Efficiency Loss in a Cournot Oligopoly with Convex Market Demand

John N. Tsitsiklis and Yunjian Xu

Laboratory or Information and Decision Systems, MIT, Cambridge, MA, 02139, {jnt@mit.edu, yunjian@mit.edu}

We consider a Cournot oligopoly model where multiple suppliers (oligopolists) compete by choosing quantities. We compare the social welfare achieved at a Cournot equilibrium to the maximum possible, for the case where the inverse market demand function is convex. We establish a lower bound on the efficiency of Cournot equilibria in terms of a scalar parameter derived from the inverse demand function, namely, the ratio of the slope of the inverse demand function at the Cournot equilibrium to the average slope of the inverse demand function between the Cournot equilibrium and a social optimum. Also, for the case of a single, monopolistic, profit maximizing supplier, or of multiple suppliers who collude to maximize their total profit, we establish a similar but tighter lower bound on the efficiency of the resulting output. Our results provide nontrivial quantitative bounds on the loss of social welfare for several convex inverse demand functions that appear in the economics literature.

Key words: Cournot oligopoly, price of anarchy

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1. Introduction

We consider a Cournot oligopoly model where multiple suppliers (oligopolists) compete by choosing quantities, with a focus on the case where the inverse market demand function is convex. Our objectives are to compare the optimal social welfare to: (i) the social welfare at a Cournot equilibrium and (ii) the social welfare achieved when the suppliers collude to maximize the total profit, or, equivalently, when there is a single supplier.

1.1. Background

In a book on oligopoly theory (see Chapter 2.4 of Friedman (1983)), Friedman raises two questions on the relation between Cournot equilibria and competitive equilibria. First, "is the Cournot

equilibrium close, in some reasonable sense, to the competitive equilibrium?" Furthermore, "will the two equilibria coincide as the number of firms goes to infinity?" The answer to the second question seems to be positive, in general. Indeed, the efficiency properties of Cournot equilibria in economies and markets with a large number of suppliers and/or consumers have been much explored. For the case of a large number of suppliers, it is known that every Cournot equilibrium is approximately a socially optimal competitive equilibrium (Gabszewicz and Vial 1972, Novshek and Sonnenschein 1978, Novshek 1980). Furthermore, Ushio (1985) derives necessary and/or sufficient conditions on the relative numbers of consumers and suppliers for the efficiency loss associated with every Cournot equilibrium to approach zero, as the number of suppliers increases to infinity.

In more recent work, attention has turned to the efficiency of Cournot equilibria in settings that involve an arbitrary (possibly small) number of suppliers or consumers. Anderson and Renault (2003) quantify the efficiency loss in Cournot oligopoly models with concave demand functions. However, most of their results focus on the relation between consumer surplus, producer surplus, and the aggregate social welfare achieved at a Cournot equilibrium, rather than on the relation between the social welfare achieved at a Cournot equilibrium and the optimal social welfare.

The concept of efficiency loss is intimately related to the concept of "price of anarchy," advanced by Koutsoupias and Papadimitriou in a seminal paper (Koutsoupias and Papadimitriou 1999); it provides a natural measure of the difference between a Cournot equilibrium and a socially optimal competitive equilibrium. In the spirit of Koutsoupias and Papadimitriou (1999), we define the efficiency of a Cournot equilibrium as the ratio of its aggregate social welfare to the optimal social welfare. Recent works have reported various efficiency bounds for Cournot oligopoly with affine demand functions. Kluberg and Perakis (2008) compare the social welfare and the aggregate profit earned by the suppliers under Cournot competition to the corresponding maximum possible, for the case where suppliers produce multiple differentiated products and demand is an affine function of the price. Closer to the present paper, Johari and Tsitsiklis (2005) establish a 2/3 lower bound on the efficiency of a Cournot equilibrium, when the inverse demand function is affine. They also show that the 2/3 lower bound applies to a monopoly model with general concave demand.

The efficiency loss in a Cournot oligopoly with some specific forms of convex inverse demand functions has received some recent attention. Corchon (2008) studies the special case of convex inverse demand functions of the form $p(q) = \alpha - \beta q^\gamma$, analyzes the efficiency loss at a Cournot equilibrium and shows that when $\gamma > 0$, the worst case efficiency loss occurs when an efficient supplier has to share the market with infinitely many inefficient suppliers. Guo and Yang (2005) consider a class of inverse demand functions that solve a certain differential equation (for example, constant elasticity inverse demand functions belong to this class), and establish efficiency lower bounds that depend on equilibrium market shares, the market demand, and the number of suppliers.

In this paper, we study the efficiency loss in a Cournot oligopoly model with general convex demand functions[1]. Convex demand functions, such as the negative exponential and the constant elasticity demand curves, have been widely used in oligopoly analysis and marketing research (Bulow and Pfleiderer 1983, Fabingeryand and Weyl 2009, Tyagi 1999). In general, a Cournot equilibrium need not exist when the inverse demand function is convex. However, it is well known that a Cournot equilibrium will exist if the inverse demand function is "not too convex" (e.g., if the inverse demand function is concave), in which case the quantities supplied by different suppliers are strategic substitutes (Bulow et al. 1985, Berry and Pakes 2003). Existence results for Cournot oligopolies for the case of strategic substitutes can be found in Novshek (1985), Gaudet and Salant (1991), Svizzero (1991), and Long and Soubeyran (2000). Note however, that the strategic substitutes condition is not necessary for the existence of Cournot equilibria. For example, using the theory of ordinally supermodular games, Amir (1996) shows that the log-concavity of inverse demand functions guarantees the existence of a Cournot equilibrium. In this paper, we will not address the case of concave inverse demand functions, which appears to be qualitatively different, as will be illustrated by an example in Section 3.5.

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[1] Since a demand function is generally nonincreasing, the convexity of a demand function implies that the corresponding inverse demand function is also convex.

1.2. Our contribution

For Cournot oligopolies with convex and nonincreasing demand functions, we establish a lower bound of the form $f(c/d)$ on the efficiency achieved at a Cournot equilibrium. Here, f is a function given in closed form; c is the absolute value of the slope of the inverse demand function at the Cournot equilibrium; and d is the absolute value of the slope of the line that agrees with the inverse demand function at the Cournot equilibrium and at a socially optimal point. For convex and nonincreasing inverse demand functions, we have $c \geq d$; for affine inverse demand functions, we have $c/d = 1$. In the latter case, our efficiency bound is $f(1) = 2/3$, which is consistent with the bound derived in Johari and Tsitsiklis (2005). More generally, the ratio c/d can be viewed as a measure of nonlinearity of the inverse demand function. As the ratio c/d goes to infinity, the lower bound converges to zero and arbitrarily high efficiency losses are possible. The usefulness of this result lies in that it allows us to lower bound the efficiency of Cournot equilibria for a large class of Cournot oligopoly models in terms of qualitative properties of the inverse demand function, without having to restrict to the special case of affine demand functions, and without having to calculate the equilibrium and the social optimum.

An interesting special case of our model arises when $N = 1$, in which case we are dealing with a single, monopolistic, supplier. The previous lower bounds for Cournot equilibria continue to hold. However, by using the additional assumption that $N = 1$, we can hope to obtain sharper (i.e., larger) lower bounds in terms of the same scalar parameter c/d. Let us also note that the case $N = 1$ also covers a setting where there are multiple suppliers who choose to collude and coordinate production so as to maximize their total profit.

1.3. Outline of the paper

The rest of the paper is organized as follows. In the next section, we formulate the model and review available results on the existence of Cournot equilibria. In Section 3, we provide some mathematical preliminaries on Cournot equilibria that will be useful later, including the fact that efficiency lower bounds can be obtained by restricting to linear cost functions. In Section 4, we consider affine

inverse demand functions and derive a refined lower bound on the efficiency of Cournot equilibria that depends on a small amount of ex post information. We also show this bound to be tight. In Section 5, we consider a more general model, involving convex inverse demand functions. We show that for convex inverse demand functions, and for the purpose of studying the worst case efficiency loss, it suffices to restrict to a special class of piecewise linear inverse demand functions. This leads to the main result of this paper, a lower bound on the efficiency of Cournot equilibria (Theorem 2). Based on this theorem, in Section 6 we derive a number of corollaries that provide efficiency lower bounds that can be calculated without detailed information on these equilibria, and apply these results to various commonly encountered convex inverse demand functions. In Section 7, we establish a lower bound on the efficiency of monopoly outputs (Theorem 3), and show by example that the social welfare at a monopoly output can be higher than that achieved at a Cournot equilibrium (Example 8). Finally, in Section 8, we make some brief concluding remarks.

2. Formulation and Background

In this section, we define the Cournot competition model that we study in this paper. We also review available results on the existence of Cournot equilibria.

We consider a market for a single homogeneous good with inverse demand function $p : [0, \infty) \to [0, \infty)$ and N suppliers. Supplier $n \in \{1, 2, \dots, N\}$ has a cost function $C_n : [0, \infty) \to [0, \infty)$. Each supplier n chooses a nonnegative real number x_n, which is the amount of the good to be supplied by her. The **strategy profile** $\mathbf{x} = (x_1, x_2, \dots, x_N)$ results in a total supply denoted by $X = \sum_{n=1}^{N} x_n$, and a corresponding market price $p(X)$. The payoff to supplier n is

$$\pi_n(x_n, \mathbf{x}_{-n}) = x_n p(X) - C_n(x_n),$$

where we have used the standard notation $\mathbf{x}_{-n}$ to indicate the vector $\mathbf{x}$ with the component x_n omitted. A strategy profile $\mathbf{x} = (x_1, x_2, \dots, x_N)$ is a Cournot (or Nash) equilibrium if

$$\pi_n(x_n, \mathbf{x}_{-n}) \geq \pi_n(x, \mathbf{x}_{-n}), \qquad \forall \, x \geq 0, \ \ \forall \, n \in \{1, 2, \dots, N\}.$$

In the sequel, we denote by f' and f'' the first and second, respectively, derivatives of a scalar function f, if they exist. For a function defined on a domain $[0, Q]$, the derivatives at the endpoints 0 and Q are defined as left and right derivatives, respectively. For points in the interior of the domain, and if the derivative is not guaranteed to exist, we use the notation $\partial_+ f$ and $\partial_- f$ to denote the right and left, respectively, derivatives of f; these are guaranteed to exist for convex or concave functions f.

2.1. Existence results

Some results on the existence of Cournot equilibrium are provided by Szidarovszky and Yakowitz (1977), but require the concavity of the inverse demand function. Mcmanus (1964) provides an existence result under minimal assumptions on the inverse demand function $p(\cdot)$, but only for the special case where all suppliers have the same cost function $C(\cdot)$ — a rather restrictive assumption. The most relevant result for our purposes is provided by Novshek (1985) who does not require the suppliers to be identical or the inverse demand functions to be concave.

PROPOSITION 1. *(Novshek 1985) Suppose that the following conditions hold:*

(a) The inverse demand function $p(\cdot)$ is continuous.

(b) There exists a real number $Q > 0$ such that $p(q) = 0$ for $q \geq Q$. Furthermore, $p(\cdot)$ is twice continuously differentiable and strictly decreasing on $[0, Q)$.

(c) For every $q \in [0, Q)$, we have $p'(q) + q p''(q) \leq 0$.

(d) The cost functions $C_n(\cdot)$, $n = 1, 2, \ldots, N$, are nondecreasing and lower-semi-continuous.

Then, there exists a Cournot equilibrium.

If the inverse demand function $p(\cdot)$ is convex, the condition (c) in the preceding proposition implies that

$$\frac{\partial^2 \pi_n}{\partial x_n \partial x_m}(X) \leq 0, \qquad \forall\, m \neq n, \ \ \forall\, X \in (0, Q),$$

i.e., that the quantities supplied by different suppliers are strategic substitutes. We finally note that Amir (1996) proves existence of a Cournot equilibrium in a setting where the strategic substitutes

condition does not hold. Instead, this reference assumes that the inverse demand function $p(\cdot)$ is strictly decreasing and log-concave.

