# RAINBOW CONNECTION OF SPARSE RANDOM GRAPHS 

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#### Abstract

An edge colored graph $G$ is rainbow edge connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connectivity of a connected graph $G$, denoted by $r c(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected.

In this work we study the rainbow connectivity of binomial random graphs at the connectivity threshold $p=\frac{\log n+\omega}{n}$ where $\omega=\omega(n) \rightarrow \infty$ and $\omega=o(\log n)$ and of random $r$-regular graphs where $r \geq 3$ is a fixed integer. Specifically, we prove that the rainbow connectivity $r c(G)$ of $G=G(n, p)$ satisfies $r c(G) \sim \max \left\{Z_{1}\right.$, diameter $\left.(G)\right\}$ with high probability (whp). Here $Z_{1}$ is the number of vertices in $G$ whose degree equals 1 and the diameter of $G$ is asymptotically equal to $\frac{\log n}{\log \log n}$ whp. Finally, we prove that the rainbow connectivity $r c(G)$ of the random $r$-regular graph $G=G(n, r)$ whp satisfies $r c(G)=O\left(\log ^{\theta_{r}} n\right)$ where $\theta_{r}=\frac{\log (r-1)}{\log (r-2)}$ when $r \geq 4$ and $r c(G)=O\left(\log ^{4} n\right)$ whp when $r=3$.


## 1. Introduction

Connectivity is a fundamental graph theoretic property. Recently, the concept of rainbow connectivity was introduced by Chartrand et al. in [7]. An edge colored graph $G$ is rainbow edge connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connectivity $r c(G)$ of a connected graph $G$ is the smallest number of colors that are needed in order to make $G$ rainbow edge connected. Notice, that by definition a rainbow edge connected graph is also connected and furthermore any connected graph has a trivial edge coloring that makes it rainbow edge connected, since one may color the edges of a given spanning tree with distinct colors. Other basic facts established in [7] are that $r c(G)=1$ if and only if $G$ is a clique and $r c(G)=|V(G)|-1$ if and only if $G$ is a tree. Besides its theoretical interest, rainbow connectivity is also of interest in applied settings, such as securing sensitive information [13], transfer and networking [5].

The concept of rainbow connectivity has attracted the interest of various researchers. Chartrand et al. [7] determine the rainbow connectivity of several special classes of graphs, including multipartite graphs. Caro et al. 4 prove that for a connected graph $G$ with $n$ vertices and minimum degree $\delta$, the rainbow connectivity satisfies $r c(G) \leq \frac{\log \delta}{\delta} n(1+f(\delta))$, where $f(\delta)$ tends to zero as $\delta$ increases. The following simpler bound was also proved in [4], $r c(G) \leq n \frac{4 \log n+3}{\delta}$. Krivelevich and Yuster [12] removed the logarithmic factor from the Caro et al. [4] upper bound. Specifically they proved that $r c(G) \leq \frac{20 n}{\delta}$. Due to a construction of a graph with minimum degree $\delta$ and diameter $\frac{3 n}{\delta+1}-\frac{\delta+7}{\delta+1}$ by Caro et al. [4], the best upper bound one can hope for is $r c(G) \leq \frac{3 n}{\delta}$. Chandran, Das, Rajendraprasad and Varma [6] have subsequently proved an upper bound of $\frac{3 n}{\delta+1}+3$, which is therefore essentially optimal.

As Caro et al. point out, the random graph setting poses several intriguing questions. Specifically, let $G=G(n, p)$ denote the binomial random graph on $n$ vertices with edge probability $p$ [8]. Caro et al. [4] proved that $p=\sqrt{\log n / n}$ is the sharp threshold for the property $r c(G(n, p)) \leq 2$. He and

[^0]Liang [9] studied further the rainbow connectivity of random graphs. Specifically, they obtain the sharp threshold for the property $r c(G) \leq d$ where $d$ is constant. For further results and references we refer the interested reader to the recent survey of Li and Sun [13]. In this work we look at the rainbow connectivity of the binomial graph at the connectivity threshold $p=\frac{\log n+\omega}{n}$ where $\omega=o(\log n)$. This range of values for $p$ poses problems that cannot be tackled with the techniques developed in the aforementioned work. Rainbow connectivity has not been studied in random regular graphs to the best of our knowledge.

Let

$$
\begin{equation*}
L=\frac{\log n}{\log \log n} \tag{1}
\end{equation*}
$$

and let $A \sim B$ denote $A=(1+o(1)) B$ as $n \rightarrow \infty$.
We establish the following theorems:
Theorem 1. Let $G=G(n, p), p=\frac{\log n+\omega}{n}, \omega \rightarrow \infty, \omega=o(\log n)$. Also, let $Z_{1}$ be the number of vertices of degree 1 in $G$. Then, with high probability (whp) 1

$$
r c(G) \sim \max \left\{Z_{1}, L\right\}
$$

It is known that whp the diameter of $G(n, p)$ is asymptotic to $L$ for $p$ as in the above range, see for example Theorem 10.17 of Bollobás [2]. Theorem 1 gives asymptotically optimal results. Our next theorem is not quite as precise.

Theorem 2. Let $G=G(n, r)$ be a random r-regular graph where $r \geq 3$ is a fixed integer. Then, whp

$$
r c(G)= \begin{cases}O\left(\log ^{4} n\right) & r=3 \\ O\left(\log ^{2 \theta_{r}} n\right) & r \geq 4\end{cases}
$$

where $\theta_{r}=\frac{\log (r-1)}{\log (r-2)}$.
All logarithms whose base is omitted are natural. It will be clear from our proofs that the colorings in the above two theorems can be constructed in a low order polynomial time. The second theorem, while weaker, contains an unexpected use of a Markov Chain Monte-Carlo (MCMC) algorithm for randomly coloring a graph.

The paper is organized as follows: After giving a sketch of our approach in Section 2, in Sections 3, 4 we prove Theorems 1, 2 respectively. Finally, in Section 5 we conclude by suggesting open problems.

## 2. Sketch of approach

The general idea in the proofs of both theorems is as follows:
(a) Randomly color the edges of the graph in question. For Theorem 1 we can (in the main) use a uniformly random coloring. The distribution for Theorem 2 is a little more complicated.
(b) To prove that this works, we have to find, for each pair of vertices $x, y$, a large collection of edge disjoint paths joining them. It will then be easy to argue that at least one of these paths is rainbow colored.

[^1](c) To find these paths we pick a typical vertex $x$. We grow a regular tree $T_{x}$ with root $x$. The depth is chosen carefully. We argue that for a typical pair of vertices $x, y$, many of the leaves of $T_{x}$ and $T_{y}$ can be put into 1-1 correspondence $f$ so that (i) the path $P_{x}$ from $x$ to leaf $v$ of $T_{x}$ is rainbow colored, (ii) the path $P_{y}$ from $y$ to the leaf $f(v)$ of $T_{y}$ is ranbow colored and (iii) $P_{x}, P_{y}$ do not share color.
(d) We argue that from most of the leaves of $T_{x}, T_{y}$ we can grow a tree of depth approximately equal to half the diameter. These latter trees themselves contain a bit more than $n^{1 / 2}$ leaves. These can be constructed so that they are vertex disjoint. Now we argue that each pair of trees, one associated with $x$ and one associated with $y$, are joined by an edge.
(e) We now have, by construction, a large set of edge disjoint paths joining leaves $v$ of $T_{x}$ to leaves $f(v)$ of $T_{y}$. A simple estimation shows that $w h p$ for at least one leaf $v$ of $T_{x}$, the path from $v$ to $f(v)$ is rainbow colored and does not use a color already used in the path from $x$ to $v$ in $T_{x}$ or the path from $y$ to $f(v)$ in $T_{y}$.
We now fill in the details of both cases.

