# Determining $L(2,1)$-Span in Polynomial Space 

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#### Abstract

A $k$ - $L(2,1)$-labeling of a graph is a function from its vertex set into the set $\{0, \ldots, k\}$, such that the labels assigned to adjacent vertices differ by at least 2 , and labels assigned to vertices of distance 2 are different. It is known that finding the smallest $k$ admitting the existence of a $k-L(2,1)$ labeling of any given graph is NP-Complete.

In this paper we present an algorithm for this problem, which works in time $O\left((9+\epsilon)^{n}\right)$ and polynomial memory, where $\epsilon$ is an arbitrarily small positive constant. This is the first exact algorithm for $L(2,1)$-labeling problem with time complexity $O\left(c^{n}\right)$ for some constant $c$ and polynomial space complexity.


## 1 Introduction

A frequency assignment problem is the problem of assigning channels of frequency (represented by nonnegative integers) to each radio transmitter, so that no transmitters interfere with each other. Hale [12] formulated this problem in terms of so-called $T$-coloring of graphs.

According to [11, Roberts was the first who proposed a modification of this problem, which is called an $L(2,1)$-labeling problem. It asks for such a labeling with nonnegative integer labels, that no vertices in distance 2 in a graph have the same label and labels of adjacent vertices differ by at least 2 .

A $k$ - $L(2,1)$-labeling problem is to determine if there exists an $L(2,1)$-labeling of a given graph with no label greater than $k$. By $\lambda(G)$ we denote an $L(2,1)$ span of $G$, which is the smallest value of $k$ that guarantees the existence of a $k$ - $L(2,1)$-labeling of $G$.

The problem of $L(2,1)$-labeling has been extensively studied (see [3, 7, 10 20] for some surveys on the problem and its generalizations). A considerable attention has been given to bounding the value of $\lambda(G)$ by some function of $G$.

Griggs and Yeh [11] proved that $\lambda(G) \leq \Delta^{2}+\Delta^{1}$ and conjectured, that

[^0]$\lambda(G) \leq \Delta^{2}$ for every graph $G$. There are several results supporting this conjecture, for example Gonçalves [9] proved that $\lambda(G) \leq \Delta^{2}+\Delta-2$ for graphs with $\Delta \geq 3$. Havet et al. [13] have settled the conjecture in affirmative for graphs with $\Delta \geq 10^{69}$. For graphs with smaller $\Delta$, the conjecture still remains open. It is interesting to note that the Petersen and Hoffmann-Singleton graphs are the only two known graphs with maximum degree greater than 2 , for which this bound is tight.

The second main branch of research in $L(2,1)$-labeling was pointed to analyzing the problem from the complexity point of view. For $k \geq 4$, the $k-L(2,1)$ labeling problem was proven to be NP-complete by Fiala et al. 6] (for $k \leq 3$ the problem is polynomial). It remains NP-complete even for regular graphs (see Fiala and Kratochvíl [8]), planar graphs (see Eggeman et al. [4]) or series-parallel graphs (see Fiala et al. [5]).

An exact algorithm for the so called Channel Assignment Problem, presented by Král' [19], implies an $O^{*}\left(4^{n}\right) \square^{2}$ algorithm for the $L(2,1)$-labeling problem. Havet et al. 14 presented an algorithm for computing $L(2,1)(G)$, which works in time $O^{*}\left(15^{\frac{n}{2}}\right)=O^{*}\left(3.8730^{n}\right)$. This algorithm has been improved [17, 18, achieving a complexity bound $O^{*}\left(3.2361^{n}\right)$. Recently, a new algorithm for $L(2,1)$-labeling with a complexity bound $O^{*}\left(2.6488^{n}\right)$ has been presented [16].

All algorithms mentioned above are based on dynamic programming approach and use exponential memory. Havet et al. [14] presented a branching algorithm for $k$-L $(2,1)$-labeling problem with a time complexity $O^{*}\left((k-2.5)^{n}\right)$ and polynomial space complexity. Until now, no algorithm for $L(2,1)$-labeling with time complexity $O\left(c^{n}\right)$ for some constant $c$ and polynomial space complexity has been presented. However, there are such algorithms for a related problem of classical graph coloring. The first one, with time complexity $O\left(5.283^{n}\right)$, was shown by Bodleander and Kratsch [2]. The best currently known algorithm for graph coloring with polynomial space complexity is by Björklund et al. [1], using the inclusion-exclusion principle. Its time complexity is $O\left(2.2461^{n}\right)$.

