# Logic of Non-Monotonic Interactive Proofs<sup>\*</sup> (Formal Theory of Temporary Knowledge Transfer)

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#### Abstract

We propose a monotonic logic of internalised *non-monotonic* or *instant* interactive proofs (LiiP) and reconstruct an existing monotonic logic of internalised monotonic or persistent interactive proofs (LiP) as a minimal conservative extension of LiiP. Instant interactive proofs effect a *fragile* epistemic impact in their intended communities of peer reviewers that consists in the *impermanent* induction of the knowledge of their proof goal by means of the knowledge of the proof with the interpreting reviewer: If my peer reviewer knew my proof then she would at least then (in that instant) know that its proof goal is true. Their impact is fragile and their induction of knowledge impermanent in the sense of being the case possibly only at the instant of learning the proof. This accounts for the important possibility of internalising proofs of statements whose truth value can vary, which, as opposed to invariant statements, cannot have persistent proofs. So instant interactive proofs effect a *temporary* transfer of certain propositional knowledge (knowable *ephemeral* facts) via the transmission of certain individual knowledge (knowable non-monotonic proofs) in distributed systems of multiple interacting agents.

**Keywords:** agents as proof- and signature-checkers; constructive Kripkesemantics; interpreted communication; multi-agent distributed systems; interactive and oracle computation; proofs as sufficient evidence.

## 1 Introduction

The subject matter of this paper is modal logic of interactive proofs, i.e., a novel logic of *non-monotonic* or *instant* interactive proofs (LiiP) as well as an existing logic of monotonic or persistent interactive proofs (LiP) [Kra12]. (We abbreviate interactivity-related adjectives with lower-case letters.) The goal

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here is to define LiiP axiomatically and semantically as well as to reconstruct LiP as a minimal conservative extension of LiiP. So for distributed and multiagent systems, whose states and thus truth of statements about states can vary, proof non-monotonicity (as in LiiP) is in a logical sense more primitive than proof monotonicity (as in LiP). In contrast, proof monotonicity is perhaps more intuitive than proof non-monotonicity within formal physical theories validated by experiment and surely within mathematical theories known to be consistent.

Rephrasing [Mak05, Section 1.1] model-theoretically, the proof modality of LiiP internalises a non-monotonic notion of proof in the sense that it can happen that a proposition  $\phi$  can be proved with a (non-monotonic) proof M to an agent a in some system state s, but not anymore in some subsequent state s' in which a will have learnt additional or lost previously learnt data M'. See Appendix B for formal application examples. Like in LiP [Kra12], we understand interactive proofs as sufficient evidence to intended resource-unbounded (though unable to guess) proof- and signature-checking agents (designated verifiers).

Instant interactive proofs effect a *fragile* epistemic impact in their intended communities  $\mathcal{C}$  of peer reviewers that consists in the *impermanent* induction of the (propositional) knowledge (not only belief) of their proof goal  $\phi$  by means of the (individual) knowledge of the proof (the sufficient evidence) M with the designated interpreting reviewer a: If a knew my proof M of  $\phi$  then she would at least then (in that instant) know that the proof goal  $\phi$  is true. By individual knowledge we mean knowledge in the sense of the transitive use of the verb "to know," here to know a message, such as the plaintext of an encrypted message. Notation:  $a \not k M$  for "agent a knows message M" (cf. Definition 1). This is the classic concept of knowledge de re ("of a thing") made explicit for messages, meaning taking them apart (analysing) and putting them together (synthesising). Whereas by propositional knowledge we mean knowledge in the sense of the use of the verb "to know" with a clause, here to know that a statement is true, such as that the plaintext of an encrypted message is (individually) unknown to potential adversaries. Notation:  $\mathsf{K}_a(\phi)$  for "agent a knows that  $\phi$ (is true)" (cf. Fact 1). This is the classic concept of knowledge de dicto ("of a fact").<sup>1</sup> (We distinguish individual and propositional knowledge with respect to the "object" of knowledge [the known], i.e., with respect to a message and clause, respectively. However, individual as well as propositional knowledge can both be individual with respect to the *subject* of knowledge [the knower], i.e., an [individual] agent.) With respect to belief, propositional knowledge essentially differs in that it is necessarily true whereas belief is possibly false, as commonly known and accepted [MV07]. The epistemic impact of our instant interactive proofs is fragile and their induction of knowledge impermanent in the sense of being the case possibly only at the instant of learning the proof. This accounts for the important possibility of internalising proofs of statements, whose truth value can vary, such as statements about system states, which, as opposed to invariant statements, cannot have persistent proofs. Proofs must (not) prove

 $<sup>^1</sup>$  In a first-order setting, knowledge  $de\ re$  and  $de\ dicto$  can be related in Barcan-laws [KR10].

true (false) statements! Standard examples of statements of variable truth value are contingent (e.g., elementary) facts (expressed as atomic formulas) and characteristic formulas of states [GO07].

In contrast [Kra12], the epistemic impact of *persistent* interactive proofs is *durable* in the sense of being the case necessarily at the instant of learning the proof *and henceforth*, where time can be present implicitly (such as here) or explicitly (in future work). In other words, when a persistent proof can prove a certain statement, the proof will always be able to *robustly* do so, independently of whether or not more messages (data) than just the proof are learnt.

In sum, our instant interactive proofs effect a transfer of propositional knowledge (knowable ephemeral facts) via the transmission of certain individual knowledge (knowable non-monotonic proofs) in multi-agent distributed systems. That is, L(i)iP is a formal theory of (temporary) knowledge transfer. The overarching motivation for L(i)iP is to serve in an intuitionistic foundation of interactive computation. See [Kra12] for a programmatic and methodological motivation.

#### 1.1 Contribution

Our technical contribution in this paper is fourfold. For LiiP, we provide an adequate axiomatisation of its oracle-computational and knowledge-constructive Kripke-semantics, and a minimal conservative extension LiiP<sup>+</sup> with a single monotonicity axiom schema making LiiP<sup>+</sup> isomorphic to LiP. For LiP, we provide a substantially simplified semantic interface and a slightly simplified axiomatisation, which is a nice side-effect of obtaining LiiP<sup>+</sup>.

The Kripke-semantics for LiiP (like for LiP [Kra12]) is knowledge-constructive in the sense that (cf. Fact 1) our interactive proofs induce the knowledge of their proof goal (say  $\phi$ ) in their intended interpreting agents (say a) such that the induced knowledge ( $\mathsf{K}_{a}(\phi)$ ) is knowledge in the sense of the standard modal logic of knowledge S5 [FHMV95, MV07, HR10]. Note that our agents here are still resource-unbounded with respect to individual and propositional knowledge, though they are still unable to guess that knowledge. (Recall that S5-agents are resource-unbounded, i.e., logically omniscient.) Thus we give an epistemic explication of proofs, i.e., an explication of proofs in terms of the epistemic impact that they effect in their intended interpreting agents (i.e., the knowledge of their proof goal). Technically, we endow the proof modality with a standard Kripke-semantics [BvB07], but whose accessibility relation  ${}_{M}\mathcal{R}_{a}^{\mathcal{C}}$  we first define constructively in terms of elementary set-theoretic constructions,<sup>2</sup> namely as  ${}_{M}\mathbf{R}_{a}^{\mathcal{C}}$ , and then match to an abstract semantic interface in standard form (which abstractly stipulates the characteristic properties of the accessibility relation [Fit07]). We will say that  ${}_{M}\!R^{\mathcal{C}}_{a}$  exemplifies (or realises)  ${}_{M}\!\mathcal{R}^{\mathcal{C}}_{a}$ . (A simple example of a constructive definition of a modal accessibility is the well-known definition of epistemic accessibility as state indistinguishability defined in terms

 $<sup>^{2}</sup>$ in loose analogy with the set-theoretically constructive rather than the purely axiomatic definition of numbers [Fef89] or ordered pairs (e.g., the now standard definition by Kuratowski, and other well-known definitions [Mos06])

of equality of state projections [FHMV95].) Recall, set-theoretically constructive is different from intuitionistically constructive! The Kripke-semantics for LiiP is oracle-computational in the sense that (cf. Definition 3) the individual proof knowledge (say M) can be thought of as being provided by an imaginary computation oracle, which thus acts as a hypothetical provider and imaginary epistemic source of our interactive proofs. The semantic interface of LiP here is simplified in the sense that we are able to eliminate all *a posteriori* constraints from the semantic interface in [Kra12] and thus to manage with only standard, *a priori* constraints, i.e., stipulations.

### 1.2 Roadmap

In the next section, we introduce our Logic of instant interactive Proofs (LiiP) axiomatically by means of a compact closure operator that induces the Hilbertstyle proof system that we seek and that allows the simple generation of application-specific extensions of LiiP (cf. Appendix B). We then prove some useful (further-used) deducible laws within the obtained system. Next, we introduce the set-theoretically constructive semantics and the abstract semantic interface for LiiP, and prove the axiomatic adequacy of the proof system with respect to this interface. In the construction of the semantics, we again make use of a closure operator, but this time on sets of proof terms. Finally in Section 3, we reconstruct LiP as a minimal conservative extension of LiiP.

## 2 Logic of instant interactive Proofs

The Logic of instant interactive Proofs (LiiP) provides a modal formula language over a generic message term language. The formula language offers the propositional constructors, a relational symbol 'k' for constructing atomic propositions about individual knowledge (e.g.,  $a \ M$ ), and a modal constructor '::' for propositions about proofs (e.g.,  $M ::_a^c \phi$ ). The message language offers term constructors for message pairing and (not necessarily, but possibly cryptographically implemented) signing. (Cryptographic signature creation and verification is polynomial-time computable [Kat10]. See [Kra12] for other cryptographic constructors such as encryption and hashing.) In brief, LiiP is a minimal modular extension of classical propositional logic with an interactively generalised additional operator (the proof modality) and proof-term language (only two constructors, agents as proof- and signature-checkers). Note, the language of LiiP is identical to the one of LiP [Kra12] modulo the proof-modality notation, which in LiP is ':'.

**Definition 1** (The language of LiiP). Let

- $\mathcal{A} \neq \emptyset$  designate a non-empty finite set of *agent names a*, *b*, *c*, etc.
- $\mathcal{C} \subseteq \mathcal{A}$  denote (finite and not necessarily disjoint) communities (sets) of agents  $a \in \mathcal{A}$  (referred to by their name)

•  $\mathcal{M} \ni M ::= a \mid B \mid \{\!\!\{M\}\!\!\}_a \mid (M, M)$  designate our language of message terms M over  $\mathcal{A}$  with (transmittable) agent names  $a \in \mathcal{A}$ , application-specific data B (left blank here), signed messages  $\{\!\!\{M\}\!\!\}_a$ , and message pairs (M, M)

(Messages must be grammatically well-formed, which yields an induction principle. So agent names a are logical term constants, the meta-variable B just signals the possibility of an extended term language  $\mathcal{M}$ ,  $\{\!\{\cdot\}\!\}_a$  with  $a \in \mathcal{A}$  is a unary functional symbol, and  $(\cdot, \cdot)$  a binary functional symbol.)

