

# The First-order Logical Environment

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**Abstract.** This paper describes the first-order logical environment **FOLE**. Institutions in general (Goguen and Burstall [4]), and logical environments in particular, give equivalent heterogeneous and homogeneous representations for logical systems. As such, they offer a rigorous and principled approach to distributed interoperable information systems via system consequence (Kent [6]). Since **FOLE** is a particular logical environment, this provides a rigorous and principled approach to distributed interoperable first-order information systems. The **FOLE** represents the formalism and semantics of first-order logic in a classification form. By using an interpretation form, a companion approach (Kent [7]) defines the formalism and semantics of first-order logical/relational database systems. In a strict sense, the two forms have transformational passages (generalized inverses) between one another. The classification form of first-order logic in the **FOLE** corresponds to ideas discussed in the Information Flow Framework (IFF [12]). The **FOLE** representation follows a conceptual structures approach, that is completely compatible with formal concept analysis (Ganter and Wille [2]) and information flow (Barwise and Seligman [1]).

**Keywords:** schema, specification, structure, logical environment.

## 1 Introduction

The paper “System Consequence” (Kent [6]) gave a general and abstract solution to the interoperation of information systems via the channel theory of information flow (Barwise and Seligman [1]). These can be expressed either formally, semantically or in a combined form. This general solution closely follows the theories of institutions (Goguen and Burstall [4]), <sup>1</sup> information flow and formal concept analysis (Ganter and Wille [2]). By following the approach of the “System Consequence” paper, this paper offers a solution to the interoperation of distributed systems expressed in terms of the formalism and semantics of first-order logic. It does this by defining **FOLE**, the first-order logical environment. <sup>2</sup> Since this paper develops a classification form of first order logic as a logical environment, the interaction of information systems expressed in first order logic

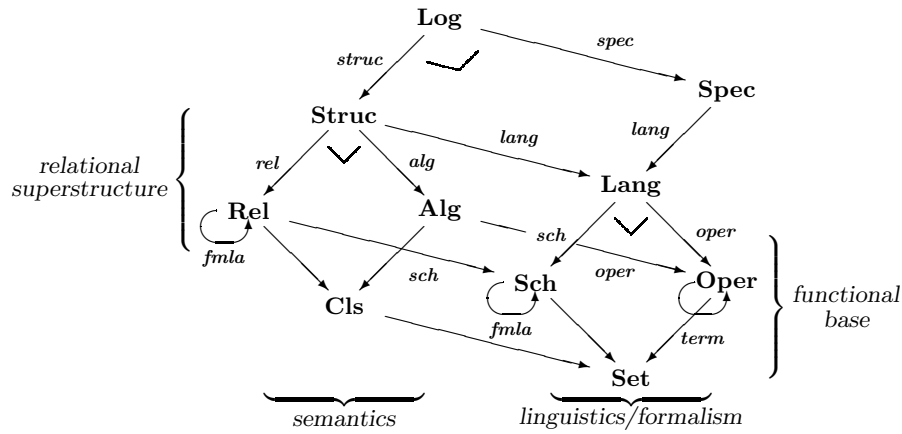
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<sup>1</sup> The technical aspect of this paper is described in the spirit of Goguen’s categorical manifesto [3] by using the terminology of mathematical context, passage and bridge in place of category, functor and natural transformation.

<sup>2</sup> A logical environment is a special and more structurally pleasing case of an institution, where the semantics is completely compatible with satisfaction.

have a firm foundation. Section 2 surveys the architecture of the first-order logical environment **FOLE**. Section 3 discusses the linguistic/formal and semantic components of **FOLE**; detailed discussions of the functional base and relational superstructure are given in Appendix A.1 and Appendix A.2, respectively. Section 4 explains how **FOLE** is a logical environment; a proof of this fact is given in Appendix A.4. Section 5 discusses **FOLE** information systems. Finally, section 6 summarizes and states future plans for work on these topics.

## 2 Architecture

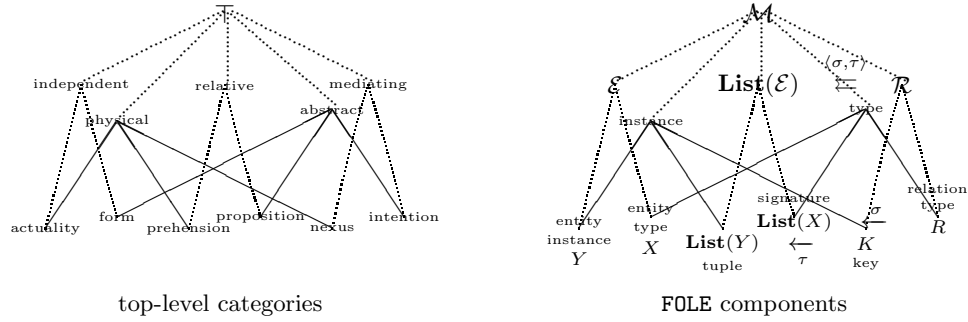


**Fig. 1.** FOLE Fibered Architecture

Figure 1 is a 3-dimensional visualization of the fibered architecture of the first-order logical environment **FOLE**. Each node of this figure is a mathematical context, whereas each edge is a passage between two contexts. There is a projection from the 2-D prism below **Struc** representing the relational superstructure (subsec. A.2) to the 2-D prism below **Alg** representing the functional base (subsec. A.1). The front diamond below **Lang** represents the linguistics/formalism, whereas the back diamond below **Struc** represents the semantics. The projective passages from semantics to linguistics/formalism represent the fibration left-to-right and the indexing right-to-left. The vee-shape at the top of each diamond states that the top mathematical context is a product of the side contexts modulo the bottom context. The mathematical contexts on the left side of each diamond form the relational aspect, whereas the mathematical contexts on the right side form the functional aspect that lifts the relational to the (first-order) logical aspect. The 2-D prism below **Log** represents the institutional architecture.

### 3 Components

The architectural components (Fig.1) divide up according to kind and aspect. The outer level describes the kind of component. The indexing kind is a language (type set, relational schema, operator domain, etc.) (front diamond Fig.1), whereas the indexed kind is either a formalism or a semantics (classification, relational structure, algebra, etc.) (back diamond Fig.1). The inner level describes the aspect of component. There are basic, relational, functional and logical aspects (bottom, left, right or top node in either Fig.1 diamond).



**Fig. 2.** Analogy

Fig.2 illustrates an analogy between the top-level ontological categories discussed in (Sowa [9]) and the components of the first-order logical environment FOLE (the relational aspect or 2-D prism below **Rel**). The pair ‘physical-abstract’, which corresponds to the Heraclitus distinction *physis-logos*, is represented in the FOLE by a classification between instances and types of various kinds. The triples (triads) ‘actuality-prehension-nexus’ and ‘form-proposition-intention’ correspond to Whitehead’s categories of existence. The latter triple, which is analogous to the ‘entity type-signature-relation type’ triple, is represented in the FOLE by a relational language (schema)  $\mathcal{S} = \langle R, \sigma, X \rangle$  (Appendix A.2.1). The former triple, which is analogous to the ‘entity instance-tuple-relation instance’ triple, is represented in the FOLE by the tuple function  $K \xrightarrow{\tau} \mathbf{List}(Y)$  (part of a FOLE structure). The firstness category of ‘independent(actuality,form)’ is represented in the FOLE by an entity classification  $\mathcal{E} = \langle X, Y, \models_{\mathcal{E}} \rangle$  (Appendix A.2.2). The thirdness category of ‘mediating(nexus,intention)’ is represented in the FOLE by a relation classification  $\mathcal{R} = \langle R, K, \models_{\mathcal{R}} \rangle$  between relational instances (keys) and relational types (or a classification between relational instances and logical formula, more generally) (Appendix A.2.2). The secondness category of ‘relative(prehension,proposition)’ is represented in the FOLE by the list construction of an entity classification  $\mathbf{List}(\mathcal{E}) = \langle \mathbf{List}(X), \mathbf{List}(Y), \models_{\mathbf{List}(\mathcal{E})} \rangle$  between tuples and signatures (Appendix A.2.2). Finally, the entire graph of the top-level

ontological categories is represented in the **FOLE** by a (model-theoretic) structure (classification form)  $\mathcal{M} = \langle \mathcal{R}, \langle \sigma, \tau \rangle, \mathcal{E} \rangle$ , where the relation  $\mathcal{R}$  and entity  $\mathcal{E}$  classifications are connected by a list designation  $\langle \sigma, \tau \rangle : \mathcal{R} \rightrightarrows \mathbf{List}(\mathcal{E})$  (Appendix A.2.2). This is appropriate, since a (model-theoretic) structure represents the knowledge in the local world of a community of discourse.

## 4 Logical Environment

The **FOLE** institution (logical system) (Kent [6]) has at its core the mathematical context of first-order logic (FOL) languages **Lang**. For any language  $\mathcal{L} = \langle \mathcal{S}, \mathcal{O} \rangle$ , there is a set of constraints  $\mathbf{fm}la(\mathcal{L})$  representing the formalism at location  $\mathcal{L}$ , and there is a mathematical context of structures  $\mathbf{struc}(\mathcal{L})$  representing the semantics at location  $\mathcal{L}$ . For any first-order logic (FOL) language morphism  $\mathcal{L}_2 = \langle \mathcal{S}_2, \mathcal{O}_2 \rangle \xrightarrow{\langle r, f, \omega \rangle} \langle \mathcal{S}_1, \mathcal{O}_1 \rangle = \mathcal{L}_1$ , there is a constraint function  $\mathbf{fm}la(\mathcal{L}_2) \xrightarrow{\mathbf{fm}la(r, f, \omega)} \mathbf{fm}la(\mathcal{L}_1)$  (Appendix A.2.1) representing flow of formalism in the forward direction, and there is a structure passage  $\mathbf{struc}(\mathcal{L}_2) \xleftarrow{\mathbf{struc}(r, f, \omega)} \mathbf{struc}(\mathcal{L}_1)$  (Appendix A.2.2) representing flow of semantics in the reverse direction. This structure passage has a relational component  $\mathbf{Rel}(\mathcal{S}_2) \xleftarrow{\mathbf{rel}(r, f)} \mathbf{Rel}(\mathcal{S}_1)$  and a functional (algebraic) component  $\mathbf{Alg}(\mathcal{O}_2) \xleftarrow{\mathbf{alg}(f, \omega)} \mathbf{Alg}(\mathcal{O}_1)$ .

**FOLE** is an institution, since the satisfaction relation is preserved during information flow along any first-order logic (FOL) language morphism  $\mathcal{L}_2 = \langle \mathcal{S}_2, \mathcal{O}_2 \rangle \xrightarrow{\langle r, f, \omega \rangle} \langle \mathcal{S}_1, \mathcal{O}_1 \rangle = \mathcal{L}_1$ :  $\mathbf{struc}(r, f, \omega)(\mathcal{M}_1) \models_{\mathcal{L}_2} (\langle I'_2, s'_2, \varphi'_2 \rangle \xrightarrow{h_2} \langle I_2, s_2, \varphi_2 \rangle)$  iff  $\mathcal{M}_1 \models_{\mathcal{L}_1} \mathbf{fm}la(\langle I'_2, s'_2, \varphi'_2 \rangle \xrightarrow{h_2} \langle I_2, s_2, \varphi_2 \rangle)$ . In short, “satisfaction is invariant under change of notation”. The institution **FOLE** is a logical environment, since for any language  $\mathcal{L} = \langle \mathcal{S}, \mathcal{O} \rangle = \langle \mathcal{R}, \sigma, X, \Omega \rangle$ , if  $\mathcal{M}_2 \xrightarrow{\langle k, g, h \rangle} \mathcal{M}_1$  is a **lang**-vertical structure morphism over  $\mathcal{L}$ , then we have the intent order  $\mathcal{M}_2 \geq_{\mathcal{L}} \mathcal{M}_1$ ; that is,  $\mathcal{M}_2 \models_{\mathcal{L}} (\varphi \vdash \psi)$  implies  $\mathcal{M}_1 \models_{\mathcal{L}} (\varphi \vdash \psi)$  for any  $\mathcal{S}$ -sequent  $(\varphi \vdash \psi)$ . In short, “satisfaction respects structure morphisms”. (See Appendix A.4 for a proof of this in the relational aspect.)

## 5 Information Systems

Following the theory of general systems, an information system consists of a collection of interconnected parts called information resources and a collection of part-part relationships between pairs of information resources called constraints. Semantic information systems have logics<sup>3</sup> as their information resources. Just as every logic has an underlying structure, so also every information system has

<sup>3</sup> A first-order logic  $\mathcal{L} = \langle \mathcal{M}, \mathcal{T} \rangle$  in **FOLE** consists of a first-order structure  $\mathcal{M}$  and a first-order specification  $\mathcal{T}$  that share a common first-order language  $\mathbf{lang}(\mathcal{M}) = \mathbf{lang}(\mathcal{T})$ . A logic enriches a first-order structure with a specification. The logic is sound when the structure  $\mathcal{M}$  satisfies every constraint in the specification  $\mathcal{T}$ .

an underlying distributed system. As such, distributed systems have structures for their component parts.

