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Abstract. For any transferable utility game in coalitional form with a nonempty core, we show that that the number of blocks required to switch from an imputation out of the core to an imputation in the core is at most $n-1$, where n is the number of players. This bound exploits the geometry of the core and is optimal. It considerably improves the upper bounds found so far by Kóczy [7], Yang [13, 14] and a previous result by ourselves [2] in which the bound was $n(n-1)/2$.

1 Introduction

1.1 Preliminaries

TU games. We consider cooperative games with transferable utility (TU-games for short). Formally, a *TU-game* is a pair (N, v) where

- $N = \{1, \ldots, n\}$ is a nonempty finite *player set*;
- $v : 2^N \longrightarrow \mathbb{R}$ is a real-valued function such that $v(\emptyset) = 0$.

A nonempty subset S of N is called a *coalition* and s stands for its cardinality. The real number $v(S)$ is interpreted as the *worth* of coalition S, i.e. the value generated by players of S when they cooperate without the help of players in $N\backslash S$. The set N is called the *grand coalition*.

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An *allocation* $x \in \mathbb{R}^n$ on N is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. We state: $x(S) = \sum_{i \in S} x_i$.

An allocation $x \in \mathbb{R}^n$ is *efficient* if $x(N) = v(N)$, *individually rational* if $x_i \geq v(\{i\})$ for each $i \in N$ and *acceptable* if $x(S) \geq v(S)$ for each $S \in 2^N$. The first two properties above can be seen as minimal requirements for an conceivable allocation. The set of efficient allocations is denoted by $E(N, v)$. An individually rational and efficient allocation is referred to as an *imputation*. The (possibly empty) set of imputations is denoted by $I(N, v) \subseteq E(N, v)$.

The core Let Γ be the class of all finite TU-games. A *solution* on Γ is a function F which assigns to each $(N, v) \in \Gamma$ a set of allocations $F(N, v)$. The most famous solution for TU-games is the core introduced by Gillies [5]. The *core* is the solution C on Γ that assigns to each TU-game $(N, v) \in \Gamma$ the possibly empty set $C(N, v)$ of all efficient and acceptable allocations, i.e.

$$C(N, v) = \left\{ x \in E(N, v) : \forall S \in 2^N, x(S) \geq v(S) \right\}.$$

Note that the core is a subset of the set of imputations. We denote by Γ^c the class of all TU-games with a nonempty core. In the rest of the article we only consider the class Γ^c. The core can be also defined using the notions of block and dominance.

Given an allocation $x \in \mathbb{R}^n$ and a coalition S, x_S denotes the restriction of x to S. For two allocations $x, y \in \mathbb{R}^n$, we write $x_S < y_S$ if $x_i < y_i$ for each $i \in S$ and $x_S \leq y_S$ if $x_i \leq y_i$ for each $i \in S$ but $x_S \neq y_S$.

Definition 1. *Assume that there exists a non empty coalition $S \in 2^N$ and two efficient allocations x and y of $E(N, v)$ such that both $y(S) \leq v(S)$ and $x_S \leq y_S$ (resp. $x_S < y_S$). In such a case, we say that S weakly (resp. strongly) blocks x, and that y weakly dominates x (resp. y strongly dominates x) via coalition S, and we denote this relation by $x \preceq_S y$ (resp. $x \prec_S y$). We write $x \preceq y$ (resp. $x \prec y$) if there exists a coalition S such that $x \preceq_S y$ (resp. $x \prec_S y$) and say that y weakly (resp. strongly) dominates x.*

The strong dominance relation indicates that it is in the interest of all players in S to switch from x to y, while the weak dominance relation only imposes that the payoff of no player in S is reduced when moving from x to y and at least one of them is strictly better off. In the rest of the article, and with the notable exception of section 3, we will use the weak dominance relation.

