Algorithmic randomness and Ramsey properties of countable homogeneous structures.

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Abstract

We study, in the context of algorithmic randomness, the closed amenable subgroups of the symmetric group S_{∞} of a countable set. In this paper we address this problem by investigating a link between the symmetries associated with Ramsey Fraissé order classes and algorithmic randomness.

Keywords: Martin-Löf randomness, topological dynamics, amenable groups, Fraïssé limits, Ramsey theory.

1 Introduction

The focus of this work is , as for example in [4, 5], on the problem of understanding the symmetries that transform a recursively presented universal structure, which in this paper will be a Fraïssé limit of finite first order structures, to a copy of such a structure which is Martin-Löf random relative to a canonical S_{∞} -invariant measure on the class of all universal structures of the given type. Here S_{∞} is the symmetric group of a countable set, with the pointwise convergence topology. This invesigation leads to a link between the symmetries associated with the so-called discernable flows in structural Ramsey theory and algorithmic randomness.

Glasner and Weiss [6] showed that there exists a unique measure on the set of linear orderings of the natural numbers (seen as a subset of Cantor space) that is invariant under the canonical action of the symmetric group of the natural numbers. The author [5] showed that this measure is computable and studied the associated Martin-Löf (ML) random points, which, due to the uniqueness of the Glasner-Weiss

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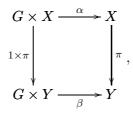
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measure, may be regarded as random linear orders. The author showed that any ML-random linear order has the order type of the rationals. Moreover, it was shown that recent work by Kechris and Sokic [12] implies that no random linear order can be the extension of the universal poset (the Fraïssé limit of finite posets) to a linear order. In [5] a study was made of so-called "randomizers". These are permutations of the natural numbers that transform a computable (Cantor) rational linear order into a random one. It was also proven in [5] that any such randomizer cannot be an automorphism of the universal poset.

The aim of this paper is to generalise these results to a broader class of Fraïssé limits \mathbb{F}_0 of Ramsey classes, the automorphism groups $\operatorname{Aut}(\mathbb{F}_0)$ not being amenable. Again, as in [5], this paper relies heavily on the groundbraking paper [10] by Kechris, Pestov and Todorcevic. The arguments in this paper require some understanding of the subtle interplay between structural Ramsey theory and topological dynamics as is beautifully explicated in the paper [10]. This paper has been written in such a way that it should be accessible to a non-specialist in Ramsey theory.

2 Preliminaries on amenable groups

Let G be a topological group and X a compact Hausdorff space. A dynamical system (X, G) (or a G-flow on X) is given by a jointly continuous action of G on X. If (Y, G) is a second dynamical system, then a G-morphism $\pi : (X, G) \to (Y, G)$ is a continuous mapping $\pi : X \to Y$ which intertwines the G-actions, i.e., the diagram



commutes with α, β being the group actions.

An isomorphism is a bijective homomorphism. A subflow of (X, G) is a G-flow on a compact subspace Y of X with the action of G on X restricted to the action on Y. A G-flow is minimal if it has no proper subflows. Every dynamical system has a minimal subflow (Zorn).

The following fact, first proven by Ellis (1949) [2], is central to the theory of dynamical systems:

Theorem 1 Let G be a Hausdorff topological group. There exists, up to G-isomorphism, a unique minimal dynamical system, denoted by (M(G), G), such that for every minimal dynamical system (X, G) there exists a G-epimorphism

$$\pi: (M,G) \longrightarrow (X,G),$$

and any two such universal systems are isomorphic.

The flow (M(G), G) is called the *universal minimal flow* of G.

We next introduce the notion of amenable groups.

Definition 1 A topological group G is amenable if, whenever X is a non-empty compact Hausdorff space and π is a continuous action of G on X, then there is a G-invariant Borel probability measure on X.

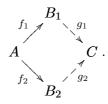
This means that, for every G-flow on a compact space X, there is a measure ν on the Borel algebra of X, such that, $\nu(X) = 1$ and, for every $g \in G$ and Borel subset U of X,

$$\nu(gU) = \nu(U).$$

3 Fraïssé limits and their recursive representations

In the sequel, \mathcal{L} will stand for the signature of a relational structure. Moreover, \mathcal{L} will always be finite and the arities of the relational symbols will all be ≥ 1 . The definitions that follow were introduced by Fraïssé in 1954.