In this paper we present the first exact algorithm for the $L(2,1)$-labeling problem with polynomially bounded space complexity. The algorithm works in time $O\left((9+\epsilon)^{n}\right)$ (where $\epsilon$ is an arbitrarily small positive constant) and is based on a divide and conquer approach.

## 2 Preliminaries

Throughout the paper we consider finite undirected graphs without multiple edges or loops. The vertex set (edge set) of a graph $G$ is denoted by $V(G)$ ( $E(G)$, respectively).

Let $\operatorname{dist}_{G}(x, y)$ be the distance between vertices $x$ and $y$ in a graph $G$, which is the length of a shortest path joining $x$ and $y$.

A set $X \subseteq V(G)$ is a 2-packing in $G$ if and only if all its vertices are in distance at least 3 from each other $\left(\forall x, y \in X \operatorname{dist}_{G}(x, y)>2\right)$.

[^1]Let $N(v)=\{u \in V(G):(u, v) \in E(G)\}$ denote the set of neighbors (the neighborhood) of a vertex $v$. The set $N[v]=N(v) \cup\{v\}$ denotes the closed neighborhood of $v$. The neighborhood of a set $X$ of vertices in $G$ is denoted by $N(X)=\bigcup_{v \in X} N(v)$ and its closed neighborhood is denoted by $N[X]=$ $N(X) \cup X$.

For a subset $X \subseteq V(G)$, we denote the subgraph of $G$ induced by the vertices in $X$ by $G[X]$. A square of a graph $G=(V, E)$ is the graph $G^{2}=(V,\{u v \in$ $\left.\left.V^{2}: \operatorname{dist}_{G}(u, v) \leq 2\right\}\right)$.
Definition 1. For a graph $G$ and sets $Y, Z, M \subseteq V(G)$, a $(k-1)-L_{Z}^{M}(Y)$ labeling of a graph $G$ is a function $c: Y \rightarrow\{0,1, \ldots, k-1\}$, such that $c^{-1}(0) \cap$ $Z=c^{-1}(k-1) \cap M=\emptyset$, and for every $v, u \in Y$ :

$$
\begin{aligned}
& |c(v)-c(u)| \geq 2 \text { if } \operatorname{dist}_{G}(u, v)=1 \\
& |c(v)-c(u)| \geq 1 \text { if } \operatorname{dist}_{G}(u, v)=2
\end{aligned}
$$

A function $c: Y \rightarrow \mathbb{N}$ is an $L_{Z}^{M}(Y)$-labeling of $G$ if there exists $k \in \mathbb{N}$ such that $c$ is a $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$
Definition 2. For $Y, Z, M \subseteq V(G)$ let $\Lambda_{Z}^{M}(Y, G)$ denote the smallest value of $k$ admitting the existence of $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$. We define $\Lambda_{Z}^{M}(\emptyset, G) \stackrel{\text { def. }}{=}$. 0 for all graphs $G$ and sets $Z, M \subseteq V(G)$.

Any $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$ with $k=\Lambda_{Z}^{M}(Y, G)$ is called optimal.
We observe that even if $c$ is an optimal $L_{Z}^{M}(Y)$-labeling of $G$, then any of the sets $c^{-1}(0)$ and $c^{-1}\left(\Lambda_{Z}^{M}(Y, G)-1\right)$ may be empty. In the extremal case, if $Z=M=Y$, then $c^{-1}(0)=c^{-1}(k-1)=\emptyset$ for all $k$ and feasible $(k-1)-L_{Z}^{M}(Y)-$ labelings $c$ of $G$.