Let x be an efficient allocation that lies out of the core. There necessarily exists an efficient allocation y such that $x(S) < y(S) \leq v(S)$. Thus, coalition S can propose to replace x by y. For instance, y can be any efficient allocation such that $y_i = x_i + (v(S) - x(S))/s$ for each $i \in S$, which makes every member of S strictly better off than in x. Thus, the players of S can threaten to split from the grand coalition since $x \prec_S y$. In a sense x fails to ensure the stability of the grand coalition. Such a situation cannot arise if x is a core allocation. Hence, the core can also be defined as the set of efficient allocation which are not (strongly of weakly) dominated i.e.

$$C(N,v) = \{x \in E(N,v) : \forall y \in E(N,v), \neg(x \preceq y)\}.$$

In other words, the maximal elements of the dominance relations over $E(N,v)$ coincide with the core allocations. As such, the core satisfies the internal stability property: elements of the core are not comparable under the weak or strong dominance relation. Nevertheless, the core is often criticized on two aspects. Firstly, it does not account for every imputation it excludes. More specifically, the core does not necessarily satisfies the external stability property: an imputation out of the core is not always dominated by an imputation of the core. Shapley [11] has proved the external stability for the class of convex TU-games.[4] Secondly, Harsanyi [6] and Chwe [3] consider that this solution concept is too myopic because it neglects the effect of successive blocks. Harsanyi [6] introduces a new indirect dominance relation, which consists of a chain of blocks, in order to cope with these lacks.

A *weak (resp. strong) chain of blocks* is a finite sequence $(x_0, x_1, \ldots, x_m)$ of efficient allocations such that, for each $k = 0, \ldots, m-1$, it holds that $x^k \preceq x^{k+1}$ (resp. $x^k \prec x^{k+1}$). The number m of allocations in the chain is called its *length*. An allocation y indirectly weakly (resp. strongly) dominates an allocation x if there exists a weak (resp. strong) chain of blocks starting at x and ending at y. Harsanyi originally applies this indirect dominance relation to study the von Neumann-Morgenstern stable sets (von Neumann and Morgenstern [12]). Sengupta and Sengupta [10] employ it to show that the core is indirectly externally stable for the weak dominance relation: starting from any imputation that stands outside the core, there always exists a weak chain of blocks which terminates in the core. In other words, the core can be considered as a von Neumann-Morgenstern stable set under the indirect weak dominance relation.

1.2 The results

This last result has initiated the literature on the accessibility of the core, on which the reader is referred to Béal et al. [2] and the references therein. The central question that has appeared is to determine a upper bound on the length of the chain of blocks needed to access the core. Several recent articles try to answer this question and this article improves on the existing answers. We only mention the two most recent approaches. In our previous article [2] we show that the core of any TU-game with n players can be accessed with a weak chain of blocks of length at most $n(n-1)/2$ blocks such that each element of this chain is an imputation. Yang [14] obtains the linear bound $2n-1$ but this result has two drawbacks. Firstly, this bound only holds for the class of cohesive TU-games i.e. the class of games in which no partition of the player set generates a larger cumulated worth than the grand coalition. Secondly, even if the starting

[4] A TU-game is convex if for each pair of coalitions $S, T \subseteq N$, it holds that $v(S \cup T) - v(S \cap T) \geq v(S) + v(T)$.

allocation is an imputation, the other allocations in the weak chain of blocks need not be efficient, which is rather far from the spirit of the original idea of Harsanyi [6].

In the present article we also obtain a linear bound: the core can be accessed in at most $n-1$ blocks. More importantly, for the first time in the literature, we are able to prove that this bound is optimal. These results can be stated as follows.

Theorem 1. *Let $(N,v) \in \Gamma^c$ be a n-player TU-game with a nonempty core with $n \geq 3$. For each imputation $x \in I(N,v)$, there exists a core element $c \in C(N,v)$ and a weak chain of blocks from x to c with length at most $n-1$ and such that each allocation of the chain is an imputation.*

This bound is optimal: for each integer $n \geq 3$, there exists a n-player TU-game $(N,v) \in \Gamma^c$ with a nonempty core and an imputation $x \in I(N,v)$ such that each weak chain of blocks from x to any core allocation in $C(N,v)$ has length at least $n-1$.

Note that in the cases $n=1$ and $n=2$ the set of imputations and the core coincide, so that the accessibility of the core is trivial. The proof of Theorem 1 relies on a procedure which is similar to the one introduced by Sengupta and Sengupta [10]. In particular, both procedures share the idea of using a core element as a reference point. Nevertheless, we introduce four major differences detailed below.

The first difference is the choice, at each step of the chain of blocks, of the blocking coalition. While Sengupta and Sengupta [10] choose a coalition among the most unsatisfied coalitions with respect to the current imputation, we select a coalition S such that the hyperplane defined by $x(S) = v(S)$ for $x \in \mathbb{R}^n$ has a nonempty intersection with the core. As a consequence, the geometry of the core plays an important role in our analysis.

