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Solutions for the stable roommates problem with payments[☆]

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ABSTRACT

The stable roommates problem with payments has as input a graph $G = (V, E)$ with an edge weighting $w : E \to \mathbb{R}_{\geq 0}$ and the problem is to find a stable solution. A solution is a matching M with a vector $p \in \mathbb{R}^V_{\geq 0}$ that satisfies $p_u + p_v = w(uv)$ for all $uv \in M$ and $p_u = 0$ for all u unmatched in M. A solution is stable if it prevents blocking pairs, i.e., pairs of adjacent vertices u and v with $p_u + p_v < w(uv)$, or equivalently, if the total blocking value $\sum_{uv \in E} \max\{0, w(uv) - (p_u + p_v)\} = 0$. By pinpointing a relationship to the accessibility of the coalition structure core of matching games, we give a constructive proof for showing that every yes-instance of the stable roommates problem with payments allows a path of linear length that starts in an arbitrary unstable solution and that ends in a stable solution. This generalizes a result of Chen, Fujishige and Yang (2011) [4] for bipartite instances to general instances. We also show that the problems Blocking Pairs and Blocking Value, which are to find a solution with a minimum number of blocking pairs or a minimum total blocking value, respectively, are NP-hard. Finally, we prove that the variant of the first problem, in which the number of blocking pairs must be minimized with respect to some fixed matching, is NP-hard, whereas this variant of the second problem is polynomial-time solvable.

1. Introduction

Consider a group of tennis players participating in a doubles tennis tournament. Each two players estimate the expected prize money they could win together by forming a pair in the tournament. Moreover, each player can negotiate his share of the prize money with his chosen partner in order to maximize his own prize money. Can the players be matched together such that no two players have an incentive to leave the matching in order to form a pair together? This example has been given by Eriksson and Karlander [6] to introduce the stable roommates problem with payments.

The stable roommates problem with payments generalizes the stable marriage problem with payments [14] and can be modeled by a weighted graph $G = (V, E)$, i.e., that has an edge weighting $w : E \to \mathbb{R}_{\geq 0}$. A vector $p \in \mathbb{R}^V$ with $p_u \geq 0$ for all $u \in V$ is said to be a *matching payoff* if there exists a matching M in G, such that $p_u + p_v = w(uv)$ for all $uv \in M$, and $p_u = 0$

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for each u that is not incident to an edge in M. We then say that p is a payoff *with respect to* M, and we call the pair (M, p) a *matching with payoffs*. A pair of adjacent vertices $\{u, v\}$, i.e., with an edge between them, is a *blocking pair* of $p \in \mathbb{R}^V_{\geq 0}$ if $p_u + p_v < w(uv)$, and their *blocking value* with respect to p is defined as $w(uv) - (p_u + p_v)$. The latter value expresses to which extent $\{u, v\}$ is a blocking pair. We define the *set of blocking pairs* of a vector $p \in \mathbb{R}^V_{\geq 0}$ as

$$B(p) = \{\{u, v\} \mid uv \in E \text{ and } p_u + p_v < w(uv)\},$$

and we define the *total blocking value* of p as

$$b(p) = \sum_{uv \in E} \max\{0, w(uv) - (p_u + p_v)\}.$$

The problem Stable Roommates with Payments is that of testing whether a weighted graph allows a *stable solution*, i.e., a matching with payoffs (M, p) such that $B(p) = \emptyset$, or equivalently, $b(p) = 0$. This problem is well known to be polynomial-time solvable (cf. [6]); recently, an $O(nm + n^2 \log n)$ time algorithm for weighted graphs on n vertices and m edges has been given [3].

We consider two natural questions in our paper:

1. Can we gradually transform an unstable solution into a stable solution assuming that a stable solution exists?
2. Can we efficiently find solutions for no-instances that are "as stable as possible"?

Question 1 is of structural importance, as it will give us some insight into the coalition formation process. A sequence of solutions starting from an unstable one and ending in a stable one is called a *path to stability*; we give a precise definition later. Question 2 is of algorithmic nature and is relevant when we consider no-instances of Stable Roommates with Payments. In order to answer it, we generalize this problem in two different ways leading to the following two decision problems. Given a weighted graph G and an integer $k \geq 0$, the Blocking Pairs problem is to test whether G allows a matching payoff p with $|B(p)| \leq k$, and the Blocking Value problem is to test whether G allows a matching payoff p with $b(p) \leq k$.

Questions 1 and 2 have been studied in two closely related settings that are well known and formed a motivation for our study. The first related setting is similar to ours except that payments are not allowed. Instead, each vertex u in an (unweighted) graph $G(V, E)$ has a linear order on its neighbors expressing a certain preference. Then two adjacent vertices u and v form a *blocking pair* relative to a matching M if either u is not matched in M or else u prefers v to its partner in M, and simultaneously, the same holds for v. This leads to the widely studied problem Stable Roommates introduced by Gale and Shapley [7]. In this setting, the results are as follows. Answering a question by Knuth [12], Roth and Vande Vate [13] showed the existence of a path to stability for any yes-instance provided that the instance is bipartite. Later, their result was generalized by Diamantoudi et al. [5] to be valid for general instances. Abraham, Biró and Manlove [1] showed that the problem of finding a matching with a minimum number of blocking pairs is NP-complete; note that the problem Blocking Value cannot be translated to this setting, due to the absence of cardinal utilities.