•  $\mathcal{P}$  designate a denumerable set of propositional variables P constrained such that for all  $a \in \mathcal{A}$  and  $M \in \mathcal{M}$ ,  $(a \nmid M) \in \mathcal{P}$  (for "a knows M") is a distinguished variable, i.e., an *atomic proposition*, (for *individual* knowledge)

(So, for  $a \in \mathcal{A}$ ,  $a \mathbf{k} \cdot \mathbf{is}$  a unary relational symbol.)

•  $\mathcal{L} \ni \phi ::= P \mid \neg \phi \mid \phi \land \phi \mid M ::_a^{\mathcal{C}} \phi$  designate our language of *logical* formulas  $\phi$ , where  $M ::_a^{\mathcal{C}} \phi$  reads "M is a  $\mathcal{C} \cup \{a\}$ -reviewable proof of  $\phi$ " in that "M can prove  $\phi$  to a (e.g., a designated verifying judge) and this is commonly accepted in the (pointed) community  $\mathcal{C} \cup \{a\}$  (e.g., for  $\mathcal{C}$  being a jury)."

Then LiiP has the following axiom and deduction-rule schemas, with greyshading indicating the difference to LiP.

**Definition 2** (The axioms and deduction rules of LiiP). Let

- $\Gamma_0$  designate an adequate set of axioms for classical propositional logic
- $\Gamma_1 := \Gamma_0 \cup \{$ 
  - a k a (knowledge of one's own name string)
  - $a \,\mathsf{k} \, M \to a \,\mathsf{k} \, \{\!\!\{M\}\!\!\}_a \quad (personal \ [\text{the same } a] \ \text{signature synthesis})$
  - $\ a \, \mathsf{k} \, \{\!\![M]\!\!\}_b \to a \, \mathsf{k} \, (M, b) \quad (universal \ [\text{any} \ a \ \text{and} \ b] \ \text{signature} \ analysis)$
  - $(a \,\mathsf{k} \, M \wedge a \,\mathsf{k} \, M') \leftrightarrow a \,\mathsf{k} \,(M, M') \quad ([un] \text{pairing})$
  - $\ (M ::_a^{\mathcal{C}} (\phi \to \phi')) \to ((M ::_a^{\mathcal{C}} \phi) \to M ::_a^{\mathcal{C}} \phi') \quad (\text{Kripke's law, K})$
  - $\ (M ::_a^{\mathcal{C}} \phi) \to (a \, \mathsf{k} \, M \to \phi) \quad (\text{epistemic truthfulness})$
  - $\bigwedge_{b \in \mathcal{C} \cup \{a\}} ((\underbrace{(M, b) ::_{a}^{\mathcal{C}} \phi}_{\text{can prove}}) \to \{\!\![M]\!\!\}_{a} ::_{b}^{\mathcal{C} \cup \{a\}} (\underbrace{a \, \mathsf{k} \, M \wedge M ::_{a}^{\mathcal{C}} \phi}_{\text{does prove}}))$

(nominal [in b] peer review)

 $- (M ::_{a}^{\mathcal{C} \cup \mathcal{C}'} \phi) \to M ::_{a}^{\mathcal{C}} \phi \quad (\text{group decomposition}) \}$ 

designate a set of axiom schemas.

Then,  $\operatorname{LiiP} := \operatorname{Cl}(\emptyset) := \bigcup_{n \in \mathbb{N}} \operatorname{Cl}^{n}(\emptyset)$ , where for all  $\Gamma \subseteq \mathcal{L}$ :  $\operatorname{Cl}^{0}(\Gamma) := \Gamma_{1} \cup \Gamma$   $\operatorname{Cl}^{n+1}(\Gamma) := \operatorname{Cl}^{n}(\Gamma) \cup$   $\{ \phi' \mid \{\phi, \phi \to \phi'\} \subseteq \operatorname{Cl}^{n}(\Gamma) \} \cup \text{ (modus ponens, MP)}$   $\{ M ::_{a}^{c} \phi \mid \phi \in \operatorname{Cl}^{n}(\Gamma) \} \cup \text{ (necessitation, N)}$   $\{ (M ::_{a}^{c} \phi) \leftrightarrow M' ::_{a}^{c} \phi \mid (a \Bbbk M \leftrightarrow a \Bbbk M') \in \operatorname{Cl}^{n}(\Gamma) \}$ (epistemic bitonicity).

We call LiiP a base theory, and  $Cl(\Gamma)$  an LiiP-theory for any  $\Gamma \subseteq \mathcal{L}$ .

Notice the logical order of LiiP, which is, due to propositions about (proofs of) propositions, higher-order propositional. Further, observe that we assume the existence of a dependable mechanism for signing messages, which we model with the above synthesis and analysis axioms. In *trusted* multi-agent systems, signatures are unforged, and thus such a mechanism is trivially given by the inclusion of the sender's name in the sent message, or by the sender's sensorial impression on the receiver when communication is immediate. In distrusted multi-agent systems (e.g., the open Internet), a practically unforgeable signature mechanism can be implemented with classical *certificate-based* or, more directly, with *identity-based* public-key cryptography [Kat10]. We also assume the existence of a pairing mechanism modelling finite sets. Such a mechanism is required by the important application of communication (not only cryptographic) protocols [And08, Chapter 3], in which concatenation of high-level data packets is associative, commutative, and idempotent. The key to the validity of K is that we understand interactive proofs as *sufficient evidence* for intended resourceunbounded proof-checking agents (who are though still unable to guess), see [Kra12, Section 3.2.2] for more details. Next, the significance of epistemic truthfulness to interactivity is that in truly distributed multi-agent systems, not all proofs are known by all agents, i.e., agents are not omniscient with respect to messages. Otherwise, why communicate with each other? So there being a proof does not imply knowledge of that proof. When an agent a does not know the proof and the agent cannot generate the proof *ex nihilo* herself by guessing it, only communication from a peer, who thus acts as an oracle, can entail the knowledge of the proof with a. That is, provability and truth are necessarily concomitant in the non-interactive setting, whereas in interactive settings they are not necessarily so [Kra12]. In nominal peer review, "can prove" suggests the proof potentiality of (M, b): "if a were to know, e.g., receive, (M, b)" (and thus know her potential interlocutor b's name). Whereas given  $\{M\}_a$  to b, e.g., in an acknowledgement from a, "does prove" suggests the proof *actuality* of M: "a does know, e.g., did receive, (M, b)", otherwise a could not have signed M. See the proof of Corollary 4.5 for a semantic justification of the raison d'être of b in (M, b). Then, the justification for the necessitation rule (schema) is that in interactive settings, validities, and thus a fortiori tautologies (in the strict sense of validities of the propositional fragment), are in some sense trivialities [Kra12]. To see why, recall that modal validities are true in *all* pointed models (cf. Definition 6), and thus not worth being communicated from one point to another in a given model, e.g., by means of specific interactive proofs. (Nothing is logically more embarrassing than talking in tautologies.) Therefore, validities deserve *arbitrary* proofs. What is worth being communicated are truths weaker than validities, namely local truths in the standard model-theoretic sense (cf. Definition 6), which may not hold universally. Otherwise why communicate with each other? Finally, observe that epistemic bitonicity is a rule of *logical modularity* that allows the modular generation of structural modal laws from equivalence term laws (cf. Theorem 1).

The grey-shading in Definition 2 indicates that the axioms and rules of LiP differ from those of LiP in exactly Kripke's law, nominal peer review, and epistemic bitonicity (cf. [Kra12] and Section 3). In LiP, these three LiP-laws correspond to the generalised Kripke-law  $(M :_a^C (\phi \to \phi')) \to ((M' :_a^C \phi) \to (M, M') :_a^C \phi')$ , (plain) peer review  $(M :_a^C \phi) \to \bigwedge_{b \in \mathcal{C} \cup \{a\}} (\{M\}_a :_b^{\mathcal{C} \cup \{a\}})$  (a k  $M \land M :_a^C \phi$ ), and epistemic antitonicity "from  $a \mathrel{k} M \to a \mathrel{k} M'$  deduce  $(M' :_a^C \phi) \to M :_a^C \phi$ ", respectively. The addition of the axiom schema

$$(M::^{\mathcal{C}}_{a}\phi)\to (M,M')::^{\mathcal{C}}_{a}\phi$$

to LiiP will result in a logic LiiP<sup>+</sup> that is isomorphic to LiP (cf. Theorem 4). So in some sense, the essential difference between instant proofs (proofs for at least an instant) and persistent proofs (proofs for eternity) is distilled in this single additional law. Following Artëmov in [Art08], this law can be interpreted as Lehrer and Paxson's indefeasibility condition for justified true belief [Kra12]. In sum, while both LiP-proofs and LiiP-proofs are indefeasible in the instant when they are learnt (they induce knowledge, not only belief), LiiP-proofs (LiP-proofs) are possibly (necessarily) (in)defeasible in the future of the instant in which they are learnt.

Now note the following macro-definitions:  $\top := a \, \mathsf{k} \, a, \perp := \neg \top, \phi \lor \phi' := \neg (\neg \phi \land \neg \phi'), \phi \to \phi' := \neg \phi \lor \phi', \text{ and } \phi \leftrightarrow \phi' := (\phi \to \phi') \land (\phi' \to \phi).$  In the sequel, ":iff" abbreviates "by definition, if and only if".

Proposition 1 (Hilbert-style proof system). Let

- $\Phi \vdash_{\text{LiiP}} \phi$  :*iff if*  $\Phi \subseteq \text{LiiP}$  *then*  $\phi \in \text{LiiP}$
- $\phi \dashv \vdash_{\text{LiiP}} \phi'$  :*iff*  $\{\phi\} \vdash_{\text{LiiP}} \phi'$  and  $\{\phi'\} \vdash_{\text{LiiP}} \phi$
- $\vdash_{\text{LiiP}} \phi$  :*iff*  $\emptyset \vdash_{\text{LiiP}} \phi$ .

In other words,  $\vdash_{\text{LiiP}} \subseteq 2^{\mathcal{L}} \times \mathcal{L}$  is a system of closure conditions in the sense of [Tay99, Definition 3.7.4]. For example:

- 1. for all axioms  $\phi \in \Gamma_1$ ,  $\vdash_{\text{LiiP}} \phi$
- 2. for modus ponens,  $\{\phi, \phi \to \phi'\} \vdash_{\text{LiiP}} \phi'$
- 3. for necessitation,  $\{\phi\} \vdash_{\text{LiiP}} M ::_a^{\mathcal{C}} \phi$

4. for epistemic bitonicity,  $\{a \mid M \leftrightarrow a \mid M'\} \vdash_{\text{LiiP}} (M ::_a^{\mathcal{C}} \phi) \leftrightarrow M' ::_a^{\mathcal{C}} \phi$ .

(In the space-saving, horizontal Hilbert-notation " $\Phi \vdash_{\text{LiiP}} \phi$ ",  $\Phi$  is not a set of hypotheses but a set of premises, cf. modus ponens, necessitation, and epistemic bitonicity.) Then  $\vdash_{\text{LiiP}}$  can be viewed as being defined by a Cl-induced Hilbert-style proof system. In fact Cl :  $2^{\mathcal{L}} \rightarrow 2^{\mathcal{L}}$  is a standard consequence operator, *i.e.*, a substitution-invariant compact closure operator.