3. Preliminaries on Cournot Equilibria

In this section, we introduce several main assumptions that we will be working with, and some definitions. In Section 3.1, we present conditions for a nonnegative vector to be a social optimum or a Cournot equilibrium. Then, in Section 3.2, we define the efficiency of a Cournot equilibrium. In Sections 3.3 and 3.4, we derive some properties of Cournot equilibria that will be useful later, but which may also be of some independent interest. For example, we show that the worst case efficiency occurs when the cost functions are linear. The proofs of all propositions in this section (except for Proposition 4) are given in Appendix A.

ASSUMPTION 1. *For any n, the cost function $C_n : [0, \infty) \to [0, \infty)$ is convex, continuous, and non-decreasing on $[0, \infty)$, and continuously differentiable on $(0, \infty)$. Furthermore, $C_n(0) = 0$.*

ASSUMPTION 2. *The inverse demand function $p : [0, \infty) \to [0, \infty)$ is continuous, nonnegative, and nonincreasing, with $p(0) > 0$. Its right derivative at 0 exists and at every $q > 0$, its left and right derivatives also exist.*

Note that we do not yet assume that the inverse demand function is convex. The reason is that some of the results to be derived in this section are valid even in the absence of such a convexity assumption. Note also that some parts of our assumptions are redundant, but are included for easy reference. For example, if $C_n(\cdot)$ is convex and nonnegative, with $C_n(0) = 0$, then it is automatically continuous and nondecreasing.

DEFINITION 1. The **optimal social welfare** is the optimal objective value in the following optimization problem,

$$
\begin{aligned}
\text{maximize} \quad & \int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n) \\
\text{subject to} \quad & x_n \geq 0, \qquad n = 1, 2, \ldots, N,
\end{aligned}
\tag{1}
$$

where $X = \sum_{n=1}^N x_n$.

In the above definition, $\int_0^X p(q)\,dq$ is the aggregate consumer surplus and $\sum_{n=1}^N C_n(x_n)$ is the total cost of the suppliers. The objective function in (1) is a measure of the social welfare across the entire economy of consumers and suppliers, the same measure as the one used in Ushio (1985) and Anderson and Renault (2003).

For a model with a nonincreasing continuous inverse demand function and continuous convex cost functions, the following assumption guarantees the existence of an optimal solution to (1), because it essentially restricts the optimization to the compact set of vectors $\mathbf{x}$ for which $x_n \leq R$, for all n.

ASSUMPTION 3. *There exists some $R > 0$ such that $p(R) \leq \min_n\{C_n'(0)\}$.*

3.1. Optimality and equilibrium conditions

We observe that under Assumption 1 and 2, the objective function in (1) is concave. Hence, we have the following *necessary and sufficient* conditions for a vector $\mathbf{x}^S$ to achieve the optimal social welfare:

$$\begin{cases} C_n'(x_n^S) = p\left(X^S\right), & \text{if } x_n^S > 0, \\ C_n'(0) \geq p\left(X^S\right), & \text{if } x_n^S = 0, \end{cases} \tag{2}$$

where $X^S = \sum_{n=1}^N x_n^S$.

The social optimization problem (1) may admit multiple optimal solutions. However, as we now show, they must all result in the same price. We note that the differentiability of the cost functions is crucial for this result to hold.

PROPOSITION 2. *Suppose that Assumptions 1 and 2 hold. All optimal solutions to (1) result in the same price.*

There are similar equilibrium conditions for a strategy profile $\mathbf{x}$. In particular, under Assumptions 1 and 2, if $\mathbf{x}$ is a Cournot equilibrium, then

$$C_n'(x_n) \leq p(X) + x_n \cdot \partial_- p(X), \quad \text{if } x_n > 0, \tag{3}$$

$$C_n'(x_n) \geq p(X) + x_n \cdot \partial_+ p(X), \tag{4}$$

where again $X = \sum_{n=1}^{N} x_n$. Note, however, that in the absence of further assumptions, the payoff of supplier n need not be a concave function of x_n and these conditions are, in general, not sufficient.

We will say that a nonnegative vector $\mathbf{x}$ is a **Cournot candidate** if it satisfies the necessary conditions (3)-(4). Note that for a given model, the set of Cournot equilibria is a subset of the set of Cournot candidates. Most of the results obtained in this section, including the efficiency lower bound in Proposition 6, apply to all Cournot candidates.

For convex inverse demand functions, the necessary conditions (3)-(4) can be further refined.

PROPOSITION 3. *Suppose that Assumptions 1 and 2 hold, and that the inverse demand function $p(\cdot)$ is convex. If $\mathbf{x}$ is a Cournot candidate with $X = \sum_{n=1}^{N} x_n > 0$, then $p(\cdot)$ must be differentiable at X, i.e.,*

$$\partial_- p(X) = \partial_+ p(X).$$

Because of the above proposition, when Assumptions 1 and 2 hold and the inverse demand function is convex, we have the following necessary (and, by definition, sufficient) conditions for a nonzero vector $\mathbf{x}$ to be a Cournot candidate:

$$\begin{cases} C_n'(x_n) = p(X) + x_n p'(X), & \text{if } x_n > 0, \\ C_n'(0) \geq p(X) + x_n p'(X), & \text{if } x_n = 0. \end{cases} \tag{5}$$

3.2. Efficiency of Cournot equilibria

As shown in Friedman (1977), if $p(0) > \min_n\{C_n'(0)\}$, then the aggregate supply at a Cournot equilibrium is positive; see Proposition 4 below for a slight generalization. If on the other hand $p(0) \leq \min_n\{C_n'(0)\}$, then the model is uninteresting, because no supplier has an incentive to produce and the optimal social welfare is zero. This motivates the assumption that follows.

ASSUMPTION 4. *The price at zero supply is larger than the minimum marginal cost of the suppliers, i.e.,*

$$p(0) > \min_n\{C_n'(0)\}.$$

PROPOSITION 4. *Suppose that Assumptions 1, 2, and 4 hold. If* $\mathbf{x}$ *is a Cournot candidate, then* $X > 0$.

Proof Suppose that $p(0) > \min_n\{C_n'(0)\}$. Then, the vector $\mathbf{x} = (0, \ldots, 0)$ violates condition (4), and cannot be a Cournot candidate. $\square$

Under Assumption 4, at least one supplier has an incentive to choose a positive quantity, which leads us to the next result.

PROPOSITION 5. *Suppose that Assumptions 1-4 hold. Then, the social welfare achieved at a Cournot candidate, as well as the optimal social welfare [cf. (1)], are positive.*

We now define the efficiency of a Cournot equilibrium as the ratio of the social welfare that it achieves to the optimal social welfare. It is actually convenient to define the efficiency of a general vector $\mathbf{x}$, not necessarily a Cournot equilibrium.

DEFINITION 2. Suppose that Assumptions 1-4 hold. The **efficiency** of a nonnegative vector $\mathbf{x} = (x_1, \ldots, x_N)$ is defined as

$$\gamma(\mathbf{x}) = \frac{\displaystyle\int_0^X p(q)\, dq - \sum_{n=1}^N C_n(x_n)}{\displaystyle\int_0^{X^S} p(q)\, dq - \sum_{n=1}^N C_n(x_n^S)}, \tag{6}$$

where $\mathbf{x}^S = (x_1^S, \ldots, x_N^S)$ is an optimal solution of the optimization problem in (1) and $X^S = \sum_{n=1}^N x_n^S$.

We note that $\gamma(\mathbf{x})$ is well defined: because of Assumption 4 and Proposition 5, the denominator on the right-hand side of (6) is guaranteed to be positive. Furthermore, even if there are multiple socially optimal solutions $\mathbf{x}^S$, the value of the denominator is the same for all such $\mathbf{x}^S$. Note that $\gamma(\mathbf{x}) \leq 1$ for every nonnegative vector $\mathbf{x}$. Furthermore, if $\mathbf{x}$ is a Cournot candidate, then $\gamma(\mathbf{x}) > 0$, by Proposition 5.

3.3. Restricting to linear cost functions

In this section, we show that in order to study the worst-case efficiency of Cournot equilibria, it suffices to consider linear cost functions. We first provide a lower bound on $\gamma(\mathbf{x})$ and then proceed

to interpret it.

PROPOSITION 6. *Suppose that Assumptions 1-4 hold and that $p(\cdot)$ is convex. Let $\mathbf{x}$ be a Cournot candidate which is not socially optimal, and let $\alpha_n = C'_n(x_n)$. Consider a modified model in which we replace the cost function of each supplier n by a new function $\overline{C}_n(\cdot)$, defined by*

$$\overline{C}_n(x) = \alpha_n x, \qquad \forall\ x \geq 0.$$

Then, for the modified model, Assumptions 1-4 still hold, the vector $\mathbf{x}$ is a Cournot candidate, and its efficiency, denoted by $\overline{\gamma}(\mathbf{x})$, satisfies $0 < \overline{\gamma}(\mathbf{x}) \leq \gamma(\mathbf{x})$.

If $\mathbf{x}$ is a Cournot equilibrium, then it satisfies Eqs. (3)-(4), and therefore is a Cournot candidate. Hence, Proposition 6 applies to all Cournot equilibria that are not socially optimal. We note that if a Cournot candidate $\mathbf{x}$ is socially optimal for the original model, then the optimal social welfare in the modified model could be zero, in which case $\gamma(\mathbf{x}) = 1$, but $\overline{\gamma}(\mathbf{x})$ is undefined; see the example that follows.

EXAMPLE 1. Consider a model involving two suppliers ($N = 2$). The cost function of supplier n is $C_n(x) = x^2$, for $n = 1, 2$. The inverse demand function is constant, with $p(q) = 1$ for any $q \geq 0$. It is not hard to see that the vector $(1/2, 1/2)$ is a Cournot candidate, which is also socially optimal. In the modified model, we have $\overline{C}_n(x) = x$, for $n = 1, 2$. The optimal social welfare achieved in the modified model is zero. □

Note that even if $\mathbf{x}$ is a Cournot equilibrium in the original model, it need not be a Cournot equilibrium in the modified model with linear cost functions, as illustrated by our next example. On the other hand, Proposition 6 asserts that a Cournot candidate in the original model remains a Cournot candidate in the modified model. Hence, to lower bound the efficiency of a Cournot equilibrium in the original model, it suffices to lower bound the efficiency achieved at a worst Cournot candidate for a modified model. Accordingly, and for the purpose of deriving lower bounds, we can (and will) restrict to the case of linear cost functions, and study the worst case efficiency over all Cournot candidates.

EXAMPLE 2. Consider a model involving only one supplier ($N = 1$). The cost function of the supplier is $C_1(x) = x^2$. The inverse demand function is given by

$$p(q) = \begin{cases} -q + 4, & \text{if } 0 \leq q \leq 4/3, \\ \max\{0, -\dfrac{1}{5}(q - 4/3) + 8/3\}, & \text{if } 4/3 < q, \end{cases}$$

which is convex and satisfies Assumption 2. It can be verified that $x_1 = 1$ maximizes the supplier's profit and thus is a Cournot equilibrium in the original model. In the modified model, $\overline{C}_1(\cdot)$ is linear with a slope of 2; the supplier can maximize its profit at $x_1 = 7/3$. Therefore, in the modified model, $x_1 = 1$ remains a Cournot candidate, but not a Cournot equilibrium. □

3.4. Other properties of Cournot candidates

In this subsection, we collect a few useful and intuitive properties of Cournot candidates. We show that at a Cournot candidate there are two possibilities: either $p(X) > p(X^S)$ and $X < X^S$, or $p(X) = p(X^S)$ (Proposition 7); in the latter case, under the additional assumption that $p(\cdot)$ is convex, a Cournot candidate is socially optimal (Proposition 8). In either case, imperfect competition can never result in a price that is less than the socially optimal price.

