solving the LP for larger values of $t$. In fact, in some cases (see Theorem 1 below), the lower and upper bound factor-revealing programs converge, that is, $\sup _{k \geq 1} z_{k}=\inf _{t \geq 1} x_{t}$.

We use a UPFRP to show that, when applied to SMFLP instances, the first algorithm of Jain et al. [10], denoted by $A 1$, is a 2.87 -approximation. For the sake of completeness, the algorithm is described in the following.

## Algorithm $A 1(C, F, c, f)$ [10]

1. Set $U:=C$, meaning that every facility starts unopened, and every city unconnected. Each city $j$ has some budget $\alpha_{j}$, initially 0 , and, at every moment, the budget that an unconnected city $j$ offers to some unopened facility $i$ equals to $\max \left(\alpha_{j}-c_{i j}, 0\right)$.
2. While $U \neq \emptyset$, the budget of each unconnected city is increased continuously until one of the following events occur:
(a) For some unconnected city $j$ and some open facility $i, \alpha_{j}=c_{i j}$. In this case, connect city $j$ to facility $i$ and remove $j$ from $U$.
(b) For some unopened facility $i, \sum_{j \in U} \max \left(\alpha_{j}-c_{i j}, 0\right)=f_{i}$. In this case, open facility $i$ and, for every unconnected city $j$ with $\alpha_{j} \geq c_{i j}$, connect $j$ to $i$ and remove $j$ from $U$.

The analysis presented by Jain et al. [10] uses the dual fitting method. That is, their algorithms produce not only a solution for the MFLP, but also a vector $\alpha=\left(\alpha_{1}, \ldots, \alpha_{|C|}\right)$ such that the value of the solution produced is equal to $\sum_{j} \alpha_{j}$. Moreover, for the first algorithm, following the dual fitting method, Jain et al. [10] proved that the vector $\alpha / 1.861$ is a feasible solution for the dual linear program presented as (3) in [10], concluding that the algorithm is a 1.861 -approximation for the MFLP. To present a similar analysis for the SMFLP, we use the same definitions and follow the steps of Jain et al. analysis. We start by adapting Lemma 3.2 from [10] for a squared metric.

Lemma 1. For every facility $i$, cities $j$ and $j^{\prime}$, and vector $\alpha$ obtained by the first algorithm of Jain et al. [10] given an instance of the SMFLP, $\sqrt{\alpha_{j}} \leq \sqrt{\alpha_{j^{\prime}}}+\sqrt{c_{i j^{\prime}}}+\sqrt{c_{i j}}$.

Proof. If $\alpha_{j} \leq \alpha_{j^{\prime}}$, the inequality obviously holds. So assume $\alpha_{j}>\alpha_{j^{\prime}}$. Let $i^{\prime}$ be the facility to which the algorithm connects city $j^{\prime}$. Thus $\alpha_{j^{\prime}} \geq c_{i^{\prime} j^{\prime}}$ and facility $i^{\prime}$ is open at time $\alpha_{j^{\prime}}<\alpha_{j}$. If $\alpha_{j}>c_{i^{\prime} j}$, then city $j$ would have connected to facility $i^{\prime}$ at some time $t \leq \max \left(\alpha_{j^{\prime}}, c_{i^{\prime} j}\right)<\alpha_{j}$, and $\alpha_{j}$ would have stopped growing then, a contradiction. Hence $\alpha_{j} \leq c_{i^{\prime} j}$. Furthermore, by the squared metric constraint, $\sqrt{c_{i^{\prime} j}} \leq \sqrt{c_{i^{\prime} j^{\prime}}}+\sqrt{c_{i j^{\prime}}}+\sqrt{c_{i j}}$. Therefore $\sqrt{\alpha_{j}} \leq \sqrt{\alpha_{j^{\prime}}}+\sqrt{c_{i j^{\prime}}}+$ $\sqrt{c_{i j}}$.

A facility $i$ is said to be $\gamma$-overtight for some positive $\gamma$ if, at the end of the algorithm,

$$
\begin{equation*}
\sum_{j} \max \left(\frac{\alpha_{j}}{\gamma}-c_{i j}, 0\right) \leq f_{i} . \tag{1}
\end{equation*}
$$

Observe that, if every facility is $\gamma$-overtight, then the vector $\alpha / \gamma$ is a feasible solution for the dual linear program presented as (3) in [10]. Jain et al. proved that, for the MFLP, every facility is 1.861 -overtight. We want to find a $\gamma$ for the SMFLP, as close to 1 as possible, for which every facility is $\gamma$-overtight.

Fix a facility $i$. Let us assume without loss of generality that $\alpha_{j} \geq \gamma c_{i j}$ only for the first $k$ cities. Following the lines of Jain et al. [10], we want to obtain the so called (lower bound) factor-revealing program. We define a set of variables $f, d_{j}$, and $\alpha_{j}$, corresponding to facility cost $f_{i}$, distance $c_{i j}$, and city contribution $\alpha_{j}$. Then, we capture the intrinsic properties of the algorithm using constraints over these variables. We assume without loss of generality that $\alpha_{1} \leq \cdots \leq \alpha_{k}$. Also, we use Lemma 3.3 from [10], that states that the total contribution offered to a facility at any time is at most its cost, that is, $\sum_{l=j}^{k} \max \left(\alpha_{j}-d_{l}, 0\right) \leq f$. Additionally, we have the inequalities from Lemma 1. Subject to all of these constraints, we want to find the minimum $\gamma$ such that the facility is $\gamma$-overtight. In terms of the defined variables, we want the maximum ratio $\sum_{j=1}^{k} \alpha_{j} /\left(f+\sum_{j=1}^{k} d_{j}\right)$. We obtain the following lower bound factor-revealing program:

$$
\begin{array}{lll}
z_{k}^{A 1}= & \max & \frac{\sum_{j=1}^{k} \alpha_{j}}{f+\sum_{j=1}^{k} d_{j}} \\
\text { s.t. } & \alpha_{j} \leq \alpha_{j+1} & \\
& \sqrt{\alpha_{j}} \leq \sqrt{\alpha_{l}}+\sqrt{d_{j}}+\sqrt{d_{l}} & \forall 1 \leq j<k  \tag{2}\\
& \sum_{l=j}^{k} \max \left(\alpha_{j}-d_{l}, 0\right) \leq f & \forall 1 \leq j \leq k \\
& \alpha_{j}, d_{j}, f \geq 0 & \forall 1 \leq j \leq k .
\end{array}
$$

The next lemma has the same statement of Lemma 3.4 in [10], but it refers to program (2). Since the proof is the same, we omit it.

Lemma 2. Let $\gamma=\sup _{k \geq 1} z_{k}^{A 1}$. Every facility is $\gamma$-overtight.
Therefore $\sup _{k \geq 1} z_{k}^{A 1}$ is an upper bound on the approximation factor of the algorithm for the SMFLP. A slight modification of the example presented in Theorem 3.5 of [10] shows that this upper bound is tight. (Take $c_{i j}=\left(\sqrt{d_{i}}+\sqrt{d_{j}}+\sqrt{\alpha_{i}}\right)^{2}$ if $k \geq i \neq j$.)

Although the constraints comming from Lemma 1 are defined by square roots, they are convex. Indeed, the next lemma, whose proof is presented in Appendix A shows that they can be expressed by an infinite set of linear inequalities.

Lemma 3. Given an instance of the SMFLP, for every facility $i$, cities $j$ and $j^{\prime}$, the vector $\alpha$ produced by the first algorithm of Jain et al. [10] is such that, for every positive $\beta, \gamma$, and $\delta$,

$$
\alpha_{j} \leq\left(1+\beta+\frac{1}{\gamma}\right) \alpha_{j^{\prime}}+\left(1+\gamma+\frac{1}{\delta}\right) c_{i j^{\prime}}+\left(1+\delta+\frac{1}{\beta}\right) c_{i j} .
$$

As a consequence, for any candidate solution for program (2) not satisfying a constraint comming from Lemma 1, there exists some violated linear inequality.

### 2.1 A first analysis

Our first step is to relax (2) into a linear program. For that, we adjust the objective function as in [10], and we approximate the inequalities with square roots using inequalities given by

Lemma3. For simplicity, here we will use only the inequalities corresponding to $\beta=\gamma=\delta=1$. With this, we will prove that $\sup _{k \geq 1} z_{k}^{41}$ is not greater than 3.236 . Later, we will improve the obtained result by using a whole set of inequalities from Lemma 3, and using a more standard factor-revealing analysis for the SMFLP. The relaxed lower factor-revealing linear program is:

$$
\begin{array}{cll}
\dot{w}_{k}=\max & \sum_{j=1}^{k} \alpha_{j} & \\
\text { s.t. } & f+\sum_{j=1}^{k} d_{j} \leq 1 & \\
& \alpha_{j} \leq \alpha_{j+1} & \forall 1 \leq j<k \\
& \alpha_{j} \leq 3 \alpha_{l}+3 d_{j}+3 d_{l} & \forall 1 \leq j, l \leq k  \tag{3}\\
& x_{j l} \geq \alpha_{j}-d_{l} & \forall 1 \leq j \leq l \leq k \\
& \sum_{l=j}^{k} x_{j l} \leq f & \forall 1 \leq j \leq k \\
& \alpha_{j}, d_{j}, f, x_{j l} \geq 0 & \forall 1 \leq j \leq l \leq k .
\end{array}
$$

As (3) is a relaxation of (2), we have that $z_{k}^{A 1} \leq \dot{w}_{k}$ and thus an upper bound on $\sup _{k \geq 1} \dot{w}_{k}$ is also an upper bound on $\sup _{k \geq 1} z_{k}^{A 1}$. Solving linear program (3) using CPLEX for $k=540$, we obtain the next lemma.

Lemma 4. $\sup _{k>1} \dot{w}_{k} \geq 3.220$.
To obtain an upper bound on their factor-revealing linear program, Jain et al. [10] presented a general dual solution of a relaxed version of the lower bound factor-revealing linear program. This solution is deduced from computational experiments and empirical results for small values of $k$. In their analysis, they guessed 2- and 3-step functions for a set of dual variables, and used a long verification to show that the value of such solution was not greater than 1.861. For the squared metric case, if we use step functions for the dual variables, the bound on the factor would be as bad as 3.625 . One can improve the obtained factor to 3.512 by guessing a piecewise function whose pieces are either constants or hyperboles.

Instead of looking for a good general dual solution, we use an alternative analysis and derive a linear minimization program from (3) whose feasible solutions are upper bounds on $\sup _{k \geq 1} \dot{w}_{k}$. Afterwards, we give an upper bound on the approximation factor by presenting a feasible solution for this program of value less than 3.236.

The idea is to determine a conical combination of the inequalities of (3) that imply inequality (1) for a $\gamma$ as small as possible. The linear minimization program will help us to choose the coefficients of such conical combination.

First, rewrite the third inequality of program (3), so that the right-hand side is zero. For each $j$ and $l$, we multiply the corresponding inequality by $\varphi_{j l}$. Denote by $A$ the sum of all these inequalities, that is,

$$
\sum_{j=1}^{k} \sum_{l=1}^{k} \varphi_{j l}\left(\alpha_{j}-3 \alpha_{l}-3 d_{l}-3 d_{j}\right) \leq 0
$$

The fourth and fifth inequalities of program (3) can be relaxed to the set of inequalities $\sum_{i=j}^{l}\left(\alpha_{j}-d_{i}\right) \leq f$, one for each $l$ such that $j \leq l \leq k$. For each $j$ and $l$, we multiply the corresponding inequality by $\theta_{j l}$ and denote by $B$ the inequality resulting of summing them up, that is,

$$
\sum_{j=1}^{k} \sum_{l=j}^{k} \theta_{j l} \sum_{i=j}^{l}\left(\alpha_{j}-d_{i}\right) \leq\left(\sum_{j=1}^{k} \sum_{l=j}^{k} \theta_{j l}\right) f .
$$

The coefficients of $\alpha_{j}$ in $A$ and $B$ are, respectively,

$$
\operatorname{coeff}_{A}\left[\alpha_{j}\right]=\sum_{l=1}^{k}\left(\varphi_{j l}-3 \varphi_{l j}\right) \quad \text { and } \quad \operatorname{coeff}_{B}\left[\alpha_{j}\right]=\sum_{l=j}^{k}(l-j+1) \theta_{j l}
$$

and the coefficients of $-d_{j}$ in $A$ and $B$ are, respectively,

$$
\operatorname{coeff}_{A}\left[-d_{j}\right]=\sum_{l=1}^{k} 3\left(\varphi_{j l}+\varphi_{l j}\right) \quad \text { and } \quad \operatorname{coeff}_{B}\left[-d_{j}\right]=\sum_{i=1}^{j} \sum_{l=j}^{k} \theta_{i l} .
$$