The second related setting originates from cooperative game theory. A *cooperative game with transferable utilities* (TU-game) is a pair (N, v), where N is a set of n *players* and a *value function* $v : 2^N \to \mathbb{R}_{\geq 0}$ with $v(\emptyset) = 0$ defined for every *coalition* S, which is a subset of N. In a *matching game* (N, v), the set N of players is the vertex set of weighted graph G, and the value of a coalition S is $v(S) = \sum_{e \in M} w(e)$, where M is a maximum weight matching in the subgraph of G induced by S. The strong relationship between the two settings stems from the fact that finding a *core allocation*, i.e., a vector $x \in \mathbb{R}^N$ with $\sum_{u \in N} x_u = v(N)$ and $\sum_{u \in S} x_u \geq v(S)$ for all $S \subseteq N$ is equivalent to solving the Stable Roommates with Payments (cf. [6]). The algorithms of Béal et al. [2] and Yang [15] applied to an n-player matching game with a nonempty core (i.e. that have at least one core allocation) find a path to stability with lengths at most $(n^2 + 4n)/4$ and $2n - 1$, respectively. For matching games, the problems Blocking Pairs and Blocking Value are formulated as the problems that are to test whether a matching game (N, E) allows an allocation x with $|B(x)| \leq k$, or $b(x) \leq k$, respectively, for some given integer k. Biró, Kern and Paulusma [3] showed that the first problem is NP-complete and that the second is polynomial-time solvable by formulating it as a linear program.

Our Results. In Section 2, we prove a structural result that provides an affirmative answer to Question 1. We show that any unstable solution for a weighted n-vertex graph G that is a yes-instance of Stable Roommates with Payments allows a path to stability of length at most $2n$. This generalizes a structural result of Chen, Fujishige and Yang [4], who show the existence of a path to stability for the aforementioned stable marriage problem with payments, which corresponds to the case when G is bipartite. In Section 3 we prove a number of computational complexity results. We first answer Question 2 by proving that Blocking Pairs and Blocking Value are NP-complete. The latter result is somewhat surprising, as the corresponding problem is polynomial-time solvable for matching games; we refer to Table 1 for a survey. In addition, we show that Blocking Value does become polynomial-time solvable if the desired matching payoff is to be with respect to some specified matching M that is part of the input, whereas this variant of Blocking Pairs turns out to be NP-complete.

2. Paths to stability

We first give a useful lemma, which immediately follows from the aforementioned fact that finding a core allocation in a matching game (N, v) defined on a weighted graph $G = (N, E)$ is equivalent to finding a stable solution for G.

Table 1
A comparison of the results for the existence of a path to stability and the problems Blocking Pairs and Blocking Value in the three different settings of stable roommates (SR), stable roommates with payments (SRwP) and matching games (MG). The three results marked by a $*$ are the new results shown in this paper.

	SR	SRwP	MG
Path to Stability	Yes	Yes*	Yes
Blocking Pairs	NP-complete	NP-complete*	NP-complete
Blocking Value	n/a	NP-complete*	P

Lemma 1 *([6]). Let G be a weighted graph that forms a yes-instance of* Stable Roommates with Payments*. Then G allows a stable solution (M^*, p^*) where M^* is a maximum weight matching of G.*

Let M be a matching in a graph $G = (V, E)$. Let $\{u, v\}$ be a blocking pair for some payoff p with respect to some matching M; note that $uv \notin M$ by definition. Let p' be a payoff with respect to a matching M'. We say that (M', p') is obtained from (M, p) by *satisfying* blocking pair $\{u, v\}$ if the following two conditions hold:

(i) $M' = (M \setminus \{xy \in M \mid x = u \text{ or } x = v\}) \cup \{uv\}$;
(ii) $p'_u \geq p_u$ and $p'_v \geq p_v$, whereas $p'_z = p_z$ if z is neither in $\{u, v\}$ nor matched to u nor to v in M (in the latter two cases $z \notin M'$ and hence $p'_z = 0$).

That is, the players of a blocking pair become matched to each other in M' by leaving their former partners in M (if these existed) unmatched (and hence with no payoff values) in M', and they share the extra utility coming from their cooperation in such a way that neither of them gets worse off. Note that at least one of them strictly improves, i.e., we have $p'_u > p_u$ or $p'_v > p_v$. This is due to the following two arguments. First, by the definition of a blocking pair, $p_u + p_v < w(uv)$. Second, $p'_u + p'_v = w(uv)$, because p' is a payoff with respect to M' and $uv \in M'$ by condition (i).

Let G be a weighted graph that forms a yes-instance of Stable Roommates with Payments. A *path to stability* for G is a sequence of matchings with payoffs

$$(M^0, p^0), (M^1, p^1), \dots, (M^k, p^k),$$

where $(M^0, p^0), \dots, (M^{k-1}, p^{k-1})$ are unstable solutions and (M^k, p^k) is a stable solution, such that (M^{i+1}, p^{i+1}) is obtained from (M^i, p^i) for $i = 0, \dots, k-1$ by satisfying some blocking pair.

A known proof technique for finding a path to stability is to make use of a so-called reference solution (see e.g. [5,2,11,15]). In our setting, this comes down to the following. We say that (M', p') is obtained from (M, p) by *satisfying* blocking pair $\{u, v\}$ *with respect to* a payoff p^* of some stable solution (M^*, p^*) that is called a *reference solution*, if in addition to conditions (i)–(ii) also the following condition is satisfied:

(iii) if $p_u \leq p^*_u$ then $p'_u \leq p^*_u$, and if $p_v \leq p^*_v$ then $p'_v \leq p^*_v$.

