A Partially Ordered Structure and a Generalization of the Canonical Partition for General Graphs with Perfect Matchings

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Abstract. This paper is concerned with structures of general graphs with perfect matchings. We first reveal a partially ordered structure among factor-components of general graphs with perfect matchings. Our second result is a generalization of Kotzig's canonical partition to a decomposition of general graphs with perfect matchings. It contains a short proof for the theorem of the canonical partition. These results give decompositions which are canonical, that is, unique to given graphs. We also show that there are correlations between these two and that these can be computed in polynomial time.

1 Introduction

This paper is concerned with matchings on graphs. For general accounts on matching theory we refer to Lovász and Plummer's book [1].

A *matching* of a graph G is a set of edges $F \subseteq E(G)$ no two of which have common vertices. A matching of cardinality $|V(G)|/2$ (resp. $|V(G)|/2-1$) is called a *perfect matching* (resp. a *near-perfect matching*). We call a graph with a perfect matching *factorizable*. An edge of a factorizable graph is called *allowed* if it is contained in a perfect matching. For a factorizable graph G, each connected component of the subgraph of G determined by all the allowed edges of it is called an *elementary component* of G. A factorizable graph which has exactly one elementary component is called *elementary*. For each elementary component H, we call $G[V(H)]$ a *factor-connected component* or *factor-component* of G, and denote the set of all the factor-components of G as $\mathcal{G}(G)$.

Matching theory is of central importance in graph theory and combinatorial optimization [2], with numerous practical applications [3]. Structure theorems that give decompositions which are canonical, namely, unique to given graphs, play important roles in matching theory. Only three theorems, i.e. the canonical partition [4–6], the Dulmage-Mendelsohn decomposition [1], and the Gallai-Edmonds structure theorem [1] have been known as such. The first two are not applicable for general graphs with perfect matchings, and the last one treats them as irreducible and does not decompose them properly, which means nothing has been known that tells non-trivial canonical structures of general graphs with perfect matchings. Therefore, in this paper, we give new canonical structure theorems for them.

By the definitions, we can view factorizable graphs as being "built" up by combining factor-components with additional edges. However it does not mean that all combinations result in graphs with desired factor-components. Thus the family of factor-components must have a certain non-trivial structure. For bipartite factorizable graphs, the *Dulmage-Mendelsohn decomposition* (in short, the *DM-decomposition*) reveals the ordered structure of their factor-components. However, as for non-bipartite graphs, no counterpart has been known.

In this paper, as our first contribution, we reveal a partially ordered structure between factor-components of general graphs with perfect matchings. It has some similar natures to the DM-decomposition, however they are distinct.

The second contribution is a generalization of the *canonical partition* [4–6]; see also [1], which is originally a decomposition of elementary graphs. Kotzig [4–6] first investigated the canonical partition of elementary graphs as the quotient set of a certain equivalence relation, and later, Lovász redefined it from the point of view of maximal barriers [1]. In this paper we generalize the canonical partition to a decomposition of general graphs with perfect matchings, based on Kotzig's way. It contains a short proof for the theorem of the canonical partition.

Note that these two results of us give canonical decompositions of graphs. We also show that there are correlations between these two and that these can be computed in polynomial time.

Any of the three existing canonical structure theorems plays significant roles in combinatorics including matching theory. The canonical partition plays a crucial role in matching theory, especially from the polyhedral point of view, that is, in the study of the matching polytope and the matching lattice [7–9]. The Dulmage-Mendelsohn decomposition is known for its application to the efficient solution of linear equations determined by large sparse matrices [1]. Additionally, it is an origin of a series of studies on submodular functions, that is, the field of the principal partition [10, 11]. The Gallai-Edmonds structure theorem is essential to the optimality of the maximum matching [1, 12]. Thus it also underlies reasonable generalizations of maximum matching problem [13, 14].

By combining the results in this paper with the Gallai-Edmonds structure theorem, we can easily obtain a refinement of the Gallai-Edmonds structure theorem, which gives a consistent view of graphs, whether they are factorizable or not, or, elementary or not [15]. Hence, we are sure that our structure theorems should be powerful tools in matching theory. In fact, the cathedral theorem [1] can be obtained from our results in a quite natural way [15].

2 Preliminaries

In this section, we list some standard definitions and well-known properties. Basics on sets, graphs, digraphs, and algorithms mostly conform to [2].

Let G be a graph and $X \subseteq V(G)$. The subgraph of G induced by X is denoted by $G[X]$. $G - X$ means $G[V(G) \setminus X]$. Given $F \subseteq E(G)$, we define the *contraction* of G by F as the graph obtained from contracting all the edges in F, and denote as G/F. Additionally, We define the *contraction* of G by X as $G/X := G/E(G[X])$. We say $H \subseteq G$ if H is a subgraph of G. If it is clear from the context, we sometimes regard a subgraph $H \subseteq G$ as the vertex set $V(H)$, a vertex v as a graph $(\{v\}, \emptyset)$.

The set of edges that has one end vertex in $X \subseteq V(G)$ and the other vertex in $Y \subseteq V(G)$ is denoted as $E_G[X, Y]$. We denote $E_G[X, V(G) \setminus X]$ as $\delta_G(X)$. We define the *set of neighbors* of X as the set of vertices in $V(G) \setminus X$ that are adjacent to vertices in X, and denote as $N_G(X)$. We sometimes denote $E_G[X, Y]$, $\delta_G(X)$, $N_G(X)$ as just $E[X, Y]$, $\delta(X)$, $N(X)$ if they are apparent from the context.

For two graphs G_1 and G_2, $G_1 + G_2 := (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ is called the *union* of them, and $G_1 \cap G_2 := (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$ the *intersection* of them.

Let $\hat{G}$ be a graph such that $G \subseteq \hat{G}$. For $e = uv \in E(\hat{G})$, $G + e$ means $(V(G) \cup \{u, v\}, E(G) \cup \{e\})$, and $G - e$ means $(V(G), E(G) \setminus \{e\})$. For a set of edges $F = \{e_i\}_{i=1}^k$, $G + F$ and $G - F$ means respectively $G + e_1 + \cdots + e_k$ and $G - e_1 - \cdots - e_k$.

For a path P and $x, y \in V(P)$, xPy means the subpath on P between x and y. For a circuit C with an orientation that makes it a dicircuit, and $x, y \in V(C)$ where $x \neq y$, xCy means the subpath in C that can be regarded as a dipath from x to y.

A vertex $v \in V(G)$ satisfying $\delta(v) \cap M = \emptyset$ is called *exposed* by M. For a matching M of G and $u \in V(G)$, u' denote the vertex to which u is matched by M. For $X \subseteq V(G)$, M_X denotes $M \cap E(G[X])$.

Let M be a matching of G. For $Q \subseteq G$, which is a path or circuit, we call Q *M-alternating* if $E(Q) \setminus M$ is a matching of Q. Let $P \subseteq G$ be an M-alternating path with end vertices u and v. If P has an even number of edges and starts with an edge in M if it is traced from u, we call it an *M-balanced path* from u to v. We regard a trivial path, that is, a path composed of one vertex and no edges as an M-balanced path. If P has an odd number of edges and $M \cap E(P)$ (resp. $E(P) \setminus M$) is a perfect matching of P, we call it *M-saturated* (resp. *M-exposed*).

Let $H \subseteq G$. We say a path $P \subseteq G$ is an *ear relative to* H if both end vertices of P are in H while internal vertices are not. So do we to a circuit if exactly one vertex of it is in H. For simplicity, we call the vertices of $V(P) \cap V(H)$ *end vertices* of P, even if P is a circuit. For an ear $R \subseteq G$ relative to H, we call it an *M-ear* if $P - V(H)$ is an M-saturated path.

A graph is called *factor-critical* if any deletion of its single vertex leaves a factorizable graph. A subgraph $G' \subseteq G$ is called *nice* if $G - V(G')$ is factorizable. The next two propositions are well-known and might be regarded as folklores.