*Proof.* Like in [Kra12]. That a Hilbert-style proof system can be viewed as induced by a compact closure operator is well-known (e.g., see [Gab95]); that Cl is indeed such an operator can be verified by inspection of the inductive definition of Cl; and substitution invariance follows from our definitional use of axiom schemas.<sup>3</sup>

Corollary 1 (Normality). LiiP is a normal modal logic.

*Proof.* Jointly by Kripke's law, *modus ponens*, necessitation (these by definition), and substitution invariance (cf. Proposition 1).  $\Box$ 

We are now going to present some useful (further-used), deducible *structural* laws of LiiP. Here, "structural" means "deducible exclusively from term axioms". The laws are enumerated in a (total) order that respects (but cannot reflect) their respective proof prerequisites. The laws are also deducible in LiP, in the same order [Kra12]. (All LiiP-deducible laws are also LiP-deducible, but not vice versa.)

Theorem 1 (Some useful deducible structural laws).

- 1.  $\vdash_{\text{LiiP}} a \, \mathsf{k}(M, M') \to a \, \mathsf{k} \, M$  (left projection, 1-way K-combinator property)
- 2.  $\vdash_{\text{LiiP}} a \, \mathsf{k} \, (M, M') \to a \, \mathsf{k} \, M' \quad (right \ projection)$
- 3.  $\vdash_{\text{LiiP}} a \, \mathsf{k}(M, M) \leftrightarrow a \, \mathsf{k} \, M$  (pairing idempotency)
- 4.  $\vdash_{\text{LiiP}} a \, \mathsf{k}(M, M') \leftrightarrow a \, \mathsf{k}(M', M)$  (pairing commutativity)
- 5.  $\vdash_{\text{LiiP}} (a \, \mathsf{k} \, M \to a \, \mathsf{k} \, M') \leftrightarrow (a \, \mathsf{k} \, (M, M') \leftrightarrow a \, \mathsf{k} \, M)$  (neutral pair elements)
- 6.  $\vdash_{\text{LiiP}} a \mathrel{\mathsf{k}}(M, a) \leftrightarrow a \mathrel{\mathsf{k}} M$  (self-neutral pair element)
- 7.  $\vdash_{\text{LiiP}} a \, \mathsf{k} \, (M, (M', M'')) \leftrightarrow a \, \mathsf{k} \, ((M, M'), M'')$  (pairing associativity)
- 8.  $\vdash_{\text{LiiP}} ((M, M) ::_{a}^{\mathcal{C}} \phi) \leftrightarrow M ::_{a}^{\mathcal{C}} \phi \quad (proof \ idempotency)$
- 9.  $\vdash_{\text{LiiP}} ((M, M') :::_a^{\mathcal{C}} \phi) \leftrightarrow (M', M) ::_a^{\mathcal{C}} \phi \quad (proof \ commutativity)$
- 10.  $\{a \, \mathsf{k} \, M \to a \, \mathsf{k} \, M'\} \vdash_{\mathrm{LiiP}} ((M, M') ::_a^{\mathcal{C}} \phi) \leftrightarrow M ::_a^{\mathcal{C}} \phi \quad (neutral \ proof \ elements)$

<sup>&</sup>lt;sup>3</sup>Alternatively to axiom schemas, we could have used axioms together with an additional substitution-rule set {  $\sigma[\phi] \mid \phi \in Cl^n(\Gamma)$  } in the definients of  $Cl^{n+1}(\Gamma)$ .

- 11.  $\vdash_{\text{LiiP}} ((M, a) :: \overset{c}{a} \phi) \leftrightarrow M :: \overset{c}{a} \phi \quad (self\text{-}neutral \ proof \ element)$
- $12. \vdash_{\text{LiiP}} ((M, (M', M'')) :: \overset{c}{\underset{a}{\circ}} \phi) \leftrightarrow ((M, M'), M'') :: \overset{c}{\underset{a}{\circ}} \phi \quad (proof \ associativity)$
- 13.  $\vdash_{\text{LiiP}} (\{\!\![M]\!\!]_a :: \stackrel{\mathcal{C}}{::} \phi) \leftrightarrow M :: \stackrel{\mathcal{C}}{:} \phi \quad (self\text{-signing idempotency})$

*Proof.* Laws 1–7 and 13 are proved like in LiP [Kra12], as LiiP and LiP have identical term axioms. Law 8, 9, 11, and 12 follows immediately from Law 3, 4, 6, and 7, respectively by epistemic bitonicity. For Law 10, suppose that  $\vdash_{\text{LiiP}} a \mathrel{k} M \rightarrow a \mathrel{k} M'$ . Hence  $\vdash_{\text{LiiP}} a \mathrel{k} (M, M') \leftrightarrow a \mathrel{k} M$  by the law of neutral pair elements and propositional logic. Hence  $\vdash_{\text{LiiP}} (M, M') ::_{a}^{c} \phi \leftrightarrow M ::_{a}^{c} \phi$  by epistemic bitonicity.

Like in LiP [Kra12], the preceding 1-way K-combinator property and the following simple corollary of Theorem 1 jointly establish the important fact that our communicating agents can be viewed as combinators in the sense of Combinatory Logic viewed in turn as a (non-equational) theory of (message or proof) term reduction [HS08]. (The converse of the above K-combinator property does not hold.)

Corollary 2 (S-combinator property).

- $1. \vdash_{\mathrm{LiiP}} a \, \mathsf{k} \left( (M, M'), M'' \right) \leftrightarrow a \, \mathsf{k} \left( M, (M'', (M', M'')) \right)$
- $\mathcal{2}. \ \vdash_{\mathrm{LiiP}} (((M,M'),M'') ::_a^{\mathcal{C}} \phi) \leftrightarrow (M,(M'',(M',M''))) ::_a^{\mathcal{C}} \phi$

*Proof.* 1 follows jointly from idempotency (copy M'''), commutativity, and associativity of pairing; and 2 follows jointly from 1 and epistemic bitonicity.  $\Box$ 

We are going to present also some useful (further-used) deducible *logical* laws of LiiP. Here, "logical" means "not structural" in the previously defined sense. Also these laws are enumerated in an order that respects their respective proof prerequisites, and are deducible in LiP in the same order [Kra12].

Theorem 2 (Some useful deducible logical laws).

- 1.  $\{\phi \to \phi'\} \vdash_{\text{LiiP}} (M ::_a^{\mathcal{C}} \phi) \to M ::_a^{\mathcal{C}} \phi' \quad (regularity)$
- 2.  $\{a \, \mathsf{k} \, M \leftrightarrow a \, \mathsf{k} \, M', \phi \to \phi'\} \vdash_{\mathrm{LiiP}} (M ::_a^{\mathcal{C}} \phi) \to M' ::_a^{\mathcal{C}} \phi' \text{ (biepistemic regul.)}$
- 3.  $\vdash_{\text{LiiP}} ((M ::_a^{\mathcal{C}} \phi) \land M ::_a^{\mathcal{C}} \phi') \leftrightarrow M ::_a^{\mathcal{C}} (\phi \land \phi') \quad (proof \ conjunctions \ bis)$
- 4.  $\vdash_{\text{LiiP}} ((M ::_a^{\mathcal{C}} \phi) \lor M ::_a^{\mathcal{C}} \phi') \to M ::_a^{\mathcal{C}} (\phi \lor \phi') \quad (proof \ disjunctions \ bis)$
- 5.  $\vdash_{\text{LiiP}} M ::_a^{\mathcal{C}} \top$  (anything can prove tautological truth)
- 6.  $\vdash_{\text{LiiP}} \{\!\!\{M\}\!\!\}_b :: \stackrel{\mathcal{C} \cup \{b\}}{a} b \, \mathsf{k} \, M \quad (authentic \, knowledge)$
- 7.  $\vdash_{\text{LiiP}} M :: \overset{\emptyset}{a} \land M \quad (self-knowledge)$
- 8.  $\vdash_{\text{LiiP}} (M ::_{a}^{\mathcal{C} \cup \mathcal{C}'} \phi) \to ((M ::_{a}^{\mathcal{C}} \phi) \land M ::_{a}^{\mathcal{C}'} \phi) \quad (group \ decomposition \ \text{bis})$

9.  $\vdash_{\text{LiiP}} (M ::_a^{\mathcal{C} \cup \{a\}} \phi) \leftrightarrow (M ::_a^{\mathcal{C}} \phi)$  (self-neutral group element).

10.  $\vdash_{\text{LiP}} M ::_{a}^{\mathcal{C}} ((M ::_{a}^{\mathcal{C}} \phi) \to \phi) \quad (self\text{-proof of truthfulness})$ 

11.  $\vdash_{\text{LiP}} M ::_{a}^{\mathcal{C}} (\neg(M ::_{a}^{\mathcal{C}} \bot)) \quad (self\text{-proof of proof consistency})$ 

12.  $\vdash_{\text{LiP}} (M ::_a^{\mathcal{C}} (M ::_a^{\mathcal{C}} \phi)) \leftrightarrow M ::_a^{\mathcal{C}} \phi \pmod{add idempotency}$ 

#### Proof. Like in LiP [Kra12].

Like in LiP, the key to the validity of modal idempotency is that each agent (e.g., a) can act herself as proof-checker, see [Kra12, Section 3.2.2] for more details.

We now continue to (re)present the constructive semantics for LiiP (cf. [Kra12, Section 2.2]) and establish some important new and further-used results about it. The essential differences to the semantics of LiP are grey-shaded.

**Definition 3** (Semantic ingredients). For the knowledge-constructive modeltheoretic study of LiiP let

- S designate the state space—a set of system states s
- msgs<sub>a</sub> :  $S \to 2^{\mathcal{M}}$  designate a *raw-data extractor* that extracts (without analysing) the (finite) set of messages from a system state *s* that agent  $a \in \mathcal{A}$  has either generated (assuming that only *a* can generate *a*'s signature) or else received *as such* (not only as a strict subterm of another message); that is, msgs<sub>a</sub>(s) is *a*'s *data base* in *s*
- $\operatorname{cl}_a^s : 2^{\mathcal{M}} \to 2^{\mathcal{M}}$  designate a *data-mining operator* such that  $\operatorname{cl}_a^s(\mathcal{D}) := \operatorname{cl}_a(\operatorname{msgs}_a(s) \cup \mathcal{D}) := \bigcup_{n \in \mathbb{N}} \operatorname{cl}_a^n(\operatorname{msgs}_a(s) \cup \mathcal{D})$ , where for all  $\mathcal{D} \subseteq \mathcal{M}$ :

 $\begin{array}{lll} \mathrm{cl}_{a}^{0}(\mathcal{D}) &:= & \{a\} \cup \mathcal{D} \\ \mathrm{cl}_{a}^{n+1}(\mathcal{D}) &:= & \mathrm{cl}_{a}^{n}(\mathcal{D}) \cup \\ & \left\{ \begin{array}{l} (M,M') \mid \{M,M'\} \subseteq \mathrm{cl}_{a}^{n}(\mathcal{D}) \end{array} \right\} \cup & (\mathrm{pairing}) \\ & \left\{ \begin{array}{l} M,M' \mid (M,M') \in \mathrm{cl}_{a}^{n}(\mathcal{D}) \end{array} \right\} \cup & (\mathrm{unpairing}) \\ & \left\{ \begin{array}{l} \{M\}_{a} \mid M \in \mathrm{cl}_{a}^{n}(\mathcal{D}) \end{array} \right\} \cup & (\mathrm{personal \ signature \ synthesis}) \\ & \left\{ \begin{array}{l} (M,b) \mid \{M\}_{b} \in \mathrm{cl}_{a}^{n}(\mathcal{D}) \end{array} \right\} & (\mathrm{universal \ signature \ analysis}) \end{array} \right. \end{array}$ 