The second difference is related to the redistribution of the excess of the chosen coalition S with respect to the current allocation in order to construct the next allocation in the chain of blocks. In this article, we split equally this excess among S's members, which means that all players in S get a strictly larger payoff. In Sengupta and Sengupta [10] only the members S who get less than their core target payoff receive a larger payoff. The choice of S and our specific redistribution of its excess allows for a suitable simplification of the procedure introduced in Sengupta and Sengupta [10].

The third difference is that the target core allocation can vary from one block to the next block in the weak chain of blocks while it is unique in Sengupta and Sengupta [10]. However, all target core allocations that are used along the weak chain of blocks have at least one common coordinate.

The fourth difference is that the result crucially relies on the use of the Davis-Maschler reduced-games (Davis and Maschler [4]). The Davis-Maschler reduced-games describe situations in which all the players agree that the left players get their core reference payoffs but continue to cooperate with the remaining players, subject to the foregoing agreement. The Davis-Maschler reduced-games are well known for being the basis of the so-called reduced-game property, which states that if an allocation is prescribed by some solution concept in a TU-game, then the restriction of this allocation to any coalition of players is also prescribed by the solution concept in the reduced-game associated with these coalition and allocation. Our previous article describes connections between a game and its Davis-Maschler reduced-games. In the current article, we explore more deeply these connections.

In Béal et al. [2], the first three aspects were not used while the fourth one was not essential to prove the results.

A parallel study using the strong dominance relation can be investigated. To the best of our knowledge, there does not exist any such study in the literature so far. This fact can be explained by our last result: the accessibility of the core is not guaranteed under the strong dominance relation.

Theorem 2. *For each $n \geq 3$, there exists a n-player TU-game $(N, v) \in \Gamma^c$ with a nonempty core and an imputation $x \in I(N, v)$ from which the core cannot be accessed by a strong chain of blocks.*

2 Optimality under the weak dominance relation

In this section, we start by stating some connections between a TU-game and its Davis-Maschler reduced-games. These intermediary results will be used later on to prove Theorem 1. Although the proofs are similar than those in Béal et al. [2], we give them for completeness. Secondly, we describe the procedure. Thirdly, we prove the first part of Theorem 1, i.e. that the core of an n-player TU-game is accessible in at most $n-1$ blocks. This part is obtained as a corollary of a more general result in which we consider efficient allocations instead of imputations. Fourthly, we show that our bound is optimal.

2.1 Reduced-games equivalences

Let $S \subset N$ be any coalition different from N and $x \in E(N, v)$ any efficient allocation. Davis and Maschler [4] introduce the *reduced-game* with respect to S and x, denoted by $(S, v_{S,x})$ and defined as

$$v_{S,x}(T) = \begin{cases} 0 & \text{if } T = \emptyset, \\ v(N) - x(N \backslash S) & \text{if } T = S, \\ \max_{R \in 2^{N \backslash S}} \big(v(T \cup R) - x(R)\big) & \text{otherwise.} \end{cases}$$

The weak dominance relation $\preceq$ will be used in the Davis-Maschler reduced-games as well. In such a case, we will specify in which game the dominance relation is applied in order to avoid any risk of confusion.

A solution F on Γ satisfies the *reduced-game property* if for each $(N, v) \in \Gamma$, each nonempty coalition $S \subset N$ and each $x \in F(N, v)$, it holds that $x_S \in F(S, v_{S,x})$. It is well-known that the core satisfies the reduced-game property. In fact, the reduced-game property is one of the axioms used by Peleg [8] in order to characterize the core.

For the class Γ^c of TU-games with a nonempty core, we will construct Davis-Maschler reduced-games with respect to core allocations only. This section establishes two interesting properties of such reduced-games.

Let $(N, v) \in \Gamma^c$ be any TU-game with a nonempty core, $S \subset N$ be any nonempty coalition and $c \in C(N, v)$ be any core allocation of (N, v). Observe that

$$v_{S,c}(S) = v(N) - c(N\backslash S) = c(N) - c(N\backslash S) = c(S).$$

The first lemma establishes connections between the sets of efficient allocations and the cores of a TU-game and of its Davis-Maschler reduced games.