Proposition 1. *Let M be a near-perfect matching of a graph G that exposes $v \in V(G)$. Then, G is factor-critical if and only if for any $u \in V(G)$ there exists an M-balanced path from u to v.*

Proposition 2. *Let G be a graph. Then G is factor-critical if and only if each block of G is factor-critical.*

Proposition 3 (implicitly stated in [16])**.** *Let G be a factor-critical graph, $v \in V(G)$, and M be a near-perfect matching that exposes v. Then for any non-loop edge $e = vu \in E(G)$, there is a nice circuit C of G which is an M-ear relative to v and contains e.*

Theorem 1 (implicitly stated in [16])**.** *Let G be a factor-critical graph. For any nice factor-critical subgraph G' of G, G/G' is factor-critical.*

Let us denote the number of odd components (i.e. connected components with odd numbers of vertices) of a graph G as $oc(G)$, and the cardinality of a maximum matching of G as $\nu(G)$. It is known as the *Berge formula* [1] that for any graph G, $|V(G)| - 2\nu(G) = \max\{oc(G-X) - |X| : X \subseteq V(G)\}$. A set of vertices that attains the maximum in the right side of the equation is called a *barrier*.

The *canonical partition* is a decomposition for elementary graphs and plays a crucial role in matching theory. First Kotzig introduced the canonical partition as a quotient set of a certain equivalence relation [4–6], and later Lovász redefined it from the point of view of barriers [1]. In fact, these are equivalent. For an elementary graph G and $u, v \in V(G)$, we say $u \sim v$ if $u = v$ or $G - u - v$ is not factorizable.

Theorem 2 (Kotzig [4–6], Lovász [1]). *Let G be an elementary graph. Then $\sim$ is an equivalence relation on $V(G)$ and the family of equivalence classes is exactly the family of maximal barriers of G.*

The family of equivalence classes of $\sim$ is called the *canonical partition* of G, and denoted by $\mathcal{P}(G)$. An *ear-decomposition* of graph G is a sequence of subgraphs $G_0, \subseteq, \cdots, \subseteq G_k = G$ such that $G_0 = (\{r\}, \emptyset)$ for some $r \in V(G)$ and for each $i \geq 1$, G_i is obtained from G_{i-1} by adding an ear P_i relative to G_{i-1}. We sometimes regard an ear-decomposition as a family of ears $\mathcal{P} = \{P_1, \ldots, P_k\}$. An ear-decomposition is called *odd* if any of its ears has an odd number of edges.

Theorem 3 (Lovász [16]). *A graph is factor-critical if and only if it has an odd ear-decomposition.*

For a factor-critical graph G and its near-perfect matching M, we call an ear-decomposition *alternating with respect to* M, or just M-alternating, if each ear is an M-ear.

Proposition 4 (Lovász [16]). *Let G be a factor-critical graph. Then for any near-perfect matching M of G, there is an M-alternating ear-decomposition of G.*

Proposition 5. *Let G be a factorizable graph, and M be a perfect matching of G. Then, for $e = xy \in E(G) \setminus M$, the followings are equivalent;*

(i) *e is allowed in G.*
(ii) *There is an M-alternating circuit containing e.*
(iii) *There is an M-saturated path between x and y.*

Proposition 6. *Let G be a graph, M be a matching of G, and $X \subseteq V(G)$ be such that M_X is a perfect matching of $G[X]$. Let P be a subgraph of G that satisfies either of the followings;*

(i) *P is an M-alternating circuit with $V(P) \cap X \neq \emptyset$,*
(ii) *for some $u \in X$, P is an M-ear relative to $\{u\}$,*
(iii) *P is an M-exposed path whose end vertices are in X, or*
(iv) *P is an M-saturated path whose end vertices are in X.*

Then, each connected component of $P - E(G[X])$ is an M-ear relative to X.

3 A Partially Ordered Structure in Factorizable Graphs

Let G be a factorizable graph. For $X \subseteq V(G)$ we call X a *separating set* if for any $H \in \mathcal{G}(G)$, $V(H) \subseteq X$ or $V(H) \cap X = \emptyset$. The next property is easy to see by the definition.

Proposition 7. *Let G be a factorizable graph, and $X \subseteq V(G)$ with $X \neq \emptyset$. The following properties are equivalent;*

(i) *X is separating.*
(ii) *There exist $H_1, \dots, H_k \in \mathcal{G}(G)$, where $k \geq 1$, such that $X = V(H_1)\dot{\cup} \cdots \dot{\cup} V(H_k)$.*
(iii) *For any perfect matching M of G, $\delta(X) \cap M = \emptyset$.*
(iv) *For any perfect matching M of G, M_X is a perfect matching of $G[X]$.*

Let $G_1, G_2 \in \mathcal{G}(G)$. We say a separating set X is a *critical-inducing set for G_1* if $V(G_1) \subseteq X$ and $G[X]/G_1$ is a factor-critical graph. Moreover, we say X is a *critical-inducing set for G_1 to G_2* if $V(G_1) \cup V(G_2) \subseteq V(G)$ and $G[X]/G_1$ is a factor-critical graph.

Definition 1. *Let G be a factorizable graph, and $G_1, G_2 \in \mathcal{G}(G)$. We say $G_1 \triangleleft G_2$ if there is a critical-inducing set for G_1 to G_2.*

Lemma 1. *Let G be a factorizable graph and M be a perfect matching of G, and let $X \subseteq V(G)$ and $G_1 \in \mathcal{G}(G)$. Then, X is a critical-inducing set for G_1 if and only if for any $x \in X \setminus V(G_1)$ there exists $y \in V(G_1)$ such that there is an M-balanced path from x to y whose vertices except y are in $X \setminus V(G_1)$.*

Proof. The claim is rather easy from Proposition 1; X is a critical-inducing set for G_1 if and only if $G[X]/G_1$ is factor-critical. Note that $M_{X \setminus V(G_1)}$ forms a near-perfect matching of $G[X]/G_1$. Therefore, $G[X]/G_1$ is factor-critical if and only if for any $x \in X$ there is an M-balanced path from x to the contracted vertex g_1 corresponding to G_1. Therefore, the claim follows. □

Proposition 8. *Let G be an elementary graph and M be a perfect matching of G. Then for any two vertices $u, v \in V(G)$ there is an M-saturated path between u and v, or an M-balanced path from u to v.*

Proof. Without loss of generality we can assume G is matching-covered, that is, every edge of G is allowed. Let $U_1 \subseteq V(G)$ be the set of vertices that can be reached from u by an M-saturated path, and $U_2 \subseteq V(G)$ be the set of vertices that can be reached from u by an M-balanced path but cannot be by any M-saturated paths. We are going to obtain the claim by showing $U := U_1 \dot\cup U_2 = V(G)$. Suppose that it fails, namely that $U \subsetneq V(G)$. Then there are $v \in U$ and $w \in V(G) \setminus U$ such that $vw \in E(G)$, since G is connected. By the definition of U, there is an M-saturated or balanced path P from u to v, which satisfies $V(P) \subseteq U$ since for each $z \in V(P)$ uPz is an M-saturated or balaned path from u to z. If P is M-saturated, therefore, $P + vw$ is an M-balanced path from u to w, which means $w \in U$, a contradiction.

Hence, hereafter we assume P is M-balanced, from u to v. Since vw is defined to be allowed, there is an M-saturated path Q between v and w by Proposition 5. Trace P from u and let x be the first vertex we encounter that is in Q; such x surely exists under the current hypotheses since $v \in V(P) \cap V(Q)$.

Claim 1. uPx is an M-balanced path.

Proof. Suppose the claim fails, which is equivalent to uPx being an M-saturated path. Then, $x' \in V(uPx)$. On the other hand, since Q is M-saturated, $x' \in V(Q)$. Therefore, $x' \in V(uPx) \cap V(Q)$, which means we counter x' before x if we trace P from u, a contradiction. □

Claim 2. xPw is an M-saturated path between x and w.