- $<_a^M \subseteq S \times S$  designate a *data preorder* on states such that for all  $s, s' \in S$ ,  $s <_a^M s'$  :iff  $cl_a^s(\{M\}) = cl_a^{s'}(\emptyset)$ , were M can be viewed as *oracle input* in addition to *a*'s *individual-knowledge base*  $cl_a^s(\emptyset)$  (cf. also [Kra12, Section 2.2])
- $<^M_{\mathcal{C}} := (\bigcup_{a \in \mathcal{C}} <^M_a)^{++}$ , where '++' designates the closure operation of so-called *generalised transitivity* in the sense that  $<^M_{\mathcal{C}} \circ <^M_{\mathcal{C}} \subseteq <^{(M,M')}_{\mathcal{C}}$
- $\equiv_a := \langle_a^a$  designate an equivalence relation of state indistinguishability

•  ${}_{M}\mathbf{R}_{a}^{\mathcal{C}} \subseteq \mathcal{S} \times \mathcal{S}$  designate a *concretely constructed* accessibility relation—short, *concrete* accessibility—for the proof modality such that for all  $s, s' \in \mathcal{S}$ ,

$$s {}_{M}\mathbf{R}_{a}^{\mathcal{C}} s' \quad :\text{iff} \quad s' \in \bigcup_{a} [\tilde{s}]_{\equiv_{a}}$$

$$s <_{\mathcal{C} \cup \{a\}}^{M} \tilde{s} \text{ and}$$

$$M \in \mathrm{cl}_{a}^{\tilde{s}}(\emptyset)$$
(iff there is  $\tilde{s} \in \mathcal{S} \text{ s.t. } s <_{\mathcal{C} \cup \{a\}}^{M} \tilde{s} \text{ and } M \in \mathrm{cl}_{a}^{\tilde{s}}(\emptyset) \text{ and } \tilde{s} \equiv_{a} s'$ )

Note that the data-mining operator  $\operatorname{cl}_a : 2^{\mathcal{M}} \to 2^{\mathcal{M}}$  is a compact closure operator, which induces a *data-derivation* relation  $\vdash_a \subseteq 2^{\mathcal{M}} \times \mathcal{M}$  such that  $\mathcal{D} \vdash_a M$  :iff  $M \in \operatorname{cl}_a(\mathcal{D})$ , which (1) has the compactness and (2) the cut property, (3) is decidable in deterministic polynomial time in the size of  $\mathcal{D}$  and M, and (4) induces a Scott information system of information tokens M [Kra12]. Fact 1 establishes the knowledge-constructiveness of our Kripke-model for LiiP (cf. Definition 5).

Fact 1 (Kripke-model knowledge-constructiveness).

for all 
$$s' \in S$$
, if  $s {}_{M}\mathbf{R}^{\mathcal{C}}_{a} s'$  then  $(\mathfrak{S}, \mathcal{V}), s' \models \phi$  if and only if  
for all  $\check{s} \in S$ , if  $s <^{\mathcal{M}}_{\mathcal{C} \cup \{a\}} \check{s}$  then  $(\mathfrak{S}, \mathcal{V}), \check{s} \models a \not{k} \underbrace{\mathcal{M}}_{M} \to \mathsf{K}_{a}(\underbrace{\phi}),$   
sufficient  
evidence induced

where the standard epistemic modality  $K_a$  is defined like in [MV07] as

$$(\mathfrak{S}, \mathcal{V}), \check{s} \models \mathsf{K}_{a}(\phi)$$
 :iff for all  $s' \in \mathcal{S}$ , if  $\check{s} \equiv_{a} s'$  then  $(\mathfrak{S}, \mathcal{V}), s' \models \phi$ .

*Proof.* By elementary-logical transformations of the definiens of  ${}_{M}\mathbf{R}_{a}^{\mathcal{C}}$ .

**Lemma 1.** If  $s <_a^M s'$  then  $s' <_a^M s'$ .

*Proof.* Consider that when  $s <_a^M s', M \in cl_a^{s'}(\emptyset)$ , and thus  $cl_a^{s'}(\{M\}) = cl_a^{s'}(\emptyset)$ .

Proposition 2 (Restricted reflexivity).

- 1.  $s <_a^a s$  (self-reflexivity)
- 2. biconditional reflexivity:
  - (a)  $s <_a^M s$  if and only if  $M \in cl_a^s(\emptyset)$ (b)  $s <_a^M s$  if and only if there is  $s' \in S$  such that  $s' <_a^M s$

*Proof.* For 1, consider that  $a \in cl_a^s(\emptyset)$ , and thus  $cl_a^s(\{a\}) = cl_a^s(\emptyset)$ . For 2.a, inspect the proof of Lemma 1. For the forward-direction of 2.b, take s as s'; and for the backward-direction apply Lemma 1.

Proposition 3 (Self-symmetry).

If 
$$s <^a_a s'$$
 then  $s' <^a_a s$ .

*Proof.* By expansion of the definition of  $(<_a^a)$  and the symmetry of equality.  $\Box$ 

Proposition 4 (Generalised transitivity).

If 
$$s <_a^M s'$$
 and  $s' <_a^{M'} s''$  then  $s <_a^{(M,M')} s''$ .

*Proof.* Let  $s, s' \in S$  and suppose that  $s <_a^M s'$  and  $s' <_a^{M'} s''$ . Thus:

- 1.  $\operatorname{cl}_a^s(\{M\}) = \operatorname{cl}_a^{s'}(\emptyset)$ ; thus  $M \in \operatorname{cl}_a^{s'}(\emptyset)$ , thus:
  - (a)  $M \in \operatorname{cl}_a^{s'}(\{M'\})$  by closure monotonicity  $(\emptyset \subseteq \{M'\})$ ,
  - (b)  $\operatorname{cl}_{a}^{s'}(\emptyset) = \operatorname{cl}_{a}^{s'}(\{M\})$ , thus  $\operatorname{cl}_{a}^{s}(\{M\}) = \operatorname{cl}_{a}^{s'}(\{M\})$ , and hence  $\operatorname{cl}_{a}^{s}(\{(M, M')\}) = \operatorname{cl}_{a}^{s'}(\{(M, M')\});$
- 2.  $\operatorname{cl}_{a}^{s'}(\{M'\}) = \operatorname{cl}_{a}^{s''}(\emptyset)$ ; thus  $M' \in \operatorname{cl}_{a}^{s''}(\emptyset)$ , thus  $\operatorname{cl}_{a}^{s''}(\emptyset) = \operatorname{cl}_{a}^{s''}(\{M'\})$ , thus  $\operatorname{cl}_{a}^{s'}(\{M'\}) = \operatorname{cl}_{a}^{s''}(\{M'\})$ , and hence  $\operatorname{cl}_{a}^{s'}(\{(M,M')\}) = \operatorname{cl}_{a}^{s''}(\{(M,M')\})$ .

Hence:

- $M \in cl_a^{s''}(\emptyset)$  by 1.a and the first assertion in 2, thus  $(M, M') \in cl_a^{s''}(\emptyset)$  by the second assertion in 2 and pairing closure, thus  $cl_a^{s''}(\emptyset) = cl_a^{s''}(\{(M, M')\});$
- $cl_a^s(\{(M, M')\}) = cl_a^{s''}(\{(M, M')\})$  by 1.b and 2.

Hence  $cl_a^s(\{(M, M')\}) = cl_a^{s''}(\emptyset)$ , and thus  $s <_a^{(M,M')} s''$  by definition.  $\Box$ Corollary 3 (Transitivity).

If  $s <_a^M s'$  and  $s' <_a^M s''$  then  $s <_a^M s''$ .

*Proof.* Directly from Proposition 4 by the fact that  $cl_a^s(\{(M, M)\}) = cl_a^s(\{M\})$ .

So as announced in Definition 3,  ${}^{\cdot}<_{a}^{M}{}^{\cdot}$  is indeed a (non-reflexive) pre-order, and  ${}^{\cdot}<_{a}{}^{a}{}^{\cdot}$  indeed an equivalence relation (cf. Proposition 2.i and 3).

**Definition 4** (Message ordering and equivalence).

- $M \sqsubseteq_a^s M'$  : iff if  $M \in cl_a^s(\emptyset)$  then  $M' \in cl_a^s(\emptyset)$
- $M \equiv_a^s M'$  : iff  $M \sqsubseteq_a^s M'$  and  $M' \sqsubseteq_a^s M$
- $M \sqsubseteq_a M'$  :iff for all  $s \in \mathcal{S}, M \sqsubseteq_a^s M'$
- $M \equiv_a M'$  : iff for all  $s \in \mathcal{S}, M \equiv_a^s M'$

**Fact 2.**  $\sqsubseteq_a^s \subseteq \mathcal{M} \times \mathcal{M}$  is a pre- but not a partial order.

Proposition 5 (Conditional stability).

If 
$$M \equiv_a M'$$
 then  $<^M_a = <^{M'}_a$ 

*Proof.* Suppose that for all  $s'' \in \mathcal{S}$ ,  $M \in \operatorname{cl}_a^{s''}(\emptyset)$  if and only if  $M' \in \operatorname{cl}_a^{s''}(\emptyset)$ , and let  $s, s' \in \mathcal{S}$ . For the  $\subseteq$ -part, suppose that  $s <_a^M s'$ , i.e.,  $\operatorname{cl}_a^s(\{M\}) = \operatorname{cl}_a^{s'}(\emptyset)$ , and thus  $M \in \operatorname{cl}_a^{s'}(\emptyset)$ . Hence:

- 1.  $M' \in \operatorname{cl}_a^{s'}(\emptyset)$  by particularisation of the first hypothesis, and  $(M, M') \in \operatorname{cl}_a^{s'}(\emptyset)$  by pairing closure; and thus  $\operatorname{cl}_a^{s'}(\{(M, M')\}) = \operatorname{cl}_a^{s'}(\emptyset)$ ;
- 2.  $M \in \operatorname{cl}_a^s(\emptyset)$  if and only if  $M' \in \operatorname{cl}_a^s(\emptyset)$  by particularisation of the first hypothesis, thus  $M \in \operatorname{cl}_a^s(\{M'\})$  if and only if  $M' \in \operatorname{cl}_a^s(\{M'\})$ , thus  $M \in \operatorname{cl}_a^s(\{M'\})$ , and thus  $\operatorname{cl}_a^s(\{M'\}) = \operatorname{cl}_a^s(\{(M,M')\})$ ;
- 3.  $\operatorname{cl}_{a}^{s'}(\{M\}) = \operatorname{cl}_{a}^{s'}(\emptyset)$ , thus  $\operatorname{cl}_{a}^{s'}(\{M\}) = \operatorname{cl}_{a}^{s}(\{M\})$ , and thus  $\operatorname{cl}_{a}^{s'}(\{(M, M')\}) = \operatorname{cl}_{a}^{s}((M, M'))$ .