Notice that $\Lambda_{\emptyset}^{\emptyset}(V(G), G)=\lambda(G)+1$ for every graph $G$.
Definition 3. For a graph $G$, a $G$-correct partition of a set $Y \subseteq V(G)$ is a triple $(A, X, B)$, such that:

1. The sets $A, X, B \subseteq Y$ form a partition of $Y$
2. $X$ is a nonempty 2-packing in $G$
3. $|A| \leq \frac{|Y|}{2}$ and $|B| \leq \frac{|Y|}{2}$

## 3 Algorithm

In this section we present a recursive algorithm for computing $\Lambda_{Z}^{M}(Y, G)$ for any graph $G$ and sets $Y, Z, M \subseteq V(G)$. It is then used to find an $L(2,1)$-span a graph $G$.

The algorithm is based on the divide and conquer approach. First, the algorithm exhaustively check if $\Lambda_{Z}^{M}(Y, G) \leq 3$. If not, the set $Y$ is partitioned into three sets $A, X, B$, which form a $G$-correct partition of $Y$. The sets $A$ and $B$ are then labeled recursively.

The labeling of the whole $Y$ is constructed from the labelings found in the recursive calls. The sets of labels used on the sets $A$ and $B$ are separated from each other by the label used for the 2 -packing $X$. This allows to solve the subproblems for $A$ and $B$ independently from each other.

Iterating over all $G$-correct partitions of $Y$, the algorithm computes the minimum $k$ admitting the existence of a $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$, which is by definition $\Lambda_{Z}^{M}(Y, G)$.

```
Algorithm 1: Find-Lambda
    Input : Graph \(G\), Sets \(Y, Z, M \subseteq V(G)\)
    if \(Y=\emptyset\) then return 0
    foreach \(c: Y \rightarrow\{0,1,2\}\) do
        for \(k \leftarrow 1\) to 3 do
                if \(c\) is a \((k-1)-L_{Z}^{M}(Y)\)-labeling of \(G\) then return \(k\)
    \(k \leftarrow \infty\)
    foreach \(G\)-correct partition \((A, X, B)\) of \(Y\) do
        if \(A \neq \emptyset\) and \(B \neq \emptyset\) then \(k_{X} \leftarrow 1\)
        if \(A=\emptyset\) and \(X \cap Z=\emptyset\) then \(k_{X} \leftarrow 1\)
        if \(A=\emptyset\) and \(X \cap Z \neq \emptyset\) then \(k_{X} \leftarrow 2\)
        if \(B=\emptyset\) and \(X \cap M=\emptyset\) then \(k_{X} \leftarrow 1\)
        if \(B=\emptyset\) and \(X \cap M \neq \emptyset\) then \(k_{X} \leftarrow 2\)
        \(k_{A} \leftarrow\) Find-Lambda \((G, A, Z, N(X))\)
        \(k_{B} \leftarrow\) Find-Lambda \((G, B, N(X), M)\)
        \(k \leftarrow \min \left(k, k_{A}+k_{X}+k_{B}\right)\)
    return \(k\)
```

Lemma 1. For a graph $G$ and sets $Y, Z, M \subseteq V(G)$, if $Y$ is a 2-packing in $G$, then $\Lambda_{Z}^{M}(Y, G) \leq 3$.

Proof. The labeling $c: Y \rightarrow\{0,1,2\}$ such that $c(v)=1$ for every $v \in Y$ is a $2-L_{Z}^{M}(Y)$ labeling of $G$.

Theorem 1. For any graph $G$ and sets $Y, Z, M \subseteq V(G)$, the algorithm call Find-Lambda $(G, Y, Z, M)$ returns $\Lambda_{Z}^{M}(Y, G)$.

Proof. If $Y=\emptyset$, the correct result is given in the line (by the definition of $\left.\Lambda_{Z}^{M}(\emptyset, G)\right)$. If $\Lambda_{Z}^{M}(Y, G) \leq 3$, the result is found by the exhaustive search in the line 4. Notice that if $|Y| \leq 1$, then by Lemma $1 \Lambda_{Z}^{M}(Y, G) \leq 3$.

Assume that the statement is true for all graphs $G^{\prime}$ and all sets $Y^{\prime}, Z^{\prime}, M^{\prime} \subseteq$ $V\left(G^{\prime}\right)$, such that $\left|Y^{\prime}\right|<n$, where $n \geq 1$.