A **FOLE** distributed system is a passage  $\mathcal{M} : \mathbf{I} \rightarrow \mathbf{Struc}$  pictured as a diagram of shape  $\mathbf{I}$  within the ambient mathematical context of first-order structures. As such, it consists of an indexed family  $\{\mathcal{M}_i \mid i \in |\mathbf{I}|\}$  of structures together with an indexed family  $\{\mathcal{M}_i \xrightarrow{m_e} \mathcal{M}_j \mid (e : i \rightarrow j) \in \mathbf{I}\}$  of structure morphisms. A **FOLE** (semantic) information system is a diagram  $\mathcal{L} : \mathbf{I} \rightarrow \mathbf{Log}$  within the mathematical context of first-order logics. This consists of an indexed family of logics  $\{\mathcal{L}_i : i \in |\mathbf{I}|\}$  and an indexed family of logic morphisms  $\{\mathcal{L}_i \xrightarrow{l_e} \mathcal{L}_j \mid (e : i \rightarrow j) \in \mathbf{I}\}$ . An information system  $\mathcal{L}$  has an underlying distributed system  $\mathcal{M} = \mathcal{L} \circ \mathbf{struc}$  of the same shape with  $\mathcal{M}_i = \mathbf{struc}(\mathcal{L}_i)$  for all  $i \in |\mathbf{I}|$ . An information channel  $\langle \gamma : \mathcal{M} \Rightarrow \Delta(\mathcal{C}), \mathcal{C} \rangle$  consists of an indexed family  $\{\mathcal{M}_i \xrightarrow{\gamma_i} \mathcal{C} \mid i \in |\mathbf{I}|\}$  of structure morphisms with a common target structure  $\mathcal{C}$  called the core of the channel. Information flows along channels. We are mainly interested in channels that cover a distributed system  $\mathcal{M} : \mathbf{I} \rightarrow \mathbf{Struc}$ , where the part-whole relationships respect the system constraints (are consistent with the part-part relationships). In this case, there exist optimal channels. An optimal core is called the sum of the distributed system, and the optimal channel components (structure morphisms) are flow links.

System interoperability is defined by moving formalism over semantics. The fusion (unification)  $\coprod \mathcal{L}$  of the information system  $\mathcal{L}$  represents the whole system in a centralized fashion. The fusion logic is defined by direct system flow: (i) direct logic flow of the component parts of the information system along the optimal channel over the underlying distributed system to a centralized location (the mathematical context of structures at the optimal channel core), and (ii) product combining the contributions of the parts into a whole. The consequence  $\mathcal{L}^\bullet$  of the information system  $\mathcal{L}$  represents the whole system in a distributed fashion. This is an information system defined by inverse system flow: (i) consequence of the fusion logic, and (ii) inverse logic flow of this consequence back along the same optimal channel, transferring the constraints of the whole system (the fusion logic) to the distributed locations (structures) of the component parts. See Kent [6] for further details.<sup>4</sup>

## 6 Summary and Future Work

In this paper we have described the first-order logical environment **FOLE** in classification form. This gives a holistic treatment of first-order logic, by the use of several novel elements: the use of signatures (type lists) for relational arities, in place of ordinal numbers; the use of abstract tuples (relational instances, keys), thus making **FOLE** compatible with relational databases; the use of classifications for

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<sup>4</sup> In light of the transformation described in Appendix A.5.2, an information system of sound logics can be regarded as a system of logical/relational databases. The system consequence of such systems represents database interoperability. Kent [6] has more details about the information flow of sound logics in an arbitrary logical environment.

both entities and relations; and the use of relational constraints for the sentences of the FOLE institution. FOLE also has an interpretation form (Kent [7]) that represents the formalism and semantics of logical/relational databases, including relational algebra. There are transformational passages between the classification form and a strict version of the interpretation form. Appendix A.5.2 briefly discusses the transformation from sound logics to logical/relational databases.

FOLE has advantages over other approaches to first-order logic: in FOLE the formalism is completely integrated into the semantics; the classification form of FOLE has a natural extension to relational/logical databases, as represented by the interpretation form of FOLE; and FOLE is a logical environment, thus allowing practitioners a rigorously defined approach towards the interoperation of online semantic systems of information resources that include relational databases.

Future work includes: finishing work on the interpretation form of FOLE; further work on defining the transformational passages between the classification and interpretation forms; developing a linearization process from FOLE to sketch-like forms of logic such as Ologs (Spivak and Kent [11]); and linking FOLE with the Common Logic standard.

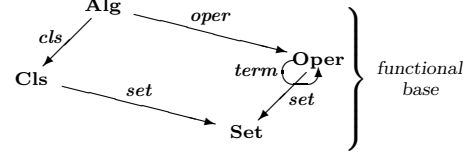
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## A Appendix

### A.1 Functional Base.

#### A.1.1 Linguistics/Formalism.



*Base Linguistics: Set.* A set (of entity types)  $X$  defines a mathematical context of type lists (signatures)  $\mathbf{List}(X) = (\mathbf{Set} \downarrow X)$ . The FOLE uses type lists for relational arities, instead of ordinal numbers.

The first subcomponent of any linguistic component is a set of entity types (sorts)  $X$ . Examples of entity types are ‘human’ representing the set of all human beings, ‘blue’ representing the set of all objects of color blue, etc. A type list (signature)  $\langle I, s \rangle$  consists of an arity set  $I$  and a type map  $I \xrightarrow{s} X$  mapping elements of the arity to entity types. This can be denoted by the list notation  $(\dots s_i \dots)$  or the type declaration notation  $(\dots i : s_i \dots)$  for  $i \in I$  and  $s_i \in X$ . For example, the type list ‘(make:String,model:String,year:Number,color:Color)’ is a type list for cars with valence 4, arity set  $\{\text{make}, \text{model}, \text{year}, \text{color}\}$ , and type map  $\{\text{make} \mapsto \mathbf{String}, \dots\}$ . A type list morphism  $\langle I_2, s_2 \rangle \xrightarrow{h} \langle I_1, s_1 \rangle$  is an arity function  $I_2 \xrightarrow{h} I_1$  that satisfies the commutative diagram  $h \cdot s_1 = s_2$ . We say that  $s_2$  is at least as general as  $s_1$ .

Given the natural numbers  $\mathbb{N} = \{0, 1, \dots\}$ , let  $\underline{\mathbb{N}}$  denote the mathematical context of finite ordinals (number sets)  $\underline{n} = \{0, 1, \dots, n-1\}$  and functions between them. This is the skeleton of the mathematical context  $\mathbf{Fin}$  of finite sets and functions. Both represent the single-sorted case where  $X = \mathbf{1}$ . We have the following inclusion of base language mathematical contexts.<sup>5</sup>

$$\begin{array}{ccccc} \underline{\mathbb{N}} & \subseteq & \mathbf{Fin} & \subseteq & \mathbf{List}^*(X) \\ \text{skeleton} & & \text{single-sorted} & & \text{many-sorted} \end{array}$$

Traditional first-order systems use the natural numbers  $\mathbb{N}$  for indexing relations. More flexible first-order systems, such as FOLE or relational database systems, use finite sets when single-sorted or type lists when many-sorted.

*Algebraic Linguistics: Oper  $\xrightarrow{set} \mathbf{Set}$ .* A functional language (operator domain) is a pair  $\langle X, \Omega \rangle$ , where  $X$  is a set of entity types (sorts) and  $\Omega$  is an  $X$ -sorted operator domain; that is,  $\Omega = \{\Omega_{x, \langle I, s \rangle} \mid x \in X, \langle I, s \rangle \in \mathbf{List}^*(X)\}$  is a collection of sets of function (operator) symbols, where  $e \in \Omega_{x, \langle I, s \rangle}$  is a function symbol of entity type (sort)  $x$  and finite arity  $\langle I, s \rangle$ ,<sup>6</sup> symbolized by  $x \xrightarrow{e} \langle I, s \rangle$ . An element  $c \in \Omega_{x, \langle \emptyset, 0_X \rangle}$  is called a constant symbol of sort  $x$ . Any operator domain  $\langle X, \Omega \rangle$  defines a mathematical context of terms  $\mathbf{Term}_{\langle X, \Omega \rangle}$ , whose objects are  $X$ -signatures  $\langle I, s \rangle$  and whose morphisms are term vectors  $\langle I', s' \rangle \xrightarrow{t} \langle I, s \rangle$ , where

<sup>5</sup> We use the mathematical context  $\mathbf{List}^*(X) = (\mathbf{Fin} \downarrow X)$  for type lists of finite arity.

<sup>6</sup> This is a slight misnomer, since  $\langle I, s \rangle$  is actually the signature of the function symbol. whereas the arity of  $e$  is the indexing set  $I$  and the valence of  $e$  is the cardinality  $|I|$ .

$t = \{s'_{i'} \xrightarrow{t_{i'}} \langle I, s \rangle \mid i' \in I'\}$  is an indexed collection (vector) of  $\langle I, s \rangle$ -ary terms. Terms and term vectors are defined by mutual induction.

A morphism of functional languages is a pair  $\langle X_2, \Omega_2 \rangle \xrightarrow{\langle f, \omega \rangle} \langle X_1, \Omega_1 \rangle$ , where  $X_2 \xrightarrow{f} X_1$  is a function of entity types (sorts) and  $\omega : \Omega_2 \rightarrow \Omega_1$  is a collection  $\{(\Omega_2)_{x_2, \langle I_2, s_2 \rangle} \xrightarrow{\omega_{x_2, \langle I_2, s_2 \rangle}} (\Omega_1)_{f(x_2), \Sigma_f(I_2, s_2)} \mid x_2 \in X_2, \langle I_2, s_2 \rangle \in \mathbf{List}^*(X_2)\}$  of maps between function symbol sets:  $\omega$  maps a function symbol  $x_2 \xrightarrow{e} \langle I_2, s_2 \rangle$  in  $\Omega_2$  to a function symbol  $f(x_2) \xrightarrow{\omega(e)} \Sigma_f(I_2, s_2) = \langle I_2, s_2 \cdot f \rangle$  in  $\Omega_1$ . Given any morphism of functional languages  $\langle X_2, \Omega_2 \rangle \xrightarrow{\langle f, \omega \rangle} \langle X_1, \Omega_1 \rangle$ , there is a term passage  $\mathbf{Term}_{\langle X_2, \Omega_2 \rangle} \xrightarrow{\mathbf{term}_{\langle f, \omega \rangle}} \mathbf{Term}_{\langle X_1, \Omega_1 \rangle}$  defined by induction. Let **Oper** denote the mathematical context of functional languages (operator domains).

Algebraic Formalism. Let  $\mathcal{O} = \langle X, \Omega \rangle$  be an operator domain. An  $\mathcal{O}$ -equation is a parallel pair of term vectors  $\langle I', s' \rangle \xrightarrow{t, t'} \langle I, s \rangle$ . We represent an equation using the traditional notation  $(t = t')$ . An equational presentation  $\langle X, \Omega, E \rangle$  consists of an operator domain  $\mathcal{O} = \langle X, \Omega \rangle$  and a set of  $\mathcal{O}$ -equations  $E$ . A congruence is any equational presentation closed under left and right term composition. Any equational presentation  $\langle X, \Omega, E \rangle$  generates a congruence  $\langle X, \Omega, E^\bullet \rangle$ , which defines a quotient mathematical context of terms  $\mathbf{Term}_{\langle X, \Omega, E \rangle}$  with a morphism  $\langle I', s' \rangle \xrightarrow{[t]} \langle I, s \rangle$  being an equivalence class of terms. There is a canonical passage  $\mathbf{Term}_{\langle X, \Omega \rangle} \xrightarrow{\parallel} \mathbf{Term}_{\langle X, \Omega, E \rangle}$ . A morphism of equational presentations  $\langle X_2, \Omega_2, E_2 \rangle \xrightarrow{\langle f, \omega \rangle} \langle X_1, \Omega_1, E_1 \rangle$  is a morphism of functional languages  $\langle X_2, \Omega_2 \rangle \xrightarrow{\langle f, \omega \rangle} \langle X_1, \Omega_1 \rangle$  that preserves equations: an  $\mathcal{O}_2$ -equation  $\langle I'_2, s'_2 \rangle \xrightarrow{t_2, t'_2} \langle I_2, s_2 \rangle$  in  $E_2$  is mapped to an  $\mathcal{O}_1$ -equation  $\Sigma_f(I'_2, s'_2) \xrightarrow{\omega^*(t), \omega^*(t')} \Sigma_f(I_2, s_2)$  in the congruence  $E_1^\bullet$ . Hence, there is a term passage  $\mathbf{Term}_{\langle X_2, \Omega_2, E_2 \rangle} \xrightarrow{\mathbf{term}_{\langle f, \omega \rangle}} \mathbf{Term}_{\langle X_1, \Omega_1, E_1 \rangle}$  that commutes with canons.