Lemma 1. *Consider any $(N, v) \in \Gamma^c$, any nonempty coalition $S \subset N$ and any $c \in C(N, v)$. Pick any allocation $x \in \mathbb{R}^n$ such that $x_{N\backslash S} = c_{N\backslash S}$. Then*

- *$x \in E(N, v)$ if and only if $x_S \in E(S, v_{S,c})$,*
- *$x \in C(N, v)$ if and only if $x_S \in C(S, v_{S,c})$.*

Proof. Firstly, suppose that $x \in E(N, v)$. It holds that

$$x_S(S) = x(N) - x(N\backslash S) = v(N) - c(N\backslash S) = c(N) - (c(N) - c(S)) = c(S) = v_{S,c}(S),$$

so that $x_S \in E(S, v_{S,c})$. Conversely, suppose that $x_S \in E(S, v_{S,c})$. Since $x_{N\backslash S} = c_{N\backslash S}$ and $x_S(S) = v_{S,c}(S) = c(S)$, we get

$$x(N) = x(S) + x(N\backslash S) = c(S) + c(N\backslash S) = c(N) = v(N),$$

proving that $x \in E(N, v)$.

Secondly, suppose that $x \in C(N, v)$. Since $x \in E(N, v)$, we have that $x_S \in E(S, v_{S,c})$. Now, choose any coalition $T \in 2^S$. It holds that:

$$v_{S,c}(T) = v(T \cup \overline{T}) - c(\overline{T}) \leq x(T \cup \overline{T}) - c(\overline{T}) = x(T) = x_S(T),$$

which means that x_S is also an acceptable allocation in $(S, v_{S,c})$. We conclude that $x_S \in C(S, v_{S,c})$.

Conversely, suppose that $x_S \in C(S, v_{S,c})$. Since $x_S \in E(S, v_{S,c})$, we have $x \in E(N, v)$. Next, choose any coalition $T \in 2^N$. The definition of $v_{S,c}$ and $x_S \in C(S, v_{S,c})$ imply that

$$\begin{aligned} v(T) &= v((T \cap S) \cup (T\backslash S)) \\ &= v((T \cap S) \cup (T\backslash S)) - c(T\backslash S) + c(T\backslash S) \\ &\leq v_{S,c}(T \cap S) + c(T\backslash S) \\ &\leq x_S(T \cap S) + c(T\backslash S) \\ &= x(T), \end{aligned}$$

from which we obtain $x \in C(N, v)$.

The second lemma describes the connections between the weak dominance relations in a TU-game and in its Davis-Maschler reduced games.

Lemma 2. *Consider any $(N, v) \in \Gamma^c$, any nonemptyset coalition $S \subset N$ and any $c \in C(N, v)$. Pick any two allocations $x, y \in E(N, v)$ such that $x_{N\setminus S} = y_{N\setminus S} = c_{N\setminus S}$ and a nonempty coalition $T \subset S$. Then:*

$$x \preceq_{T\cup\overline{T}} y \text{ in } E(N, v) \text{ if and only if } x_S \preceq_T y_S \text{ in } E(S, v_{S,c}).$$

Proof. Firstly, assume that $x \preceq_{T\cup\overline{T}} y$ in $E(N, v)$. Then $x(T \cup \overline{T}) < y(T \cup \overline{T})$ and $x_{T\cup\overline{T}} \leq y_{T\cup\overline{T}}$. In addition, $\overline{T} \in 2^{N\setminus S}$ implies $x_{\overline{T}} = y_{\overline{T}}$. It follows that $x(T) < y(T)$.

Next, $x \preceq_{T\cup\overline{T}} y$ in $E(N, v)$ also means that $y(T \cup \overline{T}) \leq v(T \cup \overline{T})$. Therefore, by definition of y, we get $v_{S,c}(T) = v(T \cup \overline{T}) - c(\overline{T}) \geq y(T \cup \overline{T}) - c(\overline{T}) = y(T)$. We conclude that $x_S \preceq_T y_S$ in $E(S, v_{S,c})$.

Secondly, assume that $x_S \preceq_T y_S$ in $E(S, v_{S,c})$. Then $x(T) < y(T)$ and $x_T \leq y_T$. Since $x_{\overline{T}} = y_{\overline{T}}$ by assumption, this implies that $x_{T\cup\overline{T}} \leq y_{T\cup\overline{T}}$.