Let $G$ be a graph and $Y, Z, M$ be subsets of $V(G)$ such that $|Y|=n$. We may assume that $\Lambda_{Z}^{M}(Y, G)>3$. Let $k$ be the value returned by the algorithm call Find-Lambda $(G, Y, Z, M)$.

First we prove that $k \geq \Lambda_{Z}^{M}(Y, G)$, i.e. there exists a $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$. Let us consider the $G$-correct partition $(A, X, B)$ of $Y$, for which the value of $k$ was set in the line 14. Since each of the sets $A$ and $B$ has less than $n$
vertices, by the inductive assumption there exists a $\left(k_{A}-1\right)-L_{Z}^{N(X)}(A)$-labeling $c_{A}$ of $G$ and a $\left(k_{B}-1\right)-L_{N(X)}^{M}(B)$-labeling $c_{B}$ of $G$.

One of the following cases occurs:

1. If $A \neq \emptyset$ and $B \neq \emptyset$, then in the line 7 the value of $k_{X}$ is set to 1 and thus $k=k_{A}+k_{B}+1$. The labeling $c$ of $Y$, defined as follows:

$$
c(v)= \begin{cases}c_{A}(v) & \text { if } v \in A \\ k_{A} & \text { if } v \in X \\ k_{A}+1+c_{B}(v) & \text { if } v \in B\end{cases}
$$

is a $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$.
2. If $A=\emptyset$ and $X \cap Z=\emptyset$, then in the line 8 the value of $k_{X}$ is set to 1 and thus $k=k_{B}+1$. The labeling $c$ of $Y$, defined as follows:

$$
c(v)= \begin{cases}0 & \text { if } v \in X \\ c_{B}(v)+1 & \text { if } v \in B\end{cases}
$$

is a $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$.
3. If $A=\emptyset$ and $X \cap Z \neq \emptyset$, then in the line 9 the value of $k_{X}$ is set to 2 and thus $k=k_{B}+2$. The labeling $c$ of $Y$, defined as follows:

$$
c(v)= \begin{cases}1 & \text { if } v \in X \\ c_{B}(v)+2 & \text { if } v \in B\end{cases}
$$

is a $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$.
4. If $B=\emptyset$ and $X \cap M=\emptyset$, then in the line 10 the value of $k_{X}$ is set to 1 and thus $k=k_{A}+1$. The labeling $c$ of $Y$, defined as follows:

$$
c(v)= \begin{cases}c_{A}(v) & \text { if } v \in A \\ k_{A} & \text { if } v \in X\end{cases}
$$

is a $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$.
5. If $B=\emptyset$ and $X \cap M \neq \emptyset$, then in line 11 the value of $k_{X}$ is set to 2 and thus $k=k_{A}+2$. The labeling $c$ of $Y$, defined as follows:

$$
c(v)= \begin{cases}c_{A}(v) & \text { if } v \in A \\ k_{A} & \text { if } v \in X\end{cases}
$$

is a $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$ (the label $k_{A}+1$ is counted as used, but no vertex is labeled with it).

The case when $X=\emptyset$ is not possible, since the partition $(A, X, B)$ is $G$ correct. The case when $A=B=\emptyset$ is not possible, since then $Y=X$ is a 2-packing in $G$ and by the Lemma $1 \Lambda_{Z}^{N}(Y, G) \leq 3$, so the algorithm would finish in the line 4.

Now let us show that $k \leq \Lambda_{Z}^{M}(Y, G)$. Let $c$ be an optimal $L_{Z}^{M}(Y)$-labeling of $G$. Let $l$ be the smallest number, such that $\left|c^{-1}(0) \cup c^{-1}(1) \cup \cdots \cup c^{-1}(l)\right| \geq \frac{|Y|}{2}$.

Let $A=c^{-1}(0) \cup \cdots \cup c^{-1}(l-1), X=c^{-1}(l)$ and $B=c^{-1}(l+1) \cup \cdots \cup$ $c^{-1}\left(\Lambda_{Z}^{M}(Y, G)-1\right)$. Notice that $X$ is a 2 -packing and $X \neq \emptyset$ by the choice of $l$. Hence we observe that the partition $(A, X, B)$ is $G$-correct, so the algorithm considers it in one of the iterations of the main loop.