## 3. Proof of Theorem 1

Observe first that $r c(G) \geq \max \left\{Z_{1}\right.$, diameter $\left.(G)\right\}$. First of all, each edge incident to a vertex of degree one must have a distinct color. Just consider a path joining two such vertices. Secondly, if the shortest distance between two vertices is $\ell$ then we need at least $\ell$ colors. Next observe that whp the diameter $D$ is asymptotically equal to $L$, see for example [2]. We break the proof of Theorem 1$]$ into several lemmas.

Let a vertex be large if $\operatorname{deg}(x) \geq \log n / 100$ and small otherwise.
Lemma 1. Whp, there do not exist two small vertices within distance at most $3 L / 4$.
Proof.

$$
\begin{aligned}
& \operatorname{Pr}\left[\exists x, y \in[n]: \operatorname{deg}(x), \operatorname{deg}(y) \leq \log n / 100 \text { and } \operatorname{dist}(x, y) \leq \frac{3 L}{4}\right] \\
& \leq\binom{ n}{2} \sum_{k=1}^{3 L / 4} n^{k-1} p^{k}\left(\sum_{i=0}^{\log n / 100}\binom{n-1-k}{i} p^{i}(1-p)^{n-1-k}\right)^{2} \\
& \leq \sum_{k=1}^{3 L / 4} n(2 \log n)^{k}\left(2\binom{n}{\log n / 100} p^{\log n / 100}(1-p)^{n-1-\log n / 100}\right)^{2} \\
& \leq \sum_{k=1}^{3 L / 4} n(2 \log n)^{k}\left(2\left(100 e^{1+o(1)}\right)^{\log n / 100} n^{-1+o(1)}\right)^{2} \\
& \leq \sum_{k=1}^{3 L / 4} n(2 \log n)^{k} n^{-1.9} \\
& \leq 2 n(2 \log n)^{3 L / 4} n^{-1.9} \\
& \leq n^{-.1} .
\end{aligned}
$$

We use the notation $e[S]$ for the number of edges induced by a given set of vertices $S$. Notice that if a set $S$ satisfies $e[S] \geq s+t$ where $t \geq 1$, the induced subgraph $G[S]$ has at least $t+1$ cycles.


Figure 1. Structure of Lemma 3
Lemma 2. Fix $t \in \mathbb{Z}^{+}$and $0<\alpha<1$. Then, whp there does not exist a subset $S \subseteq[n]$, such that $|S| \leq \alpha t L$ and $e[S] \geq|S|+t$.

Proof. For convenience, let $s=|S|$ be the cardinality of the set $S$.Then,

$$
\left.\begin{array}{rl}
\operatorname{Pr}[\exists S: s \leq \alpha t L \text { and } e[S] \geq s+t] & \leq \sum_{s \leq \alpha t L}\binom{n}{s}\left(\begin{array}{c}
s \\
2 \\
2+t
\end{array}\right)
\end{array}\right) p^{s+t} .
$$

Remark 1. Let $T$ be a rooted tree of depth at most $4 L / 7$ and let $v$ be a vertex not in $T$, but with $b$ neighbors in $T$. Let $S$ consist of $v$, the neighbors of $v$ in $T$ plus the ancestors of these neighbors. Then $|S| \leq 4 b L / 7+1 \leq 3 b L / 5$ and $e(S)=|S|+b-2$. It follows from the proof of Lemma 圆 with $\alpha=3 / 5$ and $t=8$, that we must have $b \leq 10$ with probability $1-o\left(n^{-3}\right)$.

Our next lemma shows the existence of the subgraph $G_{x, y}^{\prime}$ described next and shown in Figure 1 for a given pair of vertices $x, y$. We first deal with paths between large vertices.

Now let

$$
\begin{equation*}
\epsilon=\epsilon(n)=o(1) \text { be such that } \frac{\epsilon \log \log n}{\log 1 / \epsilon} \rightarrow \infty \text { and let } k=\epsilon L . \tag{2}
\end{equation*}
$$

Here $L$ is defined in (1) and we could take $\epsilon=1 /(\log \log n)^{1 / 2}$.
Lemma 3. Whp, for all pairs of large vertices $x, y \in[n]$ there exists a subgraph $G_{x, y}\left(V_{x, y}, E_{x, y}\right)$ of $G$ as shown in figure 1. The subgraph consists of two isomorphic vertex disjoint trees $T_{x}, T_{y}$ rooted at $x, y$ each of depth $k . T_{x}$ and $T_{y}$ both have a branching factor of $\log n / 101$. I.e. each vertex of $T_{x}, T_{y}$


Figure 2. Subgraph found in the proof of Lemma 3 ,
has at least $\log n / 101$ neighbors, excluding its parent in the tree. Let the leaves of $T_{x}$ be $x_{1}, x_{2}, \ldots, x_{\tau}$ where $\tau \geq n^{4 \epsilon / 5}$ and those of $T_{y}$ be $y_{1}, y_{2}, \ldots, y_{\tau}$. Then $y_{i}=f\left(x_{i}\right)$ where $f$ is a natural isomporphism that preserves the parent-child relation. Between each pair of leaves $\left(x_{i}, y_{i}\right), i=1,2, \ldots, \tau$ there is a path $P_{i}$ of length $(1+2 \epsilon) L$. The paths $P_{i}, i=1,2, \ldots, \tau$ are edge disjoint.

Proof. Because we have to do this for all pairs $x, y$, we note without further comment that likely (resp. unlikely) events will be shown to occur with probability $1-o\left(n^{-2}\right)$ (resp. $o\left(n^{-2}\right)$ ).

To find the subgraph shown in Figure 1 we grow tree structures as shown in Figure 2. Specifically, we first grow a tree from $x$ using BFS until it reaches depth $k$. Then, we grow a tree starting from $y$ again using BFS until it reaches depth $k$. Finally, we grow trees from the leaves of $T_{x}$ and $T_{y}$ using BFS for depth $\gamma=\left(\frac{1}{2}+\epsilon\right) L$. Now we analyze these processes. Since the argument is the same we explain it in detail for $T_{x}$ and we outline the differences for the other trees. We use the notation $D_{i}^{(\rho)}$ for the number of vertices at depth $i$ of the BFS tree rooted at $\rho$.

First we grow $T_{x}$. As we grow the tree via BFS from a vertex $v$ at depth $i$ to vertices at depth $i+1$ certain bad edges from $v$ may point to vertices already in $T_{x}$. Remark 1 shows with probability $1-o\left(n^{-3}\right)$ there can be at most 10 bad edges emanating from $v$.

Furthermore, Lemma 1 implies that there exists at most one vertex of degree less than $\frac{\log n}{100}$ at each level whp. Hence, we obtain the recursion

$$
\begin{equation*}
D_{i+1}^{(x)} \geq\left(\frac{\log n}{100}-10\right)\left(D_{i}^{(x)}-1\right) \geq \frac{\log n}{101} D_{i}^{(x)} \tag{3}
\end{equation*}
$$

Therefore the number of leaves satisfies

$$
\begin{equation*}
D_{k}^{(x)} \geq\left(\frac{\log n}{101}\right)^{\epsilon L} \geq n^{4 \epsilon / 5} \tag{4}
\end{equation*}
$$

We can make the branching factor exactly $\frac{\log n}{101}$ by pruning. We do this so that the trees $T_{x}$ are isomorphic to each other.