The age of an \mathcal{L} -structure X, written Age(X), is the class of all finite \mathcal{L} -structures (defined on finite ordinals) which can be embedded as \mathcal{L} -structures into X. The structure X is homogeneous (some authors say ultrahomogeneous) if, given any isomorphism $f: A \to B$ between finite substructures of X, there is an automorphism gof X whose restriction to A is f. A class \mathbf{K} of finite \mathcal{L} -structures has the amalgamation property if, for structures A, B_1, B_2 in \mathbf{K} and embeddings $f_i: A \to B_i$ (i = 1, 2)there is a structure C in \mathbf{K} and there are embeddings $g_i: B_i \to C$ (i = 1, 2), such that the following diagram commutes:



Suppose ${\bf K}$ is a countable class of finite ${\cal L}\mbox{-}{\rm structures},$ the domains of which are finite ordinals such that

- 1. if A is a finite \mathcal{L} -structure defined on some finite ordinal, if $B \in \mathbf{K}$ and if there is an embedding of A into B, then $A \in \mathbf{K}$;
- 2. the class **K** has the amalgamation property.

Then, Fraïssé showed that there is a countable homogeneous structure X such that $Age(X) = \mathbf{K}$. Moreover, X is unique up to isomorphism. The (essentially) unique X is called the *Fraïssé limit* K of K. Note that, conversely, the age K of a countable homogeneous structure has properties (1) and (2). We shall frequently call a countable structure which is isomorphic to a Fraïssé limit a *universal structure*.

A recursive representation of a countably infinite \mathcal{L} -structure X is a bijection $\phi : \mathbf{X} \to \mathbb{N}$ such that, for each $R \in \mathcal{L}$, if the arity of R is n, then the relation R^{ϕ} defined on \mathbb{N}^n by

$$R^{\phi}(x_1, x_2, \dots, x_n) \Longleftrightarrow R\left(\phi^{-1}(x_1), \dots, \phi^{-1}(x_n)\right),$$

is recursive. If we identify the underlying set of X with \mathbb{N} via ϕ and each R with R^{ϕ} , we call the resulting structure a *recursive* \mathcal{L} -structure on \mathbb{N} and we say it has a recursive representation on \mathbb{N} .

If X is countable and homogeneous and if Age(X) has an enumeration A_0 , A_1 , A_2 , ..., possibly with repetition, with the property that there is a recursive procedure that yields, for each $i \in \mathbb{N}$, and $R \in \mathcal{L}$, the underlying set A(i) of A_i together with the interpretation of R in A(i), then we call $(A_i : i \in \mathbb{N})$ a recursive enumeration of Age(X). It follows from the construction of Fraïssé limits from their ages, that one can construct a recursive representation of X from a recursive enumeration of its age. (Conversely, it is trivial to derive a recursive enumeration of Age(X)from a recursive representation of X.) It is therefore not difficult to find recursive representations for Fraïssé limits of classes **K** from recursive enumerations of their ages.

Theorem 2 Suppose \mathbb{C} and \mathbb{D} are countable recursively represented \mathcal{L} -structures on \mathbb{N} with the same age. Suppose that they are both homogeneous. Then there is a recursive isomorphism from \mathbb{C} to \mathbb{D} .

Proof. As mentioned in [5], the model-theoretic back-and-forth argument as discussed, for example, on pp 161-162 of Hodges [7] is constructive relative to the recursive representations of the homogeneous structures \mathbb{C} and \mathbb{D} .

4 Structural Ramsey theory in a model theoretic context

In this section we summarise the results from [10] which underly the formalisation and proof of the main theorem of this paper. Unless otherwise stated all the proofs of the statements made here can be found in [10].

Let **K** be the age of some countable \mathcal{L} -structure. For $A, \pi \in \mathbf{K}$ we denote by A^{π} the set of all the (model-theoretic) structure-preserving embeddings of π in A. For a natural number $r \geq 1$ and for $\pi, A, B \in \mathbf{K}$ we introduce the predicate $B \rightsquigarrow (A)_r^{\pi}$ (Erdős-notation) to mean:

$$B \rightsquigarrow (A)_r^{\pi} \Longleftrightarrow \left(\forall B^{\pi} \xrightarrow{\chi} r \exists A \xrightarrow{\alpha} B \xrightarrow{A^{\pi}} r \xrightarrow{\alpha_*} B^{\pi} \right).$$

Here $\alpha_* : A^{\pi} \to B^{\pi}$ is the mapping that takes an embedding $\pi \to A$ to the induced embedding $\pi \to A$.