Let $c_{A}: A \rightarrow \mathbb{N}$ be a function such that $c_{A}(v)=c(v)$ for every $v \in A$ and $c_{B}: B \rightarrow \mathbb{N}$ be a function such that $c_{B}(v)=c(v)-(l+1)$ for every $v \in B$. Notice that $c_{A}$ is an optimal $L_{Z}^{N(X)}(A)$-labeling of $G$ and $c_{B}$ is an optimal $L_{N(X)}^{M}(B)$-labeling of $G$, because otherwise $c$ would not be an optimal.

Hence by the inductive assumption the call in the line 12 returns the number $k_{A} \leq \Lambda_{Z}^{N(X)}(A, G)$ and the call in the line 13 returns the number $k_{B} \leq$ $\Lambda_{N(X)}^{M}(B, G)$.

Let $k^{\prime}$ be the value of $k_{A}+k_{X}+k_{B}$ in the iteration of the main loop when partition $(A, X, B)$ is considered.

Let us consider the following cases:

1. $A, B \neq \emptyset$. In such a case the algorithm Find-Lambda sets $k_{X}=1$ in the line 7 and

$$
\Lambda_{Z}^{M}(Y, G)=\Lambda_{Z}^{N(X)}(A, G)+\underbrace{1}_{c^{-1}(l)=X}+\Lambda_{Z}^{N(X)}(B, G) \geq k_{A}+k_{X}+k_{B}=k^{\prime}
$$

2. $A=\emptyset$ and $l=0$. In such a case $k_{A}=0$ and $X \cap Z=\emptyset$ and the algorithm Find-Lambda sets $k_{X}=1$ in the line 8 and

$$
\Lambda_{Z}^{M}(Y, G)=\underbrace{\Lambda_{Z}^{N(X)}(A, G)}_{=0}+\underbrace{1}_{c^{-1}(0)=X}+\Lambda_{Z}^{N(X)}(B, G) \geq k_{A}+k_{X}+k_{B}=k^{\prime}
$$

3. $A=\emptyset$ and $l=1$. In such a case $k_{A}=0$ and $X \cap Z \neq \emptyset$. Otherwise $c^{\prime}$ defined by $c^{\prime}(v)=c(v)-1$ for every $v \in Y$ would be a $L_{Z}^{M}(Y)$-labeling of $G$ using less labels than the optimal $L_{Z}^{M}(Y)$-labeling $c$ of $G$ - contradiction. The algorithm Find-Lambda sets $k_{X}=2$ in the line 9 and $\Lambda_{Z}^{M}(Y, G)=\underbrace{\Lambda_{Z}^{N(X)}(A, G)}_{=0}+\underbrace{1}_{c^{-1}(0)=\emptyset}+\underbrace{1}_{c^{-1}(1)=X}+\Lambda_{Z}^{N(X)}(B, G) \geq k_{A}+k_{X}+$ $k_{B}=k^{\prime}$.
4. $B=\emptyset$ and $l=\Lambda_{Z}^{M}(Y, G)-1$. In such a case $k_{B}=0$ and $X \cap M=\emptyset$, and the algorithm Find-Lambda sets $k_{X}=1$ in the line 10 and
$\Lambda_{Z}^{M}(Y, G)=\Lambda_{Z}^{N(X)}(A, G)+\underbrace{1}_{c^{-1}\left(\Lambda_{M}^{N}(Y, G)-1\right)=X}+\underbrace{\Lambda_{Z}^{N(X)}(B, G)}_{=0} \geq k_{A}+k_{X}+$ $k_{B}=k^{\prime}$.
5. $B=\emptyset$ and $l=\Lambda_{Z}^{M}(Y, G)-2$. In such a case $k_{B}=0$ and $X \cap M \neq \emptyset$ and the algorithm Find-Lambda sets $k_{X}=2$ in the line 11 and
$\Lambda_{Z}^{M}(Y, G)=\Lambda_{Z}^{N(X)}(A, G)+\underbrace{1}_{c^{-1}\left(\Lambda_{M}^{N}(Y, G)-2\right)=X}+\underbrace{1}_{c^{-1}\left(\Lambda_{M}^{N}(Y, G)\right)=\emptyset}+\underbrace{\Lambda_{Z}^{N(X)}(B, G)}_{=0} \geq$
$k_{A}+k_{X}+k_{B}=k^{\prime}$.