Furthermore, $x_S \preceq_T y_S$ in $E(S, v_{S,c})$ also means that $y(T) \leq v_{S,c}(T) = v(T \cup \overline{T}) - c(\overline{T})$. Thus, $v(T \cup \overline{T}) \geq y(T) + c(\overline{T}) = y(T \cup \overline{T})$. Therefore $x \preceq_{T\cup\overline{T}} y$ in $E(N, v)$.

One directly obtains the following corollary.

Corollary 1. *Consider any $(N, v) \in \Gamma^c$, any $S \subset N$ and any $c \in C(N, v)$. Pick any two allocations $x, y \in E(N, v)$ such that $x_{N\setminus S} = y_{N\setminus S} = c_{N\setminus S}$. If $x_S \preceq y_S$ in $E(S, v_{S,c})$, then $x \preceq x$ in $E(N, v)$.*

A key point is that this corollary does not hold if the weak dominance relation is replaced by the strong dominance relation. This explains the differences in term of accessibility of the core with respect to the weak and strong dominance relations. This aspect is formally investigated in section 3.

The intermediaries results in this section also raise a difficulty. Equivalent results cannot be stated if the considered allocations are imputations: it may happens that, for an imputation $x \in I(N, v)$, the allocation x_S is not in $I(S, v_{S,c})$. We also discuss this point latter on.

2.2 The procedure

Consider a TU-game $(N, v) \in \Gamma^c$. From now on, we fix an efficient allocation $x \in E(N, v)$. In order to exploit the results of the previous section, we will exhibit an allocation which satisfies several properties. It will be used to define the allocations along our weak chain of blocks. The construction of this allocation relies on the following result about the geometry of $C(N, v)$.

Lemma 3. *Let $B(N, v)$ denote a minimal collection, with respect to set inclusion, of coalitions in $2^N \setminus \{N\}$ such that*

$$C(N, v) = \big\{z \in E(N, v) : \forall S \in B(N, v), z(S) \geq v(S)\big\}.$$

For each $S \in B(N, v)$, there exists a core allocation $c \in C(N, v)$ such that $c(S) = v(S)$.

Proof. First note that such a minimal collection $B(N, v)$ trivially exists. By way of contradiction, assume that there exists T in $B(N, v)$ such that for each $z \in C(N, v)$ it holds that $v(T) < z(T)$. Consider the set

$$D(N, v) = \{z \in E(N, v) : \forall S \in B(N, v) \backslash \{T\}, z(S) \geq v(S)\}.$$

Since $B(N, v)$ is a minimal set with respect to inclusion, we have $C(N, v) \subset D(N, v)$. Hence, we can choose $x \in D(N, v) \backslash C(N, v)$. Pick also any $y \in C(N, v)$. By definition of x and y and by the initial assumption on T, we have $x(T) < v(T) < y(T)$. Now, define

$$\alpha = \frac{v(T) - y(T)}{x(T) - y(T)},$$

and observe that $\alpha \in (0, 1)$. Next, construct the allocation $c = \alpha x + (1 - \alpha)y$. In particular, it holds that $c(T) = v(T)$. Moreover, c belongs to $D(N, v)$ because it is a convex combination of two elements of the convex set $D(N, v)$. In addition, since both $x(S) \geq v(S)$ and $y(S) \geq v(S)$ for each $S \in B(N, v) \backslash \{T\}$, we conclude that $c \in C(N, v)$. This contradicts the initial assumption.

Because the core is a polytope, the statement of Lemma 3 can be strengthen if the core is full-dimensional, i.e. has a nonempty interior. More precisely, it is known that for each full-dimensional core, there exists a unique (up to a multiplication by positive scalars) minimal collection of constraint inequalities that determines it. Moreover, for each distinct pair of constraint inequalities, there is an element is the core that saturates the first constraint and lies strictly above the second constraint inequality. On this point we refer the reader to Schrijver [9].

Lemma 4. *Let $x \in E(N, v)$ be an efficient allocation such that $x \notin C(N, v)$. There exists an allocation y satisfying the following four properties.*

1. *$y \in E(N, v)$;*
2. *$x \preceq y$;*
3. *there exists $c \in C(N, v)$ and $i \in N$ such that $y_i = c_i$;*
4. *for each player $i \in N$, we have $y_i \geq \min(x_i, v(\{i\}))$.*

Proof. Consider any $x \in E(N, v) \backslash C(N, v)$. By Lemma 3, we can choose a coalition $S \in B(N, v)$ such that $x(S) < v(S)$, and therefore a core allocation $c \in C(N, v)$ such that $c(S) = v(S)$. We define the allocation y as follows:

$$y_i = \begin{cases} x_i + \dfrac{v(S) - x(S)}{s} & \text{if } i \in S, \\ c_i & \text{if } i \in N \backslash S. \end{cases}$$

Now we prove that y satisfies the claimed properties. Actually, we do not use the particular structure of y for players in S. The important coordinates of y are on the players in $N \backslash S$.