We define $S^*(p) = \{u \in V(G) : p_u > p^*_u\}$ to be the *set of overpaid vertices* in (M, p) with respect to (M^*, p^*). We note that when (M', p') is obtained from (M, p) by satisfying a blocking pair with respect to p^* then $S^*(p') \subseteq S^*(p)$. In order to prove the existence of a path to stability for some graph G that is a yes-instance of Stable Roommates with Payments, it may be easier to find a path to stability $(M^0, p^0), (M^1, p^1), \dots, (M^k, p^k)$, where (M^{i+1}, p^{i+1}) is obtained from (M^i, p^i) for $i = 1, \dots, k$ by satisfying some blocking pair with respect to p^*, in such a way that $S^*(p^{i+1}) \subseteq S^*(p^i)$ for $i = 0, \dots, k-1$, with strict inclusion occurring after a certain number of steps; the latter property is then to guarantee that an algorithm for solving this problem will eventually terminate in a stable solution.

We will use the approach described above in order to show that any weighted n-vertex graph G that forms a yes-instance of Stable Roommates with Payments allows a path to stability of length $2n$ that starts in an arbitrary unstable solution. Before we give the proof, we first explain in more detail how our result is connected to results from the literature. Our result is based on the work of Kóczy and Lauwers [11] on the so-called accessibility of the coalition structure core. Their result implies the existence of a path to stability for any TU-game with a nonempty core. In this setting, a path to stability is a sequence of gradual changes that transform a non-core allocation to a core allocation. Recently, Béal et al. [2] and Yang [15] built on the work of Kóczy and Lauwers [11] in order to show the accessibility of the coalition structure core in quadratic time. In particular, Yang [15] obtained a linear upper bound on the length of a path to stability for all TU-games with a nonempty core, which include the matching games with a nonempty core. We can use their proof techniques [2,15] for our setting. Our arguments are slightly different though, because for matching games (N, v) every coalition $S \subseteq N$ may be blocking instead of only pairs $\{u, v\}$ as in our setting. As a consequence, for matching games several blocking pairs may be satisfied in one step by choosing the affected vertices to form a blocking coalition. Moreover, even if the starting solution is a matching with payoffs and the final solution is a stable matching with payoffs, the intermediate solutions in a path to stability for matching games are not necessarily such allocations that can be realized as matchings with payoffs. Therefore, the arguments of Yang [15] for restricting the path length cannot be translated to obtain our linear upper bound. By pinpointing the connection to the

Input: a matching with payoffs (M^0, p^0) in a weighted stable graph G
Output: a stable solution

Set $i := 0$.

Phase 1: **while** there is a blocking pair $\{u, v\}$ for (M^i, p^i) such that $uv \in M^*$ **do**
satisfy $\{u, v\}$ with respect to p^*, $(M^{i+1}, p^{i+1}) \leftarrow (M^i, p^i)$; set $i := i + 1$.

Phase 2: **if** there is a blocking pair $\{u, v\}$ for (M^i, p^i) **then**
satisfy $\{u, v\}$ with respect to p^*, $(M^{i+1}, p^{i+1}) \leftarrow (M^i, p^i)$; set $i := i + 1$, and return to Phase 1.

Return (M^i, p^i).

Fig. 1. The algorithm for finding a path to stability. Contrary to the algorithms of Béal et al. [2] and Yang [15], we do not have to specify the payoff p^{i+1}; any vector p^{i+1} that is a payoff with respect to M^{i+1} and satisfies conditions (ii)–(iii) may be chosen.

setting of cooperative games, we are not only able to generalize the corresponding result of Chen, Fujishige and Yang [4] for the existence of a path to stability for bipartite instances (which are always yes-instances) to general yes-instances, but we could also give a simpler proof of this result with a linear upper bound on the number of blocking pairs that need to be satisfied.

Theorem 1. *Let G be a weighted n-vertex graph that forms a yes-instance of* Stable Roommates with Payments. *Let (M^0, p^0) be a matching with payoffs. Then there exists a path to stability of length at most $2n$ that starts in (M^0, p^0).*

Proof. Let G be a weighed n-vertex graph that forms a yes-instance of Stable Roommates with Payments; we also call such a graph G *stable*. Let (M^0, p^0) be a matching with payoffs. We fix a stable reference solution (M^*, p^*), where we may assume that M^* is a maximum weight matching due to Lemma 1. Note that $|M^*| \le \frac{n}{2}$ and $|M^0| \le \frac{n}{2}$. Moreover, $|S^*(p^0)| \le \frac{n}{2}$, because the vertices u and v of a pair $uv \in M^0$ cannot both belong to $S^*(p^0)$, as otherwise $p^0_u > p^*_u$, $p^0_v > p^*_v$ and $w(uv) = p^0_u + p^0_v$ would imply that $\{u, v\}$ is a blocking pair for (M^*, p^*).

To obtain a path of stability we run the algorithm displayed in Fig. 1. Recall that $S^*(p^{i+1}) \subseteq S^*(p^i)$ for any solution (M^i, p^i) for which the algorithm performs Phase 1 or 2. Now we will prove that whenever we satisfy a blocking pair $\{u, v\}$ with $uv \notin M^*$ in Phase 2 the above relation is strict. More precisely, let (M^i, p^i) be a solution after a termination of Phase 1 (so, there is no $\{u, v\}$ with $uv \in M^*$ that is a blocking pair for (M^i, p^i)) and let (M^{i+1}, p^{i+1}) be the solution obtained after satisfying a blocking pair $\{u_i, v_i\}$ with $u_iv_i \notin M^*$ for (M^i, p^i). Then we will show that $S^*(p^{i+1}) \subset S^*(p^i)$. We first show three claims, where we write $w(M) = \sum_{uv \in M} w(uv)$ for a matching M.