Hence  $cl_a^s(\{M'\}) = cl_a^{s'}(\emptyset)$  by 1, 2, and 3. And symmetrically for the  $\supseteq$ -part.  $\Box$ 

Proposition 6 (Communal lifting).

- 1. If  $\mathcal{C} \subseteq \mathcal{C}'$  then  $<^M_{\mathcal{C}} \subseteq <^M_{\mathcal{C}'}$  (communal monotonicity).
- 2. If  $M \in cl_a^s(\emptyset)$  then  $s <_{C \cup \{a\}}^M s$  (conditional reflexivity).
- 3. If  $M \equiv_a M'$  then  $<^M_{\mathcal{C} \cup \{a\}} = <^{M'}_{\mathcal{C} \cup \{a\}}$  (conditional stability).

*Proof.* 1 follows directly from definitions, 2 from 1 and Proposition 2.ii.a, and 3 from Proposition 5 and the definition of  $\binom{M}{\mathcal{C} \cup \{a\}}$  and  $\binom{M'}{\mathcal{C} \cup \{a\}}$ .

Proposition 7 (Signature property).

Proof. Let  $s, s' \in S$  and suppose that  $s <_{\mathcal{C}}^{\{\!\!\!\ m\}\!\!\!\ m\}_a} s'$ . Thus there is  $b \in \mathcal{C}$  such that  $s <_b^{\{\!\!\!\ m\}\!\!\!\ m\}_a} s'$ . Hence  $\{\!\!\!\ m\}\!\!\!\ m\}_a \in \operatorname{cl}_b^{s'}(\emptyset)$  by biconditional reflexivity (cf. Proposition 2.ii.a). But then also  $M \in \operatorname{cl}_a^{s'}(\emptyset)$  by the unforgeability of signatures (cf. the closure conditions of personal/universal signature synthesis/analysis). That is, nobody else than a can have generated  $\{\!\!\!\ m\}\!\!\!\ m\}_a$ , and thus a also knows M. (Otherwise suppose that somebody else has, and derive a contradiction.)

Corollary 4 (Concrete accessibility).

- 1. If  $C \subseteq C'$  then  ${}_{M}\mathbf{R}^{C}_{a} \subseteq {}_{M}\mathbf{R}^{C'}_{a}$  (communal monotonicity).
- 2. If  $M \equiv_a M'$  then  ${}_M R^{\mathcal{C}}_a = {}_M \cdot R^{\mathcal{C}}_a$  (conditional stability).
- 3. If  $M \in cl_a^s(\emptyset)$  then  $s_M R_a^{\mathcal{C}} s$  (conditional reflexivity).

Table 1: Satisfaction relation

$(\mathfrak{S},\mathcal{V}),s\models P$	:iff	$s \in \mathcal{V}(P)$
$(\mathfrak{S},\mathcal{V}),s\models\neg\phi$	:iff	not $(\mathfrak{S}, \mathcal{V}), s \models \phi$
$(\mathfrak{S},\mathcal{V}),s\models\phi\wedge\phi'$	:iff	$(\mathfrak{S},\mathcal{V}),s\models\phi\text{ and }(\mathfrak{S},\mathcal{V}),s\models\phi'$
$(\mathfrak{S},\mathcal{V}),s\models M::_a^{\mathcal{C}}\phi$	:iff	for all $s' \in \mathcal{S}$ , if $s \ _M \mathcal{R}_a^{\mathcal{C}} s'$ then $(\mathfrak{S}, \mathcal{V}), s' \models \phi$

- 4. If  $s_{\text{IM}} \mathbb{R}^{\mathcal{C}}_{a} s'$  then  $M \in \mathrm{cl}_{b}^{s'}(\emptyset)$  (signature property).
- 5. For all  $b \in \mathcal{C} \cup \{a\}$ ,  $({}_{[M]}_{a} \mathbf{R}_{b}^{\mathcal{C} \cup \{a\}} \circ {}_{M} \mathbf{R}_{a}^{\mathcal{C}}) \subseteq {}_{(M,b)} \mathbf{R}_{a}^{\mathcal{C}}$  (communal transitivity).

*Proof.* 1–4 follow by inspection of definitions and Proposition 6 and 7. For 5, suppose that  $b \in \mathcal{C} \cup \{a\}$  and let  $s, s', s'' \in \mathcal{S}$ . Further suppose that  $s_{[M]_a} \mathcal{R}_b^{\mathcal{C} \cup \{a\}}$  s' and  $s'_{MR_a}^{\mathcal{C}} s''$ . That is, (there is  $\tilde{s} \in \mathcal{S}$  such that  $s <_{\mathcal{C} \cup \{a\} \cup \{b\}}^{[M]_a} \tilde{s}$  and  $[M]_a \in \mathcal{C}_{b}^{\tilde{s}}(\emptyset)$  and  $\tilde{s} \equiv_b s'$ ) and (there is  $\tilde{s}' \in \mathcal{S}$  such that  $s' <_{\mathcal{C} \cup \{a\} \cup \{b\}}^{\mathcal{M}} \tilde{s}'$  and  $M \in \mathcal{cl}_a^{\tilde{s}'}(\emptyset)$  and  $\tilde{s}' \equiv_a s''$ ). Hence,  $s <_{\mathcal{C} \cup \{a\}}^{[M]_a} \tilde{s}$  by the first supposition and communal monotonicity ( $\mathcal{C} \cup \{a\} \cup \{b\} = \mathcal{C} \cup \{a\}$ ), and also  $\tilde{s} <_b^b s'$  by definition (cf. second supposition). Hence consecutively,  $\tilde{s} <_{\mathcal{C} \cup \{a\}}^{b} s'$  by the first supposition and communal monotonicity ( $\{b\} \subseteq \mathcal{C} \cup \{a\}$ ),  $s <_{\mathcal{C} \cup \{a\}}^{([M]_a, b)} s'$  by generalised transitivity,  $s <_{\mathcal{C} \cup \{a\}}^{(([M]_a, b), M)} \tilde{s}'$  by the third supposition and again generalised transitivity,  $s <_{\mathcal{C} \cup \{a\}}^{(M,b)} \tilde{s}'$  by conditional stability ((( $\{[M]_a, b), M$ ) ≡<sub>a</sub> (M, b)), and thus finally  $s_{(M,b]} \mathcal{R}_a^{\mathcal{C}} s''$  by again the third supposition. □

**Definition 5** (Kripke-model). We define the *satisfaction relation* ' $\models$ ' for LiiP in Table 1, where

•  $\mathcal{V}: \mathcal{P} \to 2^{\mathcal{S}}$  designates a usual valuation function, yet partially predefined such that for all  $a \in \mathcal{A}$  and  $M \in \mathcal{M}$ ,

$$\mathcal{V}(a \,\mathsf{k}\, M) := \{ s \in \mathcal{S} \mid M \in \mathrm{cl}_a^s(\emptyset) \}$$

(If agents are Turing-machines then a knowing M can be understood as a being able to parse M on its tape.)

•  $\mathfrak{S} := (\mathcal{S}, \{{}_{M}\mathcal{R}_{a}^{\mathcal{C}}\}_{M \in \mathcal{M}, a \in \mathcal{A}, \mathcal{C} \subseteq \mathcal{A}})$  designates a (modal) frame for LiiP with an **abstractly constrained** accessibility relation—short, **abstract** accessibility— ${}_{M}\mathcal{R}_{a}^{\mathcal{C}} \subseteq \mathcal{S} \times \mathcal{S}$  for the proof modality such that—the semantic interface:

$$- \text{ if } \mathcal{C} \subseteq \mathcal{C}' \text{ then } _{M}\mathcal{R}_{a}^{\mathcal{C}} \subseteq {}_{M}\mathcal{R}_{a}^{\mathcal{C}'}$$
$$- \text{ if } M \equiv_{a} M' \text{ then } _{M}\mathcal{R}_{a}^{\mathcal{C}} = {}_{M'}\mathcal{R}_{a}^{\mathcal{C}}$$

$$- \text{ if } M \in \operatorname{cl}_{a}^{s}(\emptyset) \text{ then } s \ _{M}\mathcal{R}_{a}^{\mathcal{C}} s$$

$$- \text{ if } s \ _{\llbracket M \rrbracket_{b}}\mathcal{R}_{a}^{\mathcal{C}} s' \text{ then } M \in \operatorname{cl}_{b}^{s'}(\emptyset)$$

$$- \text{ for all } b \in \mathcal{C} \cup \{a\}, (_{\llbracket M \rrbracket_{a}}\mathcal{R}_{b}^{\mathcal{C} \cup \{a\}} \circ _{M}\mathcal{R}_{a}^{\mathcal{C}}) \subseteq {}_{(M,b)}\mathcal{R}_{a}^{\mathcal{C}}$$

•  $(\mathfrak{S}, \mathcal{V})$  designates a (modal) *model* for LiiP.

Looking back, we recognise that Corollary 4 actually establishes the important fact that our concrete accessibility  ${}_{M}\mathbf{R}_{a}^{\mathcal{C}}$  in Definition 3 realises all the properties stipulated by our abstract accessibility  ${}_{M}\mathcal{R}_{a}^{\mathcal{C}}$  in Definition 5; we say that

 ${}_{M}\mathbf{R}_{a}^{\mathcal{C}}$  exemplifies (or realises)  ${}_{M}\mathcal{R}_{a}^{\mathcal{C}}$ 

Further, observe that LiiP (like LiP) has a Herbrand-style semantics, i.e., logical constants (agent names) and functional symbols (pairing, signing) are selfinterpreted rather than interpreted in terms of (other, semantic) constants and functions. This simplifying design choice spares our framework from the additional complexity that would arise from term-variable assignments [BG07], which in turn keeps our models propositionally modal. Our choice is admissible because our individuals (messages) are finite. (Infinitely long "messages" are non-messages; they can never be completely received, e.g., transmitting irrational numbers as such is impossible.)

**Theorem 3** (Axiomatic adequacy).  $\vdash_{\text{LiiP}}$  is adequate for  $\models$ , *i.e.*,:

- 1. if  $\vdash_{\text{LiiP}} \phi$  then  $\models \phi$  (axiomatic soundness)
- 2. if  $\models \phi$  then  $\vdash_{\text{LiiP}} \phi$  (semantic completeness).

*Proof.* Both parts can be proved with standard means: soundness follows as usual from the admissibility of the axioms and rules (cf. Appendix A.1); and completeness follows by means of the classical construction of canonical models, using Lindenbaum's construction of maximally consistent sets (cf. Appendix A.2).

## 3 LiP as an extension of LiiP

In this section, we reconstruct LiP syntactically, as a minimal conservative extension of LiiP with one simplified and one additional axiom schema, as well as semantically, with a simplified semantic interface that has none of the *a posteriori* constraints from [Kra12] but only standard, *a priori* constraints, i.e., stipulations.