Now, we sum inequalities $A$ and $B$ and obtain a new inequality $C$ :

$$
\begin{equation*}
\sum_{j=1}^{k} \operatorname{coeff}_{C}\left[\alpha_{j}\right] \alpha_{j}-\sum_{j=1}^{k} \operatorname{coeff}_{C}\left[-d_{j}\right] d_{j} \leq \operatorname{coeff}_{C}[f] f \tag{4}
\end{equation*}
$$

We want to find values for $\gamma, \theta_{j l}$, and $\varphi_{j l}$ so that the corresponding coefficients of $C$ are such that inequality (4) implies, for sufficiently large $k$, that

$$
\begin{equation*}
\sum_{j=1}^{k} \alpha_{j}-\gamma \sum_{j=1}^{k} d_{j} \leq \gamma f \tag{5}
\end{equation*}
$$

Moreover, we want $\gamma$ as small as possible. To obtain inequality (5) from inequality (4), it is enough that, for each $j$, coefficient $\operatorname{coeff}_{C}\left[\alpha_{j}\right] \geq 1, \operatorname{coeff}_{C}\left[-d_{j}\right] \leq \gamma$, and $\operatorname{coeff}_{C}[f] \leq \gamma$. Hence, this can be expressed by the following linear program.

$$
\begin{array}{lll}
y_{k}=\min & \gamma & \\
\text { s.t. } & \operatorname{coeff}_{C}\left[\alpha_{j}\right] \geq 1 & \forall 1 \leq j \leq k \\
& \operatorname{coeff}_{C}\left[-d_{j}\right] \leq \gamma & \forall 1 \leq j \leq k \\
& \operatorname{coeff}_{C}[f] \leq \gamma &  \tag{6}\\
& \varphi_{j l} \geq 0 & \forall 1 \leq j, l \leq k \\
& \theta_{j l} \geq 0 & \forall 1 \leq j \leq l \leq k .
\end{array}
$$

The interested reader may observe that program (6) is the dual of a relaxed version of the lower bound factor-revealing linear program (3). Therefore, its optimal value is an upper bound on the optimal value of (3), that is, $\dot{w}_{k} \leq y_{k}$ for every $k$.

Lemma 5. $\sup _{k \geq 1} \dot{w}_{k} \leq 3.236$.
Proof. We start by observing that $\sup _{k \geq 1} \dot{w}_{k}$ does not decrease if we restrict attention to values of $k$ that are multiples of a fixed positive integer $t$. Indeed, for an arbitrary positive integer $p$, by making $t$ replicas of a solution of (3) for $k=p$, and scaling the variables by $1 / t$, we obtain a solution of (3) for $k=p t$, that is, we deduce that $\dot{w}_{p} \leq \dot{w}_{p t}$. So we may assume that $k$ has the form $k=p t$ with $p$ and $t$ positive integers, and our goal is to prove that $\dot{w}_{k} \leq 3.236$.

We will use program (6) to obtain a tight upper bound on $\dot{w}_{k}$. The size of this program however depends on $k$, which can be arbitrarily large. So we will use a scaling argument to create another linear minimization program with a fixed number (depending only on $t$ ) of variables, and obtain a feasible solution for program (6) from a solution for this smaller program. Then, we will show that the value of the generated solution for (6) is bounded by the value of the small solution.

Consider variables $\gamma^{\prime} \in \mathbb{R}_{+}, \varphi_{j l}^{\prime} \in \mathbb{R}_{+}$for $1 \leq j, l \leq t$, and $\theta_{j l}^{\prime} \in \mathbb{R}_{+}$for $1 \leq j \leq l \leq t$. For an arbitrary $n$, let $\hat{n}=\left\lceil\frac{n}{p}\right\rceil$. We obtain a candidate solution for program (6) by taking

$$
\begin{equation*}
\varphi_{j l}=\frac{\varphi_{\hat{j} \hat{l}}^{\prime}}{p}, \quad \theta_{j l}=\frac{\theta_{\hat{j} \hat{l}}^{\prime}}{p^{2}}, \quad \text { and } \quad \gamma=\gamma^{\prime} \tag{7}
\end{equation*}
$$

Let us calculate each coefficient of $C$ for this solution.

$$
\begin{aligned}
\operatorname{coeff}_{C}\left[\alpha_{j}\right]= & \sum_{l=1}^{k}\left(\varphi_{j l}-3 \varphi_{l j}\right)+\sum_{l=j}^{k}(l-j+1) \theta_{j l} \\
= & \sum_{l=1}^{k}\left(\frac{\varphi_{\hat{j} \hat{l}}^{\prime}}{p}-3 \frac{\varphi_{\hat{\jmath} \hat{j}}^{\prime}}{p}\right)+\sum_{l=j}^{k}(l-j+1) \frac{\theta_{\hat{j} \hat{l}}^{\prime}}{p^{2}} \\
\geq & \sum_{l=1}^{p t}\left(\frac{\varphi_{\hat{j} \hat{l}}^{\prime}}{p}-3 \frac{\varphi_{\hat{l} \hat{j}}^{\prime}}{p}\right)+\sum_{l=p \hat{j}+1}^{p t}(l-p \hat{j}) \frac{\theta_{\hat{j} \hat{l}}^{\prime}}{p^{2}} \\
= & \sum_{l^{\prime}=1}^{t} p\left(\frac{\varphi_{\hat{j} l^{\prime}}^{\prime}}{p}-3 \frac{\varphi_{l^{\prime} \hat{j}}^{\prime}}{p}\right)+\sum_{l^{\prime}=\hat{j}+1}^{t} \frac{\theta_{\hat{j} l^{\prime}}^{\prime}}{p^{2}} \sum_{i=0}^{p-1}\left(p l^{\prime}-i-p \hat{j}\right) \\
= & \sum_{l^{\prime}=1}^{t}\left(\varphi_{\hat{j}^{\prime}}^{\prime}-3 \varphi_{l^{\prime} \hat{j}}^{\prime}\right)+\sum_{l^{\prime}=\hat{j}+1}^{t} \frac{\theta_{\hat{j} l^{\prime}}^{\prime}}{p^{2}}\left(p^{2} l^{\prime}-\frac{p(p-1)}{2}-p^{2} \hat{j}\right) \\
\geq & \sum_{l^{\prime}=1}^{t}\left(\varphi_{\hat{j} l^{\prime}}^{\prime}-3 \varphi_{l^{\prime} \hat{j}}^{\prime}\right)+\sum_{l^{\prime}=\hat{j}+1}^{t}\left(l^{\prime}-\hat{j}-\frac{1}{2}\right) \theta_{\hat{j} l^{\prime}}^{\prime} . \\
\operatorname{coeff}_{C}\left[-d_{j}\right]= & \sum_{l=1}^{k} 3\left(\varphi_{j l}+\varphi_{l j}\right)+\sum_{i=1}^{j} \sum_{l=j}^{k} \theta_{i l} \\
& =\sum_{l=1}^{p t} 3\left(\frac{\varphi_{\hat{j} \hat{l}}^{\prime}}{p}+\frac{\varphi_{\hat{l} \hat{j}}^{\prime}}{p}\right)+\sum_{i=1}^{j} \sum_{l=j}^{p t} \frac{\theta_{\hat{i} \hat{l}}^{\prime}}{p^{2}} \\
& \leq \sum_{l^{\prime}=1}^{t} p \cdot 3\left(\frac{\varphi_{\hat{j} l^{\prime}}^{\prime}}{p}+\frac{\varphi_{l^{\prime} \hat{j}}^{\prime}}{p}\right)+\sum_{i^{\prime}=1}^{\hat{j}} p \cdot \sum_{l^{\prime}=\hat{j}}^{t} p \cdot \frac{\theta_{i^{\prime} l^{\prime}}^{\prime}}{p^{2}} \\
& =\sum_{l^{\prime}=1}^{t} 3\left(\varphi_{\hat{j} l^{\prime}}^{\prime}+\varphi_{l^{\prime} \hat{j}}^{\prime}\right)+\sum_{i^{\prime}=1}^{\hat{j}} \sum_{l^{\prime}=\hat{j}}^{t} \theta_{i^{\prime} l^{\prime}}^{\prime} .
\end{aligned}
$$

Now, we want to find the minimum value of $\gamma^{\prime}$ and values for $\varphi_{j l}^{\prime}$ and $\theta_{j l}^{\prime}$ such that the candidate solution for program (6) is feasible. We may define the following linear program, named the upper bound factor-revealing program.

$$
\begin{array}{lll}
\dot{x}_{t}=\min & \gamma^{\prime} & \\
\mathrm{s.t.} & \sum_{l=1}^{t}\left(\varphi_{j l}^{\prime}-3 \varphi_{l j}^{\prime}\right)+\sum_{l=j+1}^{t}\left(l-j-\frac{1}{2}\right) \theta_{j l} \geq 1 & \forall 1 \leq j \leq t \\
& \sum_{l=1}^{t} 3\left(\varphi_{j l}^{\prime}+\varphi_{l j}^{\prime}\right)+\sum_{i=1}^{j} \sum_{l=j}^{t} \theta_{i l}^{\prime} \leq \gamma^{\prime} & \forall 1 \leq j \leq t  \tag{8}\\
& \sum_{j=1}^{t} \sum_{l=j}^{t} \theta_{j l}^{\prime} \leq \gamma^{\prime} & \\
& \varphi_{j l}^{\prime} \geq 0 & \forall 1 \leq j, l \leq t \\
& \theta_{j l}^{\prime} \geq 0 & \forall 1 \leq j \leq l \leq t .
\end{array}
$$

Consider an optimal solution for program (6). Replacing it in (4), that is, in inequality $C$, we obtain $\sum_{j=1}^{k} \alpha_{j}-\gamma \sum_{j=1}^{k} d_{j} \leq \gamma f$. Thus, $\dot{w}_{k} \leq \gamma=y_{k}$. Now, consider an optimal solution for program (8) and the corresponding generated solution for program (6). We obtain $y_{k} \leq \gamma=$ $\gamma=\dot{x}_{t}$ and conclude that $\dot{w}_{k} \leq \dot{x}_{t}$, and that holds for every positive integer $k$.

Using CPLEX to solve program (8), we obtained $\dot{x}_{800} \approx 3.23586<3.236$, and this concludes the proof of Lemma 5 .

### 2.2 An improved factor-revealing analysis

In Lemma 5, we obtained the minimization program (8) from a conical combination of constraints from program (3) that bounds the approximation factor. This process is similar to obtaining the dual and using a scaling argument. Indeed, we propose a systematic way to obtain an upper bound factor-revealing program.

Consider the dual program of a traditional maximization factor-revealing linear program for some $k$. Take $k$ in the form $k=p t$, for a fixed $t$. We want to create a minimization program that mimics the dual, but depends only on $t$ and bounds the dual optimal value for every $k$. The idea is to constrain the variables of the small program to obtain a feasible solution for the dual program. To obtain a linear program independent of $k$, we scale the variables by $p$. The strategy to obtain an upper bound factor-revealing program may be summarized as follows:

1. obtain the dual $P(k)$ of the lower bound factor-revealing linear program;
2. consider a block variable $x_{i}^{\prime}$ for variables $x_{(i-1) p+1}, \ldots, x_{(i-1) p+p}$ of $P(k)$;
3. identify each variable $x_{i}$ with the block variable $x_{\lceil i / p\rceil}^{\prime}$ scaled by $p$;
4. replace identified variables in $P(k)$, canceling factors $p$.

Denote the resulting program by $P^{\prime}(t)$. If $P^{\prime}(t)$ depends only on $t$, both in number of variables and constraints, then any feasible solution of $P^{\prime}(t)$ is an upper bound on the solution of $P(p t)$ for every $p$. Also, if it is the case that the value of $P(k)$ is not greater than the value of $P(k t)$, for every $t$, then a solution of $P^{\prime}(t)$ for any $t$ is also a bound on the approximation factor. Therefore, we call $P^{\prime}(t)$ an upper bound factor-revealing program.