Claim 1. $p^*_u + p^*_v = p^i_u + p^i_v$ *for all* $uv \in M^*$, *and* M^i *has maximum weight.*

We prove Claim 1 as follows. Because there is no pair of vertices $\{u, v\}$ with $uv \in M^*$ that is a blocking pair for (M^i, p^i), we have $p^*_u + p^*_v = w(uv) \le p^i_u + p^i_v$ for all $uv \in M^*$. This implies that

$$w(M^*) = \sum_{uv \in M^*} p^*_u + p^*_v \le \sum_{uv \in M^*} p^i_u + p^i_v \le w(M^i).$$

However, because M^* is a maximum weight matching, we have equality everywhere, i.e., we have $p^*_u + p^*_v = p^i_u + p^i_v$ for all $uv \in M^*$, and $w(M^i) = w(M^*)$. The latter equality implies that M^i is a maximum weight matching as well.

Claim 2. $p^i_u + p^i_v = p^*_u + p^*_v$ *for all* $uv \in M^i$.

We prove Claim 2 as follows. The stability of (M^*, p^*) implies that $p^i_u + p^i_v = w(uv) \le p^*_u + p^*_v$ for all $uv \in M^i$. This leads to

$$w(M^i) = \sum_{uv \in M^i} p^i_u + p^i_v \le \sum_{uv \in M^i} p^*_u + p^*_v \le w(M^*).$$

Together with the maximality of M^i that follows from Claim 1, this means that we have equality everywhere again, so $p^i_u + p^i_v = p^*_u + p^*_v$ for all $uv \in M^i$.

Claim 3. If a vertex t is unmatched in M^i or M^*, then $p^i_t = p^*_t = 0$.

We prove Claim 3 as follows. Suppose that t is unmatched in M^i. Then $p^i_t = 0$ by definition. We use Claim 2 and the fact that M^* and M^i are maximum weight matchings to obtain $w(M^*) = w(M^i) = \sum_{uv \in M^i}(p^i_u + p^i_v) = \sum_{uv \in M^i}(p^*_u + p^*_v)$. By definition, $w(M^*) = \sum_{u \in V} p^*_u$. Due to these two equalities, $p^*_t = 0$. The case when t is unmatched in M^* can be proven by similar arguments. This completes the proof of Claim 3.

We now consider the pair $\{u_i, v_i\}$ and write $u = u_i$ and $v = v_i$. Because $\{u, v\}$ is a blocking pair for (M^i, p^i), whereas (M^*, p^*) is a stable solution, we deduce that $p^i_u + p^i_v < w(uv) \le p^*_u + p^*_v$; note that this means that $w(uv) > 0$. If u and v are both

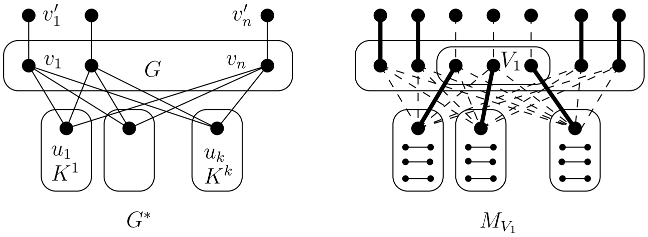

Fig. 2. The graph G^* and an example of a matching M_{V_1}. The edges within the subgraph G of G^* have not been drawn.

unmatched in M^i, then $p_u^* = p_v^* = 0$ by Claim 3. Then $w(uv) \leq 0$, which is not possible. Hence, we are left to analyze two cases.

First suppose that one of u, v, say u, is unmatched in M^i, whereas v is matched by M^i, say $vy \in M^i$. Because u is unmatched, $p_u^i = p_u^* = 0$ by Claim 3. Because we already deduced that $p_u^i + p_v^i < p_u^* + p_v^*$, this means that $p_v^i < p_v^*$. The inequality $p_v^i < p_v^*$ and the equality $p_v^i + p_y^i = p_v^* + p_y^*$ from Claim 2 imply that $p_y^i > p_y^*$, i.e., $y \in S^*(p^i)$. Because y becomes unmatched after satisfying uv by definition, we find that $p^{i+1}(y) = 0$. Hence, $S^*(p^{i+1}) \subset S^*(p^i)$.

Now suppose that both u and v are matched in M^i. Let $xu \in M^i$ and $vy \in M^i$. The equalities $p_x^i + p_u^i = p_x^* + p_u^*$ and $p_v^i + p_y^i = p_v^* + p_y^*$ from Claim 2, together with the aforementioned inequality $p_u^i + p_v^i < p_u^* + p_v^*$, imply that $p_x^i + p_y^i > p_x^* + p_y^*$. Hence, $p_x^i > p_x^*$ or $p_y^i > p_y^*$. This means that x or y is in $S^*(p^i)$. We may assume without loss of generality that $x \in S^*(p^i)$. Because x becomes unmatched after satisfying uv by definition, we find that $p^{i+1}(x) = 0$. Hence, $S^*(p^{i+1}) \subset S^*(p^i)$ also in this case.

Because the number of overpaid vertices decreases after each execution of Phase 2, the algorithm terminates and the returned solution (M^ℓ, p^ℓ) is stable. Consequently, we have shown the existence of a path to stability.

Now we set the linear upper bound for the number of steps ℓ required to reach a stable solution. Each time we satisfy a blocking pair not in M^* in Phase 2, the number of overpaid vertices decreases. Hence, we cannot satisfy more than $|S^*(p^0)| \leq \frac{n}{2}$ of them. Regarding the pairs of M^*, after the first time we satisfy a pair $uv \in M^*$ we may need to satisfy it again only if u or v is involved in a blocking pair $\{x, u\}$ or $\{u, y\}$, respectively, that is not in M^* and that is satisfied in Phase 2. Hence, the satisfaction of a pair xu not in M^* may result that at most two pairs in M^*, involving either x or u, can be subsequently satisfied in Phase 1, but all the other pairs of M^* satisfied in this execution of Phase 1 must be satisfied for the first time. Therefore we have the following upper bounds.