### A.1.2 Semantics.

Base Semantics:  $\mathbf{Cls} \xrightarrow{\text{typ}} \mathbf{Set}$ . For any entity classification  $\mathcal{E} = \langle X, Y, \models_{\mathcal{E}} \rangle$ , there is a tuple passage  $\mathbf{List}(X)^{\text{op}} \xrightarrow{\text{tup}_{\mathcal{E}}} \mathbf{Set}$  defined as the extent of the list classification  $\mathbf{List}(\mathcal{E})$ . It maps a type list (signature)  $\langle I, s \rangle \in \mathbf{List}(X)$  to its extent  $\text{tup}_{\mathcal{E}}(I, s) = \text{ext}_{\mathbf{List}(\mathcal{E})}(I, s) \subseteq \mathbf{List}(Y)$ . An entity infomorphism  $\langle f, g \rangle : \mathcal{E}_2 \rightrightarrows \mathcal{E}_1$  defines a bridge  $\text{tup}_{\mathcal{E}_2} \xleftarrow{\tau_{\langle f, g \rangle}} (\Sigma_f)^{\text{op}} \circ \text{tup}_{\mathcal{E}_1}$  between tuple passages. For any source signature  $\langle I_2, s_2 \rangle \in (\mathbf{Set} \downarrow X_2)$ , the tuple function  $\tau_{\langle f, g \rangle}(I_2, s_2) = (-) \cdot g : \text{tup}_{\mathcal{E}_1}(\Sigma_f(I_2, s_2)) \rightarrow \text{tup}_{\mathcal{E}_2}(I_2, s_2)$  is define by composition.

Algebraic Semantics:  $\mathbf{Cls} \xleftarrow{\text{cls}} \mathbf{Alg} \xrightarrow{\text{oper}} \mathbf{Oper}$ . A many-sorted algebra  $\mathcal{A} = \langle \mathcal{E}, \mathcal{O}, \langle A, \delta \rangle \rangle$  consists of an entity classification  $\mathcal{E} = \langle X, Y, \models_{\mathcal{E}} \rangle$ , an operator domain  $\mathcal{O} = \langle X, \Omega \rangle$ , and an  $\mathcal{O}$ -algebra  $\langle A, \delta \rangle$  compatible with  $\mathcal{E}$ , where  $A = \{A_x \mid$



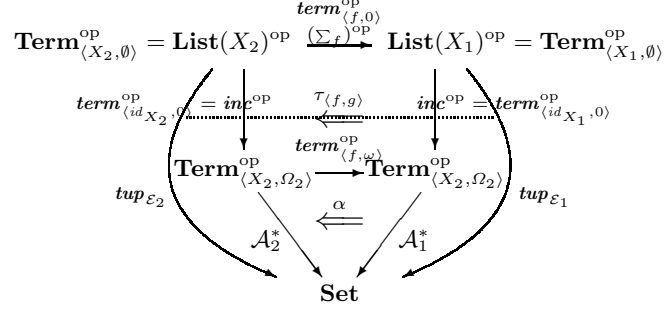


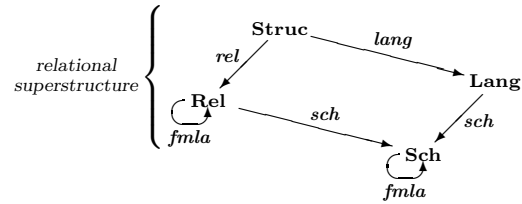
Fig. 3. Functional Base Interpretation

$x \in X$  is an  $X$ -sorted set and  $\delta$  assigns an  $\langle I, s \rangle$ -ary  $x$ -sorted function (operation)  $A_x \xleftarrow{\delta_e} A^{\langle I, s \rangle}$  to each function symbol  $x \xrightarrow{e} \langle I, s \rangle$  with  $A^{\langle I, s \rangle} = \prod_{i \in I} A_{s_i}$  the product set. A many-sorted algebra  $\mathcal{A} = \langle \mathcal{E}, \mathcal{O}, \langle A, \delta \rangle \rangle$  defines (by induction) an algebraic interpretation passage  $\mathbf{Term}_{\langle X, \Omega \rangle}^{op} \xrightarrow{\mathcal{A}^*} \mathbf{Set}$ , which extends the tuple passage  $\mathbf{tup}_{\mathcal{E}} = \mathbf{inc}^{op} \circ \mathcal{A}^*$  by compatibility. An algebra  $\mathcal{A}$  satisfies an equation  $(t = t')$ , symbolized by  $\mathcal{A} \models (t = t')$ , when the interpretation maps the terms to the same function  $\mathcal{A}^*(t) = \mathcal{A}^*(t')$ . A many-sorted algebraic homomorphism  $\mathcal{A}_2 = \langle \mathcal{E}_2, \mathcal{O}_2, \langle A_2, \delta_2 \rangle \rangle \xrightarrow{\langle f, g, \omega, h \rangle} \langle \mathcal{E}_1, \mathcal{O}_1, \langle A_1, \delta_1 \rangle \rangle = \mathcal{A}_1$  consists of an entity infomorphism  $\langle f, g \rangle : \mathcal{E}_2 \rightleftarrows \mathcal{E}_1$ , a morphism of many-sorted operator domains  $\langle f, \omega \rangle : \mathcal{O}_2 \rightarrow \mathcal{O}_1$ , and an  $\mathcal{O}_2$ -algebra morphism  $\langle A_2, \delta_2 \rangle \xleftarrow{h} \mathbf{alg}_{\langle f, \omega \rangle}(A_1, \delta_1)$  compatible with  $\langle f, g \rangle$ . A many-sorted algebraic homomorphism  $\mathcal{A}_2 \xrightarrow{\langle f, g, \omega, h \rangle} \mathcal{A}_1$  defines an algebraic bridge  $\mathcal{A}_2^* \xleftarrow{\alpha} \mathbf{term}_{\langle f, \omega \rangle}^{op} \circ \mathcal{A}_1^*$  between algebraic interpretations, which extends the tuple bridge  $\tau_{\langle f, g \rangle} = \mathbf{inc}^{op} \circ \alpha$  by compatibility. Let  $\mathbf{Alg}$  denote the mathematical context of many-sorted algebras. (The base semantics embeds into the functional semantics Fig. 3.)

## A.2 Relational Superstructure.

### A.2.1 Linguistics/Formalism.

Relational Linguistics: **Sch**.



*Schemas.* A relational language (schema)  $\mathcal{S} = \langle R, \sigma, X \rangle$  has two components: a base and a superstructure built upon the base. The base consists of a set of entity types (sorts)  $X$ , which defines the type list mathematical context  $\mathbf{List}(X)$ . The superstructure consists of a set of relation types (symbols)  $R$  and a (discrete) type list passage  $R \xrightarrow{\sigma} \mathbf{List}(X)$  mapping a relation symbol  $r \in R$  to its type list  $\sigma(r) = \langle I, s \rangle$ . A relational language (schema) morphism  $\mathcal{S}_2 = \langle R_2, \sigma_2, X_2 \rangle \xrightarrow{\langle r, f \rangle} \langle R_1, \sigma_1, X_1 \rangle = \mathcal{S}_1$  also has two components: a base and a superstructure built

upon the base. The base consists of an entity type (sort) function  $f : X_2 \rightarrow X_1$ , which defines the type list passage  $\mathbf{List}(X_2) \xrightarrow{\Sigma_f} \mathbf{List}(X_1)$  mapping a type list  $(\dots s_{i_2} \dots)$  to the type list  $(\dots f(s_{i_2}) \dots)$ . The superstructure consists of a relation type function  $r : R_2 \rightarrow R_1$  which preserves type lists, satisfying the condition  $r \cdot \sigma_1 = \sigma_2 \cdot \Sigma_f$ . Let **Sch** symbolize the mathematical context of relational languages (schemas) with type set projection passage  $\mathbf{Sch} \xrightarrow{set} \mathbf{Set}$ .

*Formulas.* For any type list  $\langle I, s \rangle$ , let  $R(I, s) \subseteq R$  denote the set of all relation types with this type list. These are called  $\langle I, s \rangle$ -ary relation symbols. Formulas form a schema  $\mathbf{fmla}(\mathcal{S}) = \langle \hat{R}, \hat{\sigma}, X \rangle$  that extends  $\mathcal{S}$ : with inductive definitions, the set of relation types is extended to a set of logical formulas  $\hat{R}$  and the relational type list function is extended to a type list function  $\hat{R} \xrightarrow{\hat{\sigma}} \mathbf{List}(X)$ . For any type list  $\langle I, s \rangle$ , let  $\hat{R}(I, s) \subseteq \hat{R}$  denote the set of all formulas with this type list. These are called  $\langle I, s \rangle$ -ary formulas. Formulas are constructed by using logical connectives within a fiber and logical flow between fibers.

- fiber:** Let  $\langle I, s \rangle$  be any type list. Any  $\langle I, s \rangle$ -ary relation symbol is an (atomic)  $\langle I, s \rangle$ -ary formula; that is,  $R(I, s) \subseteq \hat{R}(I, s)$ . For any pair of  $\langle I, s \rangle$ -ary formulas  $\varphi$  and  $\psi$ , there are the following  $\langle I, s \rangle$ -ary formulas: meet  $(\varphi \wedge \psi)$ , join  $(\varphi \vee \psi)$ , implication  $(\varphi \rightarrow \psi)$  and difference  $(\varphi \setminus \psi)$ . For any  $\langle I, s \rangle$ -ary formula  $\varphi$ , there is an  $\langle I, s \rangle$ -ary negation formula  $(\neg \varphi)$ .
- flow:** Let  $\langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle$  be any type list morphism. For any  $\langle I, s \rangle$ -ary formula  $\varphi$ , there are  $\langle I', s' \rangle$ -ary existentially/universally quantified formulas  $\Sigma_t(\varphi)$  and  $\Pi_t(\varphi)$ . For any  $\langle I', s' \rangle$ -ary formula  $\varphi'$ , there is a  $\langle I, s \rangle$ -ary substitution formula  $t^*(\varphi') = \varphi'(t)$ .

*Formula Fiber Passage.* A schema morphism  $\mathcal{S}_2 \xrightarrow{\langle r, f \rangle} \mathcal{S}_1$  can be extended to a formula schema morphism  $\mathbf{fmla}(r, f) = \langle \hat{r}, f \rangle : \mathbf{fmla}(\mathcal{S}_2) = \langle \hat{R}_2, \hat{\sigma}_2, X_2 \rangle \Rightarrow \langle \hat{R}_1, \hat{\sigma}_1, X_1 \rangle = \mathbf{fmla}(\mathcal{S}_1)$ . The formula function  $\hat{r} : \hat{R}_2 \rightarrow \hat{R}_1$ , which satisfies the condition  $inc_{\mathcal{S}_2} \cdot \hat{r} = r \cdot inc_{\mathcal{S}_1}$ , is recursively defined in Table 2.