In other words, $B \rightsquigarrow (A)_r^{\pi}$ iff: for every *r*-colouring χ of the set B^{π} consisting of the embeddings of π in *B* (copies of π in *B*), there is an embedding α of *A* into *B* such that $\chi \alpha_*$ is a constant. This means that χ assumes a constant value on all the embeddings of π into the image $A' \subset B$ of *A* under α .

We shall call an age **K** a *Ramsey age* if, for all $\pi, A \in \mathbf{K}$ with $A^{\pi} \neq \emptyset$, and all natural numbers $r \geq 1$, there is some $B \in \mathbf{K}$ such that $B \rightsquigarrow (A)_r^{\pi}$.

Assume \mathcal{L} is a countable signature containing a distinguished binary relation symbol <. An order structure A for the signature \mathcal{L} with the distinguished symbol <, is a structure A for which the interpretation $<^A$ of the symbol < in A is a total ordering.

An order class **K** for \mathcal{L} is one for which all $A \in \mathbf{K}$ are order structures (relative to the distinguished <).

Let \mathcal{L}_0 be the signature obtained by removing the distinguished symbol < from \mathcal{L} . For any \mathcal{L} -structure A, denote by A_0 the \mathcal{L}_0 -structure which is the reduct of A to \mathcal{L}_0 . This means that A_0 is the structure A where the distinguished order < interpretated as a total order <^A in A is being ignored.

Let **K** be a Fraïssé order class. Denote by \mathbf{K}_0 the class of all reducts A_0 for some $A \in \mathbf{K}$. Write \mathbb{F} for the Fraïssé limit of **K**. We now discuss when \mathbf{K}_0 is also a Fraïssé class with limit \mathbb{F}_0 the latter being the reduct of \mathbb{F} to \mathcal{L}_0 . Following Kechris, Pestov and Todorcevic [10], we say that the class **K** is *reasonable* if for every $A_0, B_0 \in \mathbf{K}_0$, and linear ordering < on A_0 such that $(A_0, <) \in \mathbf{K}$, and for an embedding $\pi : A_0 \to B_0$, there is a linear ordering $<_1$ on B_0 so that $B = (B_0, <_1,) \in \mathbf{K}$ and $\pi : A \to B$ is also an embedding. (This means that $x < y \Leftrightarrow \pi(x) <_1 \pi(y)$.)

Then Kechris et al (p 135) showed that \mathbf{K}_0 is a Fraïssé class with limit \mathbb{F}_0 , (which is the reduct of \mathbb{F} to the signature \mathcal{L}_0) iff the Fraïssé order class \mathbf{K} is reasonable.

Note that, in this case, the underlying sets of $\mathbb F$ and $\mathbb F_0$ are the same. Moreover, we can write

 $\mathbb{F} = (\mathbb{F}_0, <_0),$

for some linear ordering $<_0$ on the underlying set of \mathbb{F}_0 .

We consider the continuous action of the automorphism group $\operatorname{Aut}(\mathbb{F}_0)$ on the (topological) space of all linear orderings on the set F_0 , which is the underlying set of the structure \mathbb{F}_0 . Write $X_{\mathbf{K}} \subset \{0, 1\}^{F_0 \times F_0}$ for the orbit topological closure of the action of $\operatorname{Aut}(\mathbb{F}_0)$ on the linear ordering $<_0$, i.e.,

$$X_{\mathbf{K}} = \operatorname{Aut}(\mathbb{F}_0). <_0.$$

This set is clearly a closed, hence compact, subset of the Baire space $\{0,1\}^{F_0 \times F_0}$. Moreover, it is clearly also an $\operatorname{Aut}(\mathbb{F}_0)$ -invariant subset of $\{0,1\}^{F_0 \times F_0}$ under the natural action of $\operatorname{Aut}(\mathbb{F}_0)$ on the latter space. We have thus obtained an $\operatorname{Aut}(\mathbb{F}_0)$ -flow on $X_{\mathbf{K}}$.