Although program (2) is nonlinear, we can still use the presented strategy. If the nonlinear constraint is convex, we can approximate it by using a set of linear inequalities, and calculate the dual normally. In order to derive a better upper bound factor-revealing linear program, this time we will use a whole set of linear inequalities. Consider $m$ tuples $\left(\beta_{i}, \gamma_{i}, \delta_{i}\right)$ of positive real numbers and $B_{i}=1+\beta_{i}+\frac{1}{\gamma_{i}}, C_{i}=1+\gamma_{i}+\frac{1}{\delta_{i}}, D_{i}=1+\delta_{i}+\frac{1}{\beta_{i}}$ for $1 \leq i \leq m$. Using Lemma 3, we insert inequalities corresponding to the given tuples, replacing the nonlinear constraint, and obtain that $z_{k}^{A 1} \leq w_{k}^{A 1}$, where $w_{k}^{A 1}$ is given by

$$
\begin{array}{lll}
w_{k}^{\text {A1 }}=\max & \sum_{j=1}^{k} \alpha_{j} & \\
\text { s.t. } & f+\sum_{j=1}^{k} d_{j} \leq 1 & \\
& \alpha_{j} \leq \alpha_{j+1} & \forall 1 \leq j<k \\
& \alpha_{j} \leq B_{i} \alpha_{l}+C_{i} d_{j}+D_{i} d_{l} & \forall 1 \leq j, l \leq k, \quad 1 \leq i \leq m  \tag{9}\\
& x_{j l} \geq \alpha_{j}-d_{l} & \forall 1 \leq j \leq l \leq k \\
& \sum_{l=j}^{k} x_{j l} \leq f & \forall 1 \leq j \leq k \\
& \alpha_{j}, d_{j}, f, x_{j l} \geq 0 & \forall 1 \leq j \leq l \leq k .
\end{array}
$$

The following lemma gives a lower bound on the approximation factor of the algorithm for the SMFLP using a cutting plane insertion strategy.

Lemma 6. $\sup _{k \geq 1} z_{k}^{A 1} \geq 2.86$.
Proof. Although program (2) contains nonlinear constraints, we may use linear program packages to solve it. We start by solving program (9) with a fixed number of inequalities. Then, we employ a cutting plane insertion strategy: if the obtained solution violates some inequality with square roots of (2), we derive a cutting plane using Lemma 3, and resolve the linear program with this additional constraint. Using CPLEX with the cutting plane strategy, we obtained $z_{700}^{A 1} \approx 2.86099>2.86$.

Now, we can bound the approximation factor of the algorithm using an upper bound factorrevealing program.

Lemma 7. $\sup _{k \geq 1} z_{k}^{A 1} \leq 2.87$.
Proof. It is easy to see that, for program (9), as in the proof of Lemma 5, we can restrict attention to values of $k$ that are multiples of a fixed positive integer $t$, that is, $z_{k}^{A 1} \leq w_{k t}^{A 1}$, for every positive integer $t$. So we assume that $k$ has the form $k=p t$, with $p$ and $t$ positive integers. The dual of the linear program $(9)$ is

$$
\begin{array}{lll}
w_{k}^{A 1}=\min & \gamma & \\
\text { s.t. } & a_{j}-a_{j-1}+\sum_{i=1}^{m} \sum_{l=1}^{k} c_{j l i}-\sum_{i=1}^{m} B_{i} \sum_{l=1}^{k} c_{l j i}+\sum_{l=j}^{k} e_{j l} \geq 1 & \forall 1 \leq j \leq k \\
& \sum_{i=1}^{m} C_{i} \sum_{l=1}^{k} c_{j l i}+\sum_{i=1}^{m} D_{i} \sum_{l=1}^{k} c_{l j i}+\sum_{l=1}^{j} e_{l j} \leq \gamma & \forall 1 \leq j \leq k  \tag{10}\\
& \sum_{j=1}^{k} h_{j} \leq \gamma & \\
& e_{j l} \leq h_{j} & \forall 1 \leq j \leq l \leq k \\
& a_{0}=a_{k}=0, a_{j}, h_{j}, e_{j l}, c_{j l i} \geq 0 &
\end{array}
$$

We can derive the upper bound factor-revealing linear program. We would like to define variables as in equation (7). Just using a scale factor is not sufficient to preserve the variables $a_{j}$ in program (10). The variables $a_{j}$ correspond to the ordering restrictions of primal variables $\alpha_{j}$ in program (9), and computational experiments have indicated that removing such restrictions does not change the optimal value significantly, for large values of $k$. So, we could just set $a_{j}=0$ for all $j$. However, we want to preserve such restrictions, as they will shortly be needed to prove Lemma 9. To do this, we can interpolate the variables of the upper bound factorrevealing program to obtain the variables of the lower bound program. Again, we group sets of variables based on their indices. For that, we denote the group of a variable of index $n$ as $\hat{n}$. Let $\hat{n}=\left\lceil\frac{n}{p}\right\rceil$ and consider prime variables $\gamma^{\prime}, a_{j}^{\prime}, c_{j l i}^{\prime}, e_{j l i}^{\prime}, h_{j}^{\prime}$. We obtain a candidate solution for program 10) by defining

$$
\begin{equation*}
\gamma=\gamma^{\prime}, a_{j}=p a_{\hat{j}}^{\prime}(p \hat{j}-j)\left(a_{\hat{j}}^{\prime}-a_{\hat{j}-1}^{\prime}\right), c_{j l i}=\frac{c_{\hat{j} \hat{l} i}^{\prime}}{p}, e_{j l}=\frac{e_{\hat{j} \hat{l}}^{\prime}}{p}, \text { and } h_{j}=\frac{h_{\hat{j}}^{\prime}}{p} \tag{11}
\end{equation*}
$$

In the following, we will use definition (11) to obtain a candidate solution for program (10) from a small set of prime variables. Then, for each constraint of program (10), we obtain the expression formed by the dependent terms, and calculate it as a function of the considered variables. Notice that there is an expression for each primal variable of program (9). These expressions are analogous to the primal variables coefficients used in Lemma5, thus, for each
primal variable $x$, we say that this is the coefficient expression for $x$, and we will denote it by coeff $[x]$.

Now we create the minimization upper bound factor-revealing program. The objective value is obtained by applying definition (11) to the objective value of program (10). Then, for each group of coefficient expressions that has the same value, we include a constraint in the upper bound program that bounds the expression by the independent term. Notice that each upper bound factor-revealing linear program constraint may correspond to an arbitrarily large number of constraints of the factor-revealing linear program. In the following, we calculate and bound each coefficient expression.

First notice that $a_{j}-a_{j-1}=a_{\hat{j}}^{\prime}-a_{\hat{j}-1}^{\prime}$. To see this, it is enough to use definition (11) and consider the cases $\hat{j}=(\hat{j-1})$, and $\hat{j}=(\hat{j-1})+1$. Now we have:

$$
\begin{aligned}
\operatorname{coeff}\left[\alpha_{j}\right] & =a_{j}-a_{j-1}+\sum_{i=1}^{m} \sum_{l=1}^{k} c_{j l i}-\sum_{i=1}^{m} B_{i} \sum_{l=1}^{k} c_{l j i}+\sum_{l=j}^{k} e_{j l} \\
& =a_{\hat{j}}^{\prime}-a_{\hat{j}-1}^{\prime}+\sum_{i=1}^{m} \sum_{l=1}^{p t} \frac{c_{\hat{j} \hat{l} i}^{\prime}}{p}-\sum_{i=1}^{m} B_{i} \sum_{l=1}^{p t} \frac{c_{\hat{l} \hat{j} i}^{\prime}}{p}+\sum_{l=j}^{p t} \frac{e_{\hat{j} \hat{l}}^{\prime}}{p} \\
& \geq a_{\hat{j}}^{\prime}-a_{\hat{j}-1}^{\prime}+\sum_{i=1}^{m} \sum_{l^{\prime}=1}^{t} p \frac{c_{\hat{j} l^{\prime} i}^{\prime}}{p}-\sum_{i=1}^{m} B_{i} \sum_{l^{\prime}=1}^{t} p \frac{c_{l^{\prime} \hat{j} i}^{\prime}}{p}+\sum_{l^{\prime}=\hat{j}+1}^{t} p \frac{e_{\hat{j} l^{\prime}}^{\prime}}{p} \\
& =a_{\hat{j}}^{\prime}-a_{\hat{j}-1}^{\prime}+\sum_{i=1}^{m} \sum_{l^{\prime}=1}^{t} c^{\prime}{\hat{j} l^{\prime} i}^{m}-\sum_{i=1}^{m} B_{i} \sum_{l^{\prime}=1}^{t} c_{l^{\prime} \hat{j} i}^{\prime}+\sum_{l^{\prime}=\hat{j}+1}^{t} e_{\hat{j} l^{\prime}}^{\prime} \geq 1 .
\end{aligned}
$$

$$
\operatorname{coeff}\left[d_{j}\right]=\gamma-\sum_{i=1}^{m} C_{i} \sum_{l=1}^{k} c_{j l i}-\sum_{i=1}^{m} D_{i} \sum_{l=1}^{k} c_{l j i}-\sum_{l=1}^{j} e_{j l}
$$

$$
=\gamma^{\prime}-\sum_{i=1}^{m} C_{i} \sum_{l=1}^{p t} \frac{c_{\hat{j} \hat{j} i}^{\prime}}{p}-\sum_{i=1}^{m} D_{i} \sum_{l=1}^{p t} \frac{c_{1}^{\prime} \hat{j} \hat{i}}{p}-\sum_{l=1}^{j} \frac{e^{\prime} \hat{j} \hat{l}}{p}
$$

$$
\geq \gamma^{\prime}-\sum_{i=1}^{m} C_{i} \sum_{l^{\prime}=1}^{t} p \frac{c_{\hat{j} l^{\prime} i}^{\prime}}{p}-\sum_{i=1}^{m} D_{i} \sum_{l^{\prime}=1}^{t} p \frac{c_{l^{\prime} \hat{j} i}^{\prime}}{p}-\sum_{l^{\prime}=1}^{\hat{j}} p \frac{e_{\hat{j^{\prime}}}^{\prime}}{p}
$$

$$
=\gamma^{\prime}-\sum_{i=1}^{m} C_{i} \sum_{l^{\prime}=1}^{t} c_{\hat{j} l^{\prime} i}^{\prime}-\sum_{i=1}^{m} D_{i} \sum_{l^{\prime}=1}^{t} c_{l^{\prime} \hat{j} i}^{\prime}-\sum_{l^{\prime}=1}^{\hat{j}} e_{\hat{j l^{\prime}}}^{\prime} \geq 0
$$

$$
\operatorname{coeff}[f]=\gamma-\sum_{j=1}^{k} h_{j}=\gamma^{\prime}-\sum_{j=1}^{p t} \frac{h_{\hat{j}}^{\prime}}{p}=\gamma^{\prime}-\sum_{j^{\prime}=1}^{t} p \frac{h_{j^{\prime}}^{\prime}}{p}=\gamma^{\prime}-\sum_{j^{\prime}=1}^{t} h_{j^{\prime}}^{\prime} \geq 0 .
$$

$$
\operatorname{coeff}\left[x_{j l}\right]=h_{j}-e_{j l}=\frac{h_{\hat{j}}^{\prime}}{p}-\frac{e_{\hat{j} \hat{l}}^{\prime}}{p} \geq 0
$$

We notice that, for each primal variable, the constraint for its coefficient expression is equivalent to the constraint of any other primal variable in the same group. For example, for any pair $\alpha_{j}$ and $\alpha_{l}$ such that $\hat{j}=\hat{l}$, we need to add only one constraint to the upper bound factor-revealing program; therefore, we need only $t$ constraints for this kind of primal variable. We remark that
the constraint obtained for coeff $\left[x_{j l}\right]$ does not depend on $p$. Conjoining all different constraints, and fixing variables $a_{1}^{\prime}$ and $a_{t}^{\prime}$ to zero, we obtain program (12).

$$
\begin{array}{llll}
x_{t}^{A 1}=\min & \gamma & & \\
\text { s.t. } & a_{j}-a_{j-1}+\sum_{i=1}^{m} \sum_{l=1}^{t} c_{j l i}-\sum_{i=1}^{m} B_{i} \sum_{l=1}^{t} c_{l j i}+\sum_{l=j+1}^{t} e_{j l} \geq 1 & \forall 1 \leq j \leq t \\
& \sum_{i=1}^{m} C_{i} \sum_{l=1}^{t} c_{j l i}+\sum_{i=1}^{m} D_{i} \sum_{l=1}^{t} c_{l j i}+\sum_{l=1}^{j} e_{l j} \leq \gamma & \forall 1 \leq j \leq t  \tag{12}\\
& \sum_{j=1}^{t} h_{j} \leq \gamma & & \\
& e_{j l} \leq h_{j} & \forall 1 \leq j \leq l \leq t \\
& a_{0}=a_{t}=0, a_{j}, h_{j}, e_{j l}, c_{j l i} \geq 0 & \forall 1 \leq j, l \leq t \\
& & 1 \leq i \leq m .
\end{array}
$$

Now, we want to use Lemma 3 and choose a set of tuples $(\beta, \gamma, \delta)$ so that the squared metric is minimally relaxed. To accommodate the premises of Lemma 3, we solve the dual of the upper bound factor-revealing LP, so we may use the same cutting plane strategy used in Lemma 6 . The dual is given in the following.

$$
\begin{array}{lll}
x_{t}^{A 1}=\max & \sum_{j=1}^{t} \alpha_{j} & \\
\mathrm{s.t.} & f+\sum_{j=1}^{t} d_{j} \leq 1 & \\
& \alpha_{j} \leq \alpha_{j+1} & \forall 1 \leq j<t \\
& \alpha_{j} \leq B_{i} \alpha_{l}+C_{i} d_{j}+D_{i} d_{l} & \forall 1 \leq j, l \leq t, \quad 1 \leq i \leq m  \tag{13}\\
& x_{j l} \geq \alpha_{j}-d_{l} & \forall 1 \leq j<l \leq t \\
& \sum_{l=j}^{t} x_{j l} \leq f & \forall 1 \leq j \leq t \\
& \alpha_{j}, d_{j}, f, x_{j l} \geq 0 & \forall 1 \leq j \leq l \leq t .
\end{array}
$$

Using the cutting plane strategy with CPLEX we obtain $x_{700}^{A 1} \approx 2.8697<2.87$.
If we apply this analysis for the metric case, we obtain an upper bound factor-revealing program similar to program (13). The only difference is that, for the metric case, there are no coefficients $B_{l}, C_{l}$, and $D_{l}$. We use this modified linear program to tighten the approximation factor for the metric case.