**Proposition 1.** *There is an idempotent formula passage  $\mathbf{fmla} : \mathbf{Sch} \rightarrow \mathbf{Sch}$  that forms a monad  $\langle \mathbf{Sch}, \eta, \mathbf{fmla} \rangle$  with embedding.*

### Relational Formalism: **Fmla.**

*Constraints.* Let  $\mathcal{S} = \langle R, \sigma, X \rangle$  be a relational schema. A (binary)  $\mathcal{S}$ -sequent is a pair of formulas  $\varphi, \psi \in \hat{R}$  with the same type list  $\hat{\sigma}(\varphi) = \langle I, s \rangle = \hat{\sigma}(\psi)$ .<sup>7</sup> We represent a sequent using the turnstile notation  $\varphi \vdash \psi$ , since we want a sequent to assert logical entailment. A sequent expresses interpretation widening, with the interpretation of  $\varphi$  required to be within the interpretation of  $\psi$ . We require entailment to be a preorder, satisfying reflexivity and transitivity (Table 3). Hence,

<sup>7</sup> We regard the formulas  $\hat{R}$  to be a set of types. Since conjunction and disjunction are used in formulas, we can restrict attention to binary sequents.

$$\begin{array}{c}
\text{formula flow} \\
\text{logical aspect}
\end{array}
\left\{ \begin{array}{l}
\text{term vector} \quad \langle I', s' \rangle \xrightarrow{t} \langle I, s \rangle \quad \text{in } \mathbf{Term}_{\langle X, \Omega \rangle} \\
\text{operation} \quad \mathcal{A}^*(I', s') \xleftarrow{\mathcal{A}^*(t)} \mathcal{A}^*(I, s) \\
\text{inverse image} \quad \mathbf{Rel}_{\mathcal{A}}(I', s') \xrightarrow{t^*} \mathbf{Rel}_{\mathcal{A}}(I, s) \\
\text{quantification} \quad \mathbf{Rel}_{\mathcal{A}}(I', s') \xleftarrow[\forall_t]{\exists_t} \mathbf{Rel}_{\mathcal{A}}(I, s)
\end{array} \right.$$

$\Uparrow$  functional aspect

$$\begin{array}{c}
\text{formula flow} \\
\text{relational aspect}
\end{array}
\left\{ \begin{array}{l}
\text{type list morphism} \quad \langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle \quad \text{in } \mathbf{List}(X) = \mathbf{Term}_{\langle X, \emptyset \rangle} \\
\text{tuple map} \quad \mathbf{tup}_{\mathcal{E}}(I', s') \xleftarrow{\mathbf{tup}_{\mathcal{E}}(h)} \mathbf{tup}_{\mathcal{E}}(I, s) \\
\text{inverse image} \quad \mathbf{Rel}_{\mathcal{E}}(I', s') \xrightarrow{h^*} \mathbf{Rel}_{\mathcal{E}}(I, s) \\
\text{quantification} \quad \mathbf{Rel}_{\mathcal{E}}(I', s') \xleftarrow[\forall_h]{\exists_h} \mathbf{Rel}_{\mathcal{E}}(I, s)
\end{array} \right.$$

When the relational aspect is lifted along the functional aspect to the first-order aspect (Fig. 1 of Section 2), formula flow is lifted from being along type list morphisms  $\langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle$  to being along term vectors  $\langle I', s' \rangle \xrightarrow{t} \langle I, s \rangle$ . This holds for formula definition (above), formula function definition (Table 2), formula axiomatization (Table 3), formula classification definition (Table 4), satisfaction (Table 5), transformation to databases (Appendix A.5), etc.

**Table 1.** Lifting Flow

fiber: type list $\langle I_2, s_2 \rangle$			
<i>operator</i>			
relation	$\hat{r}(r_2)$	=	$r(r_2)$
meet	$\hat{r}(\varphi_2 \wedge_{\langle I_2, s_2 \rangle} \psi_2)$	=	$(\hat{r}(\varphi_2) \wedge_{\sum_f(I_2, s_2)} \hat{r}(\psi_2))$
join	$\hat{r}(\varphi_2 \vee_{\langle I_2, s_2 \rangle} \psi_2)$	=	$(\hat{r}(\varphi_2) \vee_{\sum_f(I_2, s_2)} \hat{r}(\psi_2))$
negation	$\hat{r}(\neg_{\langle I_2, s_2 \rangle} \varphi)$	=	$\neg_{\sum_f(I_2, s_2)} \hat{r}(\varphi)$
implication	$\hat{r}(\varphi \rightarrow_{\langle I_2, s_2 \rangle} \psi)$	=	$\hat{r}(\varphi) \rightarrow_{\sum_f(I_2, s_2)} \hat{r}(\psi)$
difference	$\hat{r}(\varphi \setminus_{\langle I_2, s_2 \rangle} \psi)$	=	$\hat{r}(\varphi) \setminus_{\sum_f(I_2, s_2)} \hat{r}(\psi)$
flow: type list morphism $\langle I'_2, s'_2 \rangle \xrightarrow{h} \langle I_2, s_2 \rangle$			
<i>operator</i>			
existential	$\hat{r}(\sum_h(\varphi_2))$	=	$\sum_h(\hat{r}(\varphi_2))$
universal	$\hat{r}(\Pi_h(\varphi_2))$	=	$\Pi_h(\hat{r}(\varphi_2))$
substitution	$\hat{r}(h^*(\varphi'_2))$	=	$h^*(\hat{r}(\varphi'_2))$

**Table 2.** Formula Function

for each type list  $\langle I, s \rangle$  there is a fiber preorder  $\mathbf{Fmla}_S(I, s) = \langle \widehat{R}, \vdash \rangle$  consisting of all  $\mathcal{S}$ -formulas with this type list. In first-order logic, we further require satisfaction of sufficient conditions (Table 3) to described the various logical operations (connectives, quantifiers, etc.) used to build formulas. An indexed  $\mathcal{S}$ -formula  $\langle I, s, \varphi \rangle$  consists of a type list  $\langle I, s \rangle$  and a formula  $\varphi$  with signature  $\langle I, s \rangle$ . An  $\mathcal{S}$ -constraint  $\langle I', s', \varphi' \rangle \xrightarrow{h} \langle I, s, \varphi \rangle$  consists of a type list morphism  $\langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle$  and a binary sequent  $(\sum_h(\varphi) \vdash \varphi')$ , or equivalently a binary sequent  $(\varphi \vdash h^*(\varphi'))$ . The mathematical context  $\mathbf{Fmla}(\mathcal{S})$  has indexed  $\mathcal{S}$ -formula as objects and  $\mathcal{S}$ -constraints as morphisms.<sup>8</sup> Let  $\mathcal{S}_2 \xrightarrow{\langle r, f \rangle} \mathcal{S}_1$  be a schema morphism. We assume that the function map  $\widehat{R}_2 \xrightarrow{\widehat{r}} \widehat{R}_1$  is monotonic (Table 3). Hence, there is a fibered formula passage  $\mathbf{Fmla}(\mathcal{S}_2) \xrightarrow{fmla_{\langle r, f \rangle}} \mathbf{Fmla}(\mathcal{S}_1)$  that commutes with the type list projections (Figure 4).

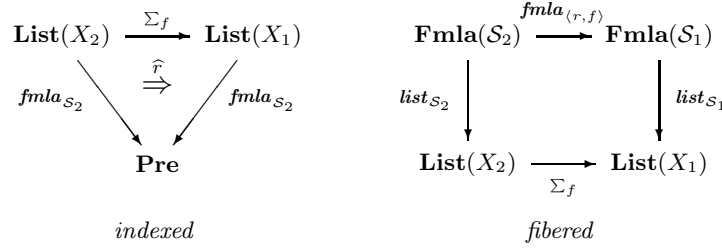


Fig. 4. Indexed-Fibered

*Specifications.* A specification  $\mathcal{T} = \langle \mathcal{S}, T \rangle$  consists of a schema  $\mathcal{S} = \langle R, \sigma, X \rangle$  and a subset  $T \subseteq \mathbf{Fmla}(\mathcal{S})$  of  $\mathcal{S}$ -constraints. As a subgraph,  $T$  extends to its consequence  $T^\bullet \subseteq \mathbf{Fmla}(\mathcal{S})$ , a mathematical subcontext, by using paths of constraints. A specification morphism  $\mathcal{T}_2 = \langle \mathcal{S}_2, T_2 \rangle \xrightarrow{\langle r, f \rangle} \langle \mathcal{S}_1, T_1 \rangle = \mathcal{T}_1$  is a schema morphism  $\mathcal{S}_2 \xrightarrow{\langle r, f \rangle} \mathcal{S}_1$  that preserves constraints: if sequent  $\varphi'_2 \vdash h^*(\varphi_2)$  is asserted in  $T_2$ , then sequent  $\widehat{r}(\varphi'_2) \vdash h^*(\widehat{r}(\varphi_2))$  is asserted in  $T_1$ .

*First-order Linguistics:*  $\mathbf{Lang} \xrightarrow[\mathbf{Sch} \times_{\mathbf{Set}} \mathbf{Oper}]{sch} \mathbf{Sch}$ . A first-order logic (FOL) language  $\mathcal{L} = \langle \mathcal{S}, \mathcal{O} \rangle$  consists of a relational schema  $\mathcal{S} = \langle R, \sigma, X \rangle$  and an operator domain  $\mathcal{O} = \langle X, \Omega \rangle$  that share a common type set  $X$ . A first-order logic (FOL) language morphism  $\mathcal{L}_2 = \langle \mathcal{S}_2, \mathcal{O}_2 \rangle \xrightarrow{\langle r, f, \omega \rangle} \langle \mathcal{S}_1, \mathcal{O}_1 \rangle = \mathcal{L}_1$  consists of a relational schema morphism  $\mathcal{S}_2 \xrightarrow{\langle r, f \rangle} \mathcal{S}_1$  and a functional language morphism  $\mathcal{O}_2 \xrightarrow{\langle f, \omega \rangle} \mathcal{O}_1$  that share a common type function  $X_2 \xrightarrow{f} X_1$ .

<sup>8</sup> In some sense, this formula/constraint approach to formalism turns the tuple calculus upside down, with atoms in the tuple calculus becoming constraints here.

schema: $\mathcal{S}$	
fiber: type list $\langle I, s \rangle$	
reflexivity :	$\varphi \vdash \varphi$
transitivity :	$\varphi \vdash \varphi'$ and $\varphi' \vdash \varphi''$ implies $\varphi \vdash \varphi''$
meet :	$\psi \vdash (\varphi \wedge \varphi')$ iff $\psi \vdash \varphi$ and $\psi \vdash \varphi'$ $(\varphi \wedge \varphi') \vdash \varphi, (\varphi \wedge \varphi') \vdash \varphi'$
join :	$(\varphi \vee \varphi') \vdash \psi$ iff $\varphi \vdash \psi$ and $\varphi' \vdash \psi$ $\varphi' \vdash (\varphi \vee \varphi), \varphi' \vdash (\varphi \vee \varphi')$
implication :	$(\varphi \wedge \varphi') \vdash \psi$ iff $\varphi \vdash (\varphi' \rightarrow \psi)$
negation :	$\neg(\neg(\varphi)) \vdash \varphi$
flow: type list morphism $\langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle$	
$\Sigma_h$ -monotonicity :	$\varphi' \vdash' \psi'$ implies $\Sigma_h(\varphi') \vdash \Sigma_h(\psi')$
$h^*$ -monotonicity :	$\varphi \vdash \psi$ implies $h^*(\varphi) \vdash' h^*(\psi)$
$\Pi_h$ -monotonicity :	$\varphi' \vdash' \psi'$ implies $\Pi_h(\varphi') \vdash \Pi_h(\psi')$
adjointness :	$\Sigma_h(\varphi') \vdash \psi$ iff $\varphi' \vdash' h^*(\psi)$ $\varphi' \vdash' h^*(\Sigma_h(\varphi')), \Sigma_h(h^*(\varphi)) \vdash \varphi$
schema morphism: $\mathcal{S}_2 \xrightarrow{\langle r, f \rangle} \mathcal{S}_1$	
$\widehat{r}$ -monotonicity :	$(\varphi_2 \vdash_2 \psi_2)$ implies $(\widehat{r}(\varphi_2) \vdash_1 \widehat{r}(\psi_2))$

Table 3. Axioms

First-order Formalism. A first-order specification  $\mathcal{T} = \langle \mathcal{S}, T, \mathcal{O}, E \rangle$  is an FOL language  $\mathcal{L} = \langle \mathcal{S}, \mathcal{O} \rangle$ , where  $\langle \mathcal{S}, T \rangle$  is a relational specification and  $\langle \mathcal{O}, E \rangle$  is an equational presentation. A first-order specification morphism  $\mathcal{T}_2 = \langle \mathcal{S}_2, T_2, \mathcal{O}_2, E_2 \rangle \xrightarrow{\langle r, f \rangle} \langle \mathcal{S}_1, T_1, \mathcal{O}_1, E_1 \rangle = \mathcal{T}_1$  is an FOL language morphism  $\mathcal{L}_2 = \langle \mathcal{S}_2, \mathcal{O}_2 \rangle \xrightarrow{\langle r, f, \omega \rangle} \langle \mathcal{S}_1, \mathcal{O}_1 \rangle = \mathcal{L}_1$ , where  $\langle \mathcal{S}_2, T_2 \rangle \xrightarrow{\langle r, f \rangle} \langle \mathcal{S}_1, T_1 \rangle$  is a relational specification morphism and  $\langle \mathcal{O}_2, E_2 \rangle \xrightarrow{\langle f, \omega \rangle} \langle \mathcal{O}_1, E_1 \rangle$  is a morphism of equational presentations. A first-order specification morphism preserves constraints: if sequent  $\varphi'_2 \vdash [t]^*(\varphi_2)$  is asserted in  $T_2$ , then sequent  $\widehat{r}(\varphi'_2) \vdash [t]^*(\widehat{r}(\varphi_2))$  is asserted in  $T_1$ .

### A.2.2 Semantics.

Relational Semantics:  $\mathbf{Rel} \xrightarrow{sch} \mathbf{Sch}$ .