This flow can be defined for any reasonable (in the technical sense as explained above) Fraïssé order class \mathbf{K} . I will call it the *discerning* flow associated with the reasonable Fraïssé order class \mathbf{K} .

Remark. The author uses the terminology *discerning* in acknowledgement of Ramseys pioneering work in developing his theorem in the context what now would be considered as a study of indiscernables in model theory.

If, in addition to being a Fraïssé order class, the class **K** is Ramsey, then every minimal subflow of the discernable flow is isomorphic to the universal minimal flow of $\operatorname{Aut}(\mathbb{F}_0)$.

The Fraïssé order class **K** is said to have the ordering property if for every $A_0 \in \mathbf{K}_0$, there is a $B_0 \in \mathbf{K}_0$ such for any linear ordering < on A_0 and every linear ordering $<_1$ on B_0 , where both $<, <_1$ are restrictions of $<_0$, there is an embedding of $(A_0, <)$ into $(B_0, <_1)$.

The discerning flow associated with the Fraïssé order class \mathbf{K} is itself minimal iff \mathbf{K} has the ordering property.

We also extract the following remark from [10].

Proposition 1 If **K** is a Fraïssé order class which is Ramsey and has the ordering property, then a total order ξ belongs to the discerning flow $X_{\mathbf{K}}$ iff for any A in the age of \mathbb{F}_0 it is the case that $\langle A \rangle$ is the restriction of ξ to A.

We shall make substantial use of this remark in the sequel.

Example. It is known (see, for example [3] that the class \mathbf{P} (finite posets, linear extensions) is Ramsey and has the ordering property. As was noted in [10], this has the implication that the discerning $\operatorname{Aut}(\mathbb{P}_0)$ -flow is thus a universal minimal flow. It acts on the space $X_{\mathbf{P}}$ consisting of the linear extensions of the universal poset \mathbb{P}_0 . Using these facts, Kechris and Sokič (2011) [12] recently showed that the automorphism group of \mathbb{P}_0 is not amenable. These results do imply that the set of linear extensions of the Fraïssé limit of finite posets are all, in a definite sense, nonrandom, at least from the point of view of algorithmic randomness as was shown in [5]. This result will be placed in a broader context in what follows.

5 Martin-Löf random countable orders

Let S_{∞} be the group of permutations of a countable set, which, without loss of generality, we may take to be \mathbb{N} . We place on S_{∞} the pointwise convergence topolopy. Let $(\mathbb{N} \times \mathbb{N})_{\neq}$ denote the set of ordered pairs (i, j) of natural numbers with $i \neq j$. Write \mathcal{M} for the set of total orders on \mathbb{N} . We identify \mathcal{M} with a subset of $\{0, 1\}^{(\mathbb{N} \times \mathbb{N})_{\neq}}$ by identifying a total order < on \mathbb{N} with the function $\xi : (\mathbb{N} \times \mathbb{N})_{\neq} \to \{0, 1\}$ given by

$$\xi(x,y) = 1 \Leftrightarrow x < y, \ x, y \in \mathbb{N}.$$

The total order associated with ξ will be denoted by $\langle \xi \rangle$. We topologise \mathcal{M} via the natural injection

$$\mathcal{M} \longrightarrow \{0,1\}^{(\mathbb{N} \times \mathbb{N})_{\neq}}$$

where the (Baire) space $\{0,1\}^{(\mathbb{N}\times\mathbb{N})\neq}$ has the product topology. As such \mathcal{M} is a closed hence compact subspace of $\{0,1\}^{(\mathbb{N}\times\mathbb{N})\neq}$.

The group S_{∞} acts continuously on \mathcal{M} if, for $\xi \in \mathcal{M}$ and $\sigma \in S_{\infty}$, we define the total order $\sigma \xi$ by:

$$x <_{\sigma\xi} y \Longleftrightarrow \sigma^{-1} x <_{\xi} \sigma^{-1} y, \ x, y \in \mathbb{N}.$$

Since S_{∞} is an amenable group, there is an S_{∞} -invariant measure on \mathcal{M} . In fact, Glasner and Weiss (2002) [6] showed that there is *exactly one* such measure (i.e., the flow on \mathcal{M} is uniquely ergodic). Their proof is based on an ergodic argument. Let us denote this measure by μ . I shall refer to this measure as in [5] as the *Glasner-Weiss measure*. We write \mathcal{M}_f for the set of finite total orders on some subset of \mathbb{N} . For $\ell \in \mathcal{M}_f$, denote by Z_ℓ the set of $\xi \in \mathcal{M}$, such that ξ is an extension of ℓ . These sets are the cylinder subsets of \mathcal{M} . Write \mathcal{Z}_0 for the class of events of the form Z_ℓ for some $\ell \in \mathcal{M}_f$ and \mathcal{Z} for the algebra generated by \mathcal{Z}_0 . Note that the σ -algebra generated by \mathcal{Z} is exactly the Borel algebra on \mathcal{M} .