PROPOSITION 7. *Suppose that Assumptions 1-4 hold. Let* $\mathbf{x}$ *and* $\mathbf{x}^S$ *be a Cournot candidate and an optimal solution to (1), respectively. If* $p(X) \neq p(X^S)$, *then* $p(X) > p(X^S)$ *and* $X < X^S$.

For the case where $p(X) = p(X^S)$, Proposition 7 does not provide any comparison between X and X^S. While one usually has $X < X^S$ (imperfect competition results in lower quantities), it is also possible that $X > X^S$, as in the following example.

EXAMPLE 3. Consider a model involving two suppliers ($N = 2$). The cost function of each supplier is linear, with slope equal to 1. The inverse demand function is convex, of the form

$$p(q) = \begin{cases} 2 - q, & \text{if } 0 \leq q \leq 1, \\ 1, & \text{if } 1 < q. \end{cases}$$

It is not hard to see that any nonnegative vector $\mathbf{x}^S$ that satisfies $x_1^S + x_2^S \geq 1$ is socially optimal; $x_1^S = x_2^S = 1/2$ is one such vector. On the other hand, it can be verified that $x_1 = x_2 = 1$ is a Cournot equilibrium. Hence, in this example, $2 = X > X^S = 1$. □

PROPOSITION 8. *Suppose that Assumptions 1-4 hold and that the inverse demand function is convex. Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and an optimal solution to (1), respectively. If $p(X) = p(X^S)$, then $p'(X) = 0$ and $\gamma(\mathbf{x}) = 1$.*

Proposition 2 shows that all social optima lead to a unique "socially optimal" price. Combining with Proposition 8, we conclude that if $p(\cdot)$ is convex, a Cournot candidate is socially optimal if and only if it results in the socially optimal price.

3.5. Concave inverse demand functions

In this section, we argue that the case of concave inverse demand functions is fundamentally different. For this reason, the study of the concave case would require a very different line of analysis, and is not considered further in this paper.

According to Proposition 8, if the inverse demand function is convex and if the price at a Cournot equilibrium equals the price at a socially optimal point, then the Cournot equilibrium is socially optimal. For nonconvex inverse demand functions, this is not necessarily true: a socially optimal price can be associated with a socially suboptimal Cournot equilibrium, as demonstrated by the following example.

EXAMPLE 4. Consider a model involving two suppliers ($N = 2$), with $C_1(x) = x$ and $C_2(x) = x^2$. The inverse demand function is concave on the interval where it is positive, of the form

$$p(q) = \begin{cases} 1, & \text{if } 0 \leq q \leq 1, \\ \max\{0, -M(q-1)+1\}, & \text{if } 1 < q, \end{cases}$$

where $M > 2$. It is not hard to see that the vector $(0.5, 0.5)$ satisfies the optimality conditions in (2), and is therefore socially optimal. We now argue that $(1/M, 1 - 1/M)$ is a Cournot equilibrium. Given the action $x_2 = 1/M$ of supplier 2, any action on the interval $[0, 1 - 1/M]$ is a best response for supplier 1. Given the action $x_1 = 1 - (1/M)$ of supplier 1, a simple calculation shows that

$$\arg\max_{x \in [0,\infty)} \left\{ x \cdot p(x + 1 - 1/M) - x^2 \right\} = 1/M.$$

Hence, $(1/M, 1 - 1/M)$ is a Cournot equilibrium. Note that $X = X^S = 1$. However, the optimal social welfare is 0.25, while the social welfare achieved at the Cournot equilibrium is $1/M - 1/M^2$. By considering arbitrarily large M, the corresponding efficiency can be made arbitrarily small. $\square$

The preceding example shows that arbitrarily high efficiency losses are possible, even if $X = X^S$. The possibility of inefficient allocations even when the price is the correct one opens up the possibility of substantial inefficiencies that are hard to bound.

4. Affine Inverse Demand Functions

We now turn our attention to the special case of affine inverse demand functions. It is already known from Johari and Tsitsiklis (2005) that $2/3$ is a tight lower bound on the efficiency of Cournot equilibria. In this section, we refine this result by providing a tighter lower bound, based on a small amount of ex post information about a Cournot equilibrium.

Throughout this section, we assume an inverse demand function of the form

$$p(q) = \begin{cases} b - aq, & \text{if } 0 \le q \le b/a, \\ 0, & \text{if } b/a < q, \end{cases} \tag{7}$$

where a and b are positive constants.[2] Under the assumption of convex costs (Assumption 1), a Cournot equilibrium is guaranteed to exist, by Proposition 1.

The main result of this section follows.

THEOREM 1. *Suppose that Assumption 1 holds (convex cost functions), and that the inverse demand function is affine, of the form* (7).[3] *Suppose also that $b > \min_n\{C'_n(0)\}$ (Assumption 4). Let $\mathbf{x}$ be a Cournot equilibrium, and let $\alpha_n = C'_n(x_n)$. Let also*

$$\beta = \frac{aX}{b - \min_n\{\alpha_n\}},$$

If $X > b/a$, then $\mathbf{x}$ is socially optimal. Otherwise:

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[2] Note that the model considered here is slightly different from that in Johari and Tsitsiklis (2005). In that work, the inverse demand function is literally affine and approaches minus infinity as the total supply increases to infinity. However, as remarked in that paper (p. 20), this difference does not affect the results.

[3] Note that Assumptions 2 and 3 hold automatically.

(a) *We have $1/2 \leq \beta < 1$.*

(b) *The efficiency of $\mathbf{x}$ satisfies,*

$$\gamma(\mathbf{x}) \geq g(\beta) = 3\beta^2 - 4\beta + 2.$$

(c) *The bound in part (b) is tight. That is, for every $\beta \in [1/2, 1)$ and every $\epsilon > 0$, there exists a model with a Cournot equilibrium whose efficiency is no more than $g(\beta) + \epsilon$.*

(d) *The function $g(\beta)$ is minimized at $\beta = 2/3$ and the worst case efficiency is $2/3$.*

Theorem 1 is proved in Appendix B.1. The lower bound $g(\beta)$ is illustrated in Fig. 1. Consider a Cournot equilibrium such that $X \leq b/a$. For the special case where all the cost functions are linear, of the form $C_n(x_n) = \alpha_n$, Theorem 1 has an interesting interpretation. We first note that a socially optimal solution is obtained when the price $b - aq$ equals the marginal cost of a "best" supplier, namely $\min_n \alpha_n$. In particular, $X^S = (b - \min_n\{\alpha_n\})/a$, and $\beta = X/X^S$. Since $p'(X) = -a < 0$, Proposition 8 implies that $p(X) \neq p(X^S)$, and Proposition 7 implies that $\beta < 1$. Theorem 1 further states that $\beta \geq 1/2$. i.e., that the total supply at a Cournot equilibrium is at least half of the socially optimal supply. Clearly, if β is close to 1 we expect the efficiency loss due to the difference $X^S - X$ to be small. However, efficiency losses may also arise if the total supply at a Cournot equilibrium is not provided by the most efficient suppliers. (As shown in Example 4, in the nonconvex case this effect can be substantial.) Our result shows that, for the convex case, β can be used to lower bound the total efficiency loss due to this second factor as well; when β is close to 1, the efficiency indeed remains close to 1. (This is in sharp contrast to the nonconvex case where we can have $X = X^S$ but large efficiency losses.) Somewhat surprisingly, the worst case efficiency also tends to be somewhat better for low β, that is, when β approaches $1/2$, as compared to intermediate values ($\beta \approx 2/3$).

5. Convex Inverse Demand Functions

In this section we study the efficiency of Cournot equilibria under more general assumptions. Instead of restricting the inverse demand function to be affine, we will only assume that it is convex. A Cournot equilibrium need not exist in general, but it does exist under some conditions (cf. Section 2.1). Our results apply whenever a Cournot equilibrium happens to exist.
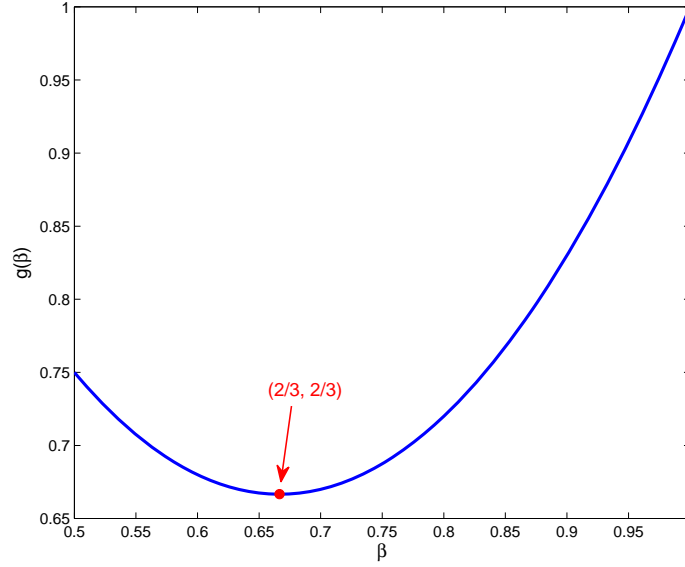
Figure 1 A tight lower bound on the efficiency of Cournot equilibria for the case of affine inverse demand functions.

We first show that a lower bound on the efficiency of a Cournot equilibrium can be established by calculating its efficiency in another model with a piecewise linear inverse demand function. Then, in Theorem 2, we establish a lower bound on the efficiency of Cournot equilibria, as a function of the ratio of the slope of the inverse demand function at the Cournot equilibrium to the average slope of the inverse demand function between the Cournot equilibrium and a socially optimal point. Then, in Section 6, we will apply Theorem 2 to specific convex inverse demand functions. Recall our definition of a Cournot candidate as a vector $\mathbf{x}$ that satisfies the necessary conditions (3)-(4).

PROPOSITION 9. *Suppose that Assumptions 1-4 hold, and that the inverse demand function is convex. Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and an optimal solution to (1), respectively. Assume that $p(X) \neq p(X^S)$ and let[4] $c = |p'(X)|$. Consider a modified model in which we replace the inverse demand function by a new function $p^0(\cdot)$, defined by*

$$
p^0(q) = \begin{cases} -c(q - X) + p(X), & \text{if} \quad 0 \leq q \leq X, \\ \max\left\{0, \dfrac{p(X^S) - p(X)}{X^S - X}(q - X) + p(X)\right\}, & \text{if} \quad X < q. \end{cases} \tag{8}
$$

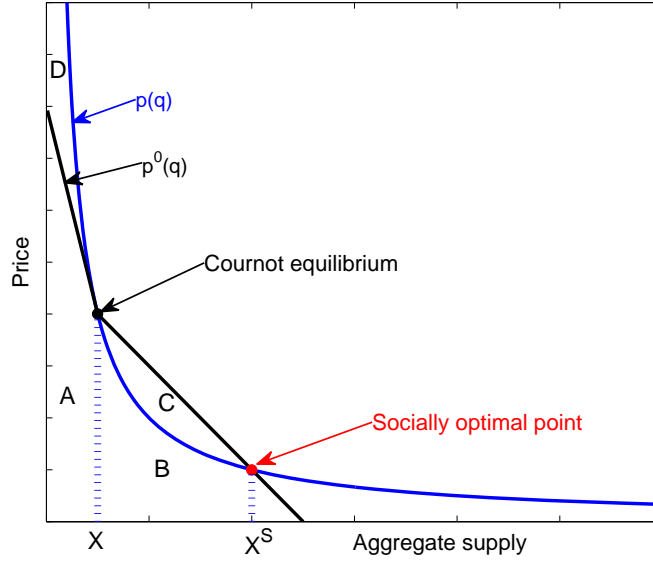[4] According to Proposition 3, $p'(X)$ must exist.