*Structures.* A (model-theoretic) relational structure (classification form) (IFF [12])  $\mathcal{M} = \langle \mathcal{R}, \langle \sigma, \tau \rangle, \mathcal{E} \rangle$  is a hypergraph of classifications — a two dimensional construction consisting of a relation classification  $\mathcal{R} = \langle R, K, \models_{\mathcal{R}} \rangle$ , an entity classification  $\mathcal{E} = \langle X, Y, \models_{\mathcal{E}} \rangle$  and a list designation  $\langle \sigma, \tau \rangle : \mathcal{R} \rightrightarrows \mathbf{List}(\mathcal{E})$ .<sup>9</sup> Hence, a

<sup>9</sup>  $\mathbf{List}(\mathcal{E}) = \langle \mathbf{List}(X), \mathbf{List}(Y), \models_{\mathbf{List}(\mathcal{E})} \rangle$  is the list construction of the entity classification. A tuple  $\langle J, t \rangle \in \mathbf{List}(Y)$  is classified by a signature  $\langle I, s \rangle \in \mathbf{List}(X)$ , symbolized by  $\langle J, t \rangle \models_{\mathbf{List}(\mathcal{E})} \langle I, s \rangle$ , when  $J = I$  and  $t_i \models_{\mathcal{E}} s_i$  for all  $i \in I$ .

structure satisfies the following condition:  $k \models_{\mathcal{R}} r$  implies  $\tau(k) \models_{\text{List}(\mathcal{E})} \sigma(r)$ . A structure  $\mathcal{M}$  has an associated schema  $\mathbf{sch}(\mathcal{M}) = \langle R, \sigma, X \rangle$ .

*Formulas.* Any structure  $\mathcal{M} = \langle \mathcal{R}, \langle \sigma, \tau \rangle, \mathcal{E} \rangle$  has an associated formula structure  $\mathbf{fmla}(\mathcal{M}) = \langle \widehat{\mathcal{R}}, \langle \widehat{\sigma}, \tau \rangle, \mathcal{E} \rangle$  with schema  $\mathbf{sch}(\mathbf{fmla}(\mathcal{M})) = \langle \widehat{\mathcal{R}}, \widehat{\sigma}, X \rangle$ . The formula classification  $\widehat{\mathcal{R}} = \langle \widehat{R}, K, \models_{\widehat{\mathcal{R}}} \rangle$ , which extends the relation classification of  $\mathcal{M}$ , is directly defined by induction in Table 4.

fiber: type list  $\langle I, s \rangle$  with interpretation  $\mathbf{tup}_{\mathcal{E}}(I, s) = \prod_{i \in I} \mathbf{ext}_{\mathcal{E}}(s_i)$

operator	definiendum	definiens
relation	$k \models_{\widehat{\mathcal{R}}} r$	$k \models_{\mathcal{R}} r$
meet	$k \models_{\widehat{\mathcal{R}}} (\varphi \wedge \psi)$	$k \models_{\widehat{\mathcal{R}}} \varphi$ and $k \models_{\widehat{\mathcal{R}}} \psi$
join	$k \models_{\widehat{\mathcal{R}}} (\varphi \vee \psi)$	$k \models_{\widehat{\mathcal{R}}} \varphi$ or $k \models_{\widehat{\mathcal{R}}} \psi$
top	$k \models_{\widehat{\mathcal{R}}} \top$	
bottom	$k \not\models_{\widehat{\mathcal{R}}} \perp$	
negation	$k \models_{\widehat{\mathcal{R}}} (\neg \varphi)$	$k \not\models_{\widehat{\mathcal{R}}} \varphi$
implication	$k \models_{\widehat{\mathcal{R}}} (\varphi \rightarrow \psi)$	if $k \models_{\widehat{\mathcal{R}}} \varphi$ then $k \models_{\widehat{\mathcal{R}}} \psi$
difference	$k \models_{\widehat{\mathcal{R}}} (\varphi \setminus \psi)$	$k \models_{\widehat{\mathcal{R}}} \varphi$ but not $k \models_{\widehat{\mathcal{R}}} \psi$

flow: type list morphism  $\langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle$  with interpretation  $\mathbf{tup}_{\mathcal{E}}(I', s') \xleftarrow{\mathbf{tup}_{\mathcal{E}}(h)} \mathbf{tup}_{\mathcal{E}}(I, s)$

operator	definiendum	definiens
existential	$k \models_{\widehat{\mathcal{R}}} \Sigma_h(\varphi)$	$\tau(k) \in \exists_h(\mathbf{R}_{\widehat{\mathcal{M}}}(\varphi))$
universal	$k \models_{\widehat{\mathcal{R}}} \Pi_h(\varphi)$	$\tau(k) \in \forall_h(\mathbf{R}_{\widehat{\mathcal{M}}}(\varphi))$
substitution	$k \models_{\widehat{\mathcal{R}}} h^*(\varphi')$	$\tau(k) \in h^{-1}(\mathbf{R}_{\widehat{\mathcal{M}}}(\varphi'))$

where  $\mathbf{R}_{\widehat{\mathcal{M}}}(\varphi) = \wp \tau(\mathbf{ext}_{\widehat{\mathcal{R}}}(\varphi))$

Table 4. Formula Classification

*Satisfaction.* Satisfaction is defined in terms of the extent order of the formula classification. For any  $\mathcal{S}$ -structure  $\mathcal{M} \in \mathbf{Rel}(\mathcal{S})$ , two formula  $\varphi, \psi \in \widehat{R}$  with the same type list  $\sigma(\varphi) = \sigma(\psi)$  satisfy the specialization-generalization order  $\varphi \leq_{\widehat{\mathcal{R}}} \psi$  when their extents satisfy the containment order  $\mathbf{ext}_{\widehat{\mathcal{R}}}(\varphi) \subseteq \mathbf{ext}_{\widehat{\mathcal{R}}}(\psi)$ . An  $\mathcal{S}$ -structure  $\mathcal{M} \in \mathbf{Rel}(\mathcal{S})$  satisfies an  $\mathcal{S}$ -sequent  $(\varphi \vdash \psi)$  when  $\varphi \leq_{\widehat{\mathcal{R}}} \psi$ .

An  $\mathcal{S}$ -structure  $\mathcal{M} \in \mathbf{Rel}(\mathcal{S})$  satisfies an  $\mathcal{S}$ -constraint  $\varphi' \xrightarrow{h} \varphi$ , symbolized by  $\mathcal{M} \models_{\mathcal{S}} (\varphi' \xrightarrow{h} \varphi)$ , when  $\mathcal{M}$  satisfies the sequent  $(\Sigma_h(\varphi) \vdash \varphi')$ ; that is, when  $\Sigma_h(\varphi) \leq_{\widehat{\mathcal{R}}} \varphi'$ ; equivalently, when  $\varphi \leq_{\widehat{\mathcal{R}}} h^*(\varphi')$ . This can be expressed in terms of implication as  $(\Sigma_h(\varphi) \rightarrow \varphi') \equiv \top$ ; equivalently,  $(\varphi \rightarrow h^*(\varphi')) \equiv \top$ . When converting structures to databases, the satisfaction relationship  $\mathcal{M} \models_{\mathcal{S}} (\varphi \xrightarrow{h} \varphi')$  determines the morphism of  $\mathcal{E}$ -relations  $\mathbf{R}_{\widehat{\mathcal{M}}}(\varphi) \xleftarrow{h} \mathbf{R}_{\widehat{\mathcal{M}}}(\varphi')$  in  $\mathbf{Rel}(\mathcal{E})$  and a morphism of  $\mathcal{E}$ -tables  $\mathbf{T}_{\widehat{\mathcal{M}}}(\varphi) \xleftarrow{\langle h, k \rangle} \mathbf{T}_{\widehat{\mathcal{M}}}(\varphi')$  in  $\mathbf{Tbl}(\mathcal{E})$ . (The operators  $\mathbf{R}_{\widehat{\mathcal{M}}}$  and  $\mathbf{T}_{\widehat{\mathcal{M}}}$  are defined in Appendix A.5.1. Satisfaction is summarized in Table 5.)

$$\begin{array}{l}
\mathcal{M} \models_{\mathcal{S}} (\varphi' \xrightarrow{h} \varphi) \\
\text{when } \Sigma_h(\varphi) \leq_{\widehat{\mathcal{R}}} \varphi' \\
\text{iff } \forall k \in K \ (k \models_{\widehat{\mathcal{R}}} (\Sigma_h(\varphi) \rightarrow \varphi')) \\
\text{iff } \forall k \in K \ (k \models_{\widehat{\mathcal{R}}} \Sigma_h(\varphi) \text{ implies } k \models_{\widehat{\mathcal{R}}} \varphi') \\
\text{implies } \exists_h(\mathbf{R}_{\widehat{\mathcal{M}}}(\varphi)) \leq \mathbf{R}_{\widehat{\mathcal{M}}}(\varphi')^a \\
\text{implies } \exists_k \left( \Sigma_h(\mathbf{T}_{\widehat{\mathcal{M}}}(\varphi)) \xrightarrow{k} \mathbf{T}_{\widehat{\mathcal{M}}}(\varphi') \right)
\end{array}$$

<sup>a</sup> For relational structure  $\mathcal{M} = \langle \mathcal{R}, \langle \sigma, \tau \rangle, \mathcal{E} \rangle$ , the fibered mathematical context  $\mathbf{Rel}(\mathcal{E})^{\text{op}} \xrightarrow{\text{list}} \mathbf{List}(X)$  of  $\mathcal{E}$ -relations is determined by the indexed preorder  $\mathbf{List}(X)^{\text{op}} \xrightarrow{\text{rel}} \mathbf{Pre}$ , which maps a type list  $\langle I, s \rangle$  to the fiber relational order  $\mathbf{Rel}_{\mathcal{E}}(I, s) = \langle \wp \mathbf{tup}_{\mathcal{E}}(I, s), \subseteq \rangle$  and maps a type list morphism  $\langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle$  to the fiber monotonic function  $\exists_h = \exists_{\mathbf{tup}_{\mathcal{E}}(h)} : \mathbf{Rel}_{\mathcal{E}}(I', s') \leftarrow \mathbf{Rel}_{\mathcal{E}}(I, s)$ . Similarly, for the fibered context  $\mathbf{Tbl}(\mathcal{E})^{\text{op}} \xrightarrow{\text{pr}} \mathbf{Term}(X)$  of  $\mathcal{E}$ -tables.

**Table 5.** Satisfaction

*Structure Morphisms.* A (model-theoretic) structure morphism (IFF [12])

$$\langle r, k, f, g \rangle : \mathcal{M}_2 = \langle \mathcal{R}_2, \langle \sigma_2, \tau_2 \rangle, \mathcal{E}_2 \rangle \rightleftharpoons \langle \mathcal{R}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{E}_1 \rangle = \mathcal{M}_1$$

is a two dimensional construction consisting of a relation infomorphism  $\langle r, k \rangle : \mathcal{R}_2 = \langle R_2, K_2, \models_{\mathcal{R}_2} \rangle \rightleftharpoons \langle R_1, K_1, \models_{\mathcal{R}_1} \rangle = \mathcal{R}_1$ , an entity infomorphism  $\langle f, g \rangle : \mathcal{E}_2 = \langle X_2, Y_2, \models_{\mathcal{E}_2} \rangle \rightleftharpoons \langle X_1, Y_1, \models_{\mathcal{E}_1} \rangle = \mathcal{E}_1$ , and a list classification square

$$\langle \langle r, k \rangle, \mathbf{List}_{\langle f, g \rangle} \rangle : \langle \mathcal{R}_2 \rightrightarrows^{\langle \sigma_2, \tau_2 \rangle} \mathbf{List}(\mathcal{E}_2) \rangle \rightleftharpoons \langle \mathcal{R}_1 \rightrightarrows^{\langle \sigma_1, \tau_1 \rangle} \mathbf{List}(\mathcal{E}_1) \rangle,$$

where the list infomorphism of the entity infomorphism is the vertical target of the list square. Hence, a structure morphism satisfies the following conditions.

infomorphisms

$$\begin{array}{lll}
k_1 \models_{\mathcal{R}_1} r(r_2) & \underline{\text{iff}} & k(k_1) \models_{\mathcal{R}_2} r_2 \\
y_1 \models_{\mathcal{E}_1} f(x_2) & \underline{\text{iff}} & g(y_1) \models_{\mathcal{E}_2} x_2 \\
t_1 \cdot g = \Sigma_g(J, t_1) \models_{\mathbf{List}(\mathcal{E}_2)} \langle I, s_2 \rangle = s_2 & \underline{\text{iff}} & t_1 = \langle J, t_1 \rangle \models_{\mathbf{List}(\mathcal{E}_1)} \Sigma_f(I, s_2) = s_2 \cdot f
\end{array}$$

list preservation

$$\begin{array}{ll}
r \cdot \sigma_1 & = \sigma_2 \cdot \Sigma_f \\
k \cdot \tau_2 & = \tau_1 \cdot \Sigma_g
\end{array}$$

Structure morphisms compose component-wise. Let  $\mathbf{Rel}$  denote the mathematical context of relational structures and structure morphisms. A structure morphism  $\langle r, k, f, g \rangle : \mathcal{M}_2 \rightleftharpoons \mathcal{M}_1$  has an associated schema morphism  $\mathbf{sch}(r, k, f, g) = \langle r, f \rangle : \mathbf{sch}(\mathcal{M}_2) = \langle R_2, \sigma_2, X_2 \rangle \Longrightarrow \langle R_1, \sigma_1, X_1 \rangle = \mathbf{sch}(\mathcal{M}_1)$ . Hence, there is a schema passage  $\mathbf{sch} : \mathbf{Rel} \rightarrow \mathbf{Sch}$ .