Proof. If $x = v$, vPx is a trivial M-balanced path from v to x. Even if $x \neq v$, so is it since x is matched by $M \cap E(P)$. Anyway, whether $x = v$ or not, vPx is an M-balanced path from v to x. Therefore, together with vPw being an M-saturated path, xPw is an M-balanced path from x to w. □

By Claims 1 and 2, $uPx + xQw$ is an M-saturated path between u and w, since $V(uPx) \cap V(xQw) = \{x\}$ by the definition of x. Hence, $w \in U$, a contradiction, and we obtain $U = V(G)$, which completes the proof. □

Proposition 9. *Let G be a factorizable graph and M be a perfect matching of G. Let $X \subseteq V(G)$, and $H \in \mathcal{G}(G)$ be such that there is an M-ear P relative to X and through H, whose end vertices are $u, v \in V(G_1)$. Let $Y := V(H) \cup V(P) \setminus \{u, v\}$. Then, for any $x \in Y$,*

(i) *there exists an internal vertex y of P such that there is an M-balanced path Q from x to y with $V(Q) \subseteq Y$ and $V(Q) \cap V(P) = \{y\}$, and*
(ii) *for w identical to either u or v, $Q + yPw$ is an M-balanced path from x to w, whose vertices except w are contained in Y.*

Proof. If $x \in V(P) \setminus \{u, v\}$, the claims are obvious. Let $x \in V(H) \setminus V(P)$. Then, by Proposition 8, for an arbitrarily chosen $z \in V(P) \cap V(H)$, there is an M-saturated or balanced path R from x to z with $V(R) \subseteq V(H)$. Trace R from x and let y be the first vertex we encounter that is in $V(P)$. Then, xRy gives a desired path in (i), and $Q := xRy + yPw$, where w is either u or v, gives one in (ii). Therefore, we are done. □

Let G be a factorizable graph and M be a perfect matching of G. We call a sequence of factor-components $S := (H_0, \ldots, H_k)$, where $k \geq 0$ and $H_i \in \mathcal{G}(G)$ for each $i = 0, \ldots, k$, an *M-ear sequence, from H_0 to H_k,* if $k = 0$ or otherwise

(i) for any $i, j \in \{0, \ldots, k\}$, $i \neq j$ yields $H_i \neq H_j$, and
(ii) for each $i = 1, \ldots, k$ there is an M-ear P_i relative to H_{i-1} and through H_i.

We call k the *length* of S. If $k \geq 1$, we call the sequence of M-ears $P := (P_1, \ldots, P_k)$ *associated with* S. If $k = 0$, an empty sequence, $P := ()$, is defined to be the M-ears associated with S, for convenience.

For S and P, we define the *sequence union* of S and P as $S \oplus P := \bigcup_{i=1}^{k} V(H_i) \cup \bigcup_{i=1}^{k} V(P_i) \setminus V(H_0)$, if $k \geq 1$. If $k = 0$, $S \oplus P := \emptyset$.

Given S and P, for any i, j with $0 \leq i \leq j \leq k$, the subsequence $(H_i, \ldots, H_j)$ is an M-ear sequence, from H_i to H_j, and we denote it as $S[i, j]$. Additionally, if $i < j$, $(P_i, \ldots, P_j)$ is a sequence of M-ears associated with $S[i, j]$, and we denote it $P[i, j]$. If $i = j$, $P = ()$ is associated with $S[i, j]$, and it is also denoted as $P[i, j]$. We denote $S[0, j] =: S^j$, and $P[0, j] =: P^j$.

Let G be a factorizable graph, and M be a perfect matching of G. Let $G_1, G_2 \in \mathcal{G}(G)$, and let $S := (G_1 = H_0, \ldots, H_k = G_2)$, where $k \geq 0$, be an M-ear sequence from G_1 to G_2, associated with M-ears P. Let us define in the following three properties for S and P:

D1(S, P): If $k \geq 2$, then by letting $P = (P_1, \ldots, P_k)$, for each $i = 2, \ldots, k$, $V(P_i)$ is disjoint from $V(H_0)$.
D2(S, P): If $k \geq 1$, by letting $P = (P_1, \ldots, P_k)$, for each $i = 1, \ldots, k$, for any $x \in S^i \oplus P^i$ there exists an internal vertex y of P_1 such that there is an M-balanced path Q from x to y with $V(Q) \subseteq S^i \oplus P^i$ and $V(Q) \cap V(P_1) = \{y\}$.
D3(S, P): If $k \geq 1$, by letting $P = (P_1, \ldots, P_k)$, for each $i = 1, \ldots, k$, for any $x \in S^i \oplus P^i$, for w which equals either of the end vertices of P_1, there is an M-balanced path R from x to w such that $V(P_1) \setminus \{w\} \subseteq S^i \oplus P^i$.

Remark 1. By their definitions, if $k = 0$, then S and P trivially satisfy D1, D2 and D3.

Remark 2. D1, D2 and D3 are closed with respect to the substructures; if S and P satisfies D1, D2 and D3, then for any $i = 0, \ldots, k$, so does S^i and P^i.

Proposition 10. *Let G be a factorizable graph and M be a perfect matching of G. Let S be an M-ear sequence, and P be a sequence of M-ears associated with S. Then, $M_{S \oplus P}$ is a perfect matching of $G[S \oplus P]$.*

Proof. If the length k of S equals zero, the claim is trivially true. Let $k \geq 1$, and let $S =: (H_0, \ldots, H_k)$ and $P =: (P_1, \ldots, P_k)$. Of course, $X := V(H_0) \dot\cup \cdots \dot\cup V(H_k)$ has a perfect matching M_X. For each P_i, the end vertices of P_i are in X and any other vertex is covered by M_{P_i}. Therefore, M contains a perfect matching of $Y := X \cup V(P_1) \cup \cdots \cup V(P_k)$. Accordingly, $S \oplus P = Y \setminus V(H_0)$ is covered by $M_{S \oplus P}$. $\square$

Lemma 2. *Let G be a factorizable graph and M be a perfect matching of G. Let $G_1 \in \mathcal{G}(G)$ and $X \subseteq V(G)$ be a critical-inducing set for G_1. Suppose there exists an M-ear P relative to X, whose end vertices are $u, v \in V(G)$, and let $I_1, \ldots, I_s \in \mathcal{G}(G)$, where $s \geq 1$, be the factor-components that have common vertices with the internal vertices of P. Then, $X \cup \bigcup_{i=1}^{s} V(I_i)$ is also a critical-inducing set for G_1.*

Proof. We prove the claim by Lemma 1; let $Y := \bigcup_{i=1}^{s} V(I_i)$. By Lemma 1,

Claim 3. for any $x \in X$ there exists $z \in V(G_1)$ such that there is an M-balanced path Q_x from x to z with $V(Q_x) \subseteq X$ and $V(Q_x) \cap V(G_1) = \{z\}$.

Claim 4. For any $y \in Y$ there exists $z \in V(G_1)$ such that there exists an M-balanced path Q_y from y to x with $V(Q_y) \subseteq X$ and $V(Q_y) \cap V(G_1) = \{y\}$.

Proof. Let $i \in \{1, \ldots, s\}$ be such that $y \in V(I_i)$. By applying Proposition 9 to X, I_i and P, for w which equals either u or v, there is an M-balanced path R from y to w such that $V(R) \setminus \{w\} \subseteq Y$. Therefore, $P + Q_w$ gives a desired path. □

Apparently by the definition $X \cup Y$ is a separating set, therefore with Claims 3 and 4 we can conclude that $X \cup Y$ is a critical-inducing set for G_1, by Lemma 1. □

Theorem 4. *Let G be a factorizable graph, M be a perfect matching of G, and $G_1, G_2 \in \mathcal{G}(G)$. Then, $G_1 \triangleleft G_2$ if and only if there exists an M-ear sequence from G_1 to G_2.*

Proof. We first prove the sufficiency. Let $G_1 \triangleleft G_2$ and $X \subseteq V(G)$ be a critical-inducing set for G_1 to G_2. Let us define the following three properties for $Y \subseteq X$:

C1(Y): Y is a critical-inducing set for G_1, and
C2(Y): for each $H \in \mathcal{G}(G)$ with $V(H) \subseteq Y$, there is an M-ear sequence from G_1 to H.

Let X' be a maximal subset of X satisfying C1 and C2. Note that $X' \neq \emptyset$ because $V(G_1)$ satisfies C1 and C2. We are going to prove the sufficiency by showing that $X' = X$. Suppose it fails, that is, $X' \subsetneq X$. Then,

Claim 5. there is an M-ear P relative to X' such that $V(P) \subseteq X$.