With a similar argument

$$
\begin{equation*}
D_{k}^{(y)} \geq n^{\frac{4}{5} \epsilon} \tag{5}
\end{equation*}
$$

The only difference is that now we also say an edge is bad if the other endpoint is in $T_{x}$. This immediately gives

$$
D_{i+1}^{(y)} \geq\left(\frac{\log n}{100}-20\right)\left(D_{i}^{(y)}-1\right) \geq \frac{\log n}{101} D_{i}^{(y)}
$$

and the required conclusion (5).
Similarly, from each leaf $x_{i} \in T_{x}$ and $y_{i} \in T_{y}$ we grow trees $\widehat{T}_{x_{i}}, \widehat{T}_{y_{i}}$ of depth $\gamma=\left(\frac{1}{2}+\epsilon\right) L$ using the same procedure and arguments as above. Remark 1 implies that there are at most 20 edges from the vertex $v$ being explored to vertices in any of the trees already constructed. At most 10 to $T_{x}$ plus any trees rooted at an $x_{i}$ and another 10 for $y$. The numbers of leaves of each $\widehat{T}_{x_{i}}$ now satisfies

$$
\widehat{D}_{\gamma}^{\left(x_{i}\right)} \geq \frac{\log n}{100}\left(\frac{\log n}{101}\right)^{\gamma} \geq n^{\frac{1}{2}+\frac{4}{5} \epsilon}
$$

Similarly for $\widehat{D}_{\gamma}^{\left(y_{i}\right)}$.
Observe next that BFS does not condition the edges between the leaves $X_{i}, Y_{i}$ of the trees $\widehat{T}_{x_{i}}$ and $\widehat{T}_{y_{i}}$. I.e., we do not need to look at these edges in order to carry out our construction. On the other hand we have conditioned on the occurence of certain events to imply a certain growth rate. We handle this technicality as follows. We go through the above construction and halt if ever we find that we cannot expand by the required amount. Let $\mathbf{A}$ be the event that we do not halt the construction i.e. we fail the conditions of Lemmas [1] or 2. We have $\operatorname{Pr}[\mathbf{A}]=1-o(1)$ and so,

$$
\operatorname{Pr}\left[\exists i: e\left(X_{i}, Y_{i}\right)=0 \mid \mathbf{A}\right] \leq \frac{\operatorname{Pr}\left[\exists i: e\left(X_{i}, Y_{i}\right)=0\right]}{\operatorname{Pr}(\mathbf{A})} \leq 2 n^{\frac{4 \epsilon}{5}}(1-p)^{n^{1+\frac{8 \epsilon}{5}}} \leq n^{-n^{\epsilon}}
$$

We conclude that whp there is always an edge between each $X_{i}, Y_{i}$ and thus a path of length at most $(1+2 \epsilon) L$ between each $x_{i}, y_{i}$.
Let $q=(1+5 \epsilon) L$ be the number of available colors. We color the edges of $G$ randomly. We show that the probability of having a rainbow path between $x, y$ in the subgraph $G_{x, y}$ of Figure 1 is at least $1-\frac{1}{n^{3}}$.

Lemma 4. Color each edge of $G$ using one color at random from $q$ available. Then, the probability of having at least one rainbow path between two fixed large vertices $x, y \in[n]$ is at least $1-\frac{1}{n^{3}}$.
Proof. We show that the subgraph $G_{x, y}$ contains such a path. We break our proof into two steps:
Before we proceed, we provide certain necessary definitions. Think of the process of coloring $T_{x}, T_{y}$ as an evolutionary process that colors edges by starting from the two roots $x, f(x)=y$ until it reaches the leaves. In the following, we call a vertex $u$ of $T_{x}\left(T_{y}\right)$ alive/living if the path $P(x, u)$ $(P(y, u))$ from $x(y)$ to $u$ is rainbow, i.e., the edges have received distinct colors. We call a pair of vertices $\{u, f(u)\}$ alive, $u \in T_{x}, f(u) \in T_{y}$ if $u, f(u)$ are both alive and the paths $P(x, u), P(y, f(u))$ share no color. Define $A_{j}=\mid\{(u, f(u)):(u, f(u))$ is alive and $\operatorname{depth}(u)=j\} \mid$ for $j=1, . ., k$.

- Step 1: Existence of at least $n^{\frac{4}{5} \epsilon}$ living pairs of leaves

Assume the pair of vertices $\{u, f(u)\}$ is alive where $u \in T_{x}, f(u) \in T_{y}$. It is worth noticing that $u, f(u)$ have the same depth in their trees. We are interested in the number of pairs of children


Figure 3. Figure shows $\frac{\log n}{101}$-ary trees $T_{x}, T_{y}$. The two roots are shown respectively at the center of the trees. In our thinking of the random coloring as an evolutionary process, the green edges incident to $x$ survive with probability 1 , the red edges incident to $y$ with probability $1-\frac{1}{q}$ and all the other edges with probability $p_{0}=\left(1-\frac{2 k}{q}\right)^{2}$. where $k$ is the depth of both trees and $q$ the number of available colors. Our analysis in Lemma 3 using these probabilities gives a lower bound on the number of alive pairs of leaves after coloring $T_{x}, T_{y}$ from the root to the leaves respectively.
$\left\{u_{i}, f\left(u_{i}\right)\right\}_{i=1, \ldots, \log n / 101}$ that will be alive after coloring the edges from depth $(u)$ to depth $(u)+1$. A living pair $\left\{u_{i}, f\left(u_{i}\right)\right\}$ by definition has the following properties: edges $\left(u, u_{i}\right) \in E\left(T_{x}\right)$ and $\left(f(u), f\left(u_{i}\right)\right) \in E\left(T_{y}\right)$ receive two distinct colors, which are different from the set of colors used in paths $P(x, u)$ and $P(y, f(u))$. Notice the latter set of colors has cardinality $2 \times \operatorname{depth}(u) \leq 2 k$.

Let $A_{j}$ be the number of living pairs at depth $j$. We first bound the size of $A_{1}$.

$$
\begin{equation*}
\operatorname{Pr}\left[A_{1} \leq \frac{\log n}{200}\right] \leq 2^{\log n / 101}\left(\frac{1}{q}\right)^{\log n / 300}=O\left(n^{-\Omega(\log \log n)}\right) \tag{6}
\end{equation*}
$$

Here $2^{\log n / 101}$ bounds the number of choices for $A_{1}$. For a fixed set $A_{1}$ there will be at least $\frac{\log n}{101}-\frac{\log n}{200} \geq \frac{\log n}{300}$ edges incident with $x$ that have the same color as their corresponding edges incident with $y$, under $f$. The factor $q^{-\log n / 300}$ bounds the probability of this event.