*Formula.* Any structure morphism  $\langle r, k, f, g \rangle : \langle \mathcal{R}_2, \langle \sigma_2, \tau_2 \rangle, \mathcal{E}_2 \rangle \rightleftharpoons \langle \mathcal{R}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{E}_1 \rangle$  has an associated formula structure morphism

$$\mathbf{fm}la(r, k, f, g) = \langle \widehat{r}, k, f, g \rangle : \mathbf{fm}la(\mathcal{M}_2) = \langle \widehat{\mathcal{R}}_2, \langle \sigma_2, \tau_2 \rangle, \mathcal{E}_2 \rangle \rightleftharpoons \langle \widehat{\mathcal{R}}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{E}_1 \rangle = \mathbf{fm}la(\mathcal{M}_1)$$

with schema morphism  $\mathbf{sch}(\mathbf{fm}la(r, k, f, g)) = \langle \hat{r}, f \rangle : \langle \hat{R}_2, \hat{\sigma}_2, X_2 \rangle \Rightarrow \langle \hat{R}_1, \hat{\sigma}_1, X_1 \rangle$ . Hence, there is a formula passage  $\mathbf{fm}la : \mathbf{Rel} \rightarrow \mathbf{Rel}$ .<sup>10</sup> Between any structure and its formula extension is an embedding structure morphism  $\eta_{\mathcal{M}} = \langle inc_{\mathcal{M}}, 1_K, 1_{\mathcal{E}} \rangle : \mathcal{M} \Rightarrow \mathbf{fm}la(\mathcal{M})$ . The formula operator commutes with embedding:  $\eta_{\mathcal{M}_2} \circ \mathbf{fm}la(r, k, f, g) = \langle r, k, f, g \rangle \circ \eta_{\mathcal{M}_1}$ . There is an embedding bridge  $\eta : \mathbf{id}_{\mathbf{Rel}} \Rightarrow \mathbf{fm}la$ .

**Proposition 2.** *There is an idempotent formula passage  $\mathbf{fm}la : \mathbf{Rel} \rightarrow \mathbf{Rel}$  that forms a monad  $\langle \mathbf{Rel}, \eta, \mathbf{fm}la \rangle$  with embedding.*

*Structure Fiber Passage.* Let  $\mathcal{S}_2 = \langle R_2, \sigma_2, X_2 \rangle \xrightarrow{\langle r, f \rangle} \langle R_1, \sigma_1, X_1 \rangle = \mathcal{S}_1$  be a schema morphism. There is a structure passage  $\mathbf{Rel}(\mathcal{S}_2) \xleftarrow{\mathbf{rel}_{\langle r, f \rangle}} \mathbf{Rel}(\mathcal{S}_1)$  defined as follows. Let  $\mathcal{M}_1 = \langle \mathcal{R}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{E}_1 \rangle \in \mathbf{Rel}(\mathcal{S}_1)$  be an  $\mathcal{S}_1$ -structure with a relation classification  $\mathcal{R}_1 = \langle R_1, K_1, \models_{\mathcal{R}_1} \rangle$ , an entity classification  $\mathcal{E}_1 = \langle X_1, Y_1, \models_{\mathcal{E}_1} \rangle$  and a list designation  $\langle \sigma_1, \tau_1 \rangle : \mathcal{R}_1 \rightrightarrows \mathbf{List}(\mathcal{E}_1)$ . Define the inverse image  $\mathcal{S}_2$ -structure  $\mathbf{rel}_{\langle r, f \rangle}(\mathcal{M}_1) = \langle r^{-1}(\mathcal{R}_1), \langle \sigma_2, \tau_1 \rangle, f^{-1}(\mathcal{E}_1) \rangle \in \mathbf{Rel}(\mathcal{S}_2)$  with  $r^{-1}(\mathcal{R}_1) = \langle R_2, K_1, \models_r \rangle$ ,  $f^{-1}(\mathcal{E}_1) = \langle X_2, Y_1, \models_f \rangle$  and a list designation  $\langle \sigma_2, \tau_1 \rangle : r^{-1}(\mathcal{R}_1) \rightrightarrows f^{-1}(\mathcal{E}_1)$ . From the definitions of inverse image classifications, we have the two logical equivalences (1)  $k_1 \models_r r_2 \iff k_1 \models_{\mathcal{E}_1} r(r_2)$  and (2)  $\langle J_1, t_1 \rangle \models_{\Sigma_f} \langle I_2, s_2 \rangle \iff \langle J_1, t_1 \rangle \models_{\mathbf{List}(\mathcal{E}_1)} \Sigma_f(I_2, s_2)$ . Hence,  $k_1 \models_r r_2$  implies  $\tau_1(k_1) \models_{\Sigma_f} \sigma_2(r_2)$ . There is a bridging structure morphism

$$\mathbf{rel}_{\langle r, f \rangle}(\mathcal{M}_1) = \langle r^{-1}(\mathcal{R}_1), \langle \sigma_2, \tau_1 \rangle, f^{-1}(\mathcal{E}_1) \rangle \xrightleftharpoons{\langle r, 1_K, f, 1_Y \rangle} \langle \mathcal{R}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{E}_1 \rangle = \mathcal{M}_1$$

with relation and entity infomorphisms  $r^{-1}(\mathcal{R}_1) \xrightleftharpoons{\langle r, 1_K \rangle} \mathcal{R}_1$  and  $f^{-1}(\mathcal{E}_1) \xrightleftharpoons{\langle f, 1_Y \rangle} \mathcal{E}_1$ .

First-order Semantics:  $\mathbf{Rel} \xleftarrow{\mathbf{rel}} \mathbf{Struc} \xrightarrow[\mathbf{Rel} \times \mathbf{Cls} \mathbf{Alg}]{\mathbf{lang}} \mathbf{Lang}$ . The mathematical context of first-order structures  $\mathbf{Struc}$  is the product of the context  $\mathbf{Rel}$  of relational structures and the context  $\mathbf{Alg}$  of algebras modulo the context  $\mathbf{Cls}$  of classifications. A first-order logic (FOL) structure is a “pair”  $\mathcal{M} = \langle \mathcal{R}, \langle \sigma, \tau \rangle, \mathcal{E}, \langle \Omega, A, \delta \rangle \rangle$  consisting of a relational structure  $\langle \mathcal{R}, \langle \sigma, \tau \rangle, \mathcal{E} \rangle$  and an algebra  $\langle \mathcal{E}, \langle \Omega, A, \delta \rangle \rangle$  that share a common entity classification  $\mathcal{E}$ . The algebra is the semantic base and the relational structure is the superstructure. Given a FOL language  $\mathcal{L} = \langle \mathcal{S}, \mathcal{O} \rangle$  and an  $\mathcal{L}$ -structure  $\mathcal{M}$  with relational  $\mathcal{S}$ -structure  $\mathbf{rel}(\mathcal{M})$  and  $\mathcal{O}$ -algebra  $\mathbf{alg}(\mathcal{M})$ .  $\mathcal{M}$  satisfies an  $\mathcal{L}$ -equation  $\langle I', s' \rangle \xrightarrow{(t=t')} \langle I, s \rangle$ , symbolized by  $\mathcal{M} \models_{\mathcal{L}} (t = t')$ , when  $\mathbf{alg}(\mathcal{M}) \models_{\mathcal{L}} (t = t')$ ; and  $\mathcal{M}$  satisfies an  $\mathcal{L}$ -constraint  $\varphi' \xrightarrow{[t]} \varphi$ , symbolized by  $\mathcal{M} \models_{\mathcal{L}} (\varphi' \xrightarrow{[t]} \varphi)$ , when  $\mathbf{rel}(\mathcal{M}) \models_{\mathcal{S}} (\varphi' \xrightarrow{t} \varphi)$  for any representative term vector  $\hat{\sigma}(\varphi') = \langle I', s' \rangle \xrightarrow{t} \langle I, s \rangle = \hat{\sigma}(\varphi)$ . A first-order logic (FOL) structure morphism  $\langle \mathcal{R}_2, \langle \sigma_2, \tau_2 \rangle, \mathcal{E}_2, \langle \Omega_2, A_2, \delta_2 \rangle \rangle \xrightarrow{\langle \langle r, k \rangle, \langle f, g \rangle, \langle \omega, h \rangle \rangle} \langle \mathcal{R}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{E}_1, \langle \Omega_1, A_1, \delta_1 \rangle \rangle$  consists a relational structure morphism

<sup>10</sup> The schema and formula passages commute:  $\mathbf{fm}la \circ \mathbf{sch} = \mathbf{sch} \circ \mathbf{fm}la$  (Fig. 1).



$\langle \mathcal{R}_2, \langle \sigma_2, \tau_2 \rangle, \mathcal{E}_2 \rangle \xrightarrow{\langle \langle r, k \rangle, \langle f, g \rangle \rangle} \langle \mathcal{R}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{E}_1 \rangle$  and an many-sorted algebraic homomorphism  $\langle \mathcal{E}_2, \mathcal{O}_2, \langle A_2, \delta_2 \rangle \rangle \xrightarrow{\langle f, g, \omega, h \rangle} \langle \mathcal{E}_1, \mathcal{O}_1, \langle A_1, \delta_1 \rangle \rangle$  that share a common entity infomorphism  $\langle f, g \rangle : \mathcal{E}_2 \rightleftarrows \mathcal{E}_1$ .

### A.3 Examples

**Conceptual Graphs:** Consider the English sentence “John is going to Boston by bus” [9]. We describe its representation in a FOLE logic language  $\mathcal{L} = \langle R, \sigma, X, \Omega \rangle$ . By representing the verb as a ternary relation, a graphical representation is

$$\begin{array}{ccc} [Person : John] & \xleftarrow{agnt} (Go) & \xrightarrow{dest} [City : Boston] \\ & \downarrow inst & \\ & [Bus] & \end{array}$$

Formally, we have the following elements: three entity types  $\mathbf{Person}, \mathbf{City}, \mathbf{Bus} \in X$ ; a relation type  $\mathbf{Go} \in R$  with signature  $\sigma(\mathbf{Go}) = \langle I, s \rangle$  having valence 3, arity  $I = \{\mathbf{agnt}, \mathbf{dest}, \mathbf{inst}\}$  and signature function  $I \xrightarrow{s} X$  mapping  $\mathbf{agnt} \mapsto \mathbf{Person}$ ,  $\mathbf{dest} \mapsto \mathbf{City}$ ,  $\mathbf{inst} \mapsto \mathbf{Bus}$ ; a constant symbol  $\mathbf{John} \in \Omega_{\mathbf{Person}, \langle \emptyset, 0_X \rangle}$  of sort  $\mathbf{Person}$  and a constant symbol  $\mathbf{Boston} \in \Omega_{\mathbf{City}, \langle \emptyset, 0_X \rangle}$  of sort  $\mathbf{City}$ .<sup>11</sup> In a conceptual graph representation, the logic language  $\mathcal{L} = \langle R, \sigma, X, \Omega \rangle$  corresponds to a CG module  $\langle X, R, C \rangle$  with type hierarchy  $X$ , relation hierarchy  $R$  and catalog of individuals  $C \subseteq \Omega$ . A CG representation is

$$\begin{array}{l} [\mathbf{Go}] - \\ (\mathbf{agnt}) \rightarrow [\mathbf{Person} : \mathbf{John}] \\ (\mathbf{dest}) \rightarrow [\mathbf{City} : \mathbf{Boston}] \\ (\mathbf{inst}) \rightarrow [\mathbf{Bus}]. \end{array}$$

Formally (compare this linear form to 11), we have the following elements: four entity types  $\mathbf{Go}, \mathbf{Person}, \mathbf{City}, \mathbf{Bus} \in X$ ; three relation types  $\mathbf{agnt}, \mathbf{dest}, \mathbf{inst} \in R$  with signatures  $\sigma(\mathbf{agnt}) = \langle \mathbf{2}, s_{\mathbf{agnt}} \rangle$ ,  $\sigma(\mathbf{dest}) = \langle \mathbf{2}, s_{\mathbf{dest}} \rangle$ ,  $\sigma(\mathbf{inst}) = \langle \mathbf{2}, s_{\mathbf{inst}} \rangle$  having valence 2, arity  $\mathbf{2} = \{0, 1\}$  and signatures  $s_{\mathbf{agnt}}, s_{\mathbf{dest}}, s_{\mathbf{inst}} : \mathbf{2} \rightarrow X$ , where  $s_{\mathbf{agnt}}(0) = s_{\mathbf{dest}}(0) = s_{\mathbf{inst}}(0) = \mathbf{Go}$ ,  $s_{\mathbf{agnt}}(1) = \mathbf{Person}$ ,  $s_{\mathbf{dest}}(1) = \mathbf{City}$ , and  $s_{\mathbf{inst}}(1) = \mathbf{Bus}$ ; and two constants as above.