Lemma 8. For the MFLP, the approximation factor of $A 1$ [10] is between 1.814 and 1.816 .
Proof. Let $\hat{z}_{k}^{A 1}$ be the optimal value of the lower bound factor-revealing program (5) in [10]. The corresponding upper bound factor-revealing program is:

$$
\begin{array}{lll}
\hat{x}_{t}^{A 1}=\max & \sum_{j=1}^{t} \alpha_{j} & \\
\text { s.t. } & f+\sum_{j=1}^{t} d_{j} \leq 1 & \\
& \alpha_{j} \leq \alpha_{j+1} & \forall 1 \leq j<t \\
& \alpha_{j} \leq \alpha_{l}+d_{j}+d_{l} & \forall 1 \leq j, l \leq t  \tag{14}\\
& x_{j l} \geq \alpha_{j}-d_{l} & \forall 1 \leq j<l \leq t \\
& \sum_{l=j}^{t} x_{j l} \leq f & \forall 1 \leq j \leq t \\
& \alpha_{j}, d_{j}, f, x_{j l} \geq 0 & \forall 1 \leq j \leq l \leq t .
\end{array}
$$

Numerical computations using CPLEX show that $\hat{z}_{1000}^{41} \approx 1.81412>1.814$, and that $\hat{x}_{1000}^{41} \approx$ $1.81584<1.816$.

We notice that the only difference between the upper and lower bound factor-revealing programs is that the upper bound factor-revealing program does not contain the restrictions $\alpha_{j}-d_{j} \leq x_{j j}$ for all $j$. We exploit the similarity between these programs to bound the gap between their optimal values. The following lemma is valid for both the metric and squared metric cases.

Lemma 9. Let $z_{k}^{A 1}$ be the optimal value of the lower bound factor-revealing program (9) (program (5) in [10]) and let $(\boldsymbol{\alpha}, \mathbf{d}, \mathbf{x}, \mathbf{f})$ be an optimal solution for program (13) (respectively program (14)) with cost value $x_{k}^{A 1}$. If $\varepsilon=\max _{j}\left\{\alpha_{j}-d_{j}\right\}$, then $z_{k}^{A 1} \geq \frac{1}{1+\varepsilon} x_{k}^{A 1}$.
Proof. Let $f^{\prime}=f+\varepsilon$ and $x^{\prime}$ be such that $x_{j l}^{\prime}=x_{j l}$ if $j \neq l$, and $x_{j j}^{\prime}=\max \left\{0, \alpha_{j}-d_{j}\right\} \geq 0=x_{j j}$. Observe that $\left(\boldsymbol{\alpha}, \mathbf{d}, \mathbf{x}^{\prime}, \mathbf{f}^{\prime}\right)$ has objective value $x_{k}^{A 1}$ and is a feasible solution for the lower bound factor-revealing program (9), except that it might violate the first restriction of program (9) (program (5) in [10], respectively). Indeed, it might be the case that $1<f^{\prime}+\sum_{j=1}^{k} d_{j} \leq 1+\varepsilon$. Now, it is enough to multiply each variable by $\frac{1}{1+\varepsilon}$, and obtain a feasible solution.

From the last lemma, it is clear that the upper and lower bound factor-revealing programs yield close values, except for an error factor that depends only on the variable values of an optimal solution for the upper bound factor-revealing program. Since the optimal value decreases as the number of variables $k$ increases, it is reasonable to expect that the value of both factorrevealing programs become very close as $k$ tends to infinity. Indeed, for the metric case, it is easy to show that this error vanishes as $k$ goes to infinity and, therefore, the upper bound and the lower bound factor-revealing programs converge to the same value, as $k$ goes to infinity.

Theorem 1. Let $\hat{z}_{k}^{41}$ be as in program (5) in [10] and let $\hat{x}_{k}^{41}$ be as in program (14). Then $\sup _{k \geq 1} \hat{z}_{k}^{A 1}=\inf _{k \geq 1} \hat{x}_{k}^{A 1}$.

Proof. First notice that, if we double a dual solution of program (14), then the obtained solution for the corresponding minimization upper bound factor-revealing is still feasible. Therefore, we may assume that $k$ is arbitrarily large. Consider an optimal solution of program (14). We have that $\alpha_{j}-d_{j} \leq \alpha_{l}+d_{l}$, for every $j$ and $l$. Let $j$ be such that $\varepsilon=\alpha_{j}-d_{j}$ is maximum and add up these inequalities for all $l$. We get $k \varepsilon=k\left(\alpha_{j}-d_{j}\right)=\sum_{l=1}^{k}\left(\alpha_{j}-d_{j}\right) \leq \sum_{l=1}^{k}\left(\alpha_{l}+d_{l}\right) \leq \hat{x}_{k}^{A 1}+1 \leq$ $1.816+1$. From Lemmas 8 and 9 , we get that $\hat{x}_{k}^{A 1} \geq \hat{z}_{k}^{A 1} \geq \frac{1}{1+\varepsilon} \hat{x}_{k}^{A 1} \geq \frac{1}{1+2.816 / k} \hat{x}_{k}^{A 1}$. Taking the limit as $k$ goes to infinity, we get that $\sup _{k \geq 1} \hat{z}_{k}^{A 1}=\inf _{k \geq 1} \hat{x}_{k}^{A 1}$.

It would be nice to bound the values of the variables of program (13), as this would suffice to show that the factor-revealing programs also converge for the squared metric case. Since the coefficients of the squared triangle inequality involved in program (13) are all greater than one, we cannot use the same approach as in Theorem1. Although experiments suggest that the value of variable $\alpha_{k}$ in an optimal solution decreases as $k$ increases, it does not seem trivial to determine whether $\alpha_{k}$ vanishes when $k$ goes to infinity.

## 3 Analysis of the second algorithm

In this section, we analyze the second algorithm of Jain et al. [10] for the squared metric case. The algorithm is essentially the same as Algorithm A1, but each connected city keeps contributing to unopened facilities. The contribution of a connected city $j$ to an unopened facility $i$ is the budget that the city would save if facility $i$ were opened. The algorithm, that is denoted by $A 2$, is described in the following.

## Algorithm $A 2(C, F, c, f)$ [10]

1. Set $U:=C$, meaning that every facility starts unopened, and every city unconnected. Each city $j$ has some budget $\alpha_{j}$, initially 0 . At every moment, for each unopened facility $i$, if city $j$ is unconnected, then $j$ offers $\max \left(\alpha_{j}-c_{i j}, 0\right)$ to $i$, and, if city $j$ is connected to facility $i^{\prime}$, then $j$ offers $\max \left(c_{i^{\prime} j}-c_{i j}, 0\right)$ to $i$.
2. While $U \neq \emptyset$, the budget of each unconnected city is increased continuously until one of the following events occur:
(a) For some unconnected city $j$ and some open facility $i, \alpha_{j}=c_{i j}$. In this case, connect city $j$ to facility $i$ and remove $j$ from $U$.
(b) For some unopened facility $i$, the total offer $i$ receives from the cities equals the cost $f_{i}$ of opening $i$. In this case, open facility $i$, connect to $i$ each city $j$ with a positive offer to $i$, and remove each connected city from $U$.

For the metric case, the approximation factor is 1.61 . With a completely analogous reasoning, we obtain the corresponding factor-revealing program (15). The variables are the same as in program (2). The new variable $r_{j l}$ corresponds to the budget $\alpha_{j}$ if city $j$ is connected at the same time as city $l$, or corresponds to the distance from $j$ to the facility to which $j$ is connected just before $l$ is connected.

$$
\begin{array}{lll}
z_{k}^{A 2}=\max & \frac{\sum_{j=1}^{k} \alpha_{j}}{f+\sum_{j=1}^{k} d_{j}} & \\
\text { s.t. } & \alpha_{j} \leq \alpha_{j+1} & \forall 1 \leq j<k \\
& r_{j l} \geq r_{j, l+1} & \forall 1 \leq j<l<k \\
& \sqrt{\alpha_{l}} \leq \sqrt{r_{j l}}+\sqrt{d_{l}}+\sqrt{d_{j}} & \forall 1 \leq j<l \leq k  \tag{15}\\
& \sum_{j=1}^{l-1} \max \left(r_{j l}-d_{j}, 0\right)+\sum_{j=l}^{k} \max \left(\alpha_{l}-d_{j}, 0\right) \leq f & \forall 1 \leq l \leq k \\
& \alpha_{j}, d_{j}, f, r_{j, l} \geq 0 & \forall 1 \leq j \leq l \leq k .
\end{array}
$$

We repeat the previous analysis to give lower and upper bounds on the approximation factor of the second algorithm for the SMFLP.

Lemma 10. $2.415 \leq \sup _{k \geq 1} z_{k}^{A 2} \leq 2.425$.
Proof. First, we obtain an upper bound factor-revealing program. See details in Appendix B. This program is exactly the same as program (15), except that the fourth constraint is replaced with

$$
\sum_{j=1}^{l-1} \max \left(r_{j l}-d_{j}, 0\right)+\sum_{j=l+1}^{k} \max \left(\alpha_{l}-d_{j}, 0\right) \leq f .
$$

Let $x_{k}^{A 2}$ be the optimal value of such a program. With CPLEX we get that $z_{500}^{A 2} \approx 2.41565>$ 2.415 , and that $x_{500}^{42} \approx 2.42473<2.425$.

Solving the upper bound factor-revealing LP obtained for the MFLP for $k=500$, we may show that the approximation factor of $A 2$ [10] is 1.602 . The lower bound factor-revealing program and the maximization upper bound factor-revealing program are essentially the same, except for the extra terms of the kind $\max \left(\alpha_{l}-d_{l}\right)$. Therefore, Lemma 9 also holds for such programs. For the metric case, using a similar analysis to that of Theorem 1, one can show that the lower and the upper bound factor-revealing programs converge.

Theorem 2. Let $\hat{z}_{k}^{A 2}$ be as in program (25) in [10] and let $\hat{x}_{k}^{A 2}$ be the optimal value of the corresponding upper bound factor-revealing program obtained by removing the terms of the kind $\max \left(\alpha_{l}-d_{l}\right)$ from the fourth restriction. Then $\sup _{k \geq 1} z_{k}^{A 2}=\inf _{k \geq 1} x_{k}^{A 2}$.

## 4 Scaling and greedy augmentation

Algorithm $A 2$ can be analyzed as a bi-factor approximation algorithm. The analysis uses a factor-revealing linear program, and is similar to the previous analysis. Mahdian, Ye, and Zhang [15] observed that, due to the asymmetry between the approximation guarantee for the opened facilities cost and the connections cost, Algorithm A2 may be used to open facilities that are very economical. This gives rise to a two-phase algorithm, denoted here by $\operatorname{A3}(\delta)$, based on scaling the cost of facilities by a constant $\delta \geq 1$, and on the greedy augmentation technique introduced by Guha and Khuller [7]. The first phase opens the most economical facilities, and the second phase greedily includes facilities that reduce the cost of the solution.

## Algorithm $A 3(\delta)(C, F, c, f)$ [15]

1. Scaling:
(a) Scale the facility costs by a factor $\delta$.
(b) Run Algorithm $A 2$ on the scaled instance.
2. Greedy augmentation:

While there are facilities that, if open, reduce the total cost:
(a) Compute the gain $g_{i}$ of opening each unopened facility $i$.
(b) Open a facility $i$ that maximizes the ratio $\frac{g_{i}}{f_{i}}$.