For $j>1$ we see that the random variable equal to the number of living pairs of children of $(u, f(u))$ stochastically dominates the random variable $X \sim \operatorname{Bin}\left(\frac{\log n}{101}, p_{0}\right)$, where $p_{0}=\left(1-\frac{2 k}{q}\right)^{2}=$ $\left(\frac{1+3 \epsilon}{1+5 \epsilon}\right)^{2}$. The colorings of the descendants of each live pair are independent and so we have using the Chernoff bounds for $2 \leq j \leq k$,

$$
\begin{align*}
\operatorname{Pr}\left[\left.A_{j}<\left(\frac{\log n}{200}\right)^{j} p_{0}^{j-1} \right\rvert\, A_{j-1}\right. & \left.\geq\left(\frac{\log n}{200}\right)^{j-1} p_{0}^{j-2}\right] \\
& \leq \exp \left\{-\frac{1}{2} \cdot\left(\frac{99}{200}\right)^{2} \cdot \frac{\log n}{101} \cdot\left(\frac{\log n}{200}\right)^{j-1} p_{0}^{j}\right\}=O\left(n^{-\Omega(\log \log n)}\right) \tag{7}
\end{align*}
$$

(6)) and (77) justify assuming that $A_{k} \geq\left(\frac{\log n}{200}\right)^{k} p_{0}^{k-1} \geq n^{\frac{4}{5} \epsilon}$.

(A)

Figure 4. Taking care of small vertices.

- Step 2: Existence of rainbow paths between $x, y$ in $G_{x, y}$

Assuming that there are $\geq n^{4 \epsilon / 5}$ living pairs of leaves $\left(x_{i}, y_{i}\right)$ for vertices $x, y$,

$$
\operatorname{Pr}(x, y \text { are not rainbow connected }) \leq\left(1-\prod_{i=0}^{2 \gamma-1}\left(1-\frac{2 k+i}{q}\right)\right)^{n^{4 \epsilon / 5}}
$$

But

$$
\prod_{i=0}^{2 \gamma-1}\left(1-\frac{2 k+i}{q}\right) \geq\left(1-\frac{2 k+2 \gamma}{q}\right)^{2 \gamma}=\left(\frac{\epsilon}{1+5 \epsilon}\right)^{2 \gamma} .
$$

So

$$
\begin{align*}
& \operatorname{Pr}(x, y \text { are not rainbow connected }) \leq \exp \left\{-n^{4 \epsilon / 5}\left(\frac{\epsilon}{1+5 \epsilon}\right)^{2 \gamma}\right\} \\
& =\exp \left\{-n^{4 \epsilon / 5-O(\log (1 / \epsilon) / \log \log n)}\right\} . \tag{8}
\end{align*}
$$

Using (2) and the union bound taking (8) over all large $x, y$ completes the proof of Lemma 4 .
We now finish the proof of Theorem 1 i.e. take care of small vertices.
We showed in Lemma 4 that whp for any two large vertices, a random coloring results in a rainbow path joining them. We divide the small vertices into two sets: vertices of degree $1, V_{1}$ and the vertices of degree at least $2, V_{2}$. Suppose that our colors are $1,2, \ldots, q$ and $V_{1}=\left\{v_{1}, v_{2}, \ldots, v_{s}\right\}$. We begin by giving the edge incident with $v_{i}$ the color $i$. Then we slightly modify the argument in Lemma 4. If $x$ is the neighbor of $v_{i} \in V_{1}$ then color $i$ cannot be used in Steps 1 and 2 of that procedure. In terms of analysis this replaces $q$ by $(q-1)\left((q-2)\right.$ if $y$ is also a neighbor of $\left.V_{1}\right)$ and the argument is essentially unchanged i.e. whp there will be a rainbow path between each pair of large vertices. Furthermore, any path starting at $v_{i}$ can only use color $i$ once and so there will be rainbow paths between $V_{1}$ and $V_{1}$ and between $V_{1}$ and the set of large vertices.

The set $V_{2}$ is treated by using only two extra colors. Assume that Red and Blue have not been used in our coloring. Then we use Red and Blue to color two of the edges incident to a vertex $u \in V_{2}$ (the remaining edges are colored arbitrarily). This is shown in Figure 4a, Suppose that $V_{2}=\left\{w_{1}, w_{2}, \ldots, w_{t}\right\}$. Then if we want a rainbow path joining $w_{i}, w_{j}$ where $i<j$ then we use the
red edge to go to its neighbor $w_{i}^{\prime}$. Then we take the already constructed rainbow path to $w_{j}^{\prime \prime}$, the neighbor of $w_{j}$ via a blue edge. Then we can continue to $w_{j}$.

## 4. Proof of Theorem 2

We first observe that simply randomly coloring the edges of $G=G(n, r)$ with $q=n^{o(1)}$ colors will not do. This is because there will whp be $\Omega\left(n q^{1-r^{2}}\right)=\Omega\left(n^{1-o(1)}\right)$ vertices $v$ where all edges at distance at most two from $v$ have the same color.

We follow a similar strategy to the proof in Theorem 1. We grow small trees $T_{x}$ from each vertex $x$. Then for a pair of vertices $x, y$ we build disjoint trees on the leaves of $T_{x}, T_{y}$ so that whp we can find edge disjoint paths between any set of leaves $S_{x}$ of $T_{x}$ and any set of leaves of $S_{y}$ of the same size. A bounded number of leaves of $T_{x}, T_{y}$ will be excluded from this statement. The main difference will come from our procedure for coloring the edges. Because of the similarities, we will give a little less detail in the common parts of our proofs. We are in effect talking about building a structure like that shown in Figure 2. There is one difference, we will have to take care of which leaves of $T_{x}$ we pair with which leaves of $T_{y}$, for a pair of vertices $x, y$.

Having grown the trees, we have the problem of coloring the edges. Instead of independently and randomly coloring the edges, we use a greedy algorithm that produces a coloring that is guaranteed to color edges differently, if they are close. This will guarantee that the edges of $T_{x}$ are rainbow, for all vertices $x$. We then argue that we can find, for each vertex pair $x, y$, a partial mapping $g$ from the leaves of $T_{x}$ to the leaves of $T_{y}$ such that the path from $x$ to leaf $v$ in $T_{x}$ and the path from $y$ to leaf $g(v)$ in $T_{y}$ do not share a color. This assumes that $v$ has an image under the partial mapping $g$. We will have to argue that $g$ is defined on enough vertices in $T_{x}$. Given this, we then consider the colors on a set of edge disjoint paths that we can construct from the leaves of $T_{x}$ to their $g$-counterpart in the leaves of $T_{y}$.

We will use the configuration model of Bollobás [3] in our proofs, see [11] or [14] for details. Let $W=[2 m=r n]$ be our set of configuration points and let $W_{i}=[(i-1) r+1, i r], i \in[n]$, partition $W$. The function $\phi: W \rightarrow[n]$ is defined by $w \in W_{\phi(w)}$. Given a pairing $F$ (i.e. a partition of $W$ into $m$ pairs) we obtain a (multi-)graph $G_{F}$ with vertex set $[n]$ and an edge $(\phi(u), \phi(v))$ for each $\{u, v\} \in F$. Choosing a pairing $F$ uniformly at random from among all possible pairings $\Omega_{W}$ of the points of $W$ produces a random (multi-)graph $G_{F}$. Each $r$-regular simple graph $G$ on vertex set $[n]$ is equally likely to be generated as $G_{F}$. Here simple means without loops of multiple edges. Furthermore, if $r=O(1)$ then $G_{F}$ is simple with a probability bounded below by a positive value independent of $n$. Therefore, any event that occurs whp in $G_{F}$ will also occur whp in $G(n, r)$.
4.1. Tree building. We will grow a Breadth First Search tree $T_{x}$ from each vertex. We will grow each tree to depth

$$
k=k_{r}= \begin{cases}\left\lceil\log _{r-2} \log n\right\rceil & r \geq 4 \\ \left\lceil 2 \log _{2} \log n-2 \log _{2} \log _{2} \log n\right\rceil & r=3\end{cases}
$$

Observe that

$$
\begin{equation*}
T_{x} \text { has at most } r\left(1+(r-1)+(r-1)^{2}+\cdots+(r-1)^{k-1}\right)=r \frac{(r-1)^{k}-1}{r-1} \text { edges. } \tag{9}
\end{equation*}
$$

It is useful to observe that
Lemma 5. Whp, no set of $s \leq \ell_{1}=\frac{1}{10} \log _{r-1} n$ vertices contains more than $s$ edges.