Proof. $G[X]/G_1$ is factor-critical and $G[X']/G_1$ is a nice factor-critical subgraph of $G[X]/G_1$ by Proposition 7. Therefore, $G[X]/X'$ is factor-critical by Theorem 1 and $M_{X \setminus X'}$ forms a near-perfect matching of $G[X]/X'$ exposing only the contracted vertex x' corresponding to X'. By Proposition 3, in $G[X]/X'$ there is an M-ear P relative to x', and in G it corresponds to an M-ear relative to X' with $V(P) \subseteq X$. Thus, the claim follows. □

Let $u, v \in X'$ be the end vertices of P. Let $I_1, \ldots, I_s \in \mathcal{G}(G)$ be the factor-components that have common vertices with internal vertices of P. We are going to prove that $X'' := X' \cup \bigcup_{i=1}^{s} V(I_i)$ satisfies C1 and C2.

Claim 6. X'' satisfies C2.

Proof. By Lemma 1, there exists an M-balanced path Q_u (resp. Q_v) from u (resp. v) to a vertex of $V(G_1)$, which is contained in X and whose vertices except the end vertex in $V(G_1)$ are disjoint from $V(G_1)$. Trace Q_u from u and let r_u be the first vertex we encounter that is contained in a factor-component I_0 which has common vertices also with Q_v; such I_0 surely exists since both Q_u and Q_v have some vertices in G_1. Trace Q_v from v and let r_v be the first vertex we encounter that is in $V(I_0)$. For each $w \in \{u, v\}$, wQ_wr_w is an M-balanced path from w to r_w such that $V(wQ_wr_w) \subseteq X'$ and $V(wQ_wr_w) \cap V(I_0) = \{r_w\}$, and it holds that $V(uQ_ur_u) \cap V(vQ_vr_v) \setminus \{r_u, r_v\} = \emptyset$. Therefore, $uQ_ur_u + P + vQ_vr_v$ is an M-ear relative to I_0 and through every $I_1, \ldots, I_s$. By the definition of X', there is an M-ear sequence from G_1 to I_0. Therefore, by adding subsequence (I_0, I_i) to it, we obtain an M-ear sequence from G_1 to I_i, for each $i = 1, \ldots, s$. Thus, we obtain the claim. □

Claim 7. X'' satisfies C1.

Proof. This is immediate by Lemma 2. □

With Claims 6 and 7, X'' contradicts the maximality of X'. Therefore, we obtain $X' = X$, accordingly the sufficiency part of the claim follows.

From now on we prove the necessity. Let $(G_1 = H_0, \ldots, H_k = G_2)$, where $k \geq 0$, be the M-ear sequence from G_1 to G_2 We are going to prove that there is a critical-inducing set for G_1 to G_2. We proceed by induction on k. For the case $k = 0$, that is, $G_1 = G_2$, the claim apparently holds by taking $V(G_1)$.

Let $k > 0$ and suppose the claim holds for $k-1$. By the induction hypothesis, for the M-ear subsequence $(H_0, \ldots, H_{k-1})$, there is a critical-inducing set X' for H_0 to H_{k-1}.

Claim 8. There is an M-ear P relative to X' and through H_k.

Proof. Let P_k the associated M-ear relative to H_{k-1} and through H_k. By Proposition 6 each connected component $P - E(G[X'])$ is an M-ear relative to X', and one of them, which we call P, is through H_k. Therefore, the claim follows. □

Let $I_1, \ldots, I_s \in \mathcal{G}(G)$, where $s \geq 1$, be the factor-components that have common vertices with the internal vertices of P, and let $Y := \bigcup_{i=1}^{s} V(I_i)$. Then, by applying Lemma 2 to the critical-inducing set X' for G_1 and the M-ear P, we obtain that $X' \cup Y$ is a critical-inducing set for G_1 to H_k. This completes the proof. □

Lemma 3. *Let G be a factorizable graph, and M be a perfect matching. Let $S := (H_0, \ldots, H_k)$, where $k \geq 1$, be an M-ear sequence, associated with M-ears $P := (P_1, \ldots, P_k)$. Suppose S^i and P^i satisfy D1, D2, and D3 for each $i = 0, \ldots, k-1$, and S and P satisfy D1. Then, S and P also satisfy D2 and D3.*

Proof. If $k = 1$, then by applying Proposition 9 to $V(H_0)$, P_1, and H_1, it holds that S and P satisfy D1, D2 and D3.

Hence hereafter let $k \geq 2$. First note that each connected component of $P_k - E(G[S^{k-1} \oplus P^{k-1}])$ is an M-ear relative to $S^{k-1} \oplus P^{k-1}$ by Proposition 6, and is disjoint from $V(H_0)$ since P_k is.

Take $x \in S \oplus P \setminus S^{k-1} \oplus P^{k-1}$ arbitrarily, and let P_k^x be a connected component of $P_k - E(G[S^{k-1} \oplus P^{k-1}])$ such that x is an internal vertex of P_k^x if $x \in V(P)$, or one through H_k if $x \in V(H_k) \setminus V(P)$.

Claim 9. There exists $y \in S^{k-1} \oplus P^{k-1}$ such that there exists an M-balanced path Q from x to y whose vertices except y are contained in $S \oplus P \setminus S^{k-1} \oplus P^{k-1}$.

Proof. By applying Proposition 9 to $S^{k-1} \oplus P^{k-1}$, P_k^x, and H_k (if $x \in V(H_k)$), we obtain an internal vertex y of P_1 and an M-balanced path Q from x to y with $V(Q) \setminus \{y\} \subseteq V(H_k) \cup V(P_k^x) \setminus S^{k-1} \oplus P^{k-1}$. Since P_k is disjoint from $V(H_0)$, we can see $V(H_k) \cup V(P_k^x) \subseteq S \oplus P$. Therefore, $V(Q) \setminus \{y\} \subseteq S \oplus P \setminus S^{k-1} \oplus P^{k-1}$, and the claim follows. □

Claim 10. S and P satisfy D2.

Proof. By the hypothesis on S^{k-1} and P^{k-1} there exists an internal vertex z of P_1 such that there is an M-balanced path R from y to z with $V(R) \subseteq S^{k-1} \oplus P^{k-1}$ and $V(R) \cap V(P_1) = \{z\}$. Therefore, by Claim 9, $Q + R$ is an M-balanced path from x to z, whose veritices are contained in $S \oplus P$ and disjoint from P_1 except z.

Since x is chosen arbitrarily from $S \oplus P \setminus S^{k-1} \oplus P^{k-1}$, we obtain that S and P satisfy D2. □

By similar arguments, we can say that S and P satisfy D3 too, and the claim follows.

Proposition 11. *Let G be a factorizable graph and M be a perfect matching. Let $G_1, G_2 \in \mathcal{G}(G)$ be such that $G_1 \lhd G_2$, and let $k \geq 0$ be the length of the shortest M-ear sequence from G_1 to G_2. Then, there exists an M-ear sequence S of shortest length, and M-ears P associated with S such that D1(S, P), D2(S, P), and D3(S,P) hold.*

Proof. We proceed by induction on k. If $k = 0$, the claim is trivially true. If $k = 1$, for any shortest M-ear sequence $S = (H_0 = G_1, H_1 = G_2)$ from G_1 to G_2 and associated M-ears $P = (P_1)$, D1(S, P) trivially holds by the definition of D1, and moreover D2(S, P) and D3(S, P) also hold by applying Proposition 9 to $V(H_0)$, P_1, and H_k.

Let $k \geq 2$, and suppose the claim is true for any two factor-components $G'_1, G'_2 \in \mathcal{G}(G)$ such that the length of the shortest M-ear sequence from G'_1 to G'_2, is $1, \ldots, k-1$.

Take arbitrarily an M-ear sequence $S = (G_1 = H_0, \ldots, H_k = G_2)$ from G_1 to G_2 of shortest length, and M-ears $P = (P_1, \ldots, P_k)$ associated with it. Let u_1, v_1 be the end vertices of P_1.

Claim 11. Without loss of generality we can assume that S and P are chosen so that for each $i = 1, \dots, k-1$, S^i and P^i satisfy D1, D2, and D3.