For the first property, the equality $v(S) = c(S)$ implies that

$$y(N) = x(S) + s\frac{v(S) - x(S)}{s} + c(N \backslash S) = v(S) + c(N \backslash S) = c(S) + c(N \backslash S) = v(N),$$

which proves that $y \in E(N, v)$.

For the second property, the claim $x \preceq y$ is obviously satisfied since $v(S) - x(S) > 0$ and $y(S) = v(S)$ imply $x \preceq_S y$.

For the third property, it is enough to show that $N \backslash S \neq \emptyset$, which is ensured by $S \in B(N, v)$ since $N \notin B(N, v)$.

For the fourth property, the inequality $x(S) < v(S)$ implies that $y_S > x_S$. Finally, for each $i \in N \backslash S$, it holds that $y_i = c_i \geq v(\{i\})$.

The selection of the unsatisfied coalition S such that $x \preceq_S y$ is the corner stone of our construction. It is a main difference with Sengupta and Sengupta [10] and Béal et al. [2]. We choose $S \in B(N, v)$ whereas Sengupta and Sengupta [10] choose S such that the positive excess $v(S) - x(S)$ is maximal, and Béal et al. [2] select a coalition among the smallest coalitions with positive excess. Selecting S in $B(N, v)$ is necessary to construct an allocation y in a simpler way than in Sengupta and Sengupta [10] and Béal et al. [2]. Another difference with Sengupta and Sengupta [10] and Béal et al. [2] is related to the use of the core allocations as a target. Both use a unique core allocation along their weak chain of blocks. Here, we take in account the whole geometry of the core since it is the chosen coalition $S \in B(N, v)$ that determines which core allocation can be used to construct the current block.

2.3 The upper bound

We now have the material to prove that the core of any n-player TU-game can be accessed in at most $n - 1$ blocks.

Proposition 1. *Let $(N, v) \in \Gamma^c$ be a n-player TU-game with a nonempty core. For each efficient allocation $x \in E(N, v)$, there exists a core element $c \in C(N, v)$ and a weak chain of blocks from x to c such that*

- *the length of this weak chain of blocks is at most $n - 1$;*
- *each blocking efficient allocation z in this weak chain of blocks satisfies the condition that, for each $j \in N$, $z_j \geq \min(x_j, v(\{j\}))$.*

Proof. Consider any arbitrary TU-game $(N, v) \in \Gamma^c$. The proof is done by induction on the number n of players in (N, v).

Initialization: For $n = 1$, the unique efficient allocation is also the unique core allocation so that the result trivially holds.

Induction hypothesis: Assume that the statement of the Proposition 1 holds for any k-player TU-game, $k \in \{1, \ldots, n - 1\}$.

Induction step: Consider any n-player TU-game $(N, v) \in \Gamma^c$ and any $x \in E(N, v)$. If $x \in C(N, v)$, then we are done. Otherwise, using the procedure described in Lemma 4, we construct an imputation y satisfying the four properties stated in that Lemma.

In particular, there exists a player $i \in N$ and a core allocation $c \in C(N, v)$ such that $y_i = c_i$. Consider the Davis-Maschler reduced-game $(N \backslash \{i\}, v_{N \backslash \{i\}, c})$. This TU-game is an $(n-1)$-player TU-game with a nonempty core since Lemma

1 states that $c_{N\setminus\{i\}} \in C(N\setminus\{i\}, v_{N\setminus\{i\},c})$. The induction hypothesis can be used: there exists a core element $d \in C(N\setminus\{i\}, v_{N\setminus\{i\},c})$ and a weak chain of blocks $(z^0, z^1, \ldots, z^m)$ in $(N\setminus\{i\}, v_{N\setminus\{i\},c})$ such that $z^0 = y_{N\setminus\{i\}}$ and $z^m = d$ with $m \leq n-2$. For each $k \in \{0, \ldots, m\}$, it also holds that $z^k \in E(N\setminus\{i\}, v_{N\setminus\{i\},c})$. Furthermore, for each $j \in N\setminus\{i\}$, we have $z_j^0 \geq \min(y_j, v_{N\setminus\{i\},c}(\{j\}))$ and, for each $k \in \{1, \ldots, m\}$, $z_j^k \geq \min(z_j^{k-1}, v_{N\setminus\{i\},c}(\{j\}))$. Hence, for each $k \in \{1, \ldots, m\}$ and each $j \in N\setminus\{i\}$, it holds that $z_j^k \geq \min(y_j, v_{N\setminus\{i\},c}(\{j\}))$.