For $Z \in \mathcal{Z}_0$ we write Z^0 for the complement of Z and Z^1 for Z. Let $(T_i)_{i \in \mathbb{N}}$ be any enumeration of the algebra \mathcal{Z} generated by $(Z_\ell)_{\ell \in \mathcal{M}_f}$ in such a way that one can effectively retrieve from a given $i \in \mathbb{N}$, the corresponding T_i as a finite union of sets T of the form

$$T = Z_{\ell_1}^{\delta_1} \cap \dots Z_{\ell_k}^{\delta_k},\tag{1}$$

where each ℓ_i is in \mathcal{M}_f and $\delta_i \in \{0, 1\}$ for $i = 1, \ldots, k$. We call any such enumeration a *recursive representation* of \mathcal{Z} .

The Glasner-Weiss measure μ is computable in the following sense:

Theorem 3 Denote by μ the Glasner-Weiss measure on the Borel-algebra of \mathcal{M} . Let $(T_i : i < \omega)$ be a recursive representation of the algebra \mathcal{Z} . There is an effective procedure that yields, for $i, k \in \mathbb{N}$, a binary rational β_k such that

$$\left|\mu\left(T_{i}\right)-\beta_{k}\right|<2^{-k}.$$

A proof of this result appears in [5].

Definition 2 A set $A \subset \mathcal{M}$ is of constructive measure 0, if, for some recursive representation of $(T_i : i \in \mathbb{N})$ of \mathcal{Z} , there is a total recursive $\phi : \mathbb{N}^2 \to \mathbb{N}$ such that

$$A \subset \bigcap_n \bigcup_m T_{\phi(n,m)}$$

and $\mu\left(\bigcup_{m} T_{\phi(n,m)}\right)$ converges effectively to 0 as $n \to \infty$.

Definition 3 A total order ξ is said to be μ -Martin-Löf random if ξ is in the complement of every subset B of \mathcal{M} of constructive measure 0.

6 The main theorem

Write $ML_{\mu} \subset \mathcal{M}$ for the set of μ -Martin-Löf random total orders.

Theorem 4 Let \mathbf{K} be a recursive Fraissé order class which is Ramsey and has the ordering property. Write

$$\mathbb{F} = (\mathbb{F}_0, <)$$

for its Fraïssé limit and $X_{\mathbf{K}}$ for the associated discerning flow. Fix some recursive representation of \mathbb{F} . Note that

$$X_{\mathbf{K}} \subset \mathcal{M}.$$

If some element of $X_{\mathbf{K}}$ is μ -Martin-Löf random, then the automorphism group $Aut(\mathbb{F}_0)$ is amenable. Equivalently, if $Aut(\mathbb{F}_0)$ is not amenable, then

$$ML_{\mu} \cap X_{\mathbf{K}} = \emptyset.$$

Consequently, if for some $\xi \in X_{\mathbf{K}}$ and some automorphism π of \mathbb{F}_0 it is the case that the linear order $\pi\xi$ is μ -Martin-Löf random, then $Aut(\mathbb{F}_0)$ is amenable.

Proof: Note that a topological group G is amenable iff its universal minimal flow M(G) has a G-invariant probability measure. Indeed, let ν be an invariant measure on M(G). Consider any G-flow on some compact Hausdorff space X. By Zorn's lemma there is a minimal subflow Y and a G-embedding i of Y into X. Therefore, there are G-morphisms

$$M(G) \xrightarrow{\pi} Y \xrightarrow{i} X$$

Let ρ be the pushout measure of ν under $i\pi$. In other words, for every Borel subset A of X, we set

$$\rho(A) = \nu(\pi^{-1}i^{-1}A).$$

Then ρ is an invariant measure on X. The converse is trivial, since M(G) is a compact G-flow.