Proof. By the induction hypothesis, there exist an M-ear sequence from H_0 to H_{k-1}, which is of shortest length, and M-ears associated with it which satisfy D1, D2, and D3; note that its length is $k-1$. Without loss of generality, we can assume S^{k-1} and P^{k-1} coincides to them. Since the conditions D1, D2, and D3 are closed with substructures, the claim follows. □

If P_k is disjoint from $V(H_0)$, namely if D1(S, P) holds, then by Lemma 3, S and P also satisfy D2 and D3, and the claim follows.

Hence hereafter suppose that might fail i.e. P_k might not be disjoint from $V(H_0)$. By Proposition 6, each connected component of $P_k - E(G[S^{k-1} \oplus P^{k-1}])$ is an M-ear relative to $S^{k-1} \oplus P^{k-1}$. Take one of them Q arbitrarily that has common vertices with H_k.

Take $x \in V(Q) \cap V(H_k)$ arbitrarily, and let u, v be the end vertices of Q. Trace xQu from x and let y be the first vertex we encounter that is in $V(H_0) \cup \{u\}$. On the other hand, trace xQv from x and let z be the first vertex we encounter that is in $V(H_0) \cap \{v\}$. Then,

Claim 12. yQz is an M-exposed path, whose internal vertices contains $x \in V(H_k)$, and whose vertices except the end vertices y and z are disjoint from $V(H_0) \cup S^{k-1} \oplus P^{k-1}$.

Claim 13. Q is disjoint from $V(H_0)$.

Proof. We are going to prove $y = u$ and $z = v$; First suppose the case where $y, z \in V(H_0)$. Then, yQz is an M-ear relative to H_0 and through H_k, which means (H_0, H_k) forms an M-ear sequence of length one, contradicting the definition of k, since $k \geq 2$.

Second suppose the case where $y \in V(H_0)$ and $z = v$. Since S^{k-1} and P^{k-1} satisfy D3, for either $w \in \{u_1, v_1\}$ there is an M-balanced path R from z to w such that $V(R) \setminus \{w\} \subseteq S^{k-1} \oplus P^{k-1}$. Therefore, $yQz + R$ is an M-ear relative to H_0 and through H_k, again letting (H_0, H_k) be an M-ear sequence, a contradiction.

In the third case where $y = u$ and $z \in V(H_0)$, by symmetric arguments we are again lead to a contradiction.

Therefore, we obtain that $y = u$ and $z = v$, which is equivalent to Q being disjoint from $V(H_0)$. □

Since S^{k-1} and P^{k-1} satisfy D3, for each $\alpha \in \{u, v\}$ there is an M-balanced path Q_α from α to r_α, where r_α equals either u_1 or v_1, such that $V(Q_\alpha) \setminus \{r_\alpha\} \subseteq S^{k-1} \oplus P^{k-1}$. Trace Q_u from u and let s be the first vertex we encounter that is contained in a factor-component, say $I \in \mathcal{G}(G)$, which has common vertices also with $V(Q_v)$; such I surely exists since both Q_u and Q_v have vertices in H_0. Trace Q_v from v and let t be the first vertex we encounter that is in $V(I)$.

Claim 14. $I \neq H_0$. Accordingly, $V(Q_u) \cup V(Q_v) \subseteq S^{k-1} \oplus P^{k-1}$.

Proof. $S^{k-1} \oplus P^{k-1} \cap V(H_0) = \emptyset$, and for each $\alpha \in \{u, v\}$, $V(Q_\alpha) \setminus \{r_\alpha\} \subseteq S^{k-1} \oplus P^{k-1}$. Therefore, $I = H_0$ only if $V(Q_u) \cap V(Q_v) = \emptyset$ or $V(Q_u) \cap V(Q_v) = \{r_u\} = \{r_v\}$. Then, $Q_u + Q + Q_v$ forms an M-ear relative to H_0 and through H_k, letting (H_0, H_k) be an M-ear sequence of length one, a contradiction. □

Claim 15. Each connected component of $uQ_us + Q + vQ_zt - E(I)$ is an M-ear relative to I, one of which, say $\hat{Q}$, is through H_k.

Proof. uQ_us and vQ_zt are M-balanced paths respectively from u to s and from v to t, and they are disjoint if $s \neq t$, or have only one common vertex $s = t$ if $s = t$. Additionally, they are both contained in $S^{k-1} \oplus P^{k-1}$ by Claim 14, while $V(Q) \cap S^{k-1} \oplus P^{k-1} = \{u, v\}$. Therefore, $uQ_us + Q + vQ_zt$ forms an M-exposed path between s and t if $s \neq t$, or an M-ear relative to $\{s\} = \{t\}$ if $s = t$, in both cases having internal vertices contained in H_k, since Q does. Hence, by Proposition 6, the claim follows. □

By the arguments up till now, I has some vertices in $S^{k-1} \oplus P^{k-1}$. Hence, I equals either of $H_1, \ldots, H_{k-1}$ or otherwise it just has common vertices other than u_1 or v_1, with either of $P_1, \ldots, P_{k-1}$.

Claim 16. If I equals either of $H_1, \ldots, H_{k-1}$, then $I = H_{k-1}$.

Proof. If $k = 2$, the claim is trivially true. Let $k \geq 3$ and suppose the claim fails, that is, $I = H_i$ for $i \in \{1, \ldots, k-2\}$. Then, $(H_0, \ldots, H_i = I, H_k)$ forms an M-ear sequence from H_0 to H_k, associated with $(P_1, \ldots, P_i, \hat{Q})$, and of length $i+1 \leq k-1$. This contradicts the definition of k, therefore we have the claim. □

Claim 17. $\hat{Q}$ is disjoint from $V(H_0)$.

Proof. By Claim 13, Q is disjoint from $V(H_0)$, and by Claim 14, Q_u and Q_v are both disjoint from $V(H_0)$. Therefore, $\hat{Q} = Q_u + Q + Q_v$ is also disjoint from $V(H_0)$. □

Therefore, with Claims 16 and 17, in the above case, namely where $I = H_{k-1}$, $S = (H_0, \ldots, H_k)$ is an M-ear sequence, which can be regarded as being associated by M-ears $P' := (P_1, \ldots, P_{k-1}, \hat{Q})$. Since S and P' satisfy D1 by Claim 17, we have that they satisfy also D2 and D3, by Lemma 3. Hence we are done for this case.

Claim 18. If I is distinct from any of $H_1, \ldots, H_{k-1}$, then P_{k-1}, an M-ear relative to H_{k-2}, is through I.

Proof. If $k = 2$, the claim apparently follows. Let $k \geq 3$ and suppose I has common vertices with P_i with $i \in \{1, \ldots, k-2\}$. Namely, P_i is an M-ear relative to H_{i-1} and through I. Hence, $(H_0, \ldots, H_{i-1}, I, H_k)$ is an M-ear sequence from H_0 to H_k, of length $i + 1 \leq k - 1$, associated with M-ears $(P_1, \ldots, P_i, \hat{Q})$. This contradicts the definition of k. Therefore we can conclude that $i = k - 1$, and the claim follows. □

Therefore, in this case, by Claim 18, $\tilde{S} := (H_0, \ldots, H_{k-2}, I, H_k)$ is an M-ear sequence associated with $\tilde{P} = (P_1, \ldots, P_{k-1}, \hat{Q})$.

– $\tilde{S}^{k-2}$ and $\tilde{P}^{k-2}$ satisfy D1, D2, and D3, since $\tilde{S}^{k-2} = S^{k-2}$ and $\tilde{P}^{k-2} = P^{k-2}$, and
– $\tilde{S}^{k-1}$ and $\tilde{P}^{k-1}$ satisfy D1, since $(k-1)$-th elements of P and $\tilde{P}$ are identical.