Proof. Indeed,

$$
\begin{align*}
\operatorname{Pr}\left(\exists S \subseteq[n],|S| \leq \ell_{1}, e[S] \geq|S|+1\right) & \leq \sum_{s=3}^{\ell_{1}}\binom{n}{s}\binom{\binom{s}{2}}{s+1}\left(\frac{r^{2}}{r n-r s}\right)^{s+1}  \tag{10}\\
& \leq \frac{r \ell_{1}}{n} \sum_{s=3}^{\ell_{1}}\binom{n}{s}\left(\begin{array}{c}
s \\
2 \\
s
\end{array}\right)\left(\frac{r^{2}}{r n-r s}\right)^{s} \\
& \leq \frac{r \ell_{1}}{n} \sum_{s=3}^{\ell_{1}}\left(\frac{n e}{s} \cdot \frac{s e}{2} \cdot \frac{2 r}{n}\right)^{s} \\
& \leq \frac{r \ell_{1}}{n} \cdot \ell_{1} \cdot\left(e^{2} r\right)^{\ell_{1}}=o(1) \tag{11}
\end{align*}
$$

Explanation of (10): The factor $\left(\frac{r^{2}}{r n-r s}\right)^{s+1}$ can be justified as follows. We can estimate

$$
\operatorname{Pr}\left(e_{1}, e_{2}, \ldots, e_{s+1} \in E\left(G_{F}\right)\right)=\prod_{i=0}^{s} \operatorname{Pr}\left(e_{i+1} \in E\left(G_{F}\right) \mid e_{1}, e_{2}, \ldots, e_{i} \in E\left(G_{F}\right)\right) \leq\left(\frac{r^{2}}{r n-r s}\right)^{s+1}
$$

if we pair up the lowest index endpoint of each $e_{i}$ in some arbitrary order. The fraction $\frac{r^{2}}{r n-r s}$ is an upper bound on the probability that this endpoint is paired with the other endpoint, regardless of previous pairings.

Denote the leaves of $T_{x}$ by $L_{x}$.
Corollary 3. Whp, $(r-1)^{k} \leq\left|L_{x}\right| \leq r(r-1)^{k-1}$ for all $x \in[n]$.
Proof. This follows from the fact that whp the vertices spanned by each $T_{x}$ span at most one cycle. This in turn follows from Lemma 5 .
Consider two vertices $x, y \in V(G)$ where $T_{x} \cap T_{y}=\emptyset$. We will show that whp we can find a subgraph $G^{\prime}\left(V^{\prime}, E^{\prime}\right), V^{\prime} \subseteq V, E^{\prime} \subseteq E$ with similar structure to that shown in Figure 2, Here $k=k_{r}$ and $\gamma=\left(\frac{1}{2}+\epsilon\right) \log _{r-1} n$ for some small positive constant $\epsilon$.
Remark 2. In our analysis we expose the pairing $F$, only as necessary. For example the construction of $T_{x}$ involves exposing all pairings involving non-leaves of $T_{x}$ and one pairing for each leaf. There can be at most one exception to this statement, for the rare case where $T_{x}$ contains a unique cycle. In particular, if we expose the point $q$ paired with a currently unpaired point $p$ of a leaf of $T_{x}$ then $q$ is chosen randomly from the remaining unpaired points.

Suppose that we have constructed $i=O(\log n)$ vertex disjoint trees of depth $\gamma$ rooted at some of the leaves of $T_{x}$. We grow the $(i+1)$ st tree $\widehat{T}_{z}$ via BFS, without using edges that go into $y$ or previously constructed trees. Let a leaf $z \in L_{x}$ be bad if we have to omit a single edge as we construct the first $\ell_{1} / 2$ levels of $\widehat{T}_{z}$. The previously constructed trees plus $y$ account for $O\left(n^{1 / 2+\epsilon}\right)$ vertices and pairings, so the probability that $z$ is bad, given all the pairings we have exposed so far, is at most $O\left((r-1)^{\ell_{1} / 2} n^{-1 / 2+\epsilon}\right)=O\left(n^{-1 / 3}\right)$. Here bad edges can only join two leaves. This probability bound holds regardless of whichever other vertices are bad. This follows from the way we build the pairing $F$, see the final statement of Remark 2, So whp there will be at most 3 bad leaves on any $T_{x}$. Indeed, $\operatorname{Pr}(\exists x: x$ has $\geq 4$ bad leaves $) \leq n\binom{O(\log n)}{4} n^{-4 / 3}=o(1)$.

If a leaf is not bad then the first $\ell_{1} / 2$ levels produce $\Theta\left(n^{1 / 20}\right)$ leaves. From this, we see that whp the next $\gamma-\ell_{1}$ levels grow at a rate $r-1-o\left(n^{-1 / 25}\right)$. Indeed, given that a level has $L$ vertices where
$n^{1 / 20} \leq L \leq n^{3 / 4}$, the number of vertices in the next level dominates $\operatorname{Bin}\left((r-1) L, 1-O\left(\frac{n^{3 / 4}}{n}\right)\right)$, after accounting for the configuration points used in building previous trees. Indeed, $(r-1) L$ configuration points associated with good leaves will be unpaired and for each of them, the probability it is paired with a point associated with a vertex in any of the trees constructed so far is $O\left(n^{1 / 2+2 \epsilon} / n\right)$. This probability bound holds regardless of the pairings of the other leaf configuration points. We can thus assert that whp we will have that all but at most three of the leaves $L_{x}$ of $T_{x}$ are roots of vertex disjoint trees $\widehat{T}_{1}, \widehat{T}_{2}, \ldots$, each with $\Theta\left(n^{1 / 2+\epsilon / 2}\right)$ leaves. Let $L_{x}^{*}$ denote these good leaves. The same analysis applies when we build trees $\widehat{T}_{1}^{\prime}, \widehat{T}_{2}^{\prime}, \ldots$, with roots at $L_{y}$.