Figure 2   The efficiency of a Cournot equilibrium cannot increase if we replace the inverse demand function by the piecewise linear function $p^0(\cdot)$. The function $p^0(\cdot)$ is tangent to the inverse demand function $p(\cdot)$ at the equilibrium point, and connects the Cournot equilibrium point with the socially optimal point.

Then, for the modified model, with inverse demand function $p^0(\cdot)$, the vector $\mathbf{x}^S$ remains socially optimal, and the efficiency of $\mathbf{x}$, denoted by $\gamma^0(\mathbf{x})$, satisfies

$$\gamma^0(\mathbf{x}) \leq \gamma(\mathbf{x}).$$

Proof   Since $p(X) \neq p(X^S)$, Proposition 7 implies that $X < X^S$, so that $p^0(\cdot)$ is well defined. Since the necessary and sufficient optimality conditions in (2) only involve the value of the inverse demand function at X^S, which has been unchanged, the vector $\mathbf{x}^S$ remains socially optimal for the modified model.

Let

$$A = \int_0^X p^0(q)\,dq,\ B = \int_X^{X^S} p(q)\,dq,\ C = \int_X^{X^S} (p^0(q) - p(q))\,dq,\ D = \int_0^X (p(q) - p^0(q))\,dq.$$

See Fig. 2 for an illustration of $p(\cdot)$ and a graphical interpretation of A, B, C, D. Note that since $p(\cdot)$ is convex, we have $C \geq 0$ and $D \geq 0$. The efficiency of $\mathbf{x}$ in the original model with inverse demand function $p(\cdot)$, is

$$0 < \gamma(\mathbf{x}) = \frac{A + D - \sum_{n=1}^{N} C_n(x_n)}{A + B + D - \sum_{n=1}^{N} C_n(x_n^S)} \leq 1,$$

where the first inequality is true because the social welfare achieved at any Cournot candidate is positive (Proposition 5). The efficiency of $\mathbf{x}$ in the modified model is

$$\gamma^0(\mathbf{x}) = \frac{A - \sum_{n=1}^{N} C_n(x_n)}{A + B + C - \sum_{n=1}^{N} C_n(x_n^S)}.$$

Note that the denominators in the above formulas for $\gamma(\mathbf{x})$ and $\gamma^0(\mathbf{x})$ are all positive, by Proposition 5.

If $A - \sum_{n=1}^{N} C_n(x_n) \leq 0$, then $\gamma^0(\mathbf{x}) \leq 0$ and the result is clearly true. We can therefore assume that $A - \sum_{n=1}^{N} C_n(x_n) > 0$. We then have

$$0 < \gamma^0(\mathbf{x}) = \frac{A - \sum_{n=1}^{N} C_n(x_n)}{A + B + C - \sum_{n=1}^{N} C_n(x_n^S)} \leq \frac{A + D - \sum_{n=1}^{N} C_n(x_n)}{A + B + C + D - \sum_{n=1}^{N} C_n(x_n^S)}$$

$$\leq \frac{A + D - \sum_{n=1}^{N} C_n(x_n)}{A + B + D - \sum_{n=1}^{N} C_n(x_n^S)} = \gamma(\mathbf{x}) \leq 1,$$

which proves the desired result. $\quad \square$

Note that unless $p(\cdot)$ happens to be linear on the interval $[X, X^S]$, the function $p^0(\cdot)$ is not differentiable at X and, according to Proposition 3, $\mathbf{x}$ cannot be a Cournot candidate for the modified model. Nevertheless, $p^0(\cdot)$ can still be used to derive a lower bound on the efficiency of Cournot candidates in the original model, as will be seen in the proof of Theorem 2, which is our main result.

THEOREM 2. *Suppose that Assumptions 1-4 hold, and that the inverse demand function is convex. Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and a solution to (1), respectively. Then, the following hold.*

(a) If $p(X) = p(X^S)$, then $\gamma(\mathbf{x}) = 1$.

(b) If $p(X) \neq p(X^S)$, let $c = |p'(X)|$, $d = |(p(X^S) - p(X))/(X^S - X)|$, and $\bar{c} = c/d$. We have $\bar{c} \geq 1$ and

$$1 > \gamma(\mathbf{x}) \geq f(\bar{c}) = \frac{\phi^2 + 2}{\phi^2 + 2\phi + \bar{c}}, \tag{9}$$
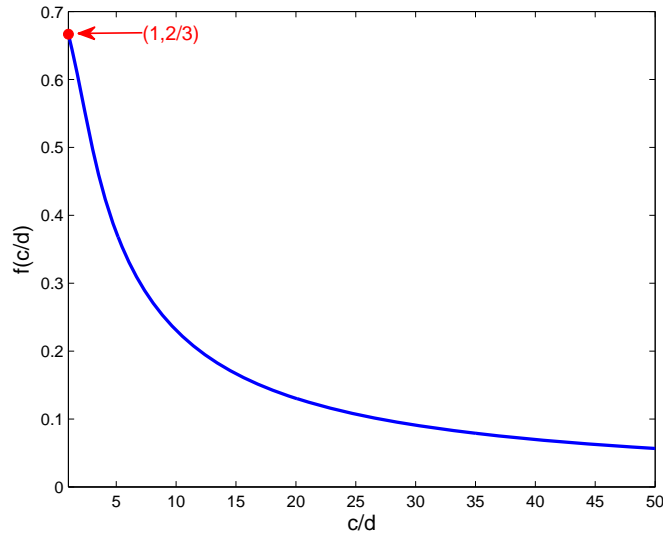
Figure 3 Plot of the lower bound on the efficiency of a Cournot equilibrium in a Cournot oligopoly with convex inverse demand functions, as a function of the ratio c/d (cf. Theorem 2).

where

$$\phi = \max \left\{ \frac{2 - \bar{c} + \sqrt{\bar{c}^2 - 4\bar{c} + 12}}{2}, 1 \right\}.$$

REMARK 1. We do not know whether the lower bound in Theorem 2 is tight. The difficulty in proving tightness is due to the fact that the vector **x** need not be a Cournot equilibrium in the modified model.

We provide the proof of Theorem 2 in Appendix B.2. Fig. 3 shows a plot of the lower bound $f(\bar{c})$ on the efficiency of Cournot equilibria, as a function of $\bar{c} = c/d$. If $p(\cdot)$ is affine, then $\bar{c} = c/d = 1$. From (9), it can be verified that $f(1) = 2/3$, which agrees with the lower bound in Johari and Tsitsiklis (2005) for the affine case. We note that the lower bound $f(\bar{c})$ is monotonically decreasing in $\bar{c}$, over the domain $[1, \infty)$. When $\bar{c} \in [1, 3)$, ϕ is at least 1, and monotonically decreasing in $\bar{c}$. When $\bar{c} \geq 3$, $\phi = 1$.

6. Applications

For a given inverse demand function $p(\cdot)$, the lower bound derived in Theorem 2 requires some knowledge on the Cournot candidate and the social optimum, namely, the aggregate supplies X

and X^S. Even so, for a large class of inverse demand functions, we can apply Theorem 2 to establish lower bounds on the efficiency of Cournot equilibria that do not require knowledge of X and X^S. With additional information on the suppliers' cost functions, the lower bounds can be further refined. At the end of this section, we apply our results to calculate nontrivial quantitative efficiency bounds for various convex inverse demand functions that have been considered in the economics literature.

COROLLARY 1. *Suppose that Assumptions 1-4 hold and that the inverse demand function is convex. Suppose also that $p(Q) = 0$ for some $Q > 0$, and that the ratio, $\mu = \partial_+ p(0)/\partial_- p(Q)$, is finite. Then, the efficiency of a Cournot candidate is at least $f(\mu)$.*

The proof of Corollary 1 can be found in Appendix B.3. For convex inverse demand functions, e.g., for negative exponential demand, with

$$p(q) = \max\{0, \alpha - \beta \log q\}, \qquad 0 < \alpha, \qquad 0 < \beta, \qquad 0 \leq q,$$

Corollary 1 does not apply, because the left derivative of $p(\cdot)$ at 0 is infinite. This motivates us to refine the lower bound in Corollary 1. By using a small amount of additional information on the cost functions, we can derive an upper bound on the total supply at a social optimum, as well as a lower bound on the total supply at a Cournot equilibrium, to strengthen Corollary 1.

COROLLARY 2. *Suppose that Assumptions 1-4 hold and that $p(\cdot)$ is convex. Let[5]*

$$s = \inf\{q \mid p(q) = \min_n C_n'(0)\}, \quad t = \inf\left\{q \mid \min_n C_n'(q) \geq p(q) + q\partial_+ p(q)\right\}. \tag{10}$$

If $\partial_- p(s) < 0$, then the efficiency of a Cournot candidate is at least $f\left(\partial_+ p(t)/\partial_- p(s)\right)$.

REMARK 2. If there exists a "best" supplier n such that $C_n'(x) \leq C_m'(x)$, for any other supplier m and any $x > 0$, then the parameters s and t depend only on $p(\cdot)$ and $C_n'(\cdot)$.

Corollary 2 is proved in Appendix B.4. We now apply Corollary 2 to three examples.

---

[5] Under Assumption 3, the existence of the real numbers defined in (10) is guaranteed.

EXAMPLE 5. Suppose that Assumptions 1, 3, and 4 hold, and that there is a best supplier, whose cost function is linear with a slope $c \geq 0$. Consider inverse demand functions of the form (cf. Eq. (6) in Bulow and Pfleiderer (1983))

$$p(q) = \max\{0, \alpha - \beta \log q\}, \qquad 0 < q, \tag{11}$$

where α and β are positive constants. Note that Corollary 1 does not apply, because the left derivative of $p(\cdot)$ at 0 is infinite.[6] Since

$$p'(q) + qp''(q) = \frac{-\beta}{q} + \frac{q\beta}{q^2} = 0, \qquad \forall q \in (0, \exp(\alpha/\beta)),$$

Proposition 1 implies that there exists a Cournot equilibrium. Through a simple calculation we obtain

$$s = \exp\left(\frac{\alpha - c}{\beta}\right), \qquad t = \exp\left(\frac{\alpha - \beta - c}{\beta}\right).$$

From Corollary 2 we obtain that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \geq f\left(\frac{\exp\left((\alpha - c)/\beta\right)}{\exp\left((\alpha - \beta - c)/\beta\right)}\right) = f\left(\exp\left(1\right)\right) \geq 0.5237. \tag{12}$$

Now we argue that the efficiency lower bound (12) holds even without the assumption that there is a best supplier associated with a linear cost function. From Proposition 6, the efficiency of any Cournot equilibrium $\mathbf{x}$ will not increase if the cost function of each supplier n is replaced by

$$\overline{C}_n(x) = C'_n(x_n)x, \qquad \forall x \geq 0.$$

Let $c = \min_n\{C'_n(x_n)\}$. Since the efficiency lower bound in (12) holds for the modified model with linear cost functions, it applies whenever the inverse demand function is of the form (11). $\square$

---

[6] In fact, $p(0)$ is undefined. This turns out to not be an issue: for a small enough $\epsilon > 0$, we can guarantee that no supplier chooses a quantity below ϵ. Furthermore, $\lim_{\epsilon \downarrow 0} \int_0^\epsilon p(q)\, dq = 0$. For this reason, the details of the inverse demand function in the vicinity of zero are immaterial as far as the chosen quantities or the resulting social welfare are concerned.