Therefore, by Lemma 3, $\tilde{S}^{k-1}$ and $\tilde{P}^{k-1}$ also satisfy D2 and D3. Moreover, by Claim 17, with Lemma 3 again applied to $\tilde{S}$ and $\tilde{P}$, we obtain, with Claim 17, that $\tilde{S}$ and $\tilde{P}$ also satisfy D1, D2, and D3. This completes the proof. □

Theorem 5. $\lhd$ *is a partial order.*

Proof. The reflexivity is obvious from the definition. The transitivity obviously follows from Theorem 4. Hence, we will prove the antisymmetry. Let $G_1, G_2 \in \mathcal{G}(G)$ be such that $G_1 \lhd G_2$ and $G_2 \lhd G_1$. Suppose that the antisymmetry fails, that is, that $G_1 \neq G_2$. Let M be a perfect matching of G. By Proposition 11, there exists an M-ear sequence from G_1 to G_2, say $S := (G_1 = H_0, \ldots, H_k = G_2)$, where $k \geq 1$, and associated M-ears $P := (P_1, \ldots, P_k)$ which D1, D2 and D3. Let u_1 and v_1 be the end vertices of P_1.

By Lemma 1 there exists $w \in V(G_2)$ such that there is an M-balanced path Q from u_1 to w. Trace Q from u_1 and let x be the first vertex we encounter that is in $(S \oplus P \cup \{v_1\}) \setminus \{u_1\}$; such a vertex surely exists since $V(G_2) \subseteq S \oplus P$.

Claim 19. Without loss of generality we can assume that $x \neq v_1$, namely, $x \in S \oplus P$ and u_1Qx is disjoint from v_1.

Proof. Suppose the claim fails, that is, $x = v_1$. Then, $u_1 \neq v_1$. If u_1Qv_1 is an M-saturated path, then $P_1 + u_1Qv_1$ forms an M-alternating circuit, containing non-allowed edges, a contradiction. Otherwise, namely if u_1Qv_1 is an M-balanced path from u_1 to v_1, then v_1Qw is an M-balanced path from v_1 to w. Now redefine x as the first vertex we encounter that is in $S \oplus P$ if we trace v_1Qw from v_1. Then, v_1Qx is an M-balanced path from v_1 to x which is disjoint from u_1. Therefore, by changing the roles of u_1 and v_1, without loss of generality, we obtain the claim. □

Therefore, hereafter let $x \in S \oplus P$, noting that u_1Qx is an M-balanced path from u_1 to x. Since $x \in S \oplus P$, by Proposition 9 there is an M-balanced path R from x to an internal vertex of P_1, say y, such that $V(R) \subseteq S \oplus P$ and $V(R) \cap V(P_1) = \{y\}$.

If u_1P_1y has an even number of edges, $u_1Qx + xRy + yP_1u_1$ is an M-alternating circuit containing non-allowed edges, a contradiction.

Hence hereafter we assume u_1P_1y has an odd number of edges. By Proposition 8, there is an M-saturated or balanced path L from v_1 to u_1 which is contained in G_1. Trace L from v_1 and let w be the first vertex on u_1Qx; note that L is disjoint from $S \oplus P$ since $V(L) \subseteq V(H_0)$ and $S \oplus P$ is disjoint from $V(H_0)$. If u_1Qw has an odd number of edges, then $wQu_1 + P_1 + v_1Lw$ is an M-alternating circuit, a contradiction. If u_1Qw has an even number of edges, then $v_1Lw + wQx + xRy + yP_1u_1$ is an M-alternating circuit, which is also a contradiction. Thus we get $G_1 = G_2$, and the claim follows. □

4 A Generalization of the Canonical Partition

For non-elementary graphs, the family of maximal barriers never gives a partition of its vertex set [1]. Therefore, to analyze the structures of general graphs with perfect matchings, we generalized the canonical partition based on Kotzig's way [4–6].

Definition 2. *Let G be a factorizable graph and $H \in \mathcal{G}(G)$. For $u, v \in V(H)$, we say $u \sim_g v$ if $u = v$ or $G - u - v$ is not factorizable.*

Theorem 6. *$\sim_g$ is an equivalence relation.*

Proof. Since the reflexivity and the symmetry are obvious from the definition, we prove the transitivity. Let M be a perfect matching of G. Let $u, v, w \in V(H)$ be such that $u \sim_g v$ and $v \sim_g w$. If any two of them are identical, clearly the claim follows. Therefore it suffices to consider the case that they are mutually distinct. Suppose that the claim fails, that is, $u \not\sim_g w$. Then there is an M-saturated path P between u and w. By Proposition 8, there is an M-balanced path Q from v to u. Trace Q from v and let x be the first vertex we encounter that in $V(Q) \cap V(P)$. If uPx has an odd number of edges, $vQx + xPu$ is an M-saturated path between u and v, a contradiction. If uPx has an even number of edges, then xPw has an odd number of edges, and by the same argument we have a contradiction. □

We call the family of equivalence classes of $\sim_g$ as the *generalized canonical partition* and denote as $\mathcal{P}_G(H)$ for each factor-component $H \in \mathcal{G}(G)$ of a factorizable graph G. Note that the notions of the canonical partition and the generalized one are coincident for an elementary graph. Thus we denote the union of equivalence classes of all the factor-components of G as $\mathcal{P}(G)$, and call it just as the *canonical partition*. Moreover our proof for Theorem 6 contains a short proof for the existence of the canonical partition. Kotzig takes three papers to prove it, thus to prove that $\sim$ is an equivalence relation "from scratch" is considered to be hard [1]. However, in fact, it can be shown in a simple way even without the premise of the Gallai-Edmonds structure theorem nor the notion of barriers. Note also that the generalized canonical partition $\mathcal{P}_G(H)$ is a refinement of $\mathcal{P}(H)$ for each $H \in \mathcal{G}(G)$.

5 Correlations between $\triangleleft$ and $\sim_g$

In this section we further analyze properties of factorizable graphs. We denote all the upper bounds of $H \in \mathcal{G}(G)$ in $(\mathcal{G}(G), \triangleleft)$ as $up^*_G(H)$ and define $up_G(H)$ as $up^*_G(H) \setminus \{H\}$. We sometimes omit the subscripts if they are apparent from the context. For simplicity, we sometimes denote the subgraph induced by the vertices in $up(H)$ (resp. $up^*(H)$) as just $G[up(H)]$ (resp. $G[up^*(H)]$), and the vertices of $up(H)$ (resp. $up^*(H)$) as just $V(up(H))$ (resp. $V(up^*(H))$).

Lemma 4. *Let G be a factorizable graph, M be a perfect matching of G, and $H \in \mathcal{G}(G)$. Let P be an M-ear relative to H with end vertices $u, v \in V(H)$. Then $u \sim_g v$.*

Proof. Suppose the claim fails, that is, $u \neq v$ and there is an M-saturated path Q between u and v. Trace Q from u and let x be the first vertex we encounter that is on $P-u$. If uPx has an even number of edges, $uQx+xPu$ is an M-alternating circuit containing non-allowed edges, a contradiction. Hence we suppose uPx has an odd number of edges. Let $I \in \mathcal{G}(G)$ be such that $x \in V(I)$. Then one of the components of $uQx + xPu - E(I)$ is an M-ear relative to I and through H, a contradiction by Theorem 4. □

Theorem 7. *Let G be a factorizable graph, and $G_0 \in \mathcal{G}(G)$. For each connected component K of $G[up(G_0)]$ there exists $T_K \in \mathcal{P}_G(G_0)$ such that $N(K) \cap V(G_0) \subseteq T_K$.*

Proof. Let M be a perfect matching of G.

Claim 20. Let $H \in up(G_0)$, and S and P be the shortest M-ear sequence from G_0 to H and associated M-ears which satisfy D1, D2 and D3. Then, there exists $T \in \mathcal{P}_G(G_0)$ such that for each factor-components H' that has common vertices with $S \oplus P$, $N(H') \cap V(G_0) \subseteq T$ holds.

Proof. Let us denote $S = (G_0 = H_0, \dots, H_k = H)$, where $k \geq 1$, and $P = (P_1, \dots, P_k)$. Let $u_1, v_1 \in V(G_0)$ be the end vertices of P_1. By Lemma 4, there exists $T \in \mathcal{P}_G(G_0)$ such that $u_1, v_1 \in T$.