**Quantification:** The universal quantification ‘ $\forall_{x \in X} P(x:X, y:Y, z:Z)$ ’ is traditionally viewed as formula flow along the type list inclusion  $\{y, z\} \subseteq \{x, y, z\}$ . FOLE handles existential/universal quantification and substitution in terms of formula flow (Table 1) along type list morphisms in the relational aspect or along term vectors in the logical aspect. Given a morphism of type

<sup>11</sup> According to (Sowa [9]), every participant of a process is an entity that plays some role in that process. There is a “linearization” procedure that converts a binary/relational logical representation (FOLE, conceptual graphs) to a unary/functional logical representation (Sketches [5], Ologs [11]). In this example, linearization would define *functional* roles, changing the ternary relation type (process) to an entity type  $\mathbf{Go} \in X$  and converting its arity elements (participant roles) to function types  $\mathbf{agnt} \in \Omega_{\mathbf{Person}, \langle \mathbf{1}, \mathbf{Go} \rangle}$ ,  $\mathbf{dest} \in \Omega_{\mathbf{City}, \langle \mathbf{1}, \mathbf{Go} \rangle}$  and  $\mathbf{inst} \in \Omega_{\mathbf{Bus}, \langle \mathbf{1}, \mathbf{Go} \rangle}$ .

lists  $\langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle$ , for any table  $\langle K, t \rangle \in \mathbf{Tbl}_{\mathcal{E}}(I, s)$ , you can get two tables  $\Sigma_h(K, t), \Pi_h(K, t) \in \mathbf{Tbl}_{\mathcal{E}}(I', s')$  as follows. Given any possible row (or better, tuple)  $t' \in \mathbf{tup}_{\mathcal{E}}(I', s')$ , you can ask either an existential or a universal question about it: for example, “Does there *exist* a key  $k \in K$  in  $T$  with image  $t'$ ?” ( $\mathbf{tup}_h(t_k) = t'$ ) or “Is it the case that *all possible* tuples  $t \in \mathbf{tup}_{\mathcal{E}}(I, s)$  with image  $t'$  are present in  $T$ ?” ([8])

**Relation/Database Joins:** The joins of  $\mathcal{E}$ -relations (or  $\mathcal{E}$ -tables) are represented in FOLE in terms of fibered products — products modulo some reference. If an  $\mathcal{S}$ -span of constraints  $\langle I_1, s_1, \varphi_1 \rangle \xleftarrow{h_1} \langle I, s, \varphi \rangle \xrightarrow{h_2} \langle I_2, s_2, \varphi \rangle$  holds in a relational structure  $\mathcal{M} = \langle \mathcal{R}, \langle \sigma, \tau \rangle, \mathcal{E} \rangle$ , it is interpreted as an opspan of  $\mathcal{E}$ -relations (or  $\mathcal{E}$ -tables). Then the join of  $\mathcal{E}$ -relations (or  $\mathcal{E}$ -tables) is represented by the formula  $\iota_1^*(\varphi_1) \wedge_{\langle \widehat{I}, \widehat{s} \rangle} \iota_2^*(\varphi_2)$ , where  $\langle I_1, s_1 \rangle \xrightarrow{\iota_1} \langle \widehat{I}, \widehat{s} \rangle \xleftarrow{\iota_2} \langle I_2, s_2 \rangle$  is the fibered sum of type lists. In general, the join of an arbitrary diagram of  $\mathcal{E}$ -relations (or  $\mathcal{E}$ -tables) is obtained by substitution followed by conjunction.

#### A.4 Logical Environment

Let  $\mathcal{S}_2 = \langle R_2, \sigma_2, X_2 \rangle \xrightarrow{\langle r, f \rangle} \langle R_1, \sigma_1, X_1 \rangle = \mathcal{S}_1$  be a schema morphism, with structure fiber passage  $\mathbf{Struc}(\mathcal{S}_2) \xleftarrow{\mathbf{struc}_{\langle r, f \rangle}} \mathbf{Struc}(\mathcal{S}_1)$  and bridging structure morphism

$$\mathbf{struc}_{\langle r, f \rangle}(\mathcal{M}_1) = \langle r^{-1}(\mathcal{R}_1), \langle \sigma_2, \tau_1 \rangle, f^{-1}(\mathcal{E}_1) \rangle \xrightarrow{\langle r, 1_K, f, 1_Y \rangle} \langle \mathcal{R}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{E}_1 \rangle = \mathcal{M}_1$$

with relation and entity infomorphisms  $r^{-1}(\mathcal{R}_1) \xrightarrow{\langle r, 1_K \rangle} \mathcal{R}_1$  and  $f^{-1}(\mathcal{E}_1) \xrightarrow{\langle f, 1_Y \rangle} \mathcal{E}_1$ .

**Proposition 3.** *The (formula) interpretation of the inverse image structure is the inverse image of the (formula) interpretation.*

**Fact 1** *The formula classification of the inverse image relation classification is the inverse image classification of the formula relation classification:*

$$r^{-1}(\widehat{\mathcal{R}}_1) = \langle R_2, \widehat{K_1}, \models_r \rangle = \langle \widehat{R}_2, K_1, \models_{\widehat{r}} \rangle = \widehat{r}^{-1}(\widehat{\mathcal{R}}_1).$$

*Proof.* The proof is by induction on formulas  $\varphi_2 \in \widehat{R}_2$ .

**Fact 2** *The formula structure morphism of the bridging structure morphism is:*

$$\langle \widehat{r}, 1_K, f, 1_Y \rangle : \langle r^{-1}(\widehat{\mathcal{R}}_1), \langle \sigma_2, \tau_1 \rangle, f^{-1}(\mathcal{E}_1) \rangle \rightleftharpoons \langle \widehat{\mathcal{R}}_1, \langle \sigma_1, \tau_1 \rangle, \mathcal{E}_1 \rangle.$$

*Its (inst-vertical) relation infomorphism*

$\langle \widehat{r}, 1_K \rangle : r^{-1}(\widehat{\mathcal{R}}_1) = \langle R_2, \widehat{K_1}, \models_r \rangle = \langle \widehat{R}_2, K_1, \models_{\widehat{r}} \rangle \rightleftharpoons \langle \widehat{R}_1, K_1, \models_{\widehat{\mathcal{R}}_1} \rangle = \widehat{\mathcal{R}}_1$   
*is the bridging infomorphism of the formula relation classification, with the infomorphism condition  $k_1 \models_{r^{-1}(\widehat{\mathcal{R}}_1)} \varphi_2 \iff k_1 \models_{\widehat{\mathcal{R}}_1} \widehat{r}(\varphi_2)$ . The extent monotonic function  $\widehat{r} : \mathbf{ext}(r^{-1}(\widehat{\mathcal{R}}_1)) \rightarrow \mathbf{ext}(\widehat{\mathcal{R}}_1)$  is an isometry:  $\varphi \leq_{r^{-1}(\widehat{\mathcal{R}}_1)} \psi \iff \widehat{r}(\varphi) \leq_{\widehat{\mathcal{R}}_1} \widehat{r}(\psi)$ .*

**Proposition 4.** *Satisfaction is invariant under change of notation; that is, for any schema morphism  $\mathcal{S}_2 = \langle R_2, \sigma_2, X_2 \rangle \xrightarrow{\langle r, f \rangle} \langle R_1, \sigma_1, X_1 \rangle = \mathcal{S}_1$  the following satisfaction condition holds:*

$$\mathbf{struc}_{\langle r, f \rangle}(\mathcal{M}_1) \models_{\mathcal{S}_2} (\varphi_2 \xrightarrow{h} \varphi'_2) \text{ iff } \mathcal{M}_1 \models_{\mathcal{S}_1} (\widehat{r}(\varphi_2) \xrightarrow{h} \widehat{r}(\varphi'_2)) = \mathbf{fmla}_{\langle r, f \rangle}(\varphi_2 \vdash \varphi'_2).$$

*Proof.* But this holds, since  $r^{-1}(\widehat{\mathcal{R}}_1) = \widehat{r}^{-1}(\widehat{\mathcal{R}}_1)$ . In more detail,

$$\begin{aligned} \mathbf{struc}_{\langle r, f \rangle}(\mathcal{M}_1) \models_{\mathcal{S}_2} (\varphi_2 \xrightarrow{h} \varphi'_2) &\text{ iff } \Sigma_h(\varphi'_2) \leq_{r^{-1}(\widehat{\mathcal{R}}_1)} \varphi_2 \\ \text{iff } \widehat{r}(\Sigma_h(\varphi'_2)) \leq_{\widehat{\mathcal{R}}_1} \widehat{r}(\varphi_2) &\text{ iff } \Sigma_h(\widehat{r}(\varphi'_2)) \leq_{\widehat{\mathcal{R}}_1} \widehat{r}(\varphi_2) \\ \text{iff } \mathcal{M}_1 \models_{\mathcal{S}_1} (\widehat{r}(\varphi_2) \xrightarrow{h} \widehat{r}(\varphi'_2)) &= \mathbf{fmla}_{\langle r, f \rangle}(\varphi_2 \vdash \varphi'_2). \end{aligned}$$

**Proposition 5.** *The institution  $\langle \mathbf{Sch}, \mathbf{fmla}, \mathbf{struc} \rangle$  is a logical environment, since it satisfies the bimodular principle “satisfaction respects structure morphisms”: given any schema  $\mathcal{S} = \langle R, \sigma, X \rangle$ , if  $\langle 1_R, k, 1_X, g \rangle : \mathcal{M}_2 \rightleftharpoons \mathcal{M}_1$  is a **sch**-vertical structure morphism over  $\mathcal{S}$ , then we have the intent order  $\mathcal{M}_2 \geq_{\mathcal{S}} \mathcal{M}_1$ ; that is,  $\mathcal{M}_2 \models_{\mathcal{S}} (\varphi \vdash \psi)$  implies  $\mathcal{M}_1 \models_{\mathcal{S}} (\varphi \vdash \psi)$  for any  $\mathcal{S}$ -sequent  $(\varphi \vdash \psi)$ .<sup>12</sup>*

*Proof.* The **typ**-vertical formula morphism  $\langle 1_{\widehat{R}}, k, 1_X, g \rangle : \widehat{\mathcal{M}}_2 \rightleftharpoons \widehat{\mathcal{M}}_1$  over  $\widehat{\mathcal{S}}$  has the **typ**-vertical relation infomorphism  $\langle 1_{\widehat{R}}, k \rangle : \widehat{\mathcal{R}}_2 \rightleftharpoons \widehat{\mathcal{R}}_1$  over  $\widehat{R}$ .  $\mathcal{M}_2 \models_{\mathcal{S}} (\varphi \vdash \psi)$  iff  $\varphi \leq_{\widehat{\mathcal{R}}_2} \psi$  implies  $\varphi \leq_{\widehat{\mathcal{R}}_1} \psi$  iff  $\mathcal{M}_1 \models_{\mathcal{S}} (\varphi \vdash \psi)$  for any  $\mathcal{S}$ -sequent  $(\varphi \vdash \psi)$ .