In [15], a factor-revealing linear program is used to analyze $\operatorname{Algorithm} \operatorname{A3}(\boldsymbol{\delta})$ with a somewhat different, but equivalent, greedy augmentation procedure. This was used to balance a bi-factor from Algorithm A2 for the MFLP. As noticed by Byrka and Aardal [2], this analysis is not restricted to Algorithm A2, and applies to any bi-factor approximation for the FLP. Therefore, since it does not depend on the cost function being a metric, we can use it to balance a bi-factor approximation for the squared metric case. This result is precisely stated as follows.

Lemma 11 ([15]). Consider a $\left(\gamma_{f}, \gamma_{c}\right)$-approximation for the FLP. Then, for every $\delta \geq 1$, Algorithm $\mathrm{A} 3(\delta)$ is a $\left(\gamma_{f}+\ln \delta+\varepsilon, 1+\frac{\gamma_{c}-1}{\delta}\right)$-approximation for the FLP.

For the metric case, it has been shown that Algorithm A2 is a (1.11, 1.78)-approximation. This and Lemma 11 give a 1.52 -approximation for the MFLP. For the SMFLP, we present an analysis based on an upper bound factor-revealing program. Using straightforward calculations, we may obtain the following:

Lemma 12. Let $\gamma_{f} \geq 1$ be a fixed value and let $\gamma_{c}=x_{k}^{A 2 c}$, where

$$
\begin{array}{lll}
x_{k}^{A 2 c}=\max & \frac{\sum_{j=1}^{k} \alpha_{j}-\gamma_{f} f}{\sum_{j=1}^{k} d_{j}} & \\
\text { s.t. } & \alpha_{l} \leq \alpha_{l+1} & \forall 1 \leq l<k \\
& r_{j l} \geq r_{j, l+1} & \forall 1 \leq j<l<k \\
& \sqrt{\alpha_{l}} \leq \sqrt{r_{j l}}+\sqrt{d_{l}}+\sqrt{d_{j}} & \forall 1 \leq j<l \leq k  \tag{16}\\
& \sum_{j=1}^{l-1} \max \left(r_{j l}-d_{j}, 0\right)+\sum_{j=l+1}^{k} \max \left(\alpha_{l}-d_{j}, 0\right) \leq f & \forall 1 \leq l \leq k \\
& \alpha_{j}, d_{j}, f, r_{j l} \geq 0 & \forall 1 \leq j \leq l \leq k .
\end{array}
$$

Then, if $\gamma_{c}<\infty$, Algorithm A2 is a $\left(\gamma_{f}, \gamma_{c}\right)$-approximation for the SMFLP.
The only difference between program (16) and the corresponding lower bound factorrevealing program is the extra term $\max \left(\alpha_{l}-d_{l}, 0\right)$ in the lower bound program, which is not in the fourth constraint of program (16). Again, having a bound for this term is sufficient to show convergence of the upper and lower bound factor-revealing programs. For the metric case, this can be done easily. Notice that we may assume $r_{j l} \leq \alpha_{j}$, so, using a similar analysis to that of Theorem 2, one can show that, if $z_{k}$ and $x_{k}$ are solutions for the lower and upper bound programs respectively, then $x_{k}-\gamma_{f} \varepsilon \leq z_{k} \leq x_{k}$, for some $\varepsilon=\mathrm{O}\left(\frac{1}{k}\right)$.

We observe that program (16) is unbounded for values of $\gamma_{f}$ close to one. This happens also for the corresponding lower bound factor-revealing program. This is in contrast to the factorrevealing programs obtained for the metric case, for which we know that Algorithm A2 is a (1,2)-approximation. In this case, the lower bound program is always bounded, but the upper bound program is unbounded for $\gamma_{f}=1$, or for values close to one. It would be interesting to strengthen this upper bound factor-revealing program, so that it could also be used in the analysis also for $\gamma_{f}=1$.

## Theorem 3. Algorithm A3 is a 2.17-approximation for the SMFLP.

Proof. Consider program (16) for $\gamma_{f}=1.45$. Numerical computations using CPLEX show that $x_{300}^{42 c} \approx 3.40339<3.4034$. From Lemma 12, we get that Algorithm A2 is a $(1.45,3.4034)$ approximation for the SMFLP. Now, for $\delta=2.0543$, Lemma 11 states that Algorithm A3 is a ( $2.169 \ldots, 2.169 \ldots$ )-approximation for the SMFLP.

In Appendix C, we summarize the results obtained with CPLEX for the analysis of algorithms $A 1, A 2$, and $A 3$.

## 5 An optimal approximation algorithm

Byrka and Aardal [2] (see also [3]) gave a 1.5-approximation for the MFLP, combining a $(1.11,1.78)$-approximation from Jain, Mahdian, and Saberi [11] and a new analysis of the LP-rounding algorithm $C S(\gamma)$ of Chudak and Shmoys [6], that leads to a $(1.6774,1.3737)$ approximation. Byrka showed that $C S(\gamma)$ has the optimal bi-factor approximation $\left(\gamma, 1+2 e^{-\gamma}\right)$ for $\gamma \geq \gamma_{0} \approx 1.6774$. By randomly selecting $\gamma$ according to a given probability distribution, Li [13] improved this result to 1.488 , that is currently the best known approximation for the MFLP.

We show that $\operatorname{CS}(\gamma)$, when applied to the SMFLP, touches its optimal bi-factor approximation curve $\left(\gamma, 1+8 e^{-\gamma}\right)$ for $\gamma \geq \gamma_{0} \approx 2.00492$. Therefore, we have an $(\alpha, \alpha)$-approximation for
the SMFLP, where $\alpha \approx 2.04011$ is the solution of equation $\gamma=1+8 e^{-\gamma}$. Since $\alpha$ is the approximation lower bound, this result implies that $C S(\alpha)$, solely used, is an optimal approximation for the SMFLP.

The natural linear program relaxation is given in the following:

$$
\begin{array}{cll}
\min & \sum_{i \in F} y_{i} f_{i}+\sum_{j \in C} \sum_{i \in F} x_{i j} c_{i j} & \\
\mathrm{s.t.} & \sum_{i \in F} x_{i j}=1 & \forall j \in C \\
& x_{i j} \leq y_{i} & \forall i \in F, j \in C \\
& x_{i j}, y_{i} \geq 0 & \forall i \in F, j \in C .
\end{array}
$$

The corresponding integer variables $y_{i}$ indicate whether facility $i$ is open, and the corresponding integer variables $x_{i j}$ indicate whether facility $i$ serves city $j$ in the solution. Algorithm $\operatorname{CS}(\gamma)$ may be summarized as follows. First, a solution $\left(x^{*}, y^{*}\right)$ of program (17) is obtained. Then, the fractional opening variables $y_{i}^{*}$ are scaled by a factor $\gamma \geq 1, \bar{y}_{i}=\gamma y_{i}^{*}$, and variables $\bar{x}_{i j}$ are defined so that city $j$ is served entirely by its closest facilities, obtaining a new solution $(\bar{x}, \bar{y})$. We may assume that this solution is complete, i.e. for every city $j$ and facility $i$, if $\bar{x}_{i j}>0$, then $\bar{x}_{i j}=\bar{y}_{i}$, and that for every $i, \bar{y}_{i} \leq 1$, since, in either case, we can split facility $i$, and obtain an equivalent instance with these properties. Finally, a clustering of some of the facilities is obtained according to a given criterion, and a probabilistic rounding procedure is used to obtain the final solution. For a detailed description of the algorithm, see [2] (also [3]).

A facility $i$ with $\bar{x}_{i j}>0$ is called a close facility of city $j$, and the set of such facilities is denoted by $C_{j}$. Similarly, a facility $i$ with $\bar{x}_{i j}=0$ but $x_{i j}^{*}>0$ is called a distant facility of $j$, and the set of such facilities is denoted by $D_{j}$. Let $F_{j}=C_{j} \cup D_{j}$. The analysis of $\operatorname{CS}(\gamma)$ uses the notion of average distance between a city $j \in C$ and a subset of facilities $F^{\prime} \subseteq F$ such that $\sum_{i \in F^{\prime}} \bar{y}_{i}>0$, defined as $d\left(j, F^{\prime}\right)=\frac{\sum_{i \in F^{\prime}} c_{i j} \cdot \bar{y}_{i}}{\sum_{i \in F^{\prime}} \bar{y}_{i}}$. For a city $j$, we also use some definitions from [3]: the average connection cost, $d_{j}=d\left(j, F_{j}\right)$; the average distance from close facilities, $d_{j}^{(c)}=d\left(j, C_{j}\right)$; the average distance from distant facilities, $d_{j}^{(d)}=d\left(j, D_{j}\right)$; the maximum distance from close facilities, $d_{j}^{(\max )}=\max _{i \in C_{j}} c_{i j}$; and the irregularity parameter $\rho_{j}$, defined as $\rho_{j}=\left(d_{j}-d_{j}^{(c)}\right) / d_{j}$ if $d_{j}>0$, and $\rho_{j}=0$ otherwise.

With these definitions, we can describe the clustering of the facilities. In each iteration, greedily select a city $j$, called the cluster center, such that the sum $d_{j}^{(c)}+d_{j}^{(\max )}$ is minimum, and build a cluster formed by $j$ and its close facilities $C_{j}$. Remove $j$ and every other city $j^{\prime}$ such that $C_{j} \cap C_{j^{\prime}}$ is not empty, and repeat this process until every city is removed. The set of facilities opened by $\operatorname{CS}(\gamma)$ is given by the following rounding procedure: for each cluster center $j$, open one facility $i$ from $C_{j}$ with probability $\bar{x}_{i j}=\bar{y}_{i}$, and, for each unclustered facility $i$, open it independently with probability $\bar{y}_{i}$. Each city is connected to its closest opened facility.

The following lemma of Byrka and Aardal [2] is used to bound the expected connection cost between a city and the closest facility from a set of facilities.

Lemma 13 ([2]). Consider a random vector $y \in\{0,1\}^{|F|}$ produced by Algorithm CS $(\gamma)$, a subset $A \subseteq F$ of facilities such that $\sum_{i \in A} \bar{y}_{i}>0$, and a city $j \in C$. Then, the following holds:

$$
E\left[\min _{i \in A, y_{i}=1} c_{i j} \mid \sum_{i \in A} y_{i} \geq 1\right] \leq d(j, A)
$$

For a given city $j$, if one facility in $C_{j}$ or $D_{j}$ is opened, then Lemma 13 states that the expected connection cost is bounded by $d_{j}^{(c)}$ and $d_{j}^{(d)}$, respectively. If no facility in $C_{j} \cup D_{j}=F_{j}$
is opened, then city $j$ can always be connected to one of the close facilities $C_{j^{\prime}}$ of the associated cluster center $j^{\prime}$, with expected connection cost $d\left(j, C_{j^{\prime}} \backslash F_{j}\right)$. Byrka and Aardal [2] showed that, for the MFLP, when $\gamma<2$, this cost is at most $d_{j}^{(d)}+d_{j^{\prime}}^{(\max )}+d_{j^{\prime}}^{(c)}$. Since for the SMFLP we need $\gamma>2$, we will use an improved version of this lemma by Li [13]. The adapted lemma for the squared metric is given in the following. The proof is the same, except that we use the squared metric property, instead of the triangle inequality.

Lemma 14. Let $j$ be a city and $j^{\prime}$ be the associated cluster center such that $C_{j} \cap C_{j^{\prime}} \neq \emptyset$. Then, $d\left(j, C_{j^{\prime}} \backslash F_{j}\right) \leq 3 \cdot\left((2-\gamma) d_{j}^{(\max )}+(\gamma-1) d_{j}^{(d)}+d_{j^{\prime}}^{(\max )}+d_{j^{\prime}}^{(c)}\right)$.
Proof. Let $d_{j j^{\prime}}=\min _{l \in F}\left(c_{l j}+c_{l j^{\prime}}\right)$, that is, the minimum connection cost of a path of length two from $j$ to $j^{\prime} \underbrace{*}$ Fix a facility $l$ such that $c_{l j}+c_{l j^{\prime}}=d_{j j^{\prime}}$. For each facility $i$ in $C_{j^{\prime}} \backslash F_{j}$, we say that a path $\left(j, l, j^{\prime}, i\right)$ is the center-path to $i$. The cost of such center-path to $i$ is defined as $d_{j j^{\prime}}+c_{i j^{\prime}}$. Notice that, using the squared metric property, $c_{i j} \leq 3\left(d_{j j^{\prime}}+c_{i j^{\prime}}\right)$, and therefore

$$
\begin{aligned}
d\left(j, C_{j^{\prime}} \backslash F_{j}\right) & =\frac{\sum_{i \in C_{j^{\prime}} \backslash F_{j}} c_{i j} \cdot \bar{y}_{i}}{\sum_{i \in C_{j^{\prime}} \backslash F_{j}} \bar{y}_{i}} \\
& \leq \frac{\sum_{i \in C_{j^{\prime}} \backslash F_{j}} 3\left(d_{j j^{\prime}}+c_{i j^{\prime}}\right) \cdot \bar{y}_{i}}{\sum_{i \in C_{j^{\prime}} \backslash F_{j}} \bar{y}_{i}} \\
& =3 \cdot\left(d_{j j^{\prime}}+d\left(j^{\prime}, C_{j^{\prime}} \backslash F_{j}\right)\right) .
\end{aligned}
$$

That is, $d\left(j, C_{j^{\prime}} \backslash F_{j}\right)$ is at most three times the average center-path cost. Following the lines of Li [13, Lemma 1], we know that $d_{j j^{\prime}}+d\left(j^{\prime}, C_{j^{\prime}} \backslash F_{j}\right) \leq(2-\gamma) d_{j}^{(\max )}+(\gamma-1) d_{j}^{(d)}+d_{j^{\prime}}^{(\max )}+$ $d_{j^{\prime}}^{(c)}$. Therefore, the lemma holds.