Let $H' \in \mathcal{G}(G)$ be such that $V(H') \cap S \oplus P \neq \emptyset$. Suppose there exists $w \in N(H') \cap V(G_0)$ and let $z \in V(H')$ be such that $wz \in E(G)$. Take $x \in V(H') \cap S \oplus P$ arbitrarily. By Proposition 8, there exists a path Q which is M-balanced from z to x or M-saturated between z and x such that $V(Q) \subseteq V(H')$. Trace Q from z and let y be the first vertex we encounter that is in $S \oplus P$. Then, zPy is an M-balanced path from z to y with $V(zPy) \subseteq V(H')$ and $V(zPy) \cap S \oplus P = \{y\}$. By D3($S$, P), for either of $r \in \{u_1, v_1\}$, there is an M-balanced path R from y to r such that $V(R) \setminus \{r\} \subseteq S \oplus P$.

Therefore, $R + zPy + wz$ forms an M-ear relative to G_0, whose end vertices are r and w. By Lemma 4, therefore, $w \in T$ and the claim follows. □

Immediately by Claim 20 we can see that for any $H \in up(G_0)$ there exists $T \in \mathcal{P}_G(G_0)$ such that $N(H) \cap V(G_0) \subseteq T$. Hence for each $T \in \mathcal{P}_G(G_0)$ we can define

$$\mathcal{K}_T := \{H \in up(G_0) : V(H) \subseteq V(K) \text{ and } N(H) \cap V(G_0) \subseteq T\}$$

and $V_T := \bigcup_{H \in \mathcal{K}_T} V(H)$. Note that $\bigcup_{T \in \mathcal{P}_G(G_0)} V_T = V(K)$.

We are going to prove the claim by showing that $|\{T \in \mathcal{P}_G(G_0) : V_T \neq \emptyset\}| = 1$. Suppose it fails; Then, since K is connected, there exist $T_1, T_2 \in \mathcal{P}_G(G_0)$ with $T_1 \neq T_2$ such that $E[V_{T_1}, V_{T_2}] \neq \emptyset$. Let $s_1 \in V_{T_1}$ and $s_2 \in V_{T_2}$ be such that $s_1 s_2 \in E[V_{T_1}, V_{T_2}]$.

Claim 21. For each $i = 1, 2$, there is an M-balanced path L_i from s_i to a vertex in T_i, say r_i, such that $V(L_i) \setminus \{r_i\} \subseteq V_{T_i}$.

Proof. Let $i \in \{1,2\}$. Let $H \in \mathcal{G}(G)$ be such that $s_i \in V(H)$. Then, $V(H) \subseteq V_{T_i}$. Take an M-ear sequence $S = (G_0 = H_0, \ldots, H_k = H)$, where $k \geq 1$, from G_0 to H and an associated M-ears $P = (P_1, \ldots, P_k)$ which satisfy D1, D2 and D3; By Claim 20, $S \oplus P \subseteq V_{T_i}$. By D3, there is an M-balanced path L_i from s_i to either of the end vertices of P_1, say $r_i \in V(G_0)$ such that $V(L_i) \setminus \{r_i\} \subseteq S \oplus P$. Therefore, $V(L_i) \setminus \{r_i\} \subseteq V_{T_i}$. □

By Claim 21, $L_1 + s_1 s_2 + L_2$ is an M-ear relative to G_0, whose end vertices are $r_1 \in T_1$ and $r_2 \in T_2$. By Lemma 4 this yields $T_1 = T_2$, a contradiction. Therefore, we can conclude that there exists $T \in \mathcal{P}_G(G_0)$ such that $V_T = V(K)$, namely the claim follows. □

By Theorem 7, we can see that upper bounds of a factor-component are each "attached" to an equivalence class of the generalized canonical partition.

Proposition 12. *Let G be a graph and M be a matching of G. Let $H_1, H_2 \subseteq G$ be factor-critical subgraphs of G such that there exists $v \in V(H_1) \cap V(H_2)$ and that for each $i = 1, 2$, M_{H_i} is a near-perfect matching of H_i exposing only v. Then, $H_1 \cup H_2$ is factor-critical.*

Proof. Apparently, $M_1 \cup M_2$ is a near-perfect matching of $H_1 \cup H_2$, exposing only v. Since H_1 and H_2 are both factor-critical, the claim follows by Proposition 1. □

Lemma 5. *Let G be a factorizable graph, and $H \in \mathcal{G}(G)$. Then, $G[up^*(H)]/H$ is factor-critical.*

Proof. Let M be a perfect matching of G. Let $\mathcal{X} \subseteq 2^{V(G)}$ be the family of separable set for H. Then, by Theorem 5, $\bigcup_{X \in \mathcal{X}} X = V(up^*(H))$. On the other hand, $G[\bigcup_{X \in \mathcal{X}} X]/H$ is factor-critical by Proposition 12. Therefore, the claim follows. □

Theorem 8. *Let G be a factorizable graph, and let $H \in \mathcal{G}(G)$ and $S \subseteq \mathcal{P}_G(H)$. Let $K_1, \ldots, K_l$, where $l \geq 1$ be some connected components of $G[up(H)]$ such that $N(K_i) \cap V(H) \subseteq S$ for $i = 1, \ldots, l$. Then, $G[V(K_1) \cup \cdots \cup V(K_l) \cup S]/S$ is factor-critical.*

Proof. First note that $G[up^*(H)]/H$ is factor-critical by Lemma 5. Let h be the contracted vertex of $G[up^*(H)]/H$. Note also that K is a connected component of $G[up(H)]$ if and only if there is a block $\widehat{K}$ of $G[up^*(H)]/H$ such that $K = \widehat{K} - h$. Therefore, by Proposition 2 the claim follows. □

Remark 3. There are factorizable graphs where $\triangleleft$ does not hold for any two factor-components, in other words, where all the factor-components are minimal in the poset. For example, we can see by Theorem 4 and Theorem 7 that bipartite factorizable graphs are such, which means Theorem 5 is not a generalization of the DM-decomposition, even though they have similar natures.

The following theorem shows that most of the factorizable graphs with $|\mathcal{G}(G)| \geq 2$, in a sense, have non-trivial structures as posets.

Theorem 9. *Let G be a factorizable graph, $G_1, G_2 \in \mathcal{G}(G)$ be factor-components for which $G_1 \lhd G_2$ does not hold, and let G_1 be minimal in the poset $(\mathcal{G}(G), \lhd)$. Then there are possibly identical complement edges e, f of G between G_1 and G_2 such that $\mathcal{G}(G+e+f) = \mathcal{G}(G)$ and $G_1 \lhd G_2$ in $(\mathcal{G}(G+e+f), \lhd)$.*

Proof. First we prove the case where there is an edge xy such that $x \in V(G_1)$ and $y \in V(G_2)$. Let M be a perfect matching of G. Choose a vertex $w \in V(G_2)$ such that $w \not\sim_g y$ in G_2, and let P be an M-saturated path between w and y. If $xw \in E(G)$, there is an M-ear $xy+P+wx$ relative to G_1 and through G_2, which means $G_1 \lhd G_2$ by Theorem 4. Thus $xw \notin E(G)$. Suppose $\mathcal{G}(G+xw) \neq \mathcal{G}(G)$. Then there is an M-alternating circuit C containing xw in $G+xw$. Give an orientation to C so that it becomes a dicircuit with the arc xx'. Trace C from x and let z be the first vertex we encounter that is in $V(G_2)$. Then $xy + xCz$ is an M-ear of G which is relative to G_2 and through G_1, which means $G_2 \lhd G_1$ by Theorem 4, a contradiction to the minimality of G_1. Thus $\mathcal{G}(G+xw) = \mathcal{G}(G)$ and we are done for this case.

Now we prove the other case, where there is no edge of G connecting G_1 and G_2. Choose any $x \in V(G_1)$ and $y \in V(G_2)$. If $\mathcal{G}(G+xy) = \mathcal{G}(G)$, we can reduce it to the first case and the claim follows. Therefore it suffices to consider the case that $\mathcal{G}(G+xy) \neq \mathcal{G}(G)$. Then, for any perfect matching M of G, there is an M-alternating circuit C in $G+xy$ containing xy. Give an orientation to C so that it becomes a dicircuit with the arc yy'. Trace C from y and let u be the first vertex of G_1, and let v be the first vertex in G_2 if we trace C from u in the opposite direction.