- We satisfy at most $\frac{n}{2}$ pairs not in M^*.
- We satisfy at most $\frac{n}{2}$ pairs of M^* for the first time.
- We satisfy pairs of M^* not for the first time at most $2 \cdot \frac{n}{2} = n$ times.

Thus we satisfy at most $\ell = \frac{n}{2} + \frac{n}{2} + n = 2n$ pairs. This completes our proof. □

Remark. Our proof of Theorem 1 is constructive. The algorithm of Fig. 1 constructs a path to stability starting in any unstable solution. Due to the linear upper bound stated in Theorem 1, its running time is $O(n^2)$ time for weighted graphs on n vertices, given a stable reference solution (M^*, p^*) which, if necessary, we can compute in $O(nm + n^2 \log n)$ [3]. We like to emphasize though that the main purpose of Theorem 1 is to show the existence of a path to stability of length at most $2n$ starting from an arbitrary solution (M^0, p^0). Moreover, the final (stable) solution on this path is not necessarily the same as the stable reference solution (M^*, p^*).

3. Blocking Pairs and Blocking Value

We start this section by showing that Blocking Pairs and Blocking Value are NP-complete. We prove the hardness of these two problems by a reduction from Independent Set, in a similar way as was done for the Blocking Pairs problem in the setting of matching games [3]. However, the latter setting and our setting are quite different in nature; in particular, we recall that the Blocking Value problem is polynomial-time solvable in the setting of matching games [3]. Hence, our hardness proof uses a number of different arguments than the hardness proof for Blocking Pairs in the setting of matching games [3].

Theorem 2. Blocking Pairs *and* Blocking Value *are* NP-*complete.*

Proof. Clearly, both problems are in NP. In order to prove NP-completeness, we reduce from the Independent Set problem. This problem takes as input a graph G with an integer k and is to test whether G contains an *independent set* of size at least k, i.e., a set S with $|S| \geq k$ such that there is no edge in G between any two vertices of S. Garey, Johnson and Stockmeyer [9] show that Independent Set is already NP-complete for the class of 3-*regular* graphs, i.e., graphs in which all vertices are of degree three. So we may assume that G is 3-regular. We also assume that $k \geq 2$. Let $n = |V|$ and let $V = \{v_1, \dots, v_n\}$.

From G we construct a weighted graph $G^* = (V^*, E^*)$ on $2n + k(4k + 3)$ vertices. First, we add a set V' of n new vertices $v_1', \dots, v_n'$, where we add an edge between v_i and v_i' for $i = 1, \dots, n$. So, every v_i' has a unique neighbor in the resulting graph, namely v_i. Now let K be a complete graph on $r = 4k+3$ vertices; note that r is odd. We add k mutually vertex-disjoint copies $K^1, \ldots, K^k$ of K to the graph constructed so far. In each copy K^i we specify a vertex u_i leading to a set $U = \{u_1, \ldots, u_k\}$. We then finish our construction of G^* by adding an edge $u_h v_i$ for all $1 \leq h \leq k$ and all $1 \leq i \leq n$; see Fig. 2. It remains to define an edge weighting w on G^*. We let $w(u_h v_i) = \frac{1}{2}$ for all $1 \leq h \leq k$ and all $1 \leq i \leq n$, whereas we assign all other edges e of G^* weight $w(e) = 1$.

We make the following observation that is important for the remainder of the proof. By our construction, there exists a matching M_{V_1} for each subset $V_1 \subseteq V$ of size k that can be decomposed as $M_{V_1} = M_1 \cup \cdots \cup M_k \cup M_{UV_1} \cup M_{V_2V_2'}$, where M_h is a perfect matching of $K^h - u_h$ for $h = 1, \ldots, k$, M_{UV_1} is a perfect matching of $G^*[U \cup V_1]$ and $M_{V_2V_2'}$ is a perfect matching of $G^*[V_2 \cup V_2']$ for $V_2 = V \setminus V_1$ and its set of neighbors V_2' in V'. We call a matching M_{V_1} as defined above a V_1-*matching*. Note that V_1 has more than one V_1-matching, because we can pick different perfect matchings for the decomposition of M_{V_1} (except for the perfect matching $M_{V_2V_2'}$ of $G[V_2, V_2']$, which is unique).

For our two NP-hardness reductions, it suffices to show that the following three statements are equivalent.

(i) G has an independent set S of size at least k.
(ii) $|B(p)| \leq k$ for some matching payoff p of G^*.
(iii) $b(p) \leq k$ for some matching payoff p of G^*.

"(i) $\Rightarrow$ (ii)" Suppose that G has an independent set S of size $|S| \geq k$. Then we may assume without loss of generality that $|S| = k$, as otherwise we could just remove some vertices from S. We pick an arbitrary S-matching M_S and define a payoff p with respect to M_S as follows. We let $p \equiv \frac{1}{2}$ on $K^1 \cup \cdots \cup K^k$, whereas we let $p \equiv 1$ on $V \setminus S$ and $p \equiv 0$ on $S \cup V'$. Because S is an independent set and $p \equiv 1$ on $V \setminus S$, no pair $\{v_i, v_j\}$ is a blocking pair. This and the definition of p ensure that $B(p) = \{\{v_i, v_i'\} \mid v_i \in S\}$, which has size k.