We introduce a number of (standard) recursion-theoretic concepts and terminology: A sequence (A_n) of sets in \mathcal{Z} is said to be *enumerable* if for each n, the set A_n is of the form $T_{\phi(n)}$ for some total recursive function $\phi: \omega \to \omega$ and some effective enumeration (T_i) of \mathcal{Z} . (Note that the sequence (A_n^c) , where A_n^c is the complement of A_n , is also an \mathcal{Z} -enumerable sequence.) In this case, we call the union $\bigcup_n A_n$ a \sum_{1}^{0} set. A set is a \prod_{1}^{0} set if it is the complement of a \sum_{1}^{0} set. It is of the form $\bigcap_n A_n$, for some \mathcal{Z} -semirecursive sequence (A_n) .

We shall also need the following observation. (In the language of algorithmic randomness, it states the well-known fact that the notion of Martin-Löf randomness is stronger than that of Kurtz randomness. A proof of this observation, in the present context, can be found in [5]. For more on Kurtz randomness, the reader is referred to the book [1] by Downey and Hirschfeldt.)

Lemma 1 If A is a Σ_1^0 subset of \mathcal{M} and if $\mu(A) = 1$, then ML_{μ} is contained in A. In particular, if B is a Π_1^0 subset of \mathcal{M} that contains some element of ML_{μ} , then $\mu(B) > 0$.

It follows from Proposition 1 that a total order ξ belongs to $X_{\mathbf{K}}$ iff for any A in the age of \mathbb{F}_0 it is the case that $\langle A \rangle$ is the restriction of ξ to A. Therefore, since \mathbf{K} is a *recursive* order class, the relation

$$\xi \in X_{\mathbf{K}}$$

is Π^0_1 definable over \mathcal{M} . It follows from Lemma 1 that, if

$$ML_{\mu} \cap X_{\mathbf{K}} \neq \emptyset,$$

then $\mu(X_{\mathbf{K}}) > 0$. This means that μ is a *nonzero* Aut(\mathbb{F}_0)-invariant measure on the flow $X_{\mathbf{K}}$.

Since **K** is a Ramsey order class, $X_{\mathbf{K}}$ is the *universal* minimal flow associated with the group $\operatorname{Aut}(\mathbb{F}_0)$. By universality we can conclude that any $\operatorname{Aut}(\mathbb{F}_0)$ -flow on a compact Hausdorff space will admit a nonzero $\operatorname{Aut}(\mathbb{F}_0)$ -invariant measure. In particular, $\operatorname{Aut}(\mathbb{F}_0)$ is an amenable topological group.

The second part now follows from the observation that if $\xi \in X_{\mathbf{K}}$ and π is an automorphism of \mathbb{F}_0 then $\pi\xi$ will also belong to $X_{\mathbf{K}}$.

Corollary 1 Fix a recursive representation of the universal poset \mathbb{P}_0 on the natural numbers \mathbb{N} . Let $\mathcal{M}(\mathbb{P}_0)$ be the class of linear extensions of \mathbb{P}_0 . Write ML_{μ} for the set of total orders on \mathbb{N} that are Martin-Löf random relative to the Glasner-Weiss probability measure μ . Then

$$ML_{\mu} \cap \mathcal{M}(\mathbb{P}_0) = \emptyset.$$

Proof: The result is a direct consequence of the fact that $Aut(\mathbb{P}_0)$ is not an amenable group. [12].

7 Open problems

The following theorem is in [5].

Theorem 5 Write Q for the set of total orders on \mathbb{N} which are isomorphic to the Cantor rational order η . Then

 $ML_{\mu} \subset \mathcal{Q}.$

In particular,

$$\mu(\mathcal{Q}) = 1.$$

This observation has the following consequence.

Theorem 6 For a total order η , set

$$S_{\mu}(\eta) := \{ \sigma \in S_{\infty} : \sigma \eta \in ML_{\mu} \}.$$

Then $S_{\mu}(\eta) \neq \emptyset$ iff η is a rational Cantor order.

Proof. By Theorem 5, if η were not rational, the corresponding set $S_{\mu}(\eta)$ must be the empty set. If η is rational, then the class Q is exactly the orbit of η under the action of S_{∞} . Since both Q and ML_{μ} have μ -measure one, it follows that

$$\mu(\mathcal{Q} \cap ML_{\mu}) = 1,$$

and, therefore, that $S_{\mu}(\eta) \neq \emptyset$.