EXAMPLE 6. Suppose that Assumptions 1, 3, and 4 hold, and that there is a best supplier, whose cost function is linear with a slope $c \geq 0$. Consider inverse demand functions of the form (cf. Eq. (5) in Bulow and Pfleiderer (1983))

$$p(q) = \max\{\alpha - \beta q^\delta, 0\}, \qquad 0 < \delta \leq 1, \tag{13}$$

where α and β are positive constants. Note that if $\delta = 1$, then $p(\cdot)$ is affine; if $0 < \delta \leq 1$, then $p(\cdot)$ is convex. Assumption 4 implies that $\alpha > \chi$. Since

$$p'(q) + qp''(q) = -\beta\delta q^{\delta-1} - \beta\delta(\delta-1)q^{\delta-1} = -\beta\delta^2 q^{\delta-1} \leq 0, \ \ 0 \leq q \leq \left(\frac{\alpha}{\beta}\right)^{1/\delta},$$

Proposition 1 implies that there exists a Cournot equilibrium. Through a simple calculation we have

$$s = \left(\frac{\alpha-c}{\beta}\right)^{1/\delta}, \ \ t = \left(\frac{\alpha-c}{\beta(\delta+1)}\right)^{1/\delta}.$$

From Corollary 2 we know that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \geq f\left(\frac{-\beta\delta t^{\delta-1}}{-\beta\delta s^{\delta-1}}\right) = f\left((\delta+1)^{\frac{1-\delta}{\delta}}\right).$$

Using the argument in Example 5, we conclude that this lower bound also applies to the case of general convex cost functions. □

As we will see in the following example, it is sometimes hard to find a closed form expression for the real number t. In such cases, since s is an upper bound for the aggregate supply at a social optimum (cf. the proof of Corollary 2 in Appendix B.4), Corollary 2 implies that the efficiency of a Cournot candidate is at least $f\left(\partial_+ p(0)/\partial_- p(s)\right)$. Furthermore, in terms of the aggregate supply at a Cournot equilibrium X, we know that $\gamma(\mathbf{x}) \geq f\left(p'(X)/\partial_- p(s)\right)$.

EXAMPLE 7. Suppose that Assumptions 1, 3, and 4 hold, and that there is a best supplier, whose cost function is linear with a slope $c \geq 0$. Consider inverse demand functions of the form (cf. p. 8 in Fabingeryand and Weyl (2009))

$$p(q) = \begin{cases} \alpha(Q-q)^\beta, & 0 < q \leq Q, \\ 0, & Q < q, \end{cases} \tag{14}$$

where $Q > 0$, $\alpha > 0$ and $\beta \geq 1$. Assumption 4 implies that $c < \alpha Q^\beta$. Note that Corollary 1 does not apply, because the right derivative of $p(\cdot)$ at Q is zero. Through a simple calculation we obtain

$$s = Q - \left(\frac{c}{\alpha}\right)^{1/\beta},$$

and

$$p'(s) = \alpha\beta \left(\frac{c}{\alpha}\right)^{(\beta-1)/\beta}, \qquad \partial_+ p(0) = \alpha\beta Q^{\beta-1}.$$

Corollary 2 implies that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \geq f\left(\frac{\partial_+ p(0)}{p'(s)}\right) = f\left(\left(\frac{\alpha Q^\beta}{c}\right)^{(\beta-1)/\beta}\right) = f\left(\left(\frac{p(0)}{c}\right)^{(\beta-1)/\beta}\right).$$

Using information on the aggregate demand at the equilibrium, the efficiency bound can be refined. Since

$$p'(X) = \alpha\beta(Q - X)^{\beta-1},$$

we have

$$\gamma(\mathbf{x}) \geq f\left(\frac{p'(X)}{p'(s)}\right) = f\left(\left(\frac{\alpha(Q-X)^\beta}{c}\right)^{(\beta-1)/\beta}\right) = f\left(\left(\frac{p(X)}{c}\right)^{(\beta-1)/\beta}\right), \tag{15}$$

so that the efficiency bound depends only on the ratio of the equilibrium price to the marginal cost of the best supplier, and the parameter β. For affine inverse demand functions, we have $\beta = 1$ and the bound in (15) equals $f(1) = 2/3$, which agrees with Theorem 1. $\quad\square$

7. Monopoly and Social Welfare

In this section we study the special case where $N = 1$, so that we are dealing with a single, monopolistic, supplier. As we explain, this case also covers a setting where multiple suppliers collude to maximize their total profit. By using the additional assumption that $N = 1$, we obtain a sharper (i.e., larger) lower bound, in Theorem 3. We then establish lower bounds on the efficiency of monopoly outputs that do not require knowledge of X and X^S.

In a Cournot oligopoly, the maximum possible profit earned by all suppliers (if they collude) is an optimal solution to the following optimization problem,

$$\begin{aligned} \text{maximize } & p\left(\sum_{n=1}^N x_n\right) \cdot \sum_{n=1}^N x_n - \sum_{n=1}^N C_n(x_n) \\ \text{subject to } & x_n \geq 0, \quad n = 1, \ldots, N. \end{aligned} \tag{16}$$

We use $\mathbf{x}^P = (x_1^P, \ldots, x_N^P)$ to denote an optimal solution to (16) (a *monopoly output*), and let $X^P = \sum_{n=1}^{N} x_n^P$.

It is not hard to see that the aggregate supply at a monopoly output, X^P, is also a Cournot equilibrium in a modified model with a single supplier ($N=1$) and a cost function given by

$$\overline{C}(X) = \inf \sum_{n=1}^{N} C_n(x_n),$$

$$\text{subject to } x_n \geq 0, \quad n = 1, \ldots, N; \quad \sum_{n=1}^{N} x_n = X. \tag{17}$$

Note that $\overline{C}(\cdot)$ is convex (linear) when the $C_n(\cdot)$ are convex (respectively, linear). Furthermore, the social welfare at the monopoly output $\mathbf{x}^P$, is the same as that achieved at the Cournot equilibrium, $x_1 = X^P$, in the modified model. Also, the socially optimal value of X, as well as the resulting social welfare is the same for the N-supplier model and the above defined modified model with $N = 1$. Therefore, the efficiency of the monopoly output equals the efficiency of the Cournot equilibrium of the modified model. To lower bound the efficiency of monopoly outputs resulting from multiple colluding suppliers, we can (and will) restrict to the case with $N = 1$.

THEOREM 3. *Suppose that Assumptions 1-4 hold, and the inverse demand function is convex. Let* $\mathbf{x}^S$ *and* $\mathbf{x}^P$ *be a social optimum and a monopoly output, respectively. Then, the following hold.*

(a) If $p(X^P) = p(X^S)$, *then* $\gamma(\mathbf{x}^P) = 1$.

(b) If $p(X^P) \neq p(X^S)$, *let* $c = |p'(X^P)|$, $d = |(p(X^S) - p(X^P))/(X^S - X^P)|$, *and* $\bar{c} = c/d$. *We have* $\bar{c} \geq 1$ *and*

$$\gamma(\mathbf{x}^P) \geq \frac{3}{3 + \bar{c}}. \tag{18}$$

(c) The bound is tight at $\bar{c} = 1$, *i.e., there exists a model with* $\bar{c} = 1$ *and a monopoly output whose efficiency is* $3/4$.

The proof for Theorem 3 can be found in Appendix C.1. Fig. 4 compares the efficiency lower bounds established for Cournot equilibria with that for monopoly outputs. For $\bar{c} = 1$, both efficiency bounds are tight and it is possible for a monopoly output to achieve a higher efficiency than that of a Cournot equilibrium, as shown in the following example.
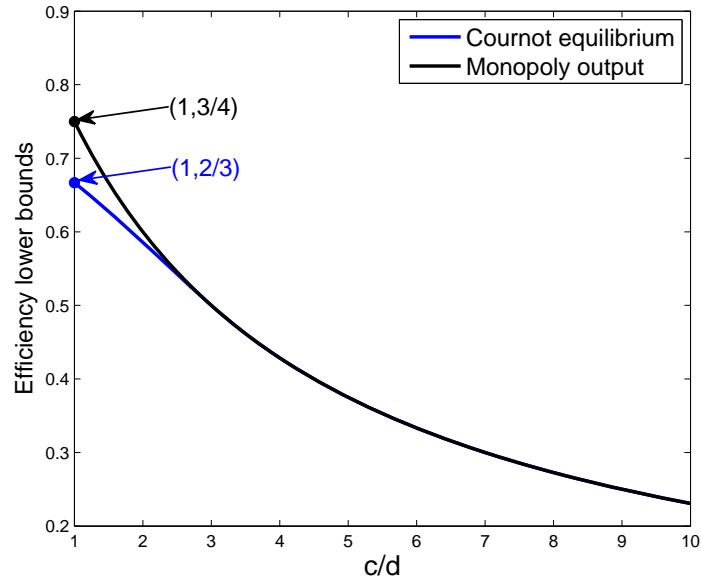
Figure 4 Comparison of the lower bounds on the efficiency of Cournot equilibria and monopoly outputs for the case of convex inverse demand functions.

EXAMPLE 8. Consider the model introduced in the proof of part (c) of Theorem 1. The inverse demand function is $p(q) = \max\{1 - q, 0\}$. The cost functions are linear:

$$C_1^N(x_1) = 0, \qquad C_n^N(x_n) = \left(1/3 - \frac{1/3}{N-1}\right) x_n, \qquad n = 2, \ldots, N.$$

If all suppliers collude to maximize the total profit, the output will be

$$x_1^P = 1/2, \qquad x_n^P = 0, \quad n = 2, \ldots, N,$$

and $\gamma(\mathbf{x}^P) = 3/4$. On the other hand, it can be verified that the vector

$$x_1 = 1/3, \qquad x_n = \frac{1/3}{N-1}, \quad n = 2, \ldots, N,$$

is a Cournot equilibrium. For any $N \geq 2$, a simple calculation shows that the associated efficiency is $(6N - 4)/(9N - 9)$. For example, in a model with $N = 10$, the efficiency of the Cournot equilibrium is less than that of the monopoly output, i.e., $\gamma(\mathbf{x}^P) = 3/4 > 56/81 = \gamma(\mathbf{x})$.   □

The above example agrees with the observation in earlier works that a monopoly output is not necessarily less efficient than an equilibrium resulting from imperfect competition (Comanor and Leibenstein 1969, Crew and Rowley 1977).

We now derive a result similar to Corollary 2, and then apply it to a numerical example with the same inverse demand function as in Example 5.

COROLLARY 3. *Suppose that Assumptions 1-4 hold, and that $p(\cdot)$ is convex. Let s and t be the real numbers defined in (10). If $\partial_- p(s) < 0$, then for any monopoly output $\mathbf{x}^P$, we have*

$$\gamma(\mathbf{x}^P) \geq \frac{\partial_- p(s)}{3\partial_- p(s) + \partial_+ p(t)}.$$

Corollary 3 is proved in Appendix C.2.

EXAMPLE 9. Suppose that Assumptions 1, 3, and 4 hold, and that there is a best supplier, whose cost function is linear with a slope $\chi \geq 0$. Consider inverse demand functions of the form in (11). Through a simple calculation we have

$$s = \exp\left(\frac{\alpha - \chi}{\beta}\right), \qquad t = \exp\left(\frac{\alpha - \beta - \chi}{\beta}\right),$$

and

$$\frac{p'(t)}{p'(s)} = \frac{\exp\left((\alpha - \chi)/\beta\right)}{\exp\left((\alpha - \beta - \chi)/\beta\right)} = \exp(1).$$

According to Corollary 3, for every monopoly output $\mathbf{x}^P$ we have,

$$\gamma(\mathbf{x}^P) \geq 3/(3 + \exp(1)) = 0.525. \tag{19}$$

Using the argument in Example 5, we conclude that this efficiency bound also applies to the case of nonlinear (convex) cost functions. $\square$

8. Conclusion

It is well known that Cournot oligopoly can yield arbitrarily high efficiency loss in general; for details, see Johari (2004). For Cournot oligopoly with convex market demand and cost functions, results such as those provided in Theorem 2 show that the efficiency loss of a Cournot equilibrium can be bounded away from zero by a function of a scalar parameter that captures quantitative properties of the inverse demand function. With additional information on the cost functions, the efficiency lower bounds can be further refined. Our results apply to various convex inverse demand functions that have been considered in the economics literature.