The next lemma follows from Lemma 14, and is straightforward.
Lemma 15. $d\left(j, C_{j^{\prime}} \backslash F_{j}\right) \leq 3\left(\gamma d_{j}+(3-\gamma) d_{j}^{(d)}\right)$.
Now, we can bound the expected facility and connection cost of a solution generated by $\operatorname{CS}(\gamma)$. The next theorem is an adapted version of Theorem 2.5 from [3].

Theorem 4. For $\gamma \geq 1$, Algorithm $\operatorname{CS}(\gamma)$ produces a solution ( $x, y$ ) for the integer program corresponding to (17) with expected facility and connection costs

$$
E\left[y_{i} f_{i}\right]=\gamma \cdot F_{i}^{*}, \text { and } E\left[\min _{i \in F, y_{i}=1} c_{i j}\right] \leq \max \left\{1+8 e^{-\gamma}, \frac{5 e^{-\gamma}+e^{-1}}{1-\frac{1}{\gamma}}\right\} \cdot C_{j}^{*}
$$

where $F_{i}^{*}=y_{i}^{*} f_{i}$ and $C_{j}^{*}=\sum_{i \in F} x_{i j}^{*} c_{i j}$.
Proof. The expected cost of facility $i$ is $E\left[y_{i} f_{i}\right]=\bar{y}_{i} f_{i}=\gamma \cdot y_{i}^{*} f_{i}=\gamma \cdot F_{i}^{*}$.
If $j$ is a cluster center, one of its close facilities is open, then the expected connection cost is $d_{j}^{(c)} \leq d_{j}=C_{j}^{*}$. We may assume that $j$ is not a cluster center. Let $p_{c}$ be the probability that the closest facility to $j$ is in $C_{j}$, and $p_{d}$ the probability that it is in $D_{j}$. If neither case occurs, then,

[^1]with probability $p_{s}=1-p_{c}-p_{d}$, the closest facility is in $C_{j^{\prime}} \backslash F_{j}$, where $j^{\prime}$ is the cluster center associated with $j$. From the definitions, we have that $d_{j}^{(c)}=\left(1-\rho_{j}\right) d_{j}, d_{j}^{(d)}=\left(1+\frac{\rho_{j}}{\gamma-1}\right) d_{j}$, and $\rho_{j} \leq 1$. Also, from [2], we know that $p_{s} \leq e^{-\gamma}$ and $p_{c} \geq 1-e^{-1}$. Combining these facts with Lemmas 13 and 15, we obtain
\[

$$
\begin{aligned}
E\left[\min _{i \in F, y_{i}=1} c_{i j}\right] & \leq p_{c} \cdot d_{j}^{(c)}+p_{d} \cdot d_{j}^{(d)}+p_{s} \cdot 3\left(\gamma d_{j}+(3-\gamma) d_{j}^{(d)}\right) \\
& =\left(\left(p_{c}+p_{d}+9 p_{s}\right)+\frac{\left(p_{c}+p_{d}+9 p_{s}\right)-\left(p_{c}+3 p_{s}\right)}{\gamma-1} \rho_{j}\right) d_{j} \\
& =\left(\left(1+8 p_{s}\right)+\frac{\left(1+8 p_{s}\right)-\left(p_{c}+3 p_{s}\right) \gamma}{\gamma-1} \rho_{j}\right) d_{j} \\
& =\left(\left(1+8 p_{s}\right)\left(1-\rho_{j}\right)+\frac{5 p_{s}+1-p_{c}}{1-\frac{1}{\gamma}} \rho_{j}\right) d_{j} \\
& \leq\left(\left(1+8 e^{-\gamma}\right)\left(1-\rho_{j}\right)+\frac{5 e^{-\gamma}+e^{-1}}{1-\frac{1}{\gamma}} \rho_{j}\right) d_{j} \\
& \leq \max \left\{1+8 e^{-\gamma}, \frac{5 e^{-\gamma}+e^{-1}}{1-\frac{1}{\gamma}}\right\} \cdot C_{j}^{*} .
\end{aligned}
$$
\]

Let $\gamma_{0}$ be the solution of equation

$$
\left(\frac{5 e^{-\gamma}+e^{-1}}{1-\frac{1}{\gamma}}\right)=\left(1+8 e^{-\gamma}\right)
$$

For $\gamma \geq \gamma_{0} \approx 2.00492$, the maximum connection cost factor is $1+8 e^{-\gamma}$, so $\operatorname{CS}(\gamma)$ touches the curve ( $\gamma, 1+8 e^{-\gamma}$ ), that is, its approximation factor is the best possible for the SMFLP, unless $\mathrm{P}=\mathrm{NP}$. The next theorem follows immediately.

Theorem 5. Let $\alpha \approx 2.04011$ be the solution of the equation $\gamma=1+8 e^{-\gamma}$. Then $\operatorname{CS}(\alpha)$ is an $\alpha$-approximation for the SMFLP and the approximation factor is the best possible unless $\mathrm{P}=\mathrm{NP}$.

Relaxed triangle inequality. We notice that the analysis of Lemma 5 and of Theorem 4 apply to a more general case of the FLP when the connection cost function satisfies $c_{i j} \leq$ $3\left(c_{i j^{\prime}}+c_{i^{\prime} j^{\prime}}+c_{i^{\prime} j}\right)$ for all facilities $i$ and $i^{\prime}$, and cities $j$ and $j^{\prime}$. Charikar et al. [4] considered a similar relaxed triangle inequality to extend their constant approximation for the $k$-medians problem with center costs to the case in which the objective is to minimize the sum of the squares of the distances of clients to their nearest centers.

We say that a connection cost function $c$ for the FLP satisfies a $\tau$-relaxed triangle inequality if $c_{i j} \leq \tau \cdot\left(c_{i j^{\prime}}+c_{i^{\prime} j^{\prime}}+c_{i^{\prime} j}\right)$, for all $i, i^{\prime} \in F$, and $j, j^{\prime} \in C$. Also, we say that the subset of the FLP that contains only instances that satisfy the $\tau$-relaxed triangle inequality is the $\tau$-RELAXED FLP. The following theorems extend Theorems 8 and 4 .

Theorem 6. Let $\gamma_{f}$ and $\gamma_{c}$ be positive constants with $\gamma_{c}<1+(3 \tau-1) e^{-\gamma_{f}}$. If there is a $\left(\gamma_{f}, \gamma_{c}\right)-$ approximation for the $\tau$-relaxed FLP , then $\mathrm{P}=\mathrm{NP}$.

Theorem 7. For $\gamma \geq 1, \operatorname{CS}(\gamma)$ is a $\left(\gamma, \max \left\{1+(3 \tau-1) e^{-\gamma}, \frac{(2 \tau-1) e^{-\gamma}+e^{-1}}{1-\gamma^{-1}}\right\}\right)$-approximation for the $\tau$-relaxed FLP.

Let $\alpha(\tau)$ be the solution of equation $\gamma=1+(3 \tau-1) e^{-\gamma}$. It is straightforward to verify that for $\tau \geq 2.620 \ldots$ we have $1+(3 \tau-1) e^{-\alpha(\tau)} \geq \frac{(2 \tau-1) e^{-\alpha(\tau)}+e^{-1}}{1-\alpha(\tau)^{-1}}$. Therefore, for $\tau \geq 2.620 \ldots$, Algorithm $\operatorname{CS}(\alpha(\tau))$ has the best approximation factor for the $\tau$-relaxed FLP.

We say that the METRIC ${ }^{\alpha}$ FLP, denoted $\mathrm{M}^{\alpha}$ FLP, is the variant of FLP that considers instances such that the connection cost function is the $\alpha^{\text {th }}$ power of a given metric. We may use the following known fact to derive approximations for $\mathrm{M}^{\alpha} \mathrm{FLP}$ using approximations for $\tau$-relaxed FLPs.

Lemma 16. Let $A, B, C$, and $D$ be non-negative numbers such that $A \leq B+C+D$, and let $\alpha \geq 1$, then $A^{\alpha} \leq 3^{\alpha-1}\left(B^{\alpha}+C^{\alpha}+D^{\alpha}\right)$.

This implies that the connection cost function that is the $\alpha^{\text {th }}$ power of a metric satisfies the $\left(3^{\alpha-1}\right)$-relaxed triangle inequality, and therefore $\mathrm{M}^{\alpha}$ FLP is a particular case of the $\left(3^{\alpha-1}\right)$ relaxed FLP.

## 6 The inapproximability threshold for SMFLP

For the MFLP, Jain et al. [10] adapted the 1.463 hardness result by Guha and Khuller [8], and showed that no algorithm is a ( $\gamma_{f}, \gamma_{c}$ )-approximation, with $\gamma_{c}<1+2 e^{-\gamma_{f}}$, unless $\mathrm{NP} \subseteq$ DTIME $\left[n^{O(\log \log n)}\right]$. Following the lines of Sviridenko (see Vygen [17, Section 4.4]), the condition is strengthened to unless $\mathrm{P}=\mathrm{NP}$. We extend these results for the SMFLP as follows.

Theorem 8. Let $\gamma_{f}$ and $\gamma_{c}$ be positive constants with $\gamma_{c}<1+8 e^{-\gamma_{f}}$. If there is a $\left(\gamma_{f}, \gamma_{c}\right)$ approximation for the SMFLP, then $\mathrm{P}=\mathrm{NP}$. In particular, let $\alpha \approx 2.04011$ be the solution of the equation $\gamma=1+8 e^{-\gamma}$, then there is no $\alpha^{\prime}$-approximation with $\alpha^{\prime}<\alpha$ for the SMFLP unless $\mathrm{P}=\mathrm{NP}$.

Adapted from [8]. For simplicity, here we show that the lower bound holds unless NP $\subseteq$ DTIME $\left[n^{\mathrm{O}(\log \log n)}\right]$. If we follow the lines of Sviridenko (see Vygen [17, Section 4.4]), the condition is changed to unless $\mathrm{P}=\mathrm{NP}$.

Assume $A$ is a $\left(\gamma_{f}, \gamma_{c}\right)$-approximation for the SMFLP with $\gamma_{c}<1+8 e^{-\gamma_{f}}$. Let $\mathscr{J}=(\mathscr{U}, \mathscr{S})$ be an instance of the Set Cover, with $\mathscr{U}$ being a set of elements, $\mathscr{S}$ a collection of subsets of $\mathscr{U}$ and $n=|\mathscr{U}|$. We will derive a $\left(d^{\prime} \ln n\right)$-approximation algorithm for the Set Cover problem, for some $d^{\prime}<1$.

Let $k$ be the optimal value of $\mathscr{J}$ for the Set Cover. If $k$ is not known, one can run this algorithm for $k=1, \ldots, n$ and output the best solution found.

The algorithm will find a solution for $\mathscr{J}$ by iteratively solving a sequence of instances of the SMFLP of the form $\mathscr{I}^{(j)}=\left(C^{(j)}, F, c, f^{(j)}\right)$, where $F=\mathscr{S}$ and the initial set $C^{(1)}=\mathscr{U}$. For each element $x_{j} \in S_{i}$, set $c_{i j}=1$, and for each $x_{j} \notin S_{i}$, set $c_{i j}=9$. Note that such $c$ is a squared metric. Let $n_{j}=\left|C^{(j)}\right|$. In the $j$ th instance, every facility cost is $f^{(j)}=\gamma_{\frac{n_{j}}{k}}$, for some positive $\gamma$ to be fixed later. For each $j$, let $S^{(j)}$ denote the solution for $\mathscr{I}^{(j)}$ produced by Algorithm $A$ and let $C^{(j+1)}$ be the elements of $C^{(j)}$ not covered by any set in $S^{(j)}$. This process stops when $C^{(j+1)}=\emptyset$ and yields the solution $S^{(1)} \cup \cdots \cup S^{(j)}$ for $\mathscr{J}$.