Theorem 4. Define the LiiP-theory

$$\operatorname{LiiP}^{+} := \operatorname{Cl}(\{\underbrace{(M::_{a}^{\mathcal{C}}\phi) \to (M,M')::_{a}^{\mathcal{C}}\phi}_{proof \ extension}\}),$$

where Cl is as in Definition 2. Then  $\text{LiiP}^+$  is isomorphic to LiP, in symbols,

$$\operatorname{LiiP}^+ \cong \operatorname{LiP}$$

In particular, the generalised Kripke law GK as mentioned before and below is deducible in LiiP<sup>+</sup>, and thus we need only stipulate the simpler standard Kripke law K for LiP, like for LiiP. Moreover, alternatively to adding the axiom schema of proof extension to LiiP, we could equivalently replace the primitive rule schema of epistemic bitonicity in LiiP with the stronger one of epistemic antitonicity.

*Proof.* The isomorphism consists in simply switching between proof-modality notations, which in LiP is ':' and in LiiP<sup>+</sup> '::'. Then, as already mentioned on Page 7, LiiP and LiP differ in the following corresponding axiom and deduction-rule schemas: Kripke's law K versus the generalised Kripke-law GK, nominal peer review (NPR) versus plain peer review, and epistemic bitonicity versus epistemic antitonicity—see below. Note that in the sequel PL abbreviates "(Classical) Propositional Logic," and  $\vdash_{\text{LiiP}^+}$  is defined similarly to  $\vdash_{\text{LiiP}}$ .

• GK (cf. Line 7) becomes deducible:

1. $\vdash_{\text{LiiP}^+} (M ::_a^{\mathcal{C}} (\phi \to \phi')) \to (M, M') ::_a^{\mathcal{C}} (\phi \to \phi')$	$proof\ extension$
$2. ~\vdash_{\mathrm{LiiP}^+} ((M,M') ::_a^{\mathcal{C}} (\phi \to \phi')) \to (((M,M') ::_a^{\mathcal{C}} \phi)$	$\to (M,M') \mathop{::}\nolimits_a^{\mathcal C} \phi') \to K$
3. $\vdash_{\mathrm{LiiP}^+} (M ::_a^{\mathcal{C}} (\phi \to \phi')) \to (((M, M') ::_a^{\mathcal{C}} \phi) \to (M)$	$(M, M') ::_a^{\mathcal{C}} \phi') 1, 2 PL$
4. $\vdash_{\operatorname{LiiP}^+} (M' ::_a^{\mathcal{C}} \phi) \to (M', M) ::_a^{\mathcal{C}} \phi$	$proof\ extension$
5. $\vdash_{\operatorname{LiiP}^+} ((M', M) ::_a^{\mathcal{C}} \phi) \leftrightarrow (M, M') ::_a^{\mathcal{C}} \phi$	proof commutativity
6. $\vdash_{\operatorname{LiiP}^+} (M' ::_a^{\mathcal{C}} \phi) \to (M, M') ::_a^{\mathcal{C}} \phi$	4, 5, PL
7. $\vdash_{\mathrm{LiiP}^+} (M ::_a^{\mathcal{C}} (\phi \to \phi')) \to ((M' ::_a^{\mathcal{C}} \phi) \to (M, M')$	$)::_{a}^{\mathcal{C}}\phi')\qquad 3,6,\mathrm{PL}.$

- plain peer review (cf. Line 3) becomes deducible:
  - $$\begin{split} 1. & \vdash_{\mathrm{LiiP}^+} \bigwedge_{b \in \mathcal{C} \cup \{a\}} ((M ::_a^{\mathcal{C}} \phi) \to (M, b) ::_a^{\mathcal{C}} \phi) & \textit{proof extension} \\ 2. & \vdash_{\mathrm{LiiP}^+} \bigwedge_{b \in \mathcal{C} \cup \{a\}} (((M, b) ::_a^{\mathcal{C}} \phi) \to \{\!\!\{M\}\!\!\}_a ::_b^{\mathcal{C} \cup \{a\}} (a \models M \land M ::_a^{\mathcal{C}} \phi)) \\ 3. & \vdash_{\mathrm{LiiP}^+} (M ::_a^{\mathcal{C}} \phi) \to \bigwedge_{b \in \mathcal{C} \cup \{a\}} (\{\!\!\{M\}\!\!\}_a ::_b^{\mathcal{C} \cup \{a\}} (a \models M \land M ::_a^{\mathcal{C}} \phi)) & 1, 2, \\ \text{PL.} \end{split}$$
- epistemic antitonicity (cf. Line 8) becomes deducible:

1.	$\vdash_{\mathrm{LiiP}^+} a k M \to a k M'$	hyp.
2.	$\vdash_{\mathrm{LiiP}^+} ((M,M') ::_a^{\mathcal{C}} \phi) \leftrightarrow M ::_a^{\mathcal{C}} \phi$	1, neutral proof elements
3.	$\vdash_{\mathrm{LiiP}^+} (M' ::_a^{\mathcal{C}} \phi) \to (M', M) ::_a^{\mathcal{C}} \phi$	$proof\ extension$
4.	$\vdash_{\mathrm{LiiP}^+} ((M',M) ::_a^{\mathcal{C}} \phi) \leftrightarrow (M,M') ::_a^{\mathcal{C}} \phi$	proof commutativity
5.	$\vdash_{\mathrm{LiiP}^+} (M' ::_a^{\mathcal{C}} \phi) \to (M, M') ::_a^{\mathcal{C}} \phi$	3, 4, PL
6.	$\vdash_{\mathrm{LiiP}^+} (M' ::_a^{\mathcal{C}} \phi) \to M ::_a^{\mathcal{C}} \phi$	2, 5, PL

7. if 
$$\vdash_{\text{LiiP}^+} a \, \mathsf{k} \, M \to a \, \mathsf{k} \, M'$$
 then  $\vdash_{\text{LiiP}^+} (M' :::_a^c \phi) \to M ::_a^c \phi$  1–6, PL  
8.  $\{a \, \mathsf{k} \, M \to a \, \mathsf{k} \, M'\} \vdash_{\text{LiiP}^+} (M' ::_a^c \phi) \to M ::_a^c \phi$  7, def.

Conversely, that is, assuming epistemic antitonicity, proof extension is directly deducible from jointly this assumption and (pair) left projection, like in LiP [Kra12].

**Corollary 5** (Simplified semantic interface for LiP). A simplified semantic interface for LiP is given by the one for LiP in Definition 5 but with the abstract accessibility  ${}_{M}\mathcal{R}_{a}^{\mathcal{C}} \subseteq \mathcal{S} \times \mathcal{S}$  being constrained

• such that if  $M \sqsubseteq_a M'$  then  ${}_M\mathcal{R}^{\mathcal{C}}_a \subseteq {}_M\mathcal{R}^{\mathcal{C}}_a$  (proof monotonicity)

 ${\bf instead} \ {\bf of} \ being \ constrained \ by \ conditional \ stability;$ 

• or alternatively such that  $(M,M')\mathcal{R}_a^{\mathcal{C}} \subseteq M\mathcal{R}_a^{\mathcal{C}}$  (pair splitting)

in addition to being constrained by conditional stability.

*Proof.* It is straightforward to check that the semantic constraints of proof monotonicity and pair splitting correspond to the syntactic laws of epistemic antitonicity and proof extension, respectively, which are interdeducible (cf. Theorem 4).  $\Box$ 

## 4 Conclusion

We have proposed LiiP with as main contributions those described in Section 1.1. The notion of non-monotonic proofs captured by LiiP has the advantage of being not only operational thanks to our proof-theoretic definition but also declarative thanks to our complementary model-theoretic definition, which gives a constructive epistemic semantics to these proofs in the sense of explicating *what* (knowledge) they effect in agents in the instant of their reception, complementing thereby the (operational) axiomatics, which explicates *how* they do so.

We conclude by mentioning [BRS12] as a piece of related work. There, the authors present a resource-bounded implicit-single-agent but dynamic logic of defeasible (and thus non-monotonic) evidence-based S4-knowledge, where they use a particular primitive Et for the implicit-agent's knowledge of evidence terms t. The authors' atomic proposition Et is a particular and strongly resource-bounded analog of my atomic proposition  $a \ M$  for an arbitrary agent a's knowledge of message terms M. Et is strongly resource-bounded in the sense that the term axioms for Et are axioms for term decomposition but not for term composition. Similar restrictions could be made for  $a \ M$ , but we opine that they would be too strong. At least some amount of term composition capabilities should be conceded also to resource-bounded agents. The authors' use of Et is crucial for their contribution, who know but must have accidentally not acknowledged the contribution of  $a \ M$  to Et. See [Kra12] for historical references of my

uses of  $a \ k M$  in logics of explicit evidence/justification/proof. The addition of atomic propositions  $a \ k M$  to languages of explicit evidence/justification/proof will probably play a similarly important role as the addition of atomic propositions  $x \in S$  to the language of first-order logic (resulting in Set Theory).

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## A Axiomatic-adequacy proof

## A.1 Axiomatic soundness

**Definition 6** (Truth & Validity [BvB07]).

- The formula  $\phi \in \mathcal{L}$  is *true* (or *satisfied*) in the model  $(\mathfrak{S}, \mathcal{V})$  at the state  $s \in \mathcal{S}$  :iff  $(\mathfrak{S}, \mathcal{V}), s \models \phi$ .
- The formula  $\phi$  is *satisfiable* in the model  $(\mathfrak{S}, \mathcal{V})$  :iff there is  $s \in \mathcal{S}$  such that  $(\mathfrak{S}, \mathcal{V}), s \models \phi$ .
- The formula  $\phi$  is globally true (or globally satisfied) in the model  $(\mathfrak{S}, \mathcal{V})$ , written  $(\mathfrak{S}, \mathcal{V}) \models \phi$ , :iff for all  $s \in \mathcal{S}$ ,  $(\mathfrak{S}, \mathcal{V}), s \models \phi$ .
- The formula  $\phi$  is *satisfiable* :iff there is a model  $(\mathfrak{S}, \mathcal{V})$  and a state  $s \in \mathcal{S}$  such that  $(\mathfrak{S}, \mathcal{V}), s \models \phi$ .
- The formula  $\phi$  is *valid*, written  $\models \phi$ , :iff for all models  $(\mathfrak{S}, \mathcal{V}), (\mathfrak{S}, \mathcal{V}) \models \phi$ .

Proposition 8 (Admissibility of LiiP-specific axioms and rules).