If $\mathcal{G}(G+uv) = \mathcal{G}(G)$, the claim follows by the same argument. Otherwise, that is, if $\mathcal{G}(G+uv) \neq \mathcal{G}(G)$, there is an M-alternating circuit D containing uv. Give an orientation to D so that it becomes a dicircuit with the arc uu'. If uDv is disjoint from the internal vertices of vCu, then $uDv + vCu$ forms an M-alternating circuit containing non-allowed edges, a contradiction. Otherwise, trace D from u and let w be the first vertex on $vCu - u$.

If wCu has an even number of edges, $wCu + uDw$ is an M-alternating circuit of G, a contradiction. Therefore, we assume wCu has an odd number of edges. Let $H \in \mathcal{G}(G)$ be such that $w \in V(H)$. Then $wCu + uDw - H$ leaves an M-ear in G which is relative to H and through G_1, contradicting the minimality of G_1. Thus this completes the proof. □

6 Algorithmic Result

In this section, we discuss the algorithmic aspects of the partial order and the generalized canonical partition. We denote by n and m respectively the number of vertices and edges of input graphs. As we work on factorizable graphs and graphs with near-perfect matchings, we can assume $m = \Omega(n)$.

We start with some materials from Edmonds' maximum matching algorithm [12], referring mainly to [1, 17]. For a tree T with a specified root vertex r, we call a vertex $v \in V(T)$ *inner* (resp. *outer*) if the unique path in T from

r to v has an odd (resp. even) number of edges. Let G be a graph and M be a matching of G. A tree $T \subseteq G$ is called *M-alternating* if exactly one vertex of it, the root, is exposed by M in G, and each inner vertex $v \in V(T)$ satisfies $|\delta(v) \cap E(T)| = 2$ and one of the edges of $\delta(v) \cap E(T)$ is contained in M.

A subgraph $S \subseteq G$ is called a *special blossom tree with respect to M (M-SBT)* if there is a partition $V(C_1)\dot\cup \cdots \dot\cup V(C_k) = V(S)$ such that

(i) $S' := S/C_1/\cdots/C_k$ is an M-alternating tree,
(ii) M_{C_i} is a near-perfect matching of C_i,
(iii) C_i is a maximal factor-critical subgraph of G if it corresponds to an outer vertex of S', and called an *outer blossom*, and
(iv) $|V(C_i)| > 1$ only if C_i is an outer blossom, for each $i = 1, \dots, k$.

Edmonds' maximum matching algorithm tells us the following facts. Let G be a graph, M be a near-perfect matching of G, and $r \in V(G)$ be the vertex exposed by M. Then an M-SBT S, with root r, can be computed, if it is carefully implemented [18,19], in $O(m)$ time. Additionally, the set of vertices from which r can be reached by an M-balanced path is exactly the set of vertices contained in the outer blossoms of S.

Thus, due to an easy reduction of the above facts, the following proposition holds; they can be regarded as a folklore. See [3]. (In [3] they are presented as those for elementary graphs, but in fact, they can be applicable for general factorizable graphs.)

Proposition 13. *Let G be a factorizable graph, M be a perfect matching of G, and $u \in V(G)$.*

(i) *The set of vertices that can be reached from u by an M-saturated path can be computed in $O(m)$ time.*
(ii) *All the allowed edges adjacent to u can be computed in $O(m)$ time.*
(iii) *All the factor-components of G can be computed in $O(nm)$ time.*

Proposition 14. *Given a factorizable graph G, one of its perfect matchings M and $\mathcal{G}(G)$, we can compute the generalized canonical partition of G in $O(nm)$ time.*

Proof. For each $H \in \mathcal{G}(G)$, we can compute $\mathcal{P}_G(H)$ in a similar way to compute the canonical partition of an elementary graph [3]. That is, for each $v \in V(H)$, compute the set of vertices U that can be reached from v by an M-saturated path, and recognize $V(H) \setminus U$ as a member of $\mathcal{P}_G(H)$. This procedure is surely compatible by Theorem 6. Thus, the claim follows by Proposition 13. $\square$

Let G be a factorizable graph and M be a perfect matching of G. We say two distinct factor-components G_1, G_2 of G with $G_1 \triangleleft G_2$ are *non-refinable* if $G_1 \triangleleft H \triangleleft G_2$ yields $G_1 = H$ or $G_2 = H$ for any $H \in \mathcal{G}(G)$. Note that if G_1 and G_2 are non-refinable, then there is an M-ear relative to G_1 and through G_2 by Theorem 4. Note also that the converse of the above fact does not hold.

Lemma 6. *Let G be a factorizable graph, M be a perfect matching of G, and $H \in \mathcal{G}(G)$. Let S be a maximal M-SBT in G/H and let C be the blossom of T containing the contracted vertex h corresponding to H. Then any non-refinable upper bound of H in $(\mathcal{G}(G), \lhd)$ has common vertices with C. Additionally, if a factor-component $I \in \mathcal{G}(G)$ has some common vertices with C, then $H \lhd I$.*

Proof. For the former part, let H' be a non-refinable upper bound of H, and P be an M-ear relative to H and through H'. Since $P - C$ is a disjoint union of M-ears relative to C, we have $P \subseteq C$ by Theorem 3 and the maximality of the outer blossoms in M-SBT. Thus the former part of the claim follows.

For the latter part, by the definition of M-SBT and Proposition 4, there is an M-alternating odd ear-decomposition $\mathcal{P} = \{P_1, \dots, P_k\}$ of C. Let $I \in \mathcal{G}(G)$ be such that $V(I) \cap V(C) \neq \emptyset$ and that $V(P_j) \cap V(I) = \emptyset$ for $j = 1, \dots, i-1$ and $V(P_i) \cap V(I) \neq \emptyset$. We proceed by induction on i. If $i = 1$, the claim obviously follows. Let $i > 1$. $G_{i-1} := P_1 + \cdots + P_{i-1}$ is factor-critical by Theorem 3, and $M_{G_{i-1}}$ is a near-perfect matching of G_{i-1}. Moreover, P_i is an M-ear relative to G_{i-1}. Therefore, with the same technique as in the proof of Theorem 4, there exists $I' \in \mathcal{G}(G)$ such that $V(I') \cap V(C) \neq \emptyset$ and that there is an M-ear relative to I' and through I. Thus, by the induction hypothesis, the latter part of the claim follows. □

Proposition 15. *Given a factorizable graph G, its perfect matching M, and $\mathcal{G}(G)$, we can compute the poset $(\mathcal{G}(G), \lhd)$ in $O(nm)$ time.*

Proof. It is sufficient to list all the non-refinable upper bounds for each factor-component of G by the following procedure:

```
D := (𝒢(G), ∅); A := ∅;
for all H ∈ 𝒢(G) do
    compute a maximal M-SBT T; let C be the blossom of T corresponding
    to its root;
    for all x ∈ V(C), which satisfies x ∈ V(I) for some I ∈ 𝒢(G) do
        A := A ∪ {(H, I)};
    end for
end for
D := (𝒢(G), A); STOP.
```

By Lemma 6, the partial order on $V(D)$ determined by the reachability corresponds to $\lhd$ after the above procedure. That is, if we define a binary relation $\prec$ on $V(D)$ so that $H' \lhd I'$ if there is a dipath from H' to I' in D, then $\prec$ and $\lhd$ coincide. For each $H \in \mathcal{G}(G)$, the above procedure costs $O(m)$ time, thus it costs $O(nm)$ time over the whole computation. □

Remark 4. Given the digraph D after the procedure in Proposition 15, we can compute all the upper bounds of a factor-component in $O(n^2)$ time. Thus, an efficient data structure that represents the poset, for example, a boolean-valued matrix L where $L[i, j] =$ true if and only if $G_i \lhd G_j$, can be obtained in additional $O(n^2)$ time.

As a maximum matching of a graph can be computed in $O(\sqrt{n}m)$ time [20, 21], we have the following, combining Propositions 13, 14, and 15.

Theorem 10. *Let G be a factorizable graph. Then the poset $(\mathcal{G}(G), \triangleleft)$ and the generalized canonical partition $\mathcal{P}(G)$ can be computed in $O(nm)$ time.*

Acknowlegements. The author is grateful to Yusuke Kobayashi and Richard Hoshino for carefully reading the paper, and Akihisa Tamura for useful discussions.

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