Following [5], note that, if $\pi \in S_{\infty}$, then

$$S_{\mu}(\eta)\pi^{-1} = S_{\mu}(\pi\eta).$$
 (2)

Indeed, for $\alpha \in S_{\mu}(\eta)$, we have $\alpha \pi^{-1}(\pi \eta) = \alpha \eta \in ML_{\mu}$ and hence $\alpha \pi^{-1} \in S_{\mu}(\pi \eta)$. Conversely, if $\tau \in S_{\mu}(\pi \eta)$, then $\tau \pi \eta \in ML_{\mu}$, i.e., $\tau \pi \in S_{\mu}(\eta)$, and, so, $\tau \in S_{\mu}(\eta)\pi^{-1}$.

If $\eta_1, \eta_2 \in \mathcal{Q}$, there is some $\pi \in S_{\infty}$ such that $\eta_2 = \pi \eta_1$. Moreover, if η_1, η_2 were both recursive, the permutation π could also be chosen to be recursive. (See Theorem 2). Write S_r for the class of recursive permutations of \mathbb{N} . We let S_r act

on the right on the class Σ of all sets of the form $S_{\mu}(\tau)$ with τ a recursive rational order on \mathbb{N} . The action is given by

$$\Sigma \times S_r \longrightarrow \Sigma,$$
$$(S_{\mu}(\tau), \pi) \mapsto S_{\mu}(\tau) \pi^{-1}, \ \pi \in S_r, \ \tau \in \mathcal{Q}_r,$$

where Q_r denotes the class of all recursive rational orders on \mathbb{N} . It follows from the preceding arguments that this S_r -action will have a single orbit, i.e, the action is transitive. Set

$$\mathcal{S} = \bigcup_{\tau \in \mathcal{Q}_r} S_\mu(\tau).$$

If we choose any fixed $\eta \in Q_r$, we also have

$$\mathcal{S} = \bigcup_{\pi \in S_r} S_\mu(\eta) \pi^{-1}.$$

We shall call the permutations in S Martin-Löf randomizers. These are the permutations that transform some recursive rational order to one which is μ -Martin-Löf random.

These arguments show that an understanding of S can be be attained from any single $S_{\mu}(\tau)$ for a single recursive rational order τ modulo the recursive permutations in S_{∞} .

Let \mathbf{K} be a recursive Fraïssé order class which is Ramsey and has the ordering property. Write again

$$\mathbb{F} = (\mathbb{F}_0, <)$$

for its Fraïssé limit and $X_{\mathbf{K}}$ for the associated discerning flow. The arguments of this paper show that, if $\tau \in \mathcal{Q}_r \cap X_{\mathbf{K}}$, then the presence of elements in $\operatorname{Aut}(\mathbb{F}_0)$ which are Martin-Löf randomizers of τ is related to the amenability of the group $\operatorname{Aut}(\mathbb{F}_0)$. Indeed, for some $\pi \in \operatorname{Aut}(\mathbb{F}_0)$ to be a Martin-Löf randomizer of any τ as above is a *generic* property, in the sense that this very fact forces the group $\operatorname{Aut}(\mathbb{F}_0)$ to be amenable! The problem still remains to identify the class of Martin-Löf randomizers.

Let \mathbf{L} be the Fraïssé order class consisting of all pairs (L, <) where L is a lattice with underlying set a finite ordinal and with < being a total order on the underlying set of L which is a linear extension of the partial order on L. As far as the author knows, it is unknown whether \mathbf{L} is Ramsey and whether it has the ordering property. The author has discussed this problem with specialists in Ramsey theory and it would appear that this problem is wide open. Writing $\mathbb{L} = (\mathbb{L}_0, <)$ for the Fraïssë limit of \mathbf{L} , it is also an interesting open problem to relate $X_{\mathbf{L}}$ to ML_{μ} and thus perhaps gaining an understanding of the amenability or not of $\operatorname{Aut}(\mathbb{L}_0)$. Note that if \mathbf{L} were Ramsey with the ordering property, then $\operatorname{Aut}(\mathbb{L})$ would be an extremely amenable group. This would mean that its universal minimal flow is a singleton.

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