- 1.  $\models a \, \mathsf{k} \, a$
- $2. \models a \,\mathsf{k} \, M \to a \,\mathsf{k} \, \{\!\![M]\!\!\}_a$

$$\begin{aligned} 3. &\models a \, \mathbf{k} \, \{\!\!\{M\}\!\!\}_b \to a \, \mathbf{k} \, (M, b) \\ 4. &\models (a \, \mathbf{k} \, M \land a \, \mathbf{k} \, M') \leftrightarrow a \, \mathbf{k} \, (M, M') \\ 5. &\models (M ::_a^C (\phi \to \phi')) \to ((M ::_a^C \phi) \to M ::_a^C \phi') \\ 6. &\models (M ::_a^C \phi) \to (a \, \mathbf{k} \, M \to \phi) \\ 7. &\models \bigwedge_{b \in \mathcal{C} \cup \{a\}} (((M, b) ::_a^C \phi) \to \{\!\!\{M\}\!\!\}_a ::_b^{\mathcal{C} \cup \{a\}} (a \, \mathbf{k} \, M \land M ::_a^C \phi)) \\ 8. &\models (M ::_a^{\mathcal{C} \cup \mathcal{C}'} \phi) \to M ::_a^C \phi \\ 9. \quad If \models \phi \ then \models M ::_a^C \phi \\ 10. \quad If \models a \, \mathbf{k} \, M \leftrightarrow a \, \mathbf{k} \, M' \ then \models (M ::_a^C \phi) \leftrightarrow M' ::_a^C \phi. \end{aligned}$$

Proof. 1–4 are immediate; 5 and 9 hold by the fact that LiiP has a standard Kripke-semantics; 6 follows directly from the conditional reflexivity of  ${}^{\prime}_{M}\mathcal{R}_{a}^{\mathcal{C}}$ , 8 directly from the communal monotonicity of  ${}^{\prime}_{M}\mathcal{R}_{a}^{\mathcal{C}}$ , and 10 directly from the conditional stability of  ${}^{\prime}_{M}\mathcal{R}_{a}^{\mathcal{C}}$ . Finally, 7 follows jointly from the signature and the communal-transitivity property of  ${}^{\prime}_{M}\mathcal{R}_{a}^{\mathcal{C}}$ —as follows: let  $(\mathfrak{S}, \mathcal{V})$  designate an arbitrary LiiP-model and let  $s \in \mathcal{S}$ . First, let  $b \in \mathcal{C} \cup \{a\}$  and suppose that  $(\mathfrak{S}, \mathcal{V}), s \models (M, b) ::_{a}^{\mathcal{C}} \phi$ . Second, let  $s' \in \mathcal{S}$  and suppose that  $s_{\{M\}_{a}}\mathcal{R}_{b}^{\mathcal{C}\cup\{a\}}s'$ . Hence  $M \in \text{cl}_{a}^{s'}(\emptyset)$  by the signature property, and thus  $(\mathfrak{S}, \mathcal{V}), s' \models a \, k \, M$  by definition. Third, let  $s'' \in \mathcal{S}$  and suppose that  $s'_{M}\mathcal{R}_{a}^{\mathcal{C}}s''$ . Hence,  $s_{(M,b)}\mathcal{R}_{a}^{\mathcal{C}}s''$  by the first, second, and third supposition and communal transitivity. Hence  $(\mathfrak{S}, \mathcal{V}), s' \models \phi$  by the first supposition. Thus  $(\mathfrak{S}, \mathcal{V}), s' \models M ::_{a}^{\mathcal{C}} \phi$  by discharge of the third supposition. Hence  $(\mathfrak{S}, \mathcal{V}), s' \models a \, k \, M \wedge M ::_{a}^{\mathcal{C}} \phi$ . Finally, consecutively discharging the remaining three suppositions,  $(\mathfrak{S}, \mathcal{V}), s \models \{M\}_{a} ::_{b}^{\mathcal{C}\cup\{a\}}(a \, k \, M \wedge M ::_{a}^{\mathcal{C}} \phi)$ , and then  $(\mathfrak{S}, \mathcal{V}), s \models \langle_{h \in \mathcal{C}\cup\{a\}}(((M, b) ::_{a}^{\mathcal{C}} \phi) \to \{M\}_{a} ::_{b}^{\mathcal{C}\cup\{a\}}(a \, k \, M \wedge M ::_{a}^{\mathcal{C}} \phi))$ .

#### A.2 Semantic completeness

For all  $\phi \in \mathcal{L}$ , if  $\models \phi$  then  $\vdash_{\text{LiiP}} \phi$ .

Proof. Let

• W designate the set of all maximally LiiP-consistent sets<sup>4</sup>

<sup>&</sup>lt;sup>4\*</sup> A set W of LiiP-formulas is maximally LiiP-consistent :iff W is LiiP-consistent and W has no proper superset that is LiiP-consistent. A set W of LiiP-formulas is LiiP-consistent :iff W is not LiiP-inconsistent. A set W of LiiP-formulas is LiiP-inconsistent :iff there is a finite  $W' \subseteq W$  such that  $((\bigwedge W') \to \bot) \in$  LiiP. Any LiiP-consistent set can be extended to a maximally LiiP-consistent set by means of the Lindenbaum Construction [Fit07, Page 90]. A set is maximally LiiP-consistent if and only if the set of logical-equivalence classes of the set is an ultrafilter of the Lindenbaum-Tarski algebra of LiiP [Ven07, Page 351]. The canonical frame is isomorphic to the ultrafilter frame of that Lindenbaum-Tarski algebra [Ven07, Page 352].

- for all  $w, w' \in \mathcal{W}, w \ _M \mathcal{C}^{\mathcal{C}}_a w'$  :iff  $\{ \phi \in \mathcal{L} \mid M :: \stackrel{\circ}{a} \phi \in w \} \subseteq w'$
- for all  $w \in \mathcal{W}, w \in \mathcal{V}_{\mathsf{C}}(P)$  :iff  $P \in w$ .

Then  $\mathfrak{M}_{\mathsf{C}} := (\mathcal{W}, \{{}_{M} \mathbb{C}_{a}^{\mathcal{C}}\}_{M \in \mathcal{M}, a \in \mathcal{A}, \mathcal{C} \subseteq \mathcal{A}}, \mathcal{V}_{\mathsf{C}})$  designates the *canonical model* for LiiP. Following Fitting [Fit07, Section 2.2], the following useful property of  $\mathfrak{M}_{\mathsf{C}}$ ,

for all 
$$\phi \in \mathcal{L}$$
 and  $w \in \mathcal{W}$ ,  $\phi \in w$  if and only if  $\mathfrak{M}_{\mathsf{C}}, w \models \phi$ ,

the so-called *Truth Lemma*, can be proved by induction on the structure of  $\phi$ :

- 1. Base case ( $\phi := P$  for  $P \in \mathcal{P}$ ). For all  $w \in \mathcal{W}$ ,  $P \in w$  if and only if  $\mathfrak{M}_{\mathsf{C}}, w \models P$ , by definition of  $\mathcal{V}_{\mathsf{C}}$ .
- 2. Inductive step ( $\phi := \neg \phi'$  for  $\phi' \in \mathcal{L}$ ). Suppose that for all  $w \in \mathcal{W}, \phi' \in w$  if and only if  $\mathfrak{M}_{\mathsf{C}}, w \models \phi'$ . Further let  $w \in \mathcal{W}$ . Then,  $\neg \phi' \in w$  if and only if  $\phi' \notin w w$  is consistent if and only if  $\mathfrak{M}_{\mathsf{C}}, w \not\models \phi'$  by the induction hypothesis if and only if  $\mathfrak{M}_{\mathsf{C}}, w \models \neg \phi'$ .
- 3. Inductive step (\$\phi := \phi' \wedge \phi'' for \$\phi', \phi'' \in \mathcal{L}\$). Suppose that for all \$w \in \mathcal{W}\$, \$\phi'' \in w\$ if and only if \$\mathbb{M}\_{C}, w\$ \$\mathbb{w}\$ \$\mathcal{e}\$ \$\phi'\$, and that for all \$w \in \mathcal{W}\$, \$\phi'' \in w\$ if and only if \$\mathbb{M}\_{C}, w\$ \$\mathbb{w}\$ \$\mathbb{w}\$, w'' \in w\$ if and only if \$\mathbb{M}\_{C}, w\$ \$\mathbb{w}\$ \$\mathbb{w}\$, because \$w\$ is maximal. Now suppose that \$\phi' \in w\$ and \$\phi'' \in w\$. Hence, \$\mathbb{M}\_{C}, w\$ \$\mathbb{w}\$ \$\mathbf{w}\$ and \$\mathbb{M}\_{C}, w\$ \$\mathbb{w}\$ \$\mathbf{w}\$ \$\mathbf{v}\$ and \$\mathbb{M}\_{C}, w\$ \$\mathbb{w}\$ \$\mathbf{w}\$ \$\mathbf{v}\$ and \$\mathbf{M}\_{C}, w\$ \$\mathbf{w}\$ \$\mathbf{w}\$ \$\mathbf{v}\$ \$\mathbf{w}\$ and \$\mathbf{M}\_{C}, w\$ \$\mathbf{w}\$ \$\mathbf{w}\$ \$\mathbf{v}\$ \$\mathbf{v}\$ and \$\mathbf{M}\_{C}, w\$ \$\mathbf{w}\$ \$\mathbf{w}\$ \$\mathbf{v}\$ \$\mathbf{v}\$ \$\mathbf{v}\$ \$\mathbf{v}\$ \$\mathbf{v}\$ \$\mathbf{v}\$ \$\mathbf{w}\$ \$\mathbf{v}\$ \$\mathb
- 4. Inductive step  $(\phi := M ::_a^{\mathcal{C}} \phi' \text{ for } M \in \mathcal{M}, a \in \mathcal{A}, \mathcal{C} \subseteq \mathcal{A}, \text{ and } \phi' \in \mathcal{L}).$

4.1	for all $w \in \mathcal{W}, \phi' \in w$ if and only if $\mathfrak{M}_{C}, w \models \phi'$	ind. hyp.
4.2	$w \in \mathcal{W}$	hyp.
4.3	$M::_a^{\mathcal{C}}\phi'\in w$	hyp.
4.4	$w' \in \mathcal{W}$	hyp.
4.5	$w \ _M\! \mathbf{C}^{\mathcal{C}}_a \ w'$	hyp.
4.6	$\{ \phi'' \in \mathcal{L} \mid M ::_a^{\mathcal{C}} \phi'' \in w \} \subseteq w'$	4.5
4.7	$\phi' \in \{ \phi'' \in \mathcal{L} \mid M :: {}^{\mathcal{C}}_{a} \phi'' \in w \}$	4.3, 4.6
4.8	$\phi' \in w'$	4.6, 4.7
4.9	$\mathfrak{M}_{C}, w' \models \phi'$	4.1,  4.4,  4.8
4.10	if $w \ _M\!\mathcal{C}^{\mathcal{C}}_a w'$ then $\mathfrak{M}_{C}, w' \models \phi'$	4.5 - 4.9
4.11	for all $w' \in \mathcal{W}$ , if $w \ _M \mathcal{C}^{\mathcal{C}}_a w'$ then $\mathfrak{M}_{C}, w' \models \phi'$	4.4 - 4.10
4.12	$\mathfrak{M}_{C},w\models M ::_a^{\mathcal{C}} \phi'$	4.11
4.13	$M ::_{a}^{\mathcal{C}} \phi' \not\in w$	hyp.