Now the probability that there is no edge joining the leaves of $\widehat{T}_{i}$ to the leaves of $\widehat{T}_{j}^{\prime}$ is at most

$$
\left(1-\frac{(r-1) \Theta\left(n^{1 / 2+\epsilon / 2}\right)}{r n}\right)^{(r-1) n^{1 / 2+\epsilon / 2}} \leq e^{-\Omega\left(n^{\epsilon}\right)}
$$

To summarise,
Remark 3. Whp we will succeed in finding in $G_{F}$ and hence in $G=G(n, r)$, for all $x, y \in V\left(G_{F}\right)$, for all $u \in L_{x}^{*}, v \in L_{y}^{*}$, a path $P_{u, v}$ from $u$ to $v$ of length $O(\log n)$ such that if $u \neq u^{\prime}$ and $v \neq v^{\prime}$ then $P_{u, v}$ and $P_{u^{\prime}, v^{\prime}}$ are edge disjoint. These paths avoid $T_{x}, T_{y}$ except at their start and endpoints.
4.2. Coloring the edges. We now consider the problem of coloring the edges of $G$. Let $H$ denote the line graph of $G$ and let $\Gamma=H^{2 k}$ denote the graph with the same vertex set as $H$ and an edge between vertices $e, f$ of $\Gamma$ if there there is a path of length at most $k$ between $e$ and $f$ in $H$. We will construct a proper coloring of $\Gamma$ using

$$
q=10(r-1)^{2 k} \sim 100 \log ^{2 \theta_{r}} n \text { where } \theta_{r}=\frac{\log (r-1)}{\log (r-2)}
$$

colors. We do this as follows: Let $e_{1}, e_{2}, \ldots, e_{m}$ be an arbitrary ordering of the vertices of $\Gamma$. For $i=1,2, \ldots, m$, color $e_{i}$ with a random color, chosen uniformly from the set of colors not currrently appearing on any neighbor in $\Gamma$. At this point only $e_{1}, e_{2}, \ldots, e_{i-1}$ will have been colored.

Suppose then that we color the edges of $G$ using the above method. Fix a pair of vertices $x, y$ of $G$. We see immediately, that no color appears twice in $T_{x}$ and no color appears twice in $T_{y}$. This is because the distance between edges in $T_{x}$ is at most $2 k$. This also deals with the case where $V\left(T_{x}\right) \cap V\left(T_{y}\right) \neq \emptyset$, for the same reason. So assume now that $T_{x}, T_{y}$ are vertex disjoint. We can find lots of paths joining $x$ and $y$. We know that the first and last $k$ edges of each path will be individually rainbow colored. We will first show that we have many choices of path where these $2 k$ edges are rainbow colored when taken together.
4.3. Case 1: $r \geq 4$ : We argue now that we can find $\sigma_{0}=(r-2)^{k-1}$ leaves $u_{1}, u_{2}, \ldots, u_{\tau} \in T_{x}$ and $\sigma_{0}$ leaves $v_{1}, v_{2}, \ldots, v_{\tau} \in T_{y}$ such for each $i$ the $T_{x}$ path from $x$ to $u_{i}$ and the $T_{y}$ path from $y$ to $v_{i}$ do not share any colors.

Lemma 6. Let $T_{1}, T_{2}$ be two vertex disjoint copies of an edge colored complete d-ary tree with $\ell$ levels, where $d \geq 3$. Let $T_{1}, T_{2}$ be rooted at $x, y$ respectively. Suppose that the colorings of $T_{1}, T_{2}$ are both rainbow. Let $\kappa=(d-1)^{\ell}$. Then there exist leaves $u_{1}, u_{2}, \ldots, u_{\kappa}$ of $T_{1}$ and leaves $v_{1}, v_{2}, \ldots v_{\kappa}$ of $T_{2}$ such that the following is true: If $P_{i}, P_{i}^{\prime}$ are the paths from $x$ to $u_{i}$ in $T_{1}$ and from $y$ to $v_{i}$ in $T_{2}$ respectively, then $P_{i} \cup P_{i}^{\prime}$ is rainbow colored for $i=1,2, \ldots, \kappa$.

Proof. Let $A_{\ell}$ be the minimum number of rainbow path pairs that we can find in any such pair of edge colored trees. We prove that $A_{\ell} \geq(d-1)^{\ell}$ by induction on $\ell$. This is true trivially for
$\ell=0$. Suppose that $x$ is incident with $x_{1}, x_{2}, \ldots, x_{d}$ and that the sub-tree rooted at $x_{i}$ is $T_{1, i}$ for $i=1,2, \ldots, d$. Define $y_{i}$ and $T_{2, i}, i=1,2, \ldots, d$ similarly with respect to $y$. Suppose that the color of the edge $\left(x, x_{i}\right)$ is $c_{i}$ for $i=1,2, \ldots, d$ and let $Q_{x}=\left\{c_{1}, c_{2}, \ldots, c_{d}\right\}$. Similarly, suppose that the color of the edge $\left(y, y_{i}\right)$ is $c_{i}^{\prime}$ for $i=1,2, \ldots, d$ and let $Q_{y}=\left\{c_{1}^{\prime}, c_{2}^{\prime}, \ldots, c_{d}^{\prime}\right\}$. Next suppose that $Q_{j}$ is the set of colors in $Q_{x}$ that appear on the edges $E\left(T_{2, j}\right) \cup\left\{\left(y, y_{j}\right)\right\}$. The sets $Q_{1}, Q_{2}, \ldots, Q_{d}$ are pair-wise disjoint. Similarly, suppose that $Q_{i}^{\prime}$ is the set of colors in $Q_{y}$ that appear on the edges $E\left(T_{1, i}\right) \cup\left\{\left(x, x_{i}\right)\right\}$. The sets $Q_{1}^{\prime}, Q_{2}^{\prime}, \ldots, Q_{d}^{\prime}$ are pair-wise disjoint.

Now define a bipartite graph $H$ with vertex set $A+B=[d]+[d]$ and an edge $(i, j)$ iff $c_{i} \notin Q_{j}$ and $c_{j}^{\prime} \notin Q_{i}^{\prime}$. We claim that if $S \subseteq A$ then its neighbor set $N_{H}(S)$ satisfies the inequality

$$
\begin{equation*}
d|S|-\left|N_{H}(S)\right|-|S| \leq|S| \cdot\left|N_{H}(S)\right| . \tag{12}
\end{equation*}
$$

Here the LHS of (12) bounds from below, the size of the set $S: N_{H}(S)$ of edges between $S$ and $N_{H}(S)$. This is because there are at most $|S|$ edges missing from $S: N_{H}(S)$ due to $i \in S$ and $j \in N_{H}(S)$ and $c_{i} \in Q_{j}$. At most $\left|N_{H}(S)\right|$ edges are missing for similar reasons. On the other hand, $d|S|$ is the number there would be without these missing edges. The RHS of (12) is a trivial upper bound.

Re-arranging we get that

$$
\left|N_{H}(S)\right|-|S| \geq\left\lceil\frac{(d-2-|S|)|S|}{|S|+1}\right\rceil \geq-1
$$

(We get -1 when $|S|=d$ ).
Thus $H$ contains a matching $M$ of size $d-1$. Suppose without loss of generality that this matching is $(i, i), i=1,2, \ldots, d-1$. We know by induction that for each $i$ we can find paths $\left(P_{i, j}, \widehat{P}_{i, j}\right), j=1,2, \ldots,(d-1)^{\ell-1}$ where $P_{i, j}$ is a root to leaf path in $T_{1, i}$ and $\widehat{P}_{i, j}$ is a root to leaf path in $T_{2, i}$ and that $P_{i, j} \cup \widehat{P}_{i, j}$ is rainbow for all $i, j$. Furthermore, $(i, i)$ being an edge of $H$, means that the edge sets $\left\{\left(x, x_{i}\right)\right\} \cup E\left(P_{i, j}\right) \cup E\left(\widehat{P}_{i, j}\right) \cup\left\{\left(y, y_{i}\right\}\right.$ are all rainbow.