## A.5 Transformation to Databases

**A.5.1 Relational Interpretation.** Let  $\mathcal{M} = \langle \mathcal{R}, \langle \sigma, \tau \rangle, \mathcal{E} \rangle$  be a (model-theoretic) relational structure. The relation classification  $\mathcal{R}$  is equivalent to the extent function  $\mathbf{ext}_{\mathcal{R}} : R \rightarrow \wp K$ , which maps a relational symbol  $r \in R$  to its  $\mathcal{R}$ -extent  $\mathbf{ext}_{\mathcal{R}}(r) \subseteq K$ . The list classification  $\mathbf{List}(\mathcal{E})$  is equivalent to the extent function  $\mathbf{ext}_{\mathbf{List}(\mathcal{E})} : \mathbf{List}(X) \rightarrow \wp \mathbf{List}(Y)$ , a restriction of the tuple passage  $\mathbf{tup}_{\mathcal{E}} : \mathbf{List}(X)^{\text{op}} \rightarrow \mathbf{Set}$ , which maps a type list  $\langle I, s \rangle \in \mathbf{List}(X)$  to its  $\mathbf{List}(\mathcal{E})$ -extent  $\mathbf{tup}_{\mathcal{E}}(I, s) \subseteq \mathbf{List}(Y)$ . The list designation satisfies the condition  $k \models_{\mathcal{R}} r$  implies  $\tau(k) \models_{\mathbf{List}(\mathcal{E})} \sigma(r)$  for all  $k \in K$  and  $r \in R$ ; so that  $k \in \mathbf{ext}_{\mathcal{R}}(r)$  implies  $\tau(k) \in \mathbf{ext}_{\mathbf{List}(\mathcal{E})}(\sigma(r)) = \mathbf{tup}_{\mathcal{E}}(\sigma(r))$ . Hence,  $\wp \tau(\mathbf{ext}_{\mathcal{R}}(r)) \subseteq \mathbf{tup}_{\mathcal{E}}(\sigma(r))$  for all  $r \in R$ . Thus, we have the function order  $\mathbf{ext}_{\mathcal{R}} \cdot \wp \tau \subseteq \sigma \cdot \mathbf{ext}_{\mathbf{List}(\mathcal{E})}$ .

The relational interpretation function  $\mathbf{R}_{\mathcal{M}} : R \rightarrow |\mathbf{Rel}(\mathcal{E})|$  maps a relational symbol  $r \in R$  with type list  $\sigma(r) = \langle I, s \rangle$  to the set of tuples  $\mathbf{R}_{\mathcal{M}}(r) = \wp \tau(\mathbf{ext}_{\mathcal{R}}(r)) \in \wp \mathbf{tup}_{\mathcal{E}}(I, s) = \mathbf{Rel}_{\mathcal{E}}(I, s)$ . The tabular interpretation function  $\mathbf{T}_{\mathcal{M}} : R \rightarrow |\mathbf{Tbl}(\mathcal{E})| = |(\mathbf{Set} \downarrow \mathbf{tup}_{\mathcal{E}})|$  maps a relational symbol  $r \in R$  with type list  $\sigma(r) = \langle I, s \rangle$  to the pair  $\mathbf{T}_{\mathcal{M}}(r) = \langle K(r), t_r \rangle$  consisting of the key set  $K(r) = \mathbf{ext}_{\mathcal{R}}(r) \subseteq K$  and the tuple function  $K(r) \xrightarrow{t_r} \mathbf{tup}_{\mathcal{E}}(I, s)$ , a restriction of the tuple function  $\tau : K \rightarrow \mathbf{List}(Y)$ , which maps a key  $k \in K_r$  to the tuple

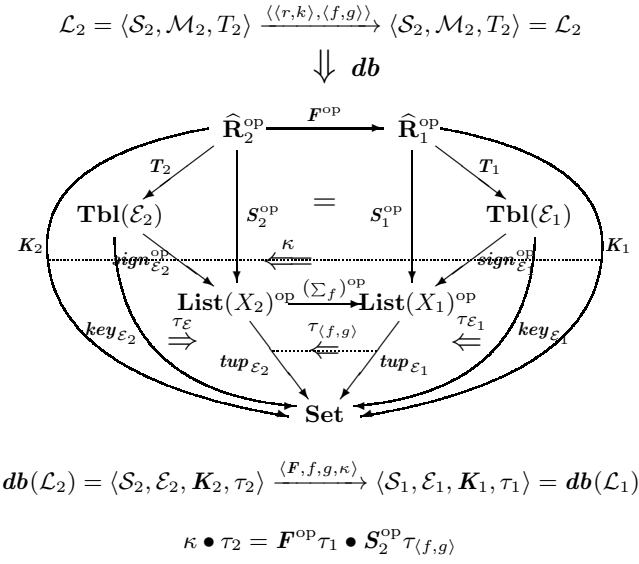
<sup>12</sup> For any classification  $\mathcal{A} = \langle X, Y, \models_{\mathcal{A}} \rangle$ , the intent order  $\mathbf{int}(\mathcal{A}) = \langle Y, \leq_{\mathcal{A}} \rangle$  is defined as follows: for two instances  $y, y' \in Y$ ,  $y \leq_{\mathcal{A}} y'$  when  $\mathbf{int}_{\mathcal{A}}(y) \supseteq \mathbf{int}_{\mathcal{A}}(y')$ ; that is, when  $y' \models_{\mathcal{A}} x$  implies  $y \models_{\mathcal{A}} x$  for each  $x \in X$ .

$t_r(k) = \tau(k) \in \mathbf{tup}_{\mathcal{E}}(I, s)$ . Applying the image passage  $\mathbf{im}_{\mathcal{E}}(I, s) : \mathbf{Tbl}_{\mathcal{E}}(I, s) \rightarrow \mathbf{Rel}_{\mathcal{E}}(I, s)$ , the image of the table interpretation is the relation interpretation  $\mathbf{im}_{\mathcal{E}}(I, s)(\mathbf{T}_{\mathcal{M}}(r)) = \mathbf{R}_{\mathcal{M}}(r)$  for any relation symbol  $r \in R$ . Using the combined image passage  $\mathbf{im}_{\mathcal{E}} : \mathbf{Tbl}(\mathcal{E}) \rightarrow \mathbf{Rel}(\mathcal{E})$ , we get the composition  $\mathbf{R}_{\mathcal{M}} = R \xrightarrow{T_{\mathcal{M}}} |\mathbf{Tbl}(\mathcal{E})| \xrightarrow{|\mathbf{im}_{\mathcal{E}}|} |\mathbf{Rel}(\mathcal{E})|$ . Note that  $t_r : K_r \rightarrow \mathbf{R}_{\mathcal{M}}(r) \rightarrow \mathbf{tup}_{\mathcal{E}}(I, s)$ , is a surjection-injection factorization of the tuple function.<sup>13</sup>

**A.5.2 Relational Logics.** A relational logic  $\mathcal{L} = \langle \mathcal{S}, \mathcal{M}, T \rangle$  consists of a relational structure  $\mathcal{M} = \langle \mathcal{R}, \langle \sigma, \tau \rangle, \mathcal{E} \rangle$  and a relational specification  $\mathcal{T} = \langle \mathcal{S}, T \rangle$  that share a common relational schema  $\mathbf{sch}(\mathcal{M}) = \mathcal{S}$ . The logic is sound when the structure  $\mathcal{M}$  satisfies every constraint in the specification  $T$ . A sound relational logic enriches a relational structure with a specification. For any sound logic  $\mathcal{L} = \langle \mathcal{S}, \mathcal{M}, T \rangle$ , there is an interpretation functor  $\widehat{\mathbf{R}}^{\text{op}} \xrightarrow{T_{\mathcal{L}}} \mathbf{Tbl}(\mathcal{E}) = (\mathbf{Set} \downarrow \mathbf{tup}_{\mathcal{E}})$ , where  $\widehat{\mathbf{R}} \subseteq \mathbf{Fmla}(\mathcal{S})$  is the consequence of  $T$ . Sound logics are important in the transformation of structures to databases (below). A relational logic morphism  $\mathcal{L}_2 = \langle \mathcal{S}_2, \mathcal{M}_2, T_2 \rangle \xrightarrow{\langle \langle r, k \rangle, \langle f, g \rangle \rangle} \langle \mathcal{S}_2, \mathcal{M}_2, T_2 \rangle = \mathcal{L}_2$  consists of a relational structure morphism  $\mathcal{M}_2 \xrightarrow{\langle \langle r, k \rangle, \langle f, g \rangle \rangle} \mathcal{M}_1$  and a relational specification morphism  $\mathcal{T}_2 = \langle \mathcal{S}_2, T_2 \rangle \xrightarrow{\langle r, f \rangle} \langle \mathcal{S}_1, T_1 \rangle = \mathcal{T}_1$  that share a common relational schema morphism  $\mathbf{sch}(\langle r, k \rangle, \langle f, g \rangle) = \mathcal{S}_2 \xrightarrow{\langle r, f \rangle} \mathcal{S}_1$ .

Any sound relational logic  $\mathcal{L} = \langle \mathcal{S}, \mathcal{M}, T \rangle$  with structure  $\mathcal{M} = \langle \mathcal{R}, \langle \sigma, \tau \rangle, \mathcal{E} \rangle$  and specification  $\mathcal{T} = \langle \mathcal{S}, T \rangle$  has an associated logical/relational database  $\mathbf{db}(\mathcal{L}) = \langle \mathcal{S}, \mathcal{E}, \mathbf{K}, \tau \rangle$  with category of formulas  $\widehat{\mathbf{R}} \subseteq \mathbf{Fmla}(\mathcal{S})$  (the consequence of  $T$ ), signature passage  $\mathbf{S} : \widehat{\mathbf{R}} \rightarrow \mathbf{List}(X)$ , entity classification  $\mathcal{E}$ , key passage  $\mathbf{K} : \widehat{\mathbf{R}}^{\text{op}} \rightarrow \mathbf{Set}$ , tuple bridge  $\tau : \mathbf{K} \Rightarrow \mathbf{S}^{\text{op}} \circ \mathbf{tup}_{\mathcal{E}}$ , and table interpretation passage  $\widehat{\mathbf{R}}^{\text{op}} \xrightarrow{T} \mathbf{Tbl}(\mathcal{E}) = (\mathbf{Set} \downarrow \mathbf{tup}_{\mathcal{E}})$ , where  $\tau = T\tau_{\mathcal{E}}$ . Any sound relational logic morphism  $\mathcal{L}_2 = \langle \mathcal{S}_2, \mathcal{M}_2, T_2 \rangle \xrightarrow{\langle \langle r, k \rangle, \langle f, g \rangle \rangle} \langle \mathcal{S}_2, \mathcal{M}_2, T_2 \rangle = \mathcal{L}_2$  with structure morphism  $\mathcal{M}_2 \xrightarrow{\langle \langle r, k \rangle, \langle f, g \rangle \rangle} \mathcal{M}_1$  and specification morphism  $\mathcal{T}_2 = \langle \mathcal{S}_2, T_2 \rangle \xrightarrow{\langle r, f \rangle} \langle \mathcal{S}_1, T_1 \rangle = \mathcal{T}_1$  has an associated (strict) logical/relational database morphism  $\mathbf{db}(\langle r, k \rangle, \langle f, g \rangle) = \langle \mathbf{F}, f, g, \kappa \rangle : \mathbf{db}(\mathcal{L}_2) = \langle \mathcal{S}_2, \mathcal{E}_2, \mathbf{K}_2, \tau_2 \rangle \rightarrow \langle \mathcal{S}_1, \mathcal{E}_1, \mathbf{K}_1, \tau_1 \rangle = \mathbf{db}(\mathcal{L}_1)$  with (strict) database schema morphism  $\langle \mathbf{F}, f \rangle : \mathcal{S}_2 \rightarrow \mathcal{S}_1$ , entity information morphism  $\langle f, g \rangle : \mathcal{E}_2 \rightrightarrows \mathcal{E}_1$ , and key natural transformation  $\kappa : \mathbf{F}^{\text{op}} \circ \mathbf{K}_1 \Rightarrow \mathbf{K}_2$ , which satisfy the condition  $\kappa \bullet \tau_2 = \mathbf{F}^{\text{op}} \tau_1 \bullet \mathbf{S}_2^{\text{op}} \tau_{\langle f, g \rangle}$ . The passage  $\widehat{\mathbf{R}}_2 \xrightarrow{\mathbf{F}} \widehat{\mathbf{R}}_1$  from formula subcontext  $\widehat{\mathbf{R}}_2 \subseteq \mathbf{Fmla}(\mathcal{S}_2)$  to formula subcontext  $\widehat{\mathbf{R}}_1 \subseteq \mathbf{Fmla}(\mathcal{S}_1)$  is a restriction of the fibered formula passage  $\mathbf{Fmla}(\mathcal{S}_2) \xrightarrow{\mathbf{fmla}_{\langle r, f \rangle}} \mathbf{Fmla}(\mathcal{S}_1)$ . (Kent [7] has more details on relational database semantics.)

<sup>13</sup> Two tables are informationally equivalent when they contain the same information; that is, when their image relations are equivalent in  $\mathbf{Rel}_{\mathcal{E}}(I, s) = \wp \mathbf{tup}_{\mathcal{E}}(I, s)$ . In particular, the table  $\mathbf{T}_{\mathcal{M}}(r)$  and relation  $\mathbf{R}_{\mathcal{M}}(r)$  of a relational symbol are informationally equivalent.



**Fig. 5.** From Sound Logics to Logical/Relational Databases