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Acknowledgments

Appendix A: Proofs of the results in Section 3

A.1. Proof of Proposition 2

Suppose not, in which case there exist two optimal solutions, $\mathbf{x}^S$ and $\overline{\mathbf{x}}^S$, such that $p(X^S) \neq p(\overline{X}^S)$. Without loss of generality, we assume that $p(X^S) > p(\overline{X}^S)$. Since $p(\cdot)$ is nonincreasing, we must have $X^S < \overline{X}^S$. For all n such that $\overline{x}_n^S > 0$, the optimality conditions (2) yield

$$C_n'(\overline{x}_n^S) = p(\overline{X}^S) < p(X^S) \leq C_n'(x_n^S).$$

Using the convexity of the cost functions, we obtain

$$\overline{x}_n^S < x_n^S, \qquad \text{if } \overline{x}_n^S > 0,$$

This contradicts the assumption that $X^S < \overline{X}^S$, and the desired result follows.

A.2. Proof of Proposition 3

Let $\mathbf{x}$ be a Cournot candidate with $X > 0$. The conditions (3)-(4) applied to some n with $x_n > 0$, imply that

$$p(X) + x_n \cdot \partial_- p(X) \geq p(X) + x_n \cdot \partial_+ p(X).$$

On the other hand, since $p(\cdot)$ is convex, we have $\partial_- p(X) \leq \partial_+ p(X)$. Hence, $\partial_- p(X) = \partial_+ p(X)$, as claimed.

A.3. Proof of Proposition 5

Using Assumption 4, we can choose some k such that $p(0) > C_k'(0)$. The right derivative with respect to x_k of the obejctive function in (1), evaluated at $\mathbf{x} = (0, 0, \ldots, 0)$, is $p(0) - C_k(0) > 0$. Hence the optimal value of the objective is strictly larger than the zero value obtained when $\mathbf{x} = (0, 0, \ldots, 0)$. Thus, the optimal social welfare is positive.

Now consider the social welfare achieved at a Cournot candidate $\mathbf{x} = (x_1, \ldots, x_N)$. Because of Assumption 4, Proposition 4 applies, and we have $X > 0$. For any supplier n such that $x_n > 0$, the necessary conditions (3) and the fact that $p(\cdot)$ is nonincreasing imply that $C_n'(x_n) \leq p(X)$. Suppose that $C_n'(x_n) = p(X)$ for every supplier n with $x_n > 0$. Then, the necessary conditions

(3)-(4) imply that $\mathbf{x}$ satisfies the sufficient optimality conditions in (2). Thus, $\mathbf{x}$ is socially optimal and the desired result follows.

Suppose now that there exists some supplier n with $x_n > 0$ and $C'_n(x_n) < p(X)$. Then,

$$\sum_{n=1}^{N} C'_n(x_n)x_n < Xp(X) \leq \int_0^X p(q)\,dq, \tag{20}$$

where the last inequality holds because the function $p(\cdot)$ is nonincreasing. Since for each n, $C_n(\cdot)$ is convex and nondecreasing, with $C_n(0) = 0$, we have

$$\sum_{n=1}^{N} C_n(x_n) \leq \sum_{n=1}^{N} C'_n(x_n)x_n < Xp(X) \leq \int_0^X p(q)\,dq. \tag{21}$$

Hence, the social welfare at the Cournot candidate, $\int_0^X p(q)\,dq - \sum_{n=1}^{N} C_n(x_n)$, is positive.

A.4. Proof of Proposition 6

We first observe that the vector $\mathbf{x}$ satisfies the necessary conditions (3)-(4) for the modified model. Hence, the vector $\mathbf{x}$ is a Cournot candidate for the modified model. It is also not hard to see that Assumptions 1 and 2 are satisfied by the modified model. Since $\alpha_n \geq C'_n(0)$ for every n, Assumption 3 also holds in the modified model.

We now show that Assumption 4 holds in the modified model, i.e., that $p(0) > \min_n\{\alpha_n\}$. Since the vector $\mathbf{x}$ is a Cournot candidate in the original model, Proposition 4 implies that $X > 0$, so that there exists some n for which $x_n > 0$. From the necessary condition (3) we have that $\alpha_n \leq p(X)$. Furthermore, if $\alpha_n = p(X)$, then $\partial_- p(X) = 0$, and the convexity of $p(\cdot)$ implies that $\partial_+ p(X) = 0$. Hence, the vector $\mathbf{x}$ satisfies the optimality condition (2), and thus is socially optimal in the original model. Under our assumption that $\mathbf{x}$ is not socially optimal in the original model, we conclude that $\alpha_n < p(X)$, which implies that Assumption 4 holds in the modified model.

Let $\mathbf{x}^S$ be an optimal solution to (1) in the original model. Since $\mathbf{x}^S$ satisfies the optimality conditions in (2) for the modified model, it remains a social optimum in the modified model. In the modified model, since Assumptions 1-4 hold, the efficiency of the vector $\mathbf{x}$ is well defined and given by

$$\overline{\gamma}(\mathbf{x}) = \frac{\int_0^X p(q)\,dq - \sum_{n=1}^{N} \alpha_n x_n}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^{N} \alpha_n x_n^S}. \tag{22}$$

Note that the denominator on the right-hand side of (22) is the optimal social welfare and the numerator is the social welfare achieved at the Cournot candidate **x**, in the modified model. Proposition 5 implies that both the denominator and the numerator on the right-hand side of (22) are positive. In particular, $\overline{\gamma}(\mathbf{x}) > 0$.

Since $C_n(\cdot)$ is convex, we have

$$C_n(x_n^S) - C_n(x_n) - \alpha_n(x_n^S - x_n) \geq 0, \qquad n = 1, \ldots, N.$$

Adding a nonnegative quantity to the denominator cannot increase the ratio and, therefore,

$$1 \geq \gamma(\mathbf{x}) = \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N C_n(x_n^S)} \geq \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \left(\alpha_n(x_n^S - x_n) + C_n(x_n)\right)} > 0. \quad (23)$$

Since $C_n(\cdot)$ is convex and nondecreasing, with $C_n(0) = 0$, we also have

$$\sum_{n=1}^N C_n(x_n) - \sum_{n=1}^N \alpha_n x_n \leq 0. \tag{24}$$

Since the right-hand side of (23) is in the interval $(0, 1]$, adding the left-hand side of Eq. (24) (a nonpositive quantity) to both the numerator and the denominator cannot increase the ratio, as long as the numerator remains nonnegative. The numerator remains indeed nonnegative because it becomes the same as the numerator in the expression (22) for $\overline{\gamma}(\mathbf{x})$. We obtain

$$
\begin{aligned}
\gamma(\mathbf{x}) &\geq \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \left(\alpha_n(x_n^S - x_n) + C_n(x_n)\right)} \\
&\geq \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n) + \left(\sum_{n=1}^N C_n(x_n) - \sum_{n=1}^N \alpha_n x_n\right)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \left(\alpha_n(x_n^S - x_n) + C_n(x_n)\right) + \left(\sum_{n=1}^N C_n(x_n) - \sum_{n=1}^N \alpha_n x_n\right)} \\
&= \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N \alpha_n x_n}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \alpha_n x_n^S} \\
&= \overline{\gamma}(\mathbf{x}).
\end{aligned}
$$

The desired result follows.

A.5. Proof of Proposition 7

By Assumption 4, $p(0) > \min_n \{C'_n(0)\}$. According to Proposition 4, we have $X > 0$. Since $p(\cdot)$ is nonincreasing, the conditions in (3) imply that

$$C'_n(x_n) \le p(X), \qquad \text{if } x_n > 0.$$

Suppose that $X \ge X^S$. Since the inverse demand function is nonincreasing and $p(X) \ne p(X^S)$, we have $p(X) < p(X^S)$ and $X > X^S$. For every supplier n with $x_n > 0$, we have

$$C'_n(x_n) \le p(X) < p(X^S) \le C'_n(x_n^S),$$

where the last inequality follows from (2). Since, $C_n(\cdot)$ is convex, the above inequality implies that

$$x_n < x_n^S, \qquad \text{if } x_n > 0,$$

from which we obtain $X < X^S$. Since we had assumed that $X \ge X^S$, we have a contradiction.

The preceding argument establishes that $X < X^S$. Since $p(\cdot)$ is nonincreasing and $p(X) \ne p(X^S)$, we must have $p(X) > p(X^S)$.

A.6. Proof of Proposition 8

Since Assumption 4 holds, Proposition 4 implies that $X > 0$. Since $p(\cdot)$ is convex, Proposition 3 shows that $p(\cdot)$ is differentiable at X and the necessary conditions in (5) are satisfied.

We will now prove that $p'(X) = 0$. Suppose not, in which case we have $p'(X) < 0$. For every n such that $x_n > 0$, from the convexity of $C_n(\cdot)$ and the conditions in (5), we have

$$C'_n(0) \le C'_n(x_n) < p(X) = p(X^S).$$

Then, the social optimality conditions (2) imply that $x_n^S > 0$. It follows that

$$C'_n(x_n) < p(X^S) = C'_n(x_n^S),$$

where the last equality follows from the optimality conditions in (2). Since $C_n(\cdot)$ is convex, we conclude that $x_n < x_n^S$. Since this is true for every n such that $x_n > 0$, we obtain $X < X^S$. Since

the function $p(\cdot)$ is nonincreasing, and we have $p'(X) < 0$ and $X < X^S$, we obtain $p(X) > p(X^S)$, which contradicts the assumption that $p(X) = p(X^S)$. The contradiction shows that $p'(X) = 0$.

Since $p'(X) = 0$ and the Cournot candidate $\mathbf{x}$ satisfies the necessary conditions in (5), it also satisfies the optimality conditions in (2). Hence, $\mathbf{x}$ is socially optimal and the desired result follows.

Appendix B: Proofs of the results in Sections 4-6

B.1. Proof of Theorem 1

We note that part (d) is an immediate consequence of the expression for $g(\beta)$, and we concentrate on the remaining parts. Since the inverse demand function is convex, Proposition 3 shows that any Cournot equilibrium satisfies the necessary conditions (5). If $X > b/a$, then $p(X) = p'(X) = 0$. In that case, the necessary conditions (5) imply the optimality conditions (2). We conclude that $\mathbf{x}$ is socially optimal.

We now assume that $X \le b/a$. Proposition 3 shows that $p'(X)$ exists, and thus $X < b/a$. Since $p'(X) = -a < 0$, Proposition 8 implies that $p(X) \ne p(X^S)$, for any social optimum $\mathbf{x}^S$. Hence, $\mathbf{x}$ is not socially optimal.

As discussed in Section 3.3, to derive a lower bound, it suffices to consider the case of linear cost functions, and obtain a lower bound on the worst case efficiency of Cournot candidates, that is, vectors that satisfy (3)-(4). We will therefore assume that $C_n(x_n) = \alpha_n x_n$ for every n. Without loss of generality, we also assume that $\alpha_1 = \min_n\{\alpha_n\}$. We consider separately the two cases where $\alpha_1 = 0$ or $\alpha_1 > 0$, respectively.

The case where $\alpha_1 = 0$

In this case, the socially optimal supply is $X^S = b/a$ and the optimal social welfare is

$$\int_0^{b/a} p(q)\, dq - 0 = \int_0^{b/a} (-ax + b)\, dx = \frac{b^2}{2a}.$$

Note also that $\beta = aX/b$.

Let $\mathbf{x}$ be a Cournot candidate. Suppose first that $x_1 = 0$. In that case, the necessary conditions $0 = \alpha_1 \ge p(X)$ imply that $p(X) = 0$. For $n \ne 1$, if $x_n > 0$, the necessary conditions yield $0 \le \alpha_n =$

$p(X) - x_n a = -x_n a$, which implies that $x_n = 0$ for all n. But then, $X = 0$, which contradicts the fact $p(X) = 0$. We conclude that $x_1 > 0$.