For each $k \in \{0, \ldots, m\}$, define the allocation $z'^k \in \mathbb{R}^n$ as $z'^k_{N\setminus\{i\}} = z^k$ and $z'^k_i = c_i$. For each $k \in \{0, \ldots, m\}$, we have from Lemma 1 that $z'^k \in E(N\setminus\{i\}, v_{N\setminus\{i\},c})$. Lemma 1 also yields that $z'^m \in C(N, v)$ since $z'^m_{N\setminus\{i\}} = z^m = d$ and $d \in C(N\setminus\{i\}, v_{N\setminus\{i\},c})$. Furthermore, from Lemma 2, the sequence of allocations $(z'^0, z'^1, \ldots, z'^m)$ is a weak chain of blocks in (N, v). This implies that $(x, z'^0, z'^1, \ldots, z'^m)$ is a weak chain of blocks of length $m+1$ in (N, v). Since $m \leq n-2$, the length of the weak chain of blocks $(x, z'^0, z'^1, \ldots, z'^m)$ is bounded by $n-1$, which proves the first part of Proposition 1.

Regarding the second part of Proposition 1, consider any player $j \in N\setminus\{i\}$. We have already proved that $z_j^k \geq \min(y_j, v_{N\setminus\{i\},c}(\{j\}))$. We also have $y_j \geq \min(x_j, v(\{j\}))$. By definition of $v_{N\setminus\{i\},c}$, it holds that $v_{N\setminus\{i\},c}(\{j\})) \geq v(\{j\})$. Altogether, this implies that

$$z_j^k \geq \min(x_j, v(\{j\})),$$

or equivalently, that $z'^k_j \geq \min(x_j, v(\{j\}))$. Lastly, the inequality $z'^k_i = c_i \geq v(\{i\}) \geq \min(x_i, v(\{i\}))$ completes the proof.

The *first part of Theorem 1* is a corollary of Proposition 1. In fact, if the set of efficient allocations is replaced by the set of imputations in the statement of Proposition 1, then the length of the weak chain of blocks required to access the core is still at most $n-1$. Moreover, for each player $j \in N$, the condition $z_j \geq \min(x_j, v(\{j\}))$ for each blocking allocation z reduces to $z_j \geq v(\{j\})$ since $x_j \geq v(\{j\})$ whenever x is an imputation. This ensures that the weak chain of blocks only contains imputations. It is nevertheless important to state Proposition 1. The reason is that, in the induction step, a blocking imputation can be lead to an efficient allocation which is not individually rational in the associated Davis-Maschler reduced-game.

2.4 Optimality

This section is devoted to the proof of the second part of Theorem 1, i.e. it is impossible to improve upon the bound $n-1$. More specifically, for each $n \geq 3$, we construct an n-player TU-game with a nonempty core and an imputation $x \in I(N, v)$ such that the length of each chain of blocks from x to any core allocation of $C(N, v)$ is at least $n-1$.

Proof (Theorem 1 — second part). Let $n \geq 3$ and (N, v) be the n-player TU-game such that $v(S) = s$ if $s \geq n-1$ and $v(S) = 0$ otherwise. If c belongs to

$C(N,v)$, then, for each player $i \in N$, it holds that

$$c_i = c(N) - c(N\backslash\{i\}) \leq v(N) - v(N\backslash\{i\}) = 1.$$

Combined with the efficiency of c, we obtain that $c_i = 1$ for each $i \in N$. Since the allocation $(1,\ldots,1)$ belongs to $C(N,v)$, we conclude this allocation is the unique core allocation.

Now, pick any $x \in I(N,v)\backslash C(N,v)$. The imputation x is acceptable for all coalitions of size at most $n-2$ and it is not acceptable for a coalition $N\backslash\{i\}$ if and only if $x_i > 1$. Then, the number $\Delta(x)$ of coalitions for which x is not acceptable is the number of players $i \in N$ such that $x_i > 1$.