Observe that an optimal solution for $\mathscr{J}$ is a solution for each $\mathscr{I}^{(j)}$ with total facility cost $k f^{(j)}$ and connection cost one for each of the $n_{j}$ cities. Therefore, $S^{(j)}$ has cost at most $\gamma_{f} k f^{(j)}+$
$\gamma_{c} n_{j}=\left(\gamma_{f} \gamma+\gamma_{c}\right) n_{j}$, because $f^{(j)}=\gamma^{n_{j}}$. Let $\beta_{j}=\left|S^{(j)}\right| / k$ and $d_{j}$ be such that $d_{j} n_{j}$ is the number of elements covered in iteration $j$, that is, the number of elements of $C^{(j)}$ in the union of the sets in $S^{(j)}$. Then the total facility cost of $S^{(j)}$ is $\beta_{j} k f^{(j)}=\beta_{j} \gamma n_{j}$. Moreover, $d_{j} n_{j}$ cities are connected with cost one and the other $n_{j}-d_{j} n_{j}=\left(1-d_{j}\right) n_{j}$ cities are connected with cost nine. Hence the total cost of $S^{(j)}$ is $\beta_{j} \gamma_{n}+d_{j} n_{j}+9\left(1-d_{j}\right) n_{j}=\left(\beta_{j} \gamma+9-8 d_{j}\right) n_{j}$. We conclude that $\gamma_{f} \gamma+\gamma_{c} \geq \beta_{j} \gamma+9-8 d_{j}$. So we have that $\gamma_{c} \geq\left(\beta_{j}-\gamma_{f}\right) \gamma+9-8 d_{j}$.

Let $d<1$ be such that $1+8 e^{-\gamma_{f} / d}>\gamma_{c}$. Suppose, for the sake of contradiction, that $d_{j} \leq$ $1-e^{-\beta_{j} / d}$ for some $j$. Then

$$
\gamma_{c} \geq\left(\beta_{j}-\gamma_{f}\right) \gamma+9-8\left(1-e^{-\beta_{j} / d}\right) .
$$

Considering $\gamma_{f}, \gamma$ and $d$ fixed, the minimum value of the right hand side is achieved when $\beta_{j}=d \ln \frac{8}{d \gamma}$. Substituting $\beta_{j}$ above, we get

$$
\gamma_{c} \geq\left(d \ln \frac{8}{d \gamma}-\gamma_{f}\right) \gamma+1+d \gamma
$$

Considering $d$ and $\gamma_{f}$ fixed, we choose the value of $\gamma$ that maximizes the right hand side, that is, $\gamma=\frac{8}{d} e^{-\frac{\gamma_{f}}{d}}$. Replacing in the inequality, we obtain $\gamma_{c} \geq 1+8 e^{-\frac{\gamma_{f}}{d}}>\gamma_{c}$, a contradiction. So $d_{j}>1-e^{-\beta_{j} / d}$ for every $j$, for this $d<1$.

Following the lines of Guha and Khuller [8], one can prove that the algorithm described above for the Set Cover is a $\left(d^{\prime} \ln n\right)$-approximation for some $d^{\prime}<1$. This implies that $\mathrm{NP} \subseteq$ DTIME $\left[n^{\mathrm{O}(\log \log n)}\right]$.

## 7 Concluding remarks

We presented a new technique for deriving upper bound factor-revealing programs, that can be solved by computer, as an alternative way to obtain an upper bound on the approximation factors of the corresponding algorithm. This technique allowed us to tighten the obtained approximation factors, and to simplify the analysis of the three primal-dual algorithms, when used for both SMFLP and MFLP instances. We hope that this technique can be employed for other problems and algorithms analyzed through factor-revealing LPs. We also showed that the algorithm of Chudak and Shmoys[6] is a 2.04 -approximation and that it has the best approximation factor for the SMFLP. This is in contrast to the MFLP, for which this algorithm is a 1.575 -approximation and the lower bound by Guha and Kuller [8] is 1.463 . Also, we note that, although there is an approximation scheme for Euclidean FLP by Arora et al. [1], the analysis of this algorithm is not valid for non-metric FLP, such as the $E^{2}$ FLP. We do not know whether $\mathrm{E}^{2}$ FLP has an approximation strictly better than 2.04 .

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## A Square root constraints

The proof of Lemma 3 is a straightforward consequence of Lemma 1 and the following result.
Lemma 17. Let $A, B, C$, and $D$ be non-negative numbers. Then $\sqrt{A} \leq \sqrt{B}+\sqrt{C}+\sqrt{D}$ if and only if $A \leq\left(1+\beta+\frac{1}{\gamma}\right) B+\left(1+\gamma+\frac{1}{\delta}\right) C+\left(1+\delta+\frac{1}{\beta}\right) D$ for every positive numbers $\beta$, $\gamma$, and $\delta$. In particular, if $\sqrt{A} \leq \sqrt{B}+\sqrt{C}+\sqrt{D}$, then $A \leq 3 B+3 C+3 D$.

Proof. Suppose $\sqrt{A} \leq \sqrt{B}+\sqrt{C}+\sqrt{D}$. As $(\sqrt{\beta B}-\sqrt{D / \beta})^{2} \geq 0$, we have that $2 \sqrt{B D} \leq$ $\beta B+D / \beta$. Similarly, $2 \sqrt{C B} \leq \gamma C+B / \gamma$ and $2 \sqrt{D C} \leq \delta D+C / \delta$. Therefore, if $\sqrt{A} \leq \sqrt{B}+$ $\sqrt{C}+\sqrt{D}$, then

$$
\begin{aligned}
A & \leq(\sqrt{B}+\sqrt{C}+\sqrt{D})^{2} \\
& =B+C+D+2 \sqrt{B D}+2 \sqrt{C B}+2 \sqrt{D C} \\
& \leq B+C+D+\beta B+D / \beta+\gamma C+B / \gamma+\delta D+C / \delta \\
& =\left(1+\beta+\frac{1}{\gamma}\right) B+\left(1+\gamma+\frac{1}{\delta}\right) C+\left(1+\delta+\frac{1}{\beta}\right) D .
\end{aligned}
$$

Choosing $\beta=\gamma=\delta=1$, we obtain $A \leq 3 B+3 C+3 D$.
Now suppose $\sqrt{A}>\sqrt{B}+\sqrt{C}+\sqrt{D}$. Let $d>0$ be such that $A=B+C+D+2 \sqrt{B D}+$ $2 \sqrt{C B}+2 \sqrt{D C}+d$. Then, $A>\left(1+\beta+\frac{1}{\gamma}\right) B+\left(1+\gamma+\frac{1}{\delta}\right) C+\left(1+\delta+\frac{1}{\beta}\right) D$ is equivalent to $\left(\beta+\frac{1}{\gamma}\right) B+\left(\gamma+\frac{1}{\delta}\right) C+\left(\delta+\frac{1}{\beta}\right) D<2 \sqrt{B D}+2 \sqrt{C B}+2 \sqrt{D C}+d$. We will analyze the cases in which none, one, two or all numbers $B, C$ and $D$ are zero. Let $\xi$ and $\xi^{\prime}$ be positive numbers such that $\xi+\xi^{\prime}<1$.
Case 1: $B, C, D>0$. Let $\beta=\sqrt{\frac{D}{B}}, \gamma=\sqrt{\frac{B}{C}}$ and $\delta=\sqrt{\frac{C}{D}}$. Then $\left(\beta+\frac{1}{\gamma}\right) B+\left(\gamma+\frac{1}{\delta}\right) C+(\delta+$ $\left.\frac{1}{\beta}\right) D=2 \sqrt{B D}+2 \sqrt{C B}+2 \sqrt{D C}<2 \sqrt{B D}+2 \sqrt{C B}+2 \sqrt{D C}+d$.
Case 2: $B=0$ and $C, D>0$. Let $\beta=\frac{D}{\xi d}, \gamma=\frac{\xi^{\prime} d}{C}$ and $\delta=\sqrt{\frac{C}{D}}$. Then $\left(\beta+\frac{1}{\gamma}\right) B+\left(\gamma+\frac{1}{\delta}\right) C+$

$$
\left(\delta+\frac{1}{\beta}\right) D=2 \sqrt{D C}+\left(\xi+\xi^{\prime}\right) d<2 \sqrt{B D}+2 \sqrt{C B}+2 \sqrt{D C}+d
$$

Case 3: $B, C=0$ and $D>0$. Let $\beta=\frac{D}{\xi d}, \gamma=1$ and $\delta=\frac{\xi^{\prime} d}{D}$. Then $\left(\beta+\frac{1}{\gamma}\right) B+\left(\gamma+\frac{1}{\delta}\right) C+$ $\left(\delta+\frac{1}{\beta}\right) D=\left(\xi+\xi^{\prime}\right) d<2 \sqrt{B D}+2 \sqrt{C B}+2 \sqrt{D C}+d$.
Case 4: $B, C, D=0$. Let $\beta=1, \gamma=1$ and $\delta=1$. Then $\left(\beta+\frac{1}{\gamma}\right) B+\left(\gamma+\frac{1}{\delta}\right) C+\left(\delta+\frac{1}{\beta}\right) D=$ $0<2 \sqrt{B D}+2 \sqrt{C B}+2 \sqrt{D C}+d$.

Observe that the lemma above is constructive in the sense that, if the given inequality with square roots is not satisfied, then shows how to determine a linear inequality that is not satisfied.

## B Upper Bound Factor-Revealing Program for A2

Consider tuples $\left(\beta_{i}, \gamma_{i}, \delta_{i}\right) \in \mathbb{R}_{+}^{* 3}$ and $B_{i}=1+\beta_{i}+\frac{1}{\gamma_{i}}, C_{i}=1+\gamma_{i}+\frac{1}{\delta_{i}}, D_{i}=1+\delta_{i}+\frac{1}{\beta_{i}}$ for $1 \leq i \leq m$. Using Lemma 17 , we insert inequalities corresponding to these tuples, replacing the nonlinear constraint, and obtain $z_{k}^{A 2} \leq w_{k}^{A 2}$, where $w_{k}^{A 2}$ is given by

$$
\begin{array}{lll}
w_{k}^{A 2}=\max & \sum_{j=1}^{k} \alpha_{j} & \\
\text { s.t. } & f+\sum_{j=1}^{k} d_{j} \leq 1 & \\
& \alpha_{j} \leq \alpha_{j+1} & \forall 1 \leq j<k \\
& r_{j l} \geq r_{j, l+1} & \forall 1 \leq j<l<k \\
& \alpha_{l} \leq B_{i} r_{j l}+C_{i} d_{l}+D_{i} d_{j} & \forall 1 \leq j<l \leq k, 1 \leq i \leq m  \tag{18}\\
& r_{j l}-d_{j} \leq x_{j l} & \forall 1 \leq j<l \leq k \\
& \alpha_{l}-d_{j} \leq x_{j l} & \forall 1 \leq l \leq j \leq k \\
& \sum_{j=1}^{k} x_{j l} \leq f & \forall 1 \leq l \leq k \\
& \alpha_{j}, d_{j}, f, r_{j l} \geq 0 & \forall 1 \leq j \leq l \leq k \\
& x_{j l} \geq 0 & \forall 1 \leq j, l \leq k .
\end{array}
$$

Now, we calculate the dual of program (18) to derive the upper bound factor-revealing linear program. After that, we calculate its dual program (22), in order to use Lemma (17, and solve the upper bound factor-revealing program inserting cutting planes. We proceed the same way as done in Lemma 7 . With similar arguments, we may see that $z_{k}^{A 2} \leq z_{k t}^{A 2}$, for any $t$, and we assume that $k$ has the form $k=p t$, for some integer $t$. The dual of linear program (18) is given in the following.

$$
\begin{array}{lll}
w_{k}^{A 2}=\min & \gamma & \\
\text { s.t. } & a_{l}-a_{l-1}+\sum_{i=1}^{m} \sum_{j=1}^{l-1} c_{j l i}+\sum_{j=l}^{k} e_{j l} \geq 1 & \forall 1 \leq l \leq k \\
& \gamma-\sum_{i=1}^{m} C_{i} \sum_{j=1}^{l-1} c_{j l i}-\sum_{i=1}^{m} D_{i} \sum_{j=l+1}^{k} c_{l j i}-\sum_{j=1}^{k} e_{l j} \geq 0 & \forall 1 \leq l \leq k \\
& \gamma-\sum_{l=1}^{k} h_{l} \geq 0 &  \tag{19}\\
& b_{j, l-1}-b_{j l}+e_{j l}-\sum_{i=1}^{m} B_{i} c_{j l i} \geq 0 & \forall 1 \leq j<l \leq k \\
& h_{l}-e_{j l} \geq 0 & \forall 1 \leq j, l \leq k \\
a_{0}=a_{k}=b_{l l}=b_{l k}=0 & \forall 1 \leq l \leq k \\
a_{l}, h_{l}, e_{j l} \geq 0 & \forall 1 \leq l, j \leq k \\
b_{j l}, c_{j l i} \geq 0 & \forall 1 \leq j<l \leq k \\
1<i<m
\end{array}
$$