Since $x_1 > 0$, the necessary conditions (5) yield $0 = \alpha_1 = b - aX - ax_1$, so that

$$x_1 = -X + \frac{b}{a}. \tag{25}$$

In particular, $X < b/a = X^S$, and $\beta < 1$. Furthermore,

$$0 \leq \sum_{n=2}^{N} x_n = X - x_1 = 2X - \frac{b}{a},$$

from which we conclude that $\beta = aX/b \geq 1/2$.

Note that for $n = 1$ we have $\alpha_n x_n = 0$. For $n \neq 1$, whenever $x_n > 0$, we have $\alpha_n = p(X) - ax_n$, so that $\alpha_n x_n = (p(X) - ax_n)x_n$. The social welfare associated with $\mathbf{x}$ is

$$
\begin{aligned}
\int_0^X p(q)\, dq - \sum_{n=1}^{N} \alpha_n x_n &= bX - \frac{1}{2}aX^2 - \sum_{n=2}^{N}(p(X) - ax_n)x_n \\
&\geq bX - \frac{1}{2}aX^2 - p(X)\sum_{n=2}^{N} x_n \\
&= bX - \frac{1}{2}aX^2 - (b - aX)(X - x_1) \\
&= bX - \frac{1}{2}aX^2 - (b - aX)\left(2X - \frac{b}{a}\right) \\
&= \frac{3}{2}aX^2 + \frac{b^2}{a} - 2bX.
\end{aligned}
\tag{26}
$$

We divide by $b^2/2a$ (the optimal social welfare) and obtain

$$\gamma(X) \geq \frac{2a}{b^2}\left(\frac{3}{2}aX^2 + \frac{b^2}{a} - 2bX\right) = 3\beta^2 - 4\beta + 2.$$

This proves the claim in part (b) of the theorem.

Tightness

We observe that the lower bound on the social welfare associated with $\mathbf{x}$ made use, in Eq. (26), of the inequality $\sum_{n=2}^{N} x_n^2 \geq 0$. This inequality becomes an equality, asymptotically, if we let $N \to \infty$ and $x_n = O(1/N)$ for $n \neq 1$. This motivates the proof of tightness (part (c) of the theorem) given below.

We are given some $\beta \in [1/2, 1)$ and construct an N-supplier model ($N \geq 2$) with $a = b = 1$, and the following linear cost functions:

$$C_1^N(x_1) = 0, \qquad C_n^N(x_n) = \left(p(X) - \frac{2X - 1}{N - 1} \right) x_n, \qquad n = 2, \ldots, N.$$

It can be verified that the variables

$$x_1 = -X + b/a, \qquad x_n = \frac{2X - b/a}{N - 1}, \quad n = 2, \ldots, N,$$

form a Cournot equilibrium. A simple calculation (consistent with the intuition given earlier) shows that as N increases to infinity, the sum $\sum_{n=2}^{N} x_n^2$ goes to zero and the associated efficiency converges to $g(\beta)$.

The case where $\alpha_1 > 0$

We now consider the case where $\alpha_n > 0$ for every n. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\min_n \{\alpha_n\} = \alpha_1 = 1$. By Assumption 4, we have $b > 1$.

At the social optimum, we must have $p(X^S) = \alpha_1 = 1$ and thus $X^S = (b - 1)/a$. The optimal social welfare is

$$\frac{(b - 1)(b + 1)}{2a} - \frac{b - 1}{a} = \frac{(b - 1)^2}{2a}.$$

Note also that $\beta = aX/(b - 1)$.

Similar to the proof for the case where $\alpha_1 = 0$, we can show that $x_1 > 0$ and therefore $1 = p(X) - ax_1 = b - aX - ax_1$, so that

$$x_1 = -X + \frac{b - 1}{a} > 0,$$

which implies that $\beta < 1$. In particular,

$$X < \frac{b - 1}{a} = X^S.$$

Furthermore,

$$0 \le \sum_{n=2}^{N} x_n = X - x_1 = 2X - \frac{b-1}{a},$$

from which we conclude that $\beta = aX/(b-1) \ge 1/2$.

A calculation similar to the one for the case where $\alpha_1 = 0$ yields

$$
\begin{aligned}
\int_0^X p(q)\, dq - \sum_{n=1}^{N} \alpha_n x_n &= bX - \frac{1}{2}aX^2 - x_1 - \sum_{n=2}^{N}(p(X) - ax_n)x_n \\
&\ge bX - \frac{1}{2}aX^2 + X - \frac{b-1}{a} - p(X)\sum_{n=2}^{N} x_n \\
&= bX - \frac{1}{2}aX^2 + X - \frac{b-1}{a} - (b - aX)(X - x_1) \\
&= bX - \frac{1}{2}aX^2 + X - \frac{b-1}{a} - (b - aX)\left(2X - \frac{b-1}{a}\right) \\
&= \frac{3}{2}aX^2 + \frac{(b-1)^2}{a} - 2(b-1)X.
\end{aligned}
$$

After dividing with the value of the social welfare, we obtain $g(\beta)$, as desired.

B.2. Proof of Theorem 2

Let $\mathbf{x}$ be a Cournot candidate. According to Proposition 8, if $p(X) = p(X^S)$, then the efficiency of the Cournot candidate must equal one, which proves part (a). To prove part (b), we assume that $p(X) \ne p(X^S)$. By Proposition 2, the Cournot candidate $\mathbf{x}$ cannot be socially optimal, and, therefore, $\gamma(\mathbf{x}) < 1$.

We have shown in Proposition 6 that if all cost functions are replaced by linear ones, the vector $\mathbf{x}$ remains a Cournot candidate, and Assumptions 1-4 still hold. Further, the efficiency of $\mathbf{x}$ cannot increase after all cost functions are replaced by linear ones. Thus, to lower bound the worst case efficiency loss, it suffices to derive a lower bound for the efficiency of Cournot candidates for the case of linear cost functions. We therefore assume that $C_n(x_n) = \alpha_n x_n$ for each n. Without loss of generality, we further assume that $\alpha_1 = \min_n\{\alpha_n\}$. Note that, by Assumption 4, we have $p(0) > \alpha_1$. We will prove the theorem by considering separately the cases where $\alpha_1 = 0$ and $\alpha_1 > 0$.

We will rely on Proposition 9, according to which the efficiency of a Cournot candidate $\mathbf{x}$ is lower bounded by the efficiency $\gamma^0(\mathbf{x})$ of $\mathbf{x}$ in a model involving the piecewise linear and convex inverse

demand function function of the form in the definition of $p^0(\cdot)$. Note that since $p(X) \neq p(X^S)$, we have that $d > 0$. For conciseness, we let $y = p(X)$ throughout the proof.

The case $\alpha_1 = 0$

Let $\mathbf{x}$ be a Cournot candidate in the original model with linear cost functions and the inverse demand function $p(\cdot)$. By Proposition 3, $\mathbf{x}$ satisfies the necessary conditions (5), with respect to the original inverse demand function $p(\cdot)$. Suppose first that $x_1 = 0$. The second inequality in (5) implies that $p(X) = 0$. On the other hand, Assumption 4 and Proposition 4 imply that $X > 0$. Thus, there exists some n such that $x_n > 0$. The first equality in (5) yields,

$$0 \leq \alpha_n = p(X) + x_n p'(X) = x_n p'(X) \leq 0,$$

which implies that $p'(X) = 0$. Then, the vector $\mathbf{x}$ satisfies the optimality conditions in (2), and is thus socially optimal in the original model. This contradicts the fact that $p(X) \neq p(X^S)$ and shows that we must have $x_1 > 0$.

If $p'(X)$ were equal to zero, then the necessary conditions (5) would imply the optimality conditions (2), and $\mathbf{x}$ would be socially optimal in the original model. Hence, we must have $p'(X) < 0$ and $c > 0$. The first equality in (5) yields $y > 0$, $x_1 = y/c$, and $X \geq y/c$. We also have

$$0 \leq \sum_{n=2}^{N} x_n = X - \frac{y}{c}. \tag{27}$$

From Proposition 9, the efficiency $\gamma^0(\mathbf{x})$ of $\mathbf{x}$ in the modified model cannot be more than its efficiency $\gamma(\mathbf{x})$ in the original model. Hence, to prove the second part of the theorem, it suffices to show that $\gamma^0(\mathbf{x}) \geq f(\bar{c})$, for any Cournot candidate with $c/d = \bar{c}$.

The optimal social welfare in the modified model is

$$\int_0^\infty p^0(q)dq - 0 = \int_0^{X+y/d} p^0(q)dq - 0 = \frac{y^2}{2d} + \frac{(2y + cX)X}{2}. \tag{28}$$

Note that for $n = 1$ we have $\alpha_n x_n = 0$. For $n \geq 2$, whenever $x_n > 0$, from the first equality in (5) we have $\alpha_n = y - x_n c$ and $\alpha_n x_n = (y - x_n c)x_n$. Hence, in the modified model, the social welfare associated with $\mathbf{x}$ is

$$
\int_0^X p^0(q)dq - \sum_{n=1}^N \alpha_n x_n = \frac{(2y + cX)X}{2} - \sum_{n=2}^N (y - x_n c)x_n
$$
$$
\geq \frac{(2y + cX)X}{2} - y\sum_{n=2}^N x_n
$$
$$
= \frac{(2y + cX)X}{2} - y(X - y/c)
$$
$$
= cX^2/2 + y^2/c.
$$

Therefore,

$$
\gamma^0(\mathbf{x}) \geq \frac{cX^2/2 + y^2/c}{y^2/(2d) + (2y + cX)X/2}. \tag{29}
$$

Note that c, d, X, and y are all positive. Substituting $\bar{c} = c/d$ and $\bar{y} = cX/y$ in (29), we obtain

$$
\gamma^0(\mathbf{x}) \geq \frac{cX^2/2 + y^2/c}{y^2/(2d) + (2y + cX)X/2} = \frac{c^2X^2/2 + y^2}{y^2c/(2d) + cXy + c^2X^2/2} = \frac{\bar{y}^2 + 2}{\bar{c} + 2\bar{y} + \bar{y}^2}. \tag{30}
$$

We have shown earlier that $X \geq y/c$, so that $\bar{y} \geq 1$. On the interval $\bar{y} \in [1, \infty)$, the minimum value of the right hand side of (30) is attained at

$$
\bar{y} = \max\left\{ \frac{2 - \bar{c} + \sqrt{\bar{c}^2 - 4\bar{c} + 12}}{2}, 1 \right\} \triangleq \phi,
$$

and thus,

$$
\gamma^0(\mathbf{x}) \geq \frac{\phi^2 + 2}{\phi^2 + 2\phi + \bar{c}} = f(\bar{c}).
$$

The case $\alpha_1 > 0$

We now consider the case where $\alpha_n > 0$ for every n. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\min_n\{\alpha_n\} = \alpha_1 = 1$. Suppose first that $x_1 = 0$. The second inequality in (5) implies that $p(X) \leq 1$. Proposition 4 also implies that $X > 0$ so that there exists some n for which $x_n > 0$. The first equality in (5) yields,

$$
\alpha_n = p(X) + x_n p'(X) \leq p(X) \leq 1.
$$

Since $\alpha_n \geq 1$, we obtain $p(X) = 1$ and $p'(X) = 0$. Then, the vector $\mathbf{x}$ satisfies the optimality conditions in (2), and thus is socially optimal in the original model. But this would contradict the fact that $p(X) \neq p(X^S)$. We conclude that $x_1 > 0$.

If $p'(X)$ were equal to zero, then the necessary conditions (5) would imply the optimality conditions (2), and $\mathbf{x}$ would be socially optimal in the modified game. Therefore, we must have $p'(X) < 0$ and $c > 0$. The first equality in (5) yields $y > 1$, $x_1 = (y-1)/c$, and $X \geq (y-1)/c$. We also have

$$0 \leq \sum_{n=2}^{N} x_n = X - \frac{y-1}{c}, \tag{31}$$

from which we conclude that $X \geq (y-1)/c$.