Let

$$
V_{1}=\left\{x: V\left(T_{x}\right) \text { contains a cycle }\right\} .
$$

When $x, y \notin V_{1}$ we apply this Lemma to $T_{x}, T_{y}$ by deleting one of the $r$ sub-trees attached to each of $x, y$ and applying the lemma directly to the $(r-1)$-ary trees that remain. This will yield $(r-2)^{k}$ pairs of paths. If $x \in V_{1}$, we delete $r-2$ sub-trees attached to $x$ leaving at least two ( $r-1$ )-ary trees of depth $k-1$ with roots adjacent to $x$. We can do the same at $y$. Let $c_{1}, c_{2}$ be the colors of the two edges from $x$ to the roots of these two trees $T_{1}, T_{2}$. Similarly, let $c_{1}^{\prime}, c_{2}^{\prime}$ be the colors of the two analogous edges from $y$ to the trees $T_{1}^{\prime}, T_{2}^{\prime}$. If color $c_{1}$ does not appear in $T_{1}^{\prime}$ then we apply the lemma to $T_{1}$ and $T_{1}^{\prime}$. Otherwise, we can apply the lemma to $T_{1}$ and $T_{2}^{\prime}$. In both cases we obtain $(r-2)^{k-1}$ pairs of paths.

Accounting for bad vertices we put

$$
\sigma=\sigma_{0}-6=(r-2)^{k-1}-6 \geq \frac{\log n}{r-2}-6
$$

and we see from Remark 3 that we can $w h p$ find $\sigma$ paths $P_{1}, P_{2}, \ldots, P_{\sigma}$ of length $O(\log n)$ from $x$ to $y$. Path $P_{i}$ goes from $x$ to a leaf $u_{i} \in L_{x}^{*}$ via $T_{x}$ and then traverses $Q_{i}=P\left(u_{i}, v_{i}\right)$ where $v_{i}=\phi\left(u_{i}\right) \in L_{y}^{*}$ and then goes from $v_{i}$ to a $y$ via $T_{y}$. Here $\phi$ is some partial map from $L_{x}^{*}$ to $L_{y}^{*}$. It is a random variable that depends on the coloring $\mathcal{C}$ of the edges of $T_{x}$ and $T_{y}$. The paths $P_{1}, P_{2}, \ldots, P_{\sigma}$ depend on the choice of $\phi$ and hence $\mathcal{C}$ and so we should write $P_{i}=P_{i}(\mathcal{C})$.

We fix the coloring $\mathcal{C}$ and hence $P_{1}, P_{2}, \ldots, P_{\sigma}$. Let $\mathcal{R}$ be the event that at least one of the paths $P_{1}, P_{2}, \ldots, P_{\sigma}$ is rainbow colored. We show that $\operatorname{Pr}(\neg \mathcal{R} \mid \mathcal{C})$ is small.

We let $c(e)$ denote the color of edge $e$ in a given coloring. We remark next that for a particular coloring $c_{1}, c_{2}, \ldots, c_{m}$ of the edges $e_{1}, e_{2}, \ldots, e_{m}$ we have

$$
\operatorname{Pr}\left(c\left(e_{i}\right)=c_{i}, i=1,2, \ldots, m\right)=\prod_{i=1}^{m} \frac{1}{a_{i}}
$$

where $q-\Delta \leq a_{i} \leq q$ is the number of colors available for the color of the edge $e_{i}$ given the coloring so far i.e. the number of colors unused by the neighbors of $e_{i}$ in $\Gamma$ when it is about to be colored.

Now fix an edge $e=e_{i}$ and the colors $c_{j}, j \neq i$. Let $C$ be the set of colors not used by the neighbors of $e_{i}$ in $\Gamma$. The choice by $e_{i}$ of its color under this conditioning is not quite random, but close. Indeed, we claim that for $c, c^{\prime} \in C$

$$
\frac{\operatorname{Pr}\left(c(e)=c \mid c\left(e_{j}\right)=c_{j}, j \neq i\right)}{\operatorname{Pr}\left(c(e)=c^{\prime} \mid c\left(e_{j}\right)=c_{j}, j \neq i\right)} \leq\left(\frac{q-\Delta}{q-\Delta-1}\right)^{\Delta}
$$

This is because, changing the color of $e_{i}$ only affects the number of colors available to neighbors of $e_{i}$, and only by at most one.

Thus, for $c \in C$, we have

$$
\operatorname{Pr}\left(c(e)=c \mid c\left(e_{j}\right)=c_{j}, j \neq i\right) \leq \frac{1}{q-\Delta}\left(\frac{q-\Delta}{q-\Delta-1}\right)^{\Delta}
$$

Now $\Delta \leq(r-1)^{2 k}=q / 10$ and we deduce that

$$
\operatorname{Pr}\left(c(e)=c \mid c\left(e_{j}\right)=c_{j}, j \neq i\right) \leq \frac{2}{q}
$$

It follows that for $i \in[\sigma]$,

$$
\operatorname{Pr}\left(P_{i} \text { is rainbow colored } \mid \mathcal{C}, \text { coloring of } \bigcup_{j \neq i} Q_{j}\right) \geq\left(1-\frac{4(k+\gamma)}{q}\right)^{2 \gamma}
$$

This is because when we consider the coloring of $Q_{i}$ there will always be at most $2 k+2 \gamma$ colors forbidden by non-neighboring edges, if it is to be rainbow colored.

It then follows that

$$
\begin{aligned}
\operatorname{Pr}(\neg \mathcal{R} \mid \mathcal{C}) & \leq\left(1-\left(1-\frac{4(k+\gamma)}{q}\right)^{2 \gamma}\right)^{\sigma} \\
& \leq\left(\frac{8 \gamma(k+\gamma)}{q}\right)^{\sigma} \\
& \leq\left(\frac{(2+10 \epsilon) \log _{r-1}^{2} n}{10 \log ^{\theta_{r}} n}\right)^{\sigma}=o\left(n^{-2}\right)
\end{aligned}
$$

This completes the proof of Theorem 2 when $r \geq 4$.
Case 2: $r=3$ :
When $r=3$ we can't use $(r-2)^{k}$ to any effect. Also, we need to increase $q$ to $\log ^{4} n$. This necessary for a variety of reasons. One reason is that we will reduce $\sigma$ to $2^{k / 2}$. We want this to be $\Omega(\log n)$ and this will force $k$ to (roughly) double what it would have been if we had followed the recipe for $r \geq 4$. This makes $\Delta$ close to $\log ^{4} n$ and we need $q \gg \Delta$.

And we need to modify the argument based on Lemma 6. Instead of inducting on the trees at depth one from the roots $x, y$, we now induct on the trees at depth two. Assume first that $x, y \notin V_{1}$. After ignoring one branch for $T_{x}$ and $T_{y}$ we now consider the sub-trees $T_{x, i}, T_{y, i}, i=1,2,3,4$ of $T_{x}, T_{y}$ whose roots $x_{1}, \ldots, x_{4}$ and $y_{1}, \ldots, y_{4}$ are at depth two. We cannot necessarily make this construction when $x \in V_{1}$. Let $P_{i}$ be the path from $x$ to $x_{i}$ in $T_{x}$ and let $\widehat{P}_{j}$ be the path from $y$ to $y_{j}$ in $T_{y}$. Next suppose that $\widehat{Q}_{j}$ is the set of colors in $Q$ that appear on the edges $E\left(T_{y, j}\right) \cup E\left(\widehat{P}_{j}\right)$. Similarly, suppose that $Q_{i}^{\prime}$ is the set of colors in $Q^{\prime}$ that appear on the edges $\left\{E\left(T_{x, i}\right) \cup E\left(P_{i}\right)\right\}$.