"(ii) $\Rightarrow$ (iii)" Suppose that $|B(p)| \leq k$ for some matching payoff p of G^*. Then $b(p) \leq k$, because each blocking pair in $B(p)$ can contribute at most a value of 1 to the total blocking value $b(p)$ as the maximum value of w is 1.

"(iii) $\Rightarrow$ (i)" Suppose that $b(p) \leq k$ for some matching payoff p of G^*. Assume that $b(p)$ is minimum over all matching payoffs. Let M be the associated matching. We first show three useful claims.

Claim 1. *For all $1 \leq h \leq k$, every $z \in V_{K^h} \setminus \{u_h\}$ is matched by M.*

We prove Claim 1 as follows. Suppose that there exists some complete graph K^h that contains a nonempty subset $D \subseteq V_{K^h} \setminus \{u_h\}$ of vertices that are unmatched in M. Assume that D contains all such vertices of $V_{K^h} \setminus \{u_h\}$. Let $A = V_{K^h} \setminus \{u_h \cup D\}$. Then, by definition, A contains exactly those vertices of $V_{K^h} \setminus \{u_h\}$ that are matched in M. We write $\alpha = |A|$ and $\delta = |D|$. By our construction, the vertices in A can only be matched by M via edges with both end-vertices in $K^h[A]$. By definition, $p_z + p_{z'} = 1$ for all $zz' \in M$ with $z, z' \in A$. This means that $\sum_{z \in A} p_z = \frac{1}{2}\alpha$. Moreover, $p \equiv 0$ on D by definition, and $\delta \geq 1$ by our assumption. We let E_1 be the set of edges with one end-vertex in A and the other one in D. We let E_2 be the set of edges with both end-vertices in D. By using the properties of A and D, we find that

$$
\begin{aligned}
k \geq b(p) &\geq \sum_{zz' \in E_1} (1 - p_z - p_{z'}) + \sum_{zz' \in E_2} (1 - p_z - p_{z'}) \\
&= \delta \sum_{z \in A} (1 - p_z) + \sum_{zz' \in E_2} 1 \\
&= \alpha\delta - \frac{1}{2}\alpha\delta + \frac{1}{2}\delta(\delta - 1) \\
&= \frac{1}{2}\alpha\delta + \frac{1}{2}\delta(\delta - 1).
\end{aligned}
$$

Recall that $\delta \geq 1$. We distinguish three cases. If $\delta = 1$, then $\alpha = r - \delta - 1 = r - 2$. Then our deduction implies that $k \geq \frac{1}{2}\alpha = \frac{1}{2}(r - 2)$, which is equivalent to $r \leq 2k + 2$. If $\delta = 2$, then $\alpha = r - 3$, and we find that $k \geq \alpha + 1 = r - 2$, which is equivalent to $r \leq k + 2$. If $\delta \geq 3$, then we find that $k \geq \frac{3}{2}\alpha + \delta \geq \alpha + \delta = r - 1$, which is equivalent to $r \leq k + 1$. Hence, in all three cases, we find that $r \leq 2k + 2$. This is not possible, because $r = 4k + 3 > 2k + 2$. We conclude that $D = \emptyset$. Hence, we have proven Claim 1.

Claim 2. *There exists a subset $V_1 \subseteq V$ such that the restriction of M to the edges of $G^*[V_1 \cup U]$ is a perfect matching.*

We prove Claim 2 as follows. First suppose that there exists some u_h that is unmatched in M. Then $p_{u_h} = 0$ by definition. Let $A = V_{K^h} \setminus \{u_h\}$. Note that $|A| = r - 1$ is even, because r is odd. Claim 1 tells us that the vertices of A are matched by edges of M. By construction, these matching edges must have both end-vertices in A. Because $p_z + p_{z'} = 1$ for all $zz' \in M$ and $p \geq 0$, this means that there are at least $\frac{1}{2}(r - 1)$ vertices in A, whose payoff is at most $\frac{1}{2}$. We consider the edges between v and those vertices and deduce that $k \geq b(p) \geq \frac{1}{2}(r - 1)(1 - \frac{1}{2} - 0)$, which is equivalent to $r \leq 4k + 1$. This is not possible, because $r = 4k + 3$. Hence, every u_h is matched by M.

Now suppose that u_h forms a matching edge of M together with some other vertex z of K^h. Then M cannot cover all vertices of K^h, because r is odd. This is not possible due to Claim 1. Hence, every u_h forms a matching edge of M with some vertex v_i from V. This gives us the set V_1, and we have proven Claim 2.

Claim 3. *$p(u) = \frac{1}{2}$ for all $u \in U$.*

We prove Claim 3 as follows. Suppose that $p_{u_h} \neq \frac{1}{2}$ for some $1 \leq h \leq k$. By Claim 2, u_h forms a matching edge of M with some vertex v_i. Then $p_{u_h} + p_{v_i} = w(u_h v_i) = \frac{1}{2}$. This means that $p_{u_h} < \frac{1}{2}$ and $p_{v_i} = \epsilon > 0$. We modify p into a new payoff p' with respect to M by increasing the payoff to u_h with ϵ and decreasing the payoff of v_i to zero. Because G is 3-regular, v_i has 3 neighbors in G. As in the proof of Claim 2, there are at least $\frac{1}{2}(r-1)$ vertices in $K^h - u_h$, whose payoff is at most $\frac{1}{2}$. Hence, taking into account the other neighbors of v_i in G^* as well, our modification of p decreases the total blocking value by at most $(k+4)\epsilon$ but at the same time increases it by at least $\frac{1}{2}(r-1)\epsilon$. Hence, $b(p') \geq b(p) - (k+4)\epsilon + \frac{1}{2}(r-1)\epsilon = b(p) + (\frac{1}{2}(r-1) - (k+4))\epsilon > b(p)$, where the latter inequality follows from the fact that $r \geq 4k+2 \geq 2k+5$, as we assume that $k \geq 2$. However, $b(p') > b(p)$ contradicts the minimality of $b(p)$. Hence, we have proven Claim 3.