From Proposition 9, the efficiency $\gamma^0(\mathbf{x})$ of the vector $\mathbf{x}$ in the modified model cannot be more than its efficiency $\gamma(\mathbf{x})$ in the original model. So, it suffices to consider the efficiency of $\mathbf{x}$ in the modified model. From the optimality conditions (2), we have that $p^0(X^S) = 1$, and thus, using the definition of d,

$$X^S = X + \frac{y-1}{d}.$$

The optimal social welfare in the modified model is

$$\int_0^{X^S} p^0(q)\,dq - X^S = \frac{y^2 - 1}{2d} + \frac{(2y + cX)X}{2} - X - \frac{y-1}{d} = \frac{(y-1)^2}{2d} + X(y-1) + \frac{cX^2}{2}.$$

Note that for $n = 1$ we have $\alpha_n x_n = x_1$. For $n \geq 2$ and whenever $x_n > 0$, from the first equality in (5) we have $\alpha_n = y - x_n c$ and $\alpha_n x_n = (y - x_n c)x_n$. Hence, in the modified model, the social welfare associated with $\mathbf{x}$ is

$$\begin{aligned}
\int_0^X p^0(q)\,dq - \sum_{n=1}^N \alpha_n x_n &= Xy + cX^2/2 - x_1 - \sum_{n=2}^N (y - x_n c)x_n \\
&\geq Xy + cX^2/2 - x_1 - y\sum_{n=2}^N x_n \\
&= X(y-1) + cX^2/2 - (y-1)\sum_{n=2}^N x_n \\
&= X(y-1) + cX^2/2 - (y-1)(X - (y-1)/c) \\
&= cX^2/2 + (y-1)^2/c.
\end{aligned}$$

Therefore,

$$\gamma^0(\mathbf{x}) \geq \frac{cX^2/2 + (y-1)^2/c}{(y-1)^2/(2d) + X(y-1) + cX^2/2}. \tag{32}$$

Note that c, d, X, and $y - 1$ are all positive. Substituting $\bar{c} = c/d$ and $\bar{y} = (cX)/(y-1)$ in (32), we obtain

$$\gamma^0(\mathbf{x}) \geq \frac{2\bar{y}^2 + 1}{\bar{c}\,\bar{y}^2 + 2\bar{y} + 1}. \tag{33}$$

From (31) we have that $\bar{y} \geq 1$. On the interval $\bar{y} \in [1, \infty)$, the minimum value of the right hand side of (33) is attained at

$$\bar{y} = \min\left\{ \frac{2 - \bar{c} + \sqrt{\bar{c}^2 - 4\bar{c} + 12}}{2}, 1 \right\} \triangleq \phi,$$

and thus,

$$\gamma^0(\mathbf{x}) \geq \frac{\phi^2 + 2}{\phi^2 + 2\phi + \bar{c}} = f(\bar{c}).$$

B.3. Proof of Corollary 1

Let $\mathbf{x}$ be a Cournot candidate. Since the inverse demand function is convex, we have that $\mu \geq 1$. If $X > Q$, then $p(X) = p'(X) = 0$. The necessary conditions (3)-(4) imply the optimality condition in (2), and thus $\gamma(\mathbf{x}) = 1 > f(\mu)$.

Now consider the case $X \leq Q$. If $p(X) = p(X^S)$ for some social optimum $\mathbf{x}^S$, then Proposition 8 implies that $\gamma(\mathbf{x}) = 1 > f(\mu)$. Otherwise, for any social optimum $\mathbf{x}^S$, we have that $\bar{c} = c/d \leq \mu$. Theorem 2 shows that the efficiency of every Cournot candidate cannot be less than $f(\bar{c})$. The desired result then follows from the fact that $f(\bar{c})$ is decreasing in $\bar{c}$.

B.4. Proof of Corollary 2

Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and a social optimum, respectively. If $p(X) = p(X^S)$, for some social optimum $\mathbf{x}^S$, then $\gamma(\mathbf{x}) = 1$ and the desired result holds trivially. Now suppose that $p(X) \neq p(X^S)$. We first derive an upper bound on the aggregate supply at a social optimum, and then establish a lower bound on the aggregate supply at a Cournot candidate. The desired results will follow from the fact that the function $f(\cdot)$ is strictly decreasing.

Step 1: There exists a social optimum with an aggregate supply no more than s.

According to Proposition 5 we have $X^S > 0$ and there exists a supplier n such that $x_n^S > 0$. From

the optimality conditions (2) we have $p(X^S) = C'_n(x^S_n)$, which implies that $p(X^S) \geq C'_n(0)$, due to the convexity of the cost functions. We conclude that

$$p(X^S) = C'_n(x^S_n) \geq C'_n(0) \geq \min_n C'_n(0). \tag{34}$$

If $p(X^S) > \min_n C'_n(0)$, then from the definition of s in (10), and the assumption that $p(\cdot)$ is nonincreasing, we have that $X^S < s$.

If $p(X^S) = \min_n C'_n(0)$, by (34) we know that for any n such that $x^S_n > 0$, we must have $C'_n(x^S_n) = C'_n(0) = p(X^S)$. Since $C_n(\cdot)$ is convex, we conclude that $C_n(\cdot)$ is actually linear on the interval $[0, x^S_n]$. We now argue that there exists a social optimum $\mathbf{x}^S$ such that $X^S \leq s$. If $X^S \leq s$, then we are done. Otherwise, we have $X^S > s$. Let $\mathcal{N}$ be the set of the indices of suppliers who produce a positive quantity at $\mathbf{x}^S$. Since $p(\cdot)$ is nonincreasing, and $p(s) = \min_n C'_n(0) = p(X^S)$, we know that for any $q \in [s, X^S]$, $p(q) = C'_n(0)$ for every $n \in \mathcal{N}$. Combing with the fact that for each supplier n in the set $\mathcal{N}$, $C_n(\cdot)$ is linear on the interval $[0, x^S_n]$, we have

$$\int_s^{X^S} p(q) \, dq = (X^S - s) C'_n(x), \qquad \forall n \in \mathcal{N}, \ \ \forall x \in [0, x^S_n),$$

from which we conclude that the vector, $(s/X^S) \cdot \mathbf{x}^S$, yields the same social welfare as $\mathbf{x}^S$, and thus is socially optimal. Note that the aggregate supply at $(s/X^S) \cdot \mathbf{x}^S$ is s.

If $p(X) = p(X^S)$, then $\gamma(\mathbf{x}) = 1$ and the desired result holds trivially. Otherwise, since $p(\cdot)$ is nonincreasing and convex, we have

$$|\partial_- p(s)| \leq \left|(p(X^S) - p(X))/(X^S - X)\right| = d. \tag{35}$$

Step 2: The aggregate supply at a Cournot candidate $\mathbf{x}$ is at least t.

Since $p(\cdot)$ is convex, $\mathbf{x}$ satisfies the necessary conditions in (5). Therefore,

$$C'_n(x_n) \geq p(X) + x_n p'(X), \qquad \forall n. \tag{36}$$

Since $X \geq x_n$, we have

$$C'_n(x_n) \leq C'_n(X), \qquad X p'(X) \leq x_n p'(X), \tag{37}$$

where the first inequality follows from the convexity of the cost functions, and the second one is true because $p'(X) \leq 0$. Combing (36) and (37), we have

$$C_n'(X) \geq p(X) + X p'(X), \qquad \forall n,$$

which implies that $X \geq t$. Since $p(\cdot)$ is nonincreasing and convex, we have

$$c = |p'(X)| \leq |\partial_+ p(t)|. \tag{38}$$

Since $\partial_- p(s) < 0$, Eqs. (35) and (38) yield

$$\bar{c} = c/d \leq \partial_+ p(t)/\partial_- p(s).$$

The desired result follows from Theorem 2, and the fact that $f(\cdot)$ is strictly decreasing.

Appendix C: Proofs of the results in Section 7

C.1. Proof of Theorem 3

According to the discussion in Section 7, we only need to lower bound the efficiency of a Cournot equilibrium $\mathbf{x}$ in a model with $N = 1$. Since $N = 1$, we can identify the vectors $\mathbf{x}$ and $\mathbf{x}^S$ with the scalars X and X^S. If $p(X) = p(X^S)$, then according to Proposition 8, the efficiency of the Cournot equilibrium, X, must equal one, which establishes part (a).

We now turn to the proof of part (b), and we assume that $p(X) \neq p(X^S)$. According to Proposition 2, we know that $\mathbf{x}$ cannot be socially optimal. We will consider separately the cases where $\alpha_1 = 0$ and $\alpha_1 > 0$.

We will again rely on Proposition 9, according to which the efficiency of a Cournot candidate $\mathbf{x}$ is lower bounded by the efficiency $\gamma^0(\mathbf{x})$ of $\mathbf{x}$ in a model involving the piecewise linear and convex inverse demand function $p^0(\cdot)$. Note that since $p(X) \neq p(X^S)$, we have that $d > 0$. As shown in the proof of Theorem 2, we have $p'(X) < 0$, i.e., $c > 0$. For conciseness, we let $y = p(X)$ throughout the proof.

The case $\alpha_1 = 0$

Applying conditions (5) to the supplier we have $X = y/c$. From Proposition 9, it suffices to show that $\gamma^0(\mathbf{x}) \geq 3/(3 + \bar{c})$. The optimal social welfare in the modified model is

$$\int_0^\infty p^0(q)\,dq - 0 = \int_0^{X+y/d} p^0(q)\,dq - 0 = \frac{y^2}{2d} + \frac{(2y+cX)X}{2}. \tag{39}$$

In the modified model, the social welfare associated with $\mathbf{x}$ is

$$\int_0^{X^P} p^0(q)\,dq - 0 = \frac{(2y+cX)X}{2}.$$

Therefore,

$$\gamma^0(\mathbf{x}) = \frac{(2y+cX)X/2}{y^2/(2d) + (2y+cX)X/2} = \frac{3}{3+\bar{c}},$$

where the last equality is true because $xc = y$.

Tightness

Consider the model introduced in the proof of part (c) of Theorem 1. The inverse demand function is $p(q) = \max\{1 - q, 0\}$. The supplier's cost function is identically zero, i.e., $C_1(x_1) = 0$. The profit maximizing output is $x_1 = 1/2$. We observe that $\gamma(\mathbf{x}) = 3/4$.

The case $\alpha_1 > 0$

We now consider the case where $\alpha_1 > 0$. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\alpha_1 = 1$. Applying conditions (5) to the supplier, we obtain $X = (y-1)/c$.

According to Proposition 9, it suffices to show that the efficiency of $\mathbf{x}$ in the modified model, $\gamma^0(\mathbf{x})$, is at least $3/(3 + \bar{c})$. From the optimality conditions (2) we have that $p^0(X^S) = 1$, and therefore,

$$X^S = X + (y-1)/d.$$

The optimal social welfare achieved in the modified model is

$$\int_0^{X^S} p^0(q)\,dq - X^S = \frac{y^2 - 1}{2d} + \frac{(2y+cX)X}{2} - X - \frac{y-1}{d} = \frac{(y-1)^2}{2d} + X^P(y-1) + \frac{c(X)^2}{2}.$$

In the modified model, the social welfare associated with $\mathbf{x}$ is

$$\int_0^X p^0(q)\, dq - X = X(y-1) + \frac{cX^2}{2}.$$

Since $cX = y - 1$, we have

$$\gamma^0(\mathbf{x}) = \frac{3}{3+\bar{c}}.$$

C.2. Proof of Corollary 3

Note that the efficiency of a monopoly output equals the efficiency of a Cournot equilibrium in a modified model with $N = 1$. Therefore, the desired result follows from the proof of Corollary 2, except that the general lower bound $f(\cdot)$ is replaced by the tighter one, $3/(3+\bar{c})$, provided by Theorem 3.