Now, we may derive the upper bound factor-revealing linear program. Let $\hat{n}=\left\lceil\frac{n}{p}\right\rceil$ and consider prime variables $\gamma^{\prime}, a_{l}^{\prime}, b_{j l}^{\prime}, c_{j l i}^{\prime}, e_{j l}^{\prime}, h_{l}^{\prime}$. We obtain a candidate solution for program (19) by defining:

$$
\begin{gather*}
\gamma=\gamma^{\prime}, \quad a_{l}=p a_{\hat{l}}^{\prime}-(p \hat{l}-l)\left(a_{\hat{l}}^{\prime}-a_{\hat{\hat{l}}-1}^{\prime}\right), \quad b_{j l}=b_{\hat{j}, \hat{l}}^{\prime}-\frac{p \hat{l}-l}{p}\left(b_{\hat{j} \hat{l}}^{\prime}-b_{\hat{j}, \hat{l}-1}^{\prime}\right), \\
c_{j l l}=\frac{c_{\hat{\hat{j}} \hat{l}}^{\prime}}{p}, \quad e_{j l}=\frac{e^{\prime} \hat{l}}{p} \quad \text { and } \quad h_{l}=\frac{h_{\hat{l}}^{\hat{1}}}{p} . \tag{20}
\end{gather*}
$$

In the following, we apply definition (20) and calculate each coefficient expression for program (19). Again, notice that $a_{l}-a_{l-1}=a_{\hat{l}}^{\prime}-a_{\hat{l}-1}^{\prime}$, and that $b_{j, l-1}-b_{j l}=\left(b_{\hat{j}, \hat{l}-1}^{\prime}-b_{\hat{j} l}^{\prime}\right) / p$. Also, fix variables $c_{l l i}^{\prime}$ at zero.

$$
\begin{aligned}
& \operatorname{coeff}\left[\alpha_{l}\right]=a_{l}-a_{l-1}+\sum_{i=1}^{m} \sum_{j=1}^{l-1} c_{j l i}+\sum_{j=l}^{k} e_{j l} \\
& =a_{\hat{l}}^{\prime}-a_{\hat{l}-1}^{\prime}+\sum_{i=1}^{m} \sum_{j=1}^{l-1} \frac{c_{\hat{j} \hat{l} i}^{\prime}}{p}+\sum_{j=l}^{p t} \frac{e_{\hat{j} \hat{l}}^{\prime}}{p} \\
& \geq a_{\hat{l}}^{\prime}-a_{\hat{l}-1}^{\prime}+\sum_{i=1}^{m} \sum_{j^{\prime}=1}^{\hat{l}-1} p \frac{c^{\prime} j^{\prime} \hat{l} i}{p}+\sum_{j^{\prime}=\hat{l}+1}^{t} p \frac{e^{\prime}{ }_{j^{\prime} \hat{l}}}{p} \\
& =a_{\hat{l}}^{\prime}-a_{\hat{l}-1}^{\prime}+\sum_{i=1}^{m} \sum_{j^{\prime}=1}^{\hat{l}-1} c_{j^{\prime} \hat{l} i}^{\prime}+\sum_{j^{\prime}=\hat{l}+1}^{t} e_{j^{\prime} \hat{l}}^{\prime} \geq 1 \\
& \operatorname{coeff}\left[d_{l}\right]=\gamma-\sum_{i=1}^{m} \sum_{j=1}^{l-1} C_{i} c_{j l i}-\sum_{i=1}^{m} \sum_{j=l+1}^{k} D_{i} c_{l j i}-\sum_{j=1}^{k} e_{l j} \\
& =\gamma^{\prime}-\sum_{i=1}^{m} C_{i} \sum_{j=1}^{l-1} \frac{c_{\hat{j} \hat{l} i}^{\prime}}{p}-\sum_{i=1}^{m} D_{i} \sum_{j=l+1}^{k} \frac{c_{\hat{l} \hat{j} i}^{\prime}}{p}-\sum_{j=1}^{k} \frac{e_{\hat{l} \hat{j}}^{\prime}}{p} \\
& \geq \gamma^{\prime}-\sum_{i=1}^{m} C_{i} \sum_{j^{\prime}=1}^{\hat{l}} p \frac{c_{j^{\prime} \hat{l} i}^{\prime}}{p}-\sum_{i=1}^{m} D_{i} \sum_{j^{\prime}=\hat{l}}^{t} p \frac{c_{\hat{l} j^{\prime}, i}^{\prime}}{p}-\sum_{j^{\prime}=1}^{t} p \frac{e_{\hat{l} j^{\prime}}^{\prime}}{p} \\
& =\gamma^{\prime}-\sum_{i=1}^{m} C_{i} \sum_{j^{\prime}=1}^{\hat{l}-1} c_{j^{\prime} \hat{l} i}^{\prime}-\sum_{i=1}^{m} D_{i} \sum_{j^{\prime}=\hat{l}+1}^{t} c_{\hat{l} j^{\prime} i}^{\prime}-\sum_{j^{\prime}=1}^{t} e_{\hat{l} j^{\prime}}^{\prime} \geq 0 \\
& \operatorname{coeff}[f]=\gamma-\sum_{l=1}^{k} h_{l}=\gamma^{\prime}-\sum_{l=1}^{k} \frac{h_{\hat{l}}^{\prime}}{p}=\gamma^{\prime}-\sum_{l^{\prime}=1}^{t} p \cdot \frac{h_{l^{\prime}}^{\prime}}{p}=\gamma^{\prime}-\sum_{l^{\prime}=1}^{t} h_{l^{\prime}}^{\prime} \geq 0 \\
& \operatorname{coeff}\left[r_{j, l}\right]=b_{j, l-1}-b_{j l}+e_{j l}-\sum_{i=1}^{r} B_{i} c_{j l i}=\frac{b_{\hat{j}, \hat{l}-1}^{\prime}-b_{\hat{j} \hat{l}}^{\prime}}{p}+\frac{e_{\hat{j} \hat{l}}^{\prime}}{p}-\sum_{i=1}^{r} B_{i} \frac{c_{\hat{j} \hat{l} i}^{\prime}}{p} \geq 0 . \\
& \operatorname{coeff}\left[x_{j l}\right]=h_{l}-e_{j, l}=\frac{h_{\hat{l}}^{\prime}}{p}-\frac{e_{\hat{j} \hat{l}}^{\prime}}{p} \geq 0 .
\end{aligned}
$$

Conjoining all constraints, the obtained upper bound factor-revealing linear program is:

$$
\begin{array}{lll}
x_{t}^{A 2}=\min & \gamma & \\
\text { s.t. } & a_{l}-a_{l-1}+\sum_{i=1}^{m} \sum_{j=1}^{l-1} c_{j l i}+\sum_{j=l+1}^{t} e_{j l} \geq 1 & \forall 1 \leq l \leq t \\
& \gamma-\sum_{i=1}^{m} C_{i} \sum_{j=1}^{l-1} c_{j l i}-\sum_{i=1}^{m} D_{i} \sum_{j=l+1}^{t} c_{l j i}-\sum_{j=1}^{t} e_{l j} \geq 0 & \forall 1 \leq l \leq t \\
& \gamma-\sum_{l=1}^{t} h_{l} \geq 0 &  \tag{21}\\
& b_{j, l-1}-b_{j l}+e_{j l}-\sum_{i=1}^{m} B_{i} c_{j l i} \geq 0 & \forall 1 \leq j<l \leq t \\
h_{l}-e_{j l} \geq 0 & \forall 1 \leq j, l \leq t \\
a_{0}=a_{t}=b_{l l}=b_{l t}=0 & \forall 1 \leq l \leq t \\
a_{l}, h_{l}, e_{j l} \geq 0 & \forall 1 \leq l, j \leq t \\
b_{j l}, c_{j l i} \geq 0 & \forall 1 \leq j<l \leq k \\
& & 1 \leq i \leq m .
\end{array}
$$

Finally, calculating the dual of program (21), we obtain program (22).

$$
\begin{array}{lll}
x_{t}^{A 2}=\max & \sum_{j=1}^{t} \alpha_{j} & \\
\text { s.t. } & f+\sum_{j=1}^{t} d_{j} \leq 1 & \\
& \alpha_{j} \leq \alpha_{j+1} & \forall 1 \leq j<t \\
& r_{j l} \geq r_{j, l+1} & \forall 1 \leq j<l<t \\
& \alpha_{l} \leq B_{i} r_{j l}+C_{i} d_{l}+D_{i} d_{j} & \forall 1 \leq j<l \leq t, 1 \leq i \leq m  \tag{22}\\
& r_{j l}-d_{j} \leq x_{j l} & \forall 1 \leq j<l \leq t \\
& \alpha_{l}-d_{j} \leq x_{j l} & \forall 1 \leq l<j \leq t \\
& \sum_{j=1}^{t} x_{j l} \leq f & \forall 1 \leq l \leq t \\
& \alpha_{j}, d_{j}, f, r_{j l} \geq 0 & \forall 1 \leq j \leq l \leq t \\
& x_{j l} \geq 0 & \forall 1 \leq j, l \leq t .
\end{array}
$$

## C Experimental results

In Table 1, we present computational results using CPLEX for the lower bound (column $z_{k}^{A 1}$ ) and upper bound (column $x_{k}^{A 1}$ ) for the approximation factor of Algorithm A1. In Table 2, we present lower and upper bounds on the approximation factor of Algorithm A2 (columns $z_{k}^{A 2}$ and $x_{k}^{A 2}$, respectively). In Table 3, we present computational results for program (15) when $\gamma_{f}=1.45$, and the approximation factor obtained from Lemma 11 . The chosen $\delta$ is given by the solution of equation $\gamma_{f}+\ln \delta=1+\frac{\gamma_{c}-1}{\delta}$, that is, $\delta=e^{W_{0}\left(\left(\gamma_{c}-1\right) e^{\gamma_{f}-1}\right)-\left(\gamma_{f}-1\right)}$. Figure 1 shows the trade-off between connection and facility costs approximation guarantees for the Algorithm $A 2$, and Figure 2 shows the trend of obtained factor for Algorithm A3 as we vary the value of $\gamma_{f}$, when $k=50$.

Table 1: Solutions of the factorrevealing programs for A1.

| $k$ | $z_{k}^{A 1}$ | $x_{k}^{A 1}$ |
| ---: | :---: | :---: |
| 10 | 2.57261 | 3.18162 |
| 20 | 2.71704 | 3.01717 |
| 50 | 2.80540 | 2.92579 |
| 100 | 2.83534 | 2.89553 |
| 200 | 2.85034 | 2.88046 |
| 300 | 2.85532 | 2.87543 |
| 400 | 2.85782 | 2.87292 |
| 500 | 2.85930 | 2.87142 |
| 600 | 2.86029 | 2.87041 |
| 700 | 2.86099 | 2.86970 |

Table 2: Solutions of the factorrevealing programs for A2.

| $k$ | $z_{k}^{A 2}$ | $x_{k}^{A 2}$ |
| ---: | :---: | :---: |
| 10 | 2.20702 | 2.65131 |
| 20 | 2.30987 | 2.53301 |
| 50 | 2.37551 | 2.46544 |
| 100 | 2.39773 | 2.44278 |
| 200 | 2.40894 | 2.43150 |
| 300 | 2.41267 | 2.42775 |
| 400 | 2.41453 | 2.42586 |
| 500 | 2.41565 | 2.42473 |

Table 3: Solutions of connection factor-revealing programs for $A 2$, and obtained factor for A3.

| $k$ | $x_{k}^{A 2 c}$ | best $\delta$ | factor |
| ---: | :---: | :---: | :---: |
| 10 | 4.02931 | 2.33433 | 2.29772 |
| 20 | 3.64790 | 2.16561 | 2.22270 |
| 50 | 3.48465 | 2.09159 | 2.18792 |
| 100 | 3.43524 | 2.06895 | 2.17704 |
| 200 | 3.41127 | 2.05793 | 2.17170 |
| 300 | 3.40339 | 2.05430 | 2.16993 |



Figure 1: Trade-off between connection and facility approximation factors.


Figure 2: Trend of the obtained balanced approximation factors.


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[^1]:    ${ }^{*}$ In [13], the connection cost $c$ is extended to a distance between $j$ and $j^{\prime}$, and the triangle inequality is then used to bound this distance with the connection cost of any path of length two. Here, we make a more explicit definition to avoid confusion, since the squared metric property is not sufficient for this purpose.