We are now ready to argue how to find an independent set of size at least k in G. Let V_1 be the set from Claim 2. By Claim 3 and the fact that the weights $w(e)$ of every edge e between U and V is set to $\frac{1}{2}$, we find that $p \equiv 0$ on V_1. Due to Claim 2, no vertex v_i' with $v_i \in V_1$ can be matched by M. Hence, $p_{v_i'} = 0$ for every $v_i \in V_1$. Because $|U| = k$, we find that $|V_1| = k$. Let E_1' denote the set of edges $v_i v_i'$ with $v_i \in V_1$. Because $|V_1| = k$, we obtain $|E_1'| = k$. Suppose that V_1 contains two adjacent vertices v_i and v_j. Then $b(p) \geq \sum_{zz' \in E_1'} (1 - p_z - p_{z'}) + (1 - p_{v_i} - p_{v_j}) = k+1$. This is not possible, because $b(p) \leq k$. Hence, no two vertices in V_1 are adjacent. In other words, V_1 is an independent set of size $|V_1| = k$, as desired. This completes the proof of Theorem 2. □

The problems Restricted Blocking Pairs and Restricted Blocking Value take as input a graph G, an integer k, and a matching M of G, and are to decide whether G has a payoff p with respect to M such that $|B(p)| \leq k$ or $b(p) \leq k$, respectively.

Theorem 3. *The* Restricted Blocking Value *problem is polynomial-time solvable, whereas the* Restricted Blocking Pairs *problem is* NP*-complete even for graphs with unit edge weights.*

Proof. We first consider the Restricted Blocking Value problem. Let $G = (V, E)$ be a graph with an edge weighting w. Let M be a given matching of G. We let V_M denote the set of vertices of G matched by M. Then we can formulate the Restricted Blocking Value problem as the linear program

$$
\begin{array}{lll}
\text{(RBV)} & \min \sum_{uv \in E \setminus M} z_{uv} & \\
& \text{s.t. } p_u + p_v = w(uv) & (uv \in M) \\
& p_u + p_v + z_{uv} \geq w(uv) & (uv \in E \setminus M) \\
& p_u \geq 0 & (u \in V_M) \\
& p_u = 0 & (u \in V \setminus V_M) \\
& z_{uv} \geq 0 & (uv \in E \setminus M).
\end{array}
$$

Consequently, Restricted Blocking Value can be solved in polynomial time by the ellipsoid method [10].

We now consider the Restricted Blocking Pairs problem. Clearly, this problem is in NP. In order to prove NP-completeness, we reduce from the 3-Satisfiability problem, which is NP-complete (cf. [8]).

Given an instance of 3-Satisfiability with Boolean variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$, we construct a graph G as follows (see Fig. 3).

- For $i = 1, \ldots, n$, construct adjacent vertices $x_i, \bar{x}_i$ that correspond to the literals over x_i.
- For $j = 1, \ldots, m$, construct pairwise adjacent vertices $u_j^{(1)}, u_j^{(2)}, u_j^{(3)}$ and pairwise adjacent vertices $v_j^{(1)}, v_j^{(2)}, v_j^{(3)}$, then add the edges $u_j^{(1)} v_j^{(1)}, u_j^{(2)} v_j^{(2)}, u_j^{(3)} v_j^{(3)}$.
- For $j = 1, \ldots, m$, let $C_j = z_1 \vee z_2 \vee z_3$. Join $u_j^{(1)}, u_j^{(2)}, u_j^{(3)}$ with the vertices that correspond to the literals z_1, z_2, z_3 by edges respectively.
- Construct $m+1$ vertices $w_0, \ldots, w_m$; for $s = 0, \ldots, m$ and $i = 1, \ldots, n$, add the edges $w_s x_i$ and $w_s \bar{x}_i$, and for $s = 0, \ldots, m$ and $j = 1, \ldots, m$ add the edges $w_s u_j^{(1)}, w_s u_j^{(2)}, w_s u_j^{(3)}, w_s v_j^{(1)}, w_s v_j^{(2)}, w_s v_j^{(3)}$.

Finally, we define

$$M = \{x_i \bar{x}_i \mid 1 \leq i \leq n\} \cup \{u_j^{(r)} v_j^{(r)} \mid 1 \leq i \leq n, 1 \leq r \leq 3\},$$

and $k = (n + 3m)(m+1) + m$. We prove that the formula $\phi = C_1 \wedge \cdots \wedge C_m$ can be satisfied if and only if there is a payoff p for G with respect to M with $|B(p)| \leq k$.

First suppose that ϕ can be satisfied, i.e., that the variables $x_1, \ldots, x_n$ are assigned values such that $\phi = true$. We define a vector p as follows.

- For $s = 0, \ldots, m$, we set $p_{w_s} = 0$.
- For $i = 1, \ldots, n$, we set $p_{x_i} = 1, p_{\bar{x}_i} = 0$ if $x_i = true$, and $p_{x_i} = 0, p_{\bar{x}_i} = 1$ otherwise.
- For $j = 1, \ldots, m$, if $C_j = z_1 \vee z_2 \vee z_3$, then we choose a literal $z_r = true$ for some $r \in \{1, 2, 3\}$, and we set $p_{u_j^{(r)}} = 0, p_{v_j^{(r)}} = 1$ and $p_{u_j^{(h)}} = 1, p_{v_j^{(h)}} = 0$ for $h \in \{1, 2, 3\} \setminus \{r\}$.