Consider any imputation $y \in I(N,v)$ such that $x \preceq y$. Note that $x \preceq_S y$ is only possible if $S = N\backslash\{i\}$ for some $i \in N$ such that $x_i > 1$. Consider any such block. By definition of a block, we get that $y(N\backslash\{i\}) \leq v(N\backslash\{i\})$. Now, consider any other coalition S for which x is not acceptable, which means that $S = N\backslash\{j\}$ for some $j \in N\backslash\{i\}$. Since $y_j \geq x_j$ and $y, x \in E(N,v)$, it holds that

$$y(N\backslash\{j\}) \leq x(N\backslash\{j\}) < v(N\backslash\{j\}).$$

In other words, if x is not acceptable for a coalition other than $N\backslash\{i\}$, then y is also not acceptable for this coalition. Thus, $\Delta(y) \geq \Delta(x) - 1$. It follows that any weak chain of blocks of length p and starting from x terminates in an allocation z such that $\Delta(z) \geq \Delta(x) - p$.

Now suppose that the starting imputation x is given $x_1 = 0$ and, for each $i \in N\backslash\{1\}$, $x_i = 1 + 1/(n-1)$. It holds that $\Delta(x) = n-1$. This implies that each weak chain of blocks of length at most $n-2$ and starting at x terminates in an allocation z, such that $\Delta(z) \geq n-1-(n-2) = 1$. In other words, each weak chain of blocks of length at most $n-2$ and starting at x cannot access the core $C(N,v)$. We conclude that a weak chain of blocks of length at most $n-1$ is necessary to access the core if x is the initial imputation.

3 An impossibility result under the strong dominance relation

In this section, we prove Theorem 2 stating that the core is not always accessible if the blocks are constructed from the strong dominance relation. For each $n \geq 3$, the proof relies on the n-player TU-game depicted in section 2.4, which is used to show that there exists an imputation from which the core is not accessible through a strong chain of blocks.

Proof (Theorem 2). For each $n \geq 3$, consider the n-player TU-game (N,v) introduced in section 2.4. Consider the set $A(N,v)$ of imputations of (N,v) such that $x \in A(N,v)$ if the following two conditions are satisfied:

- for each $i \in N\backslash\{1\}$, $x_i \geq 1$;
- there exists at most one $i \in N\backslash\{1\}$ such that $x_i = 1$.

Note that if $x \in A(N,v)$, these two conditions imply that $0 \leq x_1 < 1$. The set $A(N,v)$ is clearly nonempty since it contains the allocation x constructed at the end of the proof of the second part of Theorem 1 in the section 2.4. Now consider any imputation $x \in A(N,v)$. Pick any $i \in N$ and any efficient allocation $y \in E(N,v)$ such that $x \prec_{N\backslash\{i\}} y$. Recall that no coalition of size at most $n-2$ can block x. Furthermore, observe that $i \neq 1$ since $x(N\backslash\{1\}) > n-1 = v(N\backslash\{1\})$. By definition, for each player $j \in N\backslash\{1,i\}$, it holds that $y_j > x_j \geq 1$. Moreover, we have $y_i = v(N) - y(N\backslash\{i\}) \geq v(N) - v(N\backslash\{i\}) = 1$. This proves that y belongs to $A(N,v)$.

As a consequence, any strong block starting in $A(N,v)$ also ends in $A(N,v)$. It follows that any strong chain of blocks starting in $A(N,v)$ also terminates in $A(N,v)$. Since the unique core allocation $(1,\dots,1)$ does not belong to $A(N,v)$, we can conclude that the core of the TU-game (N,v) is not accessible from $A(N,v)$ by means of the strong dominance relation.

It is worth to mention that this result does not contradict Proposition 1, even if the procedure used in Proposition 1 relies on the strong block constructed in Lemma 4. The explanation is that a block which is strong in a Davis-Maschler reduced game is not necessarily strong in the original game.

4 Conclusion

This article proves that the optimal number of blocks required to access the core of any n-player TU-game is $n-1$. Somehow, our result provides a definitive answer to the question of the accessibility of the core. A challenging issue is to investigate whether this result still holds for the related concept of coalition structure core. From Yang [14] and Béal et al. [1] we know that the number of blocks required to access the coalition structure core is at most quadratic in the number of players.

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