Since those are all possible cases and $k$ is the minimum over values of $k^{\prime}$ for all correct partitions, clearly $k \leq \Lambda_{Z}^{M}(Y, G)$.

Observation 1. By the definition of $\Lambda_{Z}^{M}(Y, G)$, the algorithm call Find-Lambda $(G, V(G), \emptyset, \emptyset)$ returns $\lambda(G)+1$.

Lemma 2. Let $G$ be a graph on $n$ vertices, $Y, Z, M \subseteq V(G)$ and let $y=|Y|$. If $G^{2}$ is computed in advance, the algorithm Find-Lambda finds $\Lambda_{Z}^{M}(Y, G)$ in the time $O\left(C^{\log y} y^{3 \log y} 9^{y}\right)$ and polynomial space, where $C$ is a positive constant.

Proof. Having $G^{2}$ computed, checking if any two vertices in $V(G)$ are in distance at most 2 from each other in $G$ takes a constant time. Hence verifying if a given $X \subseteq Y$ is a 2-packing in $G$ can be performed in the time $O\left(y^{2}\right)$. Moreover, we can check if a given function $c: Y \rightarrow \mathbb{N}$ is an $L_{Z}^{M}(Y)$-labeling of $G$ in the time $O\left(y^{2}\right)$.

Let $y=|Y|$ be the measure of the size of the problem. Let $T(y)$ denote the running time of the algorithm Find-Lambda applied to a graph $G$ and $Y, Z, M \subseteq V(G)$.

Ale algorithm Find-Lambda first checks in constant time if $Y=\emptyset$. Then it exhaustively checks if there exists a $(k-1)-L_{Z}^{M}(Y)$-labeling of $G$ for $k \in\{1,2,3\}$. There are $3^{y}$ functions $c: Y \rightarrow\{0,1,2\}$, so this step is performed in the time $O\left(y^{2} \cdot 3^{y}\right)$.

Then for every $G$-correct partition of $Y$ the algorithm is called recursively for two sets of size at most $\frac{y}{2}$. Notice that there are at most $3^{y}$ considered partitions. Checking if a partition of $Y$ is $G$-correct can be performed in time $O\left(y^{2}\right)$. Hence we obtain the following inequality for the complexity $\left(C_{1}\right.$ and $C_{2}$ are positive constants):

$$
T(y) \leq C_{1} y^{2} 4^{y}+C_{2} y^{3} 3^{y} 2 \cdot T\left(\frac{y}{2}\right)
$$

Let $C=\max \left(C_{1}, 2 C_{2}\right)$, then

$$
T(y) \leq C y^{2} 4^{y}+C y^{3} 3^{y} \cdot T\left(\frac{y}{2}\right)
$$

It is not difficult to verify that $T(y) \leq D \cdot C^{\log y} y^{3 \log y} 9^{y}=O\left(C^{\log y} y^{3 \log y} g^{y}\right)$, where $D$ is a positive constant.

The space complexity of the algorithm is clearly polynomial.
Theorem 2. For a graph $G$ on $n$ vertices $\lambda(G)$ can be found in the time $O((9+$ $\epsilon)^{n}$ ) and polynomial space, where $\epsilon$ is an arbitrarily small positive constant.

Proof. The square of a graph $G$ can be found in the time $O\left(n^{3}\right)$. By the Observation 1 and Lemma 2, the algorithm Find-Lambda applied to $G, Y=V(G)$ and $Z=M=\emptyset$ finds $\Lambda_{\emptyset}^{\emptyset}(V(G), G)=\lambda(G)-1$ in the time $O\left(C^{\log n} n^{3 \log n} 9^{n}\right)=$ $O\left((9+\epsilon)^{n}\right)$ and polynomial space.

## Remark

We have just learned that results similar to those included in this paper were independently obtained (but not published) by Havet, Klazar, Kratochvíl, Kratsch and Liedloff [15].

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[^0]:    ${ }^{1} \Delta$ denotes the largest vertex degree in a graph

[^1]:    ${ }^{2}$ In the $O^{*}$ notation we omit polynomially bounded terms.