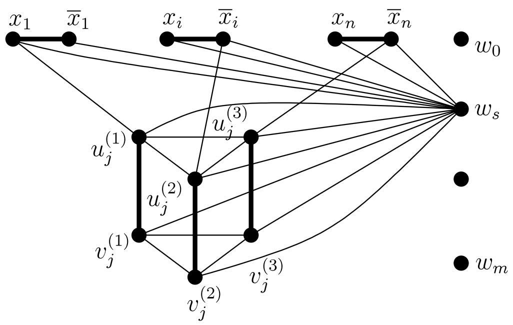

Fig. 3. The construction of G. For clarity, only one clause has been displayed, which in this example is the clause $C_j = \{x_1, \bar{x}_i, \bar{x}_n\}$, and moreover, the edges incident to w_i for $i \neq s$ have not been drawn. The edges that belong to M are shown by thick lines.

It is straightforward to check that p is a payoff with respect to M. Observe that for all $s \in \{0, \ldots, m\}$ and all $ab \in M$, exactly one of the pairs $\{w_s, a\}$, $\{w_s, b\}$ is a blocking pair. We also have for all $1 \le j \le m$ that $\{v_j^{(h)}, v_j^{(h')}\} \in B(p)$ for $\{h, h'\} = \{1, 2, 3\} \setminus \{r\}$. Moreover, all other pairs of adjacent vertices are not blocking pairs. Hence, $|B(p)| = (n+3m)(m+1)+m = k$.

Now suppose that p is a payoff for G with respect to M such that $|B(p)| \le k$. By definition, $p_{w_s} = 0$ for $s = 0, \ldots, m$. Hence, for each $ab \in M$, we have at least one of $\{w_s, a\}$, $\{w_s, b\}$ is in $B(p)$, and if $p_a < 1$ and $p_b < 1$, then both $\{w_s, a\}$ and $\{w_s, b\}$ are in $B(p)$. This observation together with the inequality $|B(p)| \le (n+3m)(m+1)+m$ yields that for each $ab \in M$, either $p_a = 1$, $p_b = 0$ or $p_a = 0$, $p_b = 1$. As a consequence, exactly $(n+3m)(m+1)$ blocking pairs include the vertices $w_0, \ldots, w_m$.

Consider an index $j \in \{1, \ldots, m\}$. If $p_{u_j^{(1)}} = p_{u_j^{(2)}} = p_{u_j^{(3)}} = 1$, then $p_{v_j^{(1)}} = p_{v_j^{(2)}} = p_{v_j^{(3)}} = 0$ and $\{v_j^{(h)}, v_j^{(h')}\} \in B(p)$ for all $1 \le h < h' \le 3$. Similarly, if $p_{u_j^{(1)}} = p_{u_j^{(2)}} = p_{u_j^{(3)}} = 0$ and $p_{v_j^{(1)}} = p_{v_j^{(2)}} = p_{v_j^{(3)}} = 1$, then $\{u_j^{(h)}, u_j^{(h')}\} \in B(p)$ for all $1 \le h < h' \le 3$. If there exist indices $h, h' \in \{1, 2, 3\}$ with $h \neq h'$ such that $p_{u_j^{(h)}} = 0$ and $p_{v_j^{(h')}} = 0$, then exactly one of the pairs from the set

$$\left\{\{u_j^{(1)}, u_j^{(2)}\}, \{u_j^{(2)}, u_j^{(3)}\}, \{u_j^{(1)}, u_j^{(3)}\}, \{v_j^{(1)}, v_j^{(2)}\}, \{v_j^{(2)}, v_j^{(3)}\}, \{v_j^{(1)}, v_j^{(3)}\}\right\}$$

is a blocking pair. Since G can have only $k - (n+3m)(m+1) = m$ blocking pairs of this type, we conclude that for all $j \in \{1, \ldots, m\}$, there is an index $h \in \{1, 2, 3\}$ such that $p_{u_j^{(h)}} = 0$, and moreover, if x_i or $\bar{x}_i$ is adjacent to $u_j^{(h)}$, then $\{x_i, u_j^{(h)}\}$ or $\{\bar{x}_i, u_j^{(h)}\}$, respectively, is not a blocking pair.

Now for $i = 1, \ldots, n$, we set the variable $x_i = true$ if $p_{x_i} = 1$, and $x_i = false$ otherwise. Consider a clause $C_j = z_1 \vee z_2 \vee z_3$. There is an index $h \in \{1, 2, 3\}$ such that $p_{u_j^{(h)}} = 0$. First suppose that $z_h = x_i$ for some $1 \le i \le n$. Then vertex $u_j^{(h)}$ is adjacent to the vertex x_i, and $\{x_i, u_j^{(h)}\} \notin B(p)$. Then $p_{x_i} = 1$ and the variable $x_i = true$. Now suppose that $z_h = \bar{x}_i$ for some $1 \le i \le n$. Then vertex $u_j^{(h)}$ is adjacent to the vertex $\bar{x}_i$ and $p_{\bar{x}_i} = 1$, i.e., the variable $x_i = false$. In both cases C_j contains a literal with the value *true*. It follows that $\phi = true$. This completes the proof of Theorem 3. □

4. Future work

Very recently, Bock, Köneman, Peis and Sanità (personal communication, August 2012) announced that Blocking Pairs is NP-complete even for graphs with unit edge weights. Their NP-hardness reduction does not work for the Blocking Value problem. Hence, we finish our paper by stating the following open problem. What is the computational complexity of Blocking Value restricted to input graphs with unit edge weights?

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