Re-define $H$ to be the bipartite graph with vertex set $A+B=[4]+[4]$. The edges of $H$ are as before: $(i, j)$ exists iff $c_{i} \notin Q_{j}$ and $c_{j}^{\prime} \notin \widehat{Q}_{i}$. This time we can only say that a color is in at most two $\widehat{Q}_{i}$ 's and similarly for the $Q_{j}^{\prime}$ 's. The effect of this is to replace (12) by

$$
4|S|-2\left(\left|N_{H}(S)\right|+|S|\right) \leq|S| \cdot\left|N_{H}(S)\right|
$$

from which we can deduce that

$$
|S|-\left|N_{H}(S)\right| \leq \frac{|S| \cdot\left|N_{H}(S)\right|}{2} \leq 2\left|N_{H}(S)\right|
$$

It follows that $\left|N_{H}(S)\right| \geq\lceil|S| / 3\rceil \geq|S|-2$ and so $H$ contains a matching of size two. An inductive argument then shows that we are able to find $2{ }^{\lfloor k / 2\rfloor}$ rainbow pairs of paths. The proof now continues as in the case $r \geq 4$, arguing about the coloring of paths $P_{1}, P_{2}, \ldots, P_{\sigma}$ where now $\sigma=2^{\lfloor k / 2\rfloor}$.

We finally deal with the vertices in $V_{1}$. We classify them according to the size of the cycle $C_{x}$ that is contained in $V\left(T_{x}\right)$. If $T_{x}$ contains a cycle $C_{x}$ then necessarily $\left|C_{x}\right| \leq 2 k$ and so there are at most $2 k$ types in our classification. It follows from Lemma 5 that if $x, y \in V_{1}$ and $T_{x} \cap T_{y} \neq \emptyset$ then $C_{x}=C_{y} w h p$. Note next that the distance from $x$ to $C_{x}$ is at most $k-\left|C_{x}\right| / 2$. If $C$ is a cycle of length at most $2 k$, let $V_{C}=\left\{x: C=C_{x}\right\}$ and let $E_{C}$ be the set of edges contained in $V_{C}$. We have

$$
\begin{equation*}
\left|V_{C}\right|=O\left(|C| 2^{k-|C| / 2}\right)=O\left(2^{k}\right)=O\left(\log ^{2} n / \log \log n\right) \tag{13}
\end{equation*}
$$

We introduce $2 k$ new sets $\widehat{Q}_{i}, i=3,4, \ldots, 2 k$ of $O\left(\log ^{2} n / \log \log n\right)$ colors, distinct from $Q$. Thus we introduce $O\left(\log ^{2} n\right)$ new colors overall. We re-color each $E_{C}$ with the colors from $\widehat{Q}_{|C|}$. It is important to observe that if $|C|=\left|C^{\prime}\right|$ then the graphs induced by $V_{C}$ and $V_{C^{\prime}}$ are isomorphic and so we can color them isomorphically. By the latter we mean that we choose some isomorphism $f$ from $V_{C}$ to $V_{C^{\prime}}$ and then if $e$ is an edge of $V_{C}$ then we color $e$ and $f(e)$ with the same color. After this re-coloring, we see that if $T_{x}$ and $T_{y}$ are not vertex disjoint, then they are contained in the same $V_{C}$. The edges of $V_{C}$ are rainbow colored and so now we only need to concern ourselves with $x, y \in V_{1}$ such that $T_{x}$ and $T_{y}$ are vertex disjoint. Assume now that $x, y \in V_{1}$.

Assume first that $x, y$ are of the same type and that they are at the same distance from $C_{x}, C_{y}$ respectively. Our aim now is to define binary trees $T_{x}^{\prime}, T_{y}^{\prime}$ "contained " in $T_{x}, T_{y}$ that can be used as in Lemma 6. If we delete an edge $e=(u, v)$ of $C_{x}$ then the graph that remains on $V\left(T_{x}\right)$ is a tree with at most two vertices $u, v$ of degree two. Now delete one of the three sub-trees of $T_{x}$. If there are vertices of degree two, make sure one of them is in this sub-tree. If necessary, shrink the path of length two with the remaining vertex of degree two in the middle to an edge $e_{x}$. It has leaves at depth $k-1$ and leaves at depth $k-2$. The resulting binary tree will be our $T_{x}^{\prime}$. The leaves at depth $k-1$ come in pairs. Delete one vertex from each pair and shrink the paths of length two through the vertex at depth $k-2$ to an edge.

The edges that are obtained by shrinking paths of length two will have two colors. Because $x, y$ are at the same distance from their cycles, we can delete $f(e)$ from $C_{y}$ and do the construction so that $T_{x}^{\prime}$ and $T_{y}^{\prime}$ will be isomorphically colored.

It is now easy to find $2^{k-2}$ pairs of paths whose unions are rainbow colored. Each leaf of $T_{x}, T_{y}$ can be labelled by a $\{0,1\}$ string of length $k-2$. We pair string $\xi_{1} \xi_{2} \cdots \xi_{k-1} \xi_{k-2}$ in $T_{x}$ with ( $\left.1-\xi_{1}\right) \xi_{2} \cdots \xi_{k-1} \xi_{k-2}$ in $T_{y}$. The associated paths will have a rainbow union. The proof now continues as in the case $r \geq 4$, arguing about the coloring of paths $P_{1}, P_{2}, \ldots, P_{\sigma}$ where now $\sigma=2^{k-2}$.

If $x$ is further from $C_{x}$ than $y$ is from $C_{y}$ then let $z$ be the vertex on the path from $x$ to $C_{x}$ at the same distance from $C_{x}$ as $y$ is from $C_{y}$. We have a rainbow path from $z$ to $y$ and adding the $T_{x}$ path from $x$ to $z$ gives us a rainbow path from $x$ to $y$. This relies on the fact that $V_{C_{x}}$ and $V_{C_{y}}$ are isomorphically colored.

If $x, y$ are of a different type, then $T_{x}$ and $T_{y}$ are re-colored with distinct colors and we can proceed as as in the case $r \geq 4$, arguing about the coloring of paths $P_{1}, P_{2}, \ldots, P_{\sigma}$ where now $\sigma=2^{k}$, using Corollary 3 .

If $x \in V_{1}$ and $y \notin V_{1}$ then we can proceed as if both are not in $V_{1}$. This is because of the re-coloring of the edges of $T_{x}$. We can proceed as as in the case $r \geq 4$, arguing about the coloring of paths $P_{1}, P_{2}, \ldots, P_{\sigma}$ where now $\sigma=2^{k}$, using Corollary 3,

This completes our proof of Theorem 2.

## 5. Conclusion

In this work we have given an aymptotically tight result on the rainbow connectivity of $G=G(n, p)$ at the connectivity threshold. It is reasonable to conjecture that this could be tightened:

Conjecture: $W h p, r c(G)=\max \left\{Z_{1}\right.$, $\left.\operatorname{diameter}(G(n, p))\right\}$.
Our result on random regular graphs is not so tight. It is still reasonable to believe that the above conjecture also holds in this case. (Of course $Z_{1}=0$ here).
It is worth mentioning that if the degree $r$ in Theorem 2 is allowed to grow as fast as $\log n$ then one can prove a result closer to that of Theorem [1.

## References

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[^1]:    ${ }^{1}$ An event $A_{n}$ holds with high probability $(w h p)$ if $\lim _{n \rightarrow+\infty} \operatorname{Pr}\left[A_{n}\right]=1$.