4.14	$\mathcal{F} = \{ \phi'' \in \mathcal{L} \mid M ::_a^{\mathcal{C}} \phi'' \in w \} \cup \{\neg \phi'\} $ hyp
4.15	$\mathcal{F}$ is LiiP-inconsistent hyp
4.16	there is $\{M ::_a^{\mathcal{C}} \phi_1, \ldots, M ::_a^{\mathcal{C}} \phi_n\} \subseteq w$ such that
	$\vdash_{\text{LiiP}} (\phi_1 \land \ldots \land \phi_n \land \neg \phi') \to \bot \qquad 4.14, 4.15$
4.17	$\{M :: \stackrel{\mathcal{C}}{\underset{a}{a}} \phi_1, \dots, M :: \stackrel{\mathcal{C}}{\underset{a}{a}} \phi_n\} \subseteq w \text{ and}$
	$\vdash_{\mathrm{LiiP}} (\phi_1 \wedge \ldots \wedge \phi_n \wedge \neg \phi') \to \bot \qquad \qquad \text{hyp}$
4.18	$\vdash_{\text{LiiP}} (\phi_1 \land \dots \land \phi_n) \to \phi' \tag{4.1}$
4.19	$\vdash_{\text{LiiP}} (M ::_a^c (\phi_1 \land \ldots \land \phi_n)) \to M ::_a^c \phi'  4.18, \text{ regularity}$
4.20	$\vdash_{\text{LiiP}} \left( \left( M ::_{a}^{\mathcal{C}} \phi_{1} \right) \land \ldots \land \left( M ::_{a}^{\mathcal{C}} \phi_{n} \right) \right) \to M ::_{a}^{\mathcal{C}} \phi' \qquad 4.19$
4.21	$M ::_{a}^{\mathcal{C}} \phi' \in w \qquad \qquad 4.17,  4.20,  w \text{ is maxima}$
4.22	false 4.13, 4.2
4.23	false 4.16, 4.17–4.22
4.24	$\mathcal{F}$ is LiiP-consistent 4.15–4.23
4.25	there is $w' \supseteq \mathcal{F}$ s.t. $w'$ is maximally LiiP-consistent 4.24
4.26	$\mathcal{F} \subseteq w'$ and $w'$ is maximally LiiP-consistent hyp
4.27	$\{ \phi'' \in \mathcal{L} \mid M ::_a^{\mathcal{C}} \phi'' \in w \} \subseteq \mathcal{F} $ $4.14$
4.28	$\{ \phi'' \in \mathcal{L} \mid M ::_a^{\mathcal{C}} \phi'' \in w \} \subseteq w' $ $4.26, 4.27$
4.29	$w \ _{M}\mathbf{C}_{a}^{\mathcal{C}} \ w'$ 4.28
4.30	$w' \in \mathcal{W}$ 4.20
4.31	$\neg \phi' \in \mathcal{F}$ 4.14
4.32	$\neg \phi' \in w' \tag{4.26, 4.3}$
4.33	$\phi' \notin w'$ 4.26 (w' is LiiP-consistent), 4.32
4.34	$\mathfrak{M}_{C}, w' \not\models \phi' \tag{4.1, 4.3}$
4.35	there is $w' \in \mathcal{W}$ s.t. $w \ _M \mathcal{C}^{\mathcal{C}}_a w'$ and $\mathfrak{M}_{C}, w' \not\models \phi' 4.29, 4.34$
4.36	$\mathfrak{M}_{C}, w \not\models M ::_{a}^{\mathcal{C}} \phi' $ $4.35$
4.37	$\mathfrak{M}_{C}, w \not\models M ::_{a}^{\mathcal{C}} \phi'$ 4.25, 4.26–4.30
4.38	$\mathfrak{M}_{C}, w \not\models M ::_{a}^{\mathcal{C}} \phi' $ $4.14 - 4.33$
4.39	$M ::_a^{\mathcal{C}} \phi' \in w$ if and only if $\mathfrak{M}_{C}, w \models M ::_a^{\mathcal{C}} \phi'  4.3-4.12, \ 4.13-4.38$

4.40 for all  $w \in \mathcal{W}, M ::_a^{\mathcal{C}} \phi' \in w$  if and only if  $\mathfrak{M}_{\mathsf{C}}, w \models M ::_a^{\mathcal{C}} \phi' 4.2 - 4.39$ 

With the Truth Lemma we can now prove that for all  $\phi \in \mathcal{L}$ , if  $\not\vdash_{\text{LiiP}} \phi$  then  $\not\models \phi$ . Let  $\phi \in \mathcal{L}$ , and suppose that  $\not\vdash_{\text{LiiP}} \phi$ . Thus,  $\{\neg \phi\}$  is LiiP-consistent, and can be extended to a maximally LiiP-consistent set w, i.e.,  $\neg \phi \in w \in \mathcal{W}$ . Hence  $\mathfrak{M}_{\mathsf{C}}, w \models \neg \phi$ , by the Truth Lemma. Thus:  $\mathfrak{M}_{\mathsf{C}}, w \not\models \phi, \mathfrak{M}_{\mathsf{C}} \not\models \phi$ , and  $\not\models \phi$ . That is,  $\mathfrak{M}_{\mathsf{C}}$  is a *universal* (for all  $\phi \in \mathcal{L}$ ) counter-model (if  $\phi$  is a non-theorem then  $\mathfrak{M}_{\mathsf{C}}$  falsifies  $\phi$ ).

We are left to prove that  $\mathfrak{M}_{\mathsf{C}}$  is also an *LiiP-model*. So let us instantiate our data mining operator  $\mathrm{cl}_a$  (cf. Page 10) on  $\mathcal{W}$  by letting for all  $w \in \mathcal{W}$ 

$$\mathrm{msgs}_a(w) := \{ M \mid a \, \mathsf{k} \, M \in w \},\$$

and let us prove that:

- 1. if  $\mathcal{C} \subseteq \mathcal{C}'$  then  ${}_{M}\mathbf{C}_{a}^{\mathcal{C}} \subseteq {}_{M}\mathbf{C}_{a}^{\mathcal{C}'}$
- 2. if  $M \equiv_a M'$  then  ${}_M C^{\mathcal{C}}_a = {}_M' C^{\mathcal{C}}_a$
- 3. if  $M \in cl_a^w(\emptyset)$  then  $w M C_a^{\mathcal{C}} w$
- 4. if  $w_{\text{IMB}} C_a^{\mathcal{C}} w'$  then  $M \in \text{cl}_b^{w'}(\emptyset)$
- 5. for all  $b \in \mathcal{C} \cup \{a\}$ ,  $({}_{[M]} {}_{a} C_{b}^{\mathcal{C} \cup \{a\}} \circ {}_{M} C_{a}^{\mathcal{C}}) \subseteq {}_{(M,b)} C_{a}^{\mathcal{C}}$ .

For (1), let  $\mathcal{C}' \subseteq \mathcal{A}$  and suppose that  $\mathcal{C} \subseteq \mathcal{C}'$ . That is,  $\mathcal{C} \cup \mathcal{C}' = \mathcal{C}'$ . Further, let  $w, w' \in \mathcal{W}$  and suppose that  $w_{M}C_{a}^{\mathcal{C}}w'$ . That is, for all  $\phi \in \mathcal{L}$ , if  $M ::_{a}^{\mathcal{C}} \phi \in w$  then  $\phi \in w'$ . Furthermore, let  $\phi \in \mathcal{L}$  and suppose that  $M ::_{a}^{\mathcal{C}'} \phi \in w$ . Thus  $M ::_{a}^{\mathcal{C} \cup \mathcal{C}'} \phi \in w$  by the first supposition. Since w is maximal,

$$(M ::_{a}^{\mathcal{C} \cup \mathcal{C}'} \phi) \to M ::_{a}^{\mathcal{C}} \phi \in w \quad (\text{group decomposition}).$$

Hence  $M ::_a^{\mathcal{C}} \phi \in w$  by modus ponens, and thus  $\phi \in w'$  by the second supposition.

For (2), suppose that  $M \equiv_a M'$ . That is, for all  $w \in \mathcal{W}$ ,  $M \in \operatorname{cl}_a^w(\emptyset)$  if and only if  $M' \in \operatorname{cl}_a^w(\emptyset)$ . Hence for all  $w \in \mathcal{W}$ ,  $a \ltimes M \in w$  if and only if  $a \ltimes M' \in w$ due to the maximality of w', which contains all the term axioms corresponding to the defining clauses of  $\operatorname{cl}_a^w$ . Hence for all  $w \in \mathcal{W}$ ,  $\mathfrak{M}_{\mathsf{C}}$ ,  $w \models a \ltimes M$  if and only if  $\mathfrak{M}_{\mathsf{C}}$ ,  $w \models a \ltimes M'$ , by the Truth Lemma. Thus for all  $w \in \mathcal{W}$ ,  $\mathfrak{M}_{\mathsf{C}}$ ,  $w \models a \ltimes M \leftrightarrow$  $a \ltimes M'$ . Hence for all  $w \in \mathcal{W}$ ,  $a \ltimes M \leftrightarrow a \ltimes M' \in w$  by the Truth Lemma. Hence the following intermediate result, called IR,

for all  $w \in \mathcal{W}$  and  $\phi \in \mathcal{L}$ ,  $(M ::_a^{\mathcal{C}} \phi) \leftrightarrow M' ::_a^{\mathcal{C}} \phi \in w$ ,

by epistemic bitonicity. Further, let  $w, w' \in \mathcal{W}$ . Hence,

- $w {}_{M}C_{a}^{\mathcal{C}} w'$  by definition if and only if
- (for all  $\phi \in \mathcal{L}$ , if  $M ::_a^c \phi \in w$  then  $\phi \in w'$ ) by IR if and only if
- (for all  $\phi \in \mathcal{L}$ , if  $M' ::_a^c \phi \in w$  then  $\phi \in w'$ ) by definition if and only if
- $w _{M'} \mathcal{C}^{\mathcal{C}}_a w'$ .

For (3), let  $w \in \mathcal{W}$  and suppose that  $M \in \operatorname{cl}_a^w(\emptyset)$ . Hence  $a \, \mathsf{k} \, M \in w$  due to the maximality of w, which contains all the term axioms corresponding to the defining clauses of  $\operatorname{cl}_a^w$ . Further suppose that  $M ::_a^{\mathcal{C}} \phi \in w$ . Since w is maximal,

 $(M ::_a^{\mathcal{C}} \phi) \to (a \operatorname{\mathsf{k}} M \to \phi) \in w$  (epistemic truthfulness).

Hence,  $a \not k M \rightarrow \phi \in w$ , and  $\phi \in w$ , by consecutive modus ponens.

For (4), let  $w, w' \in \mathcal{W}$  and suppose that  $w_{\{M\}_b} C_a^{\mathcal{C}} w'$ . That is, for all  $\phi \in \mathcal{L}$ , if  $\{M\}_b :: {}^{\mathcal{C}}_a \phi \in w$  then  $\phi \in w'$ . Since w is maximal,

$$[\![M]\!]_b ::_a^{\mathcal{C} \cup \{b\}} b \,\mathsf{k} \, M \in w \quad (\text{authentic knowledge})$$

$$(\{\!\![M]\!]_b ::_a^{\mathcal{C} \cup \{b\}} b \, \mathsf{k} \, M) \to \{\!\![M]\!]_b ::_a^{\mathcal{C}} b \, \mathsf{k} \, M \in w \quad \text{(group decomposition)}.$$

Hence,  $\{\![M]\!]_b :: {}^{\mathcal{C}}_a b \mathrel{k} M \in w$  by modus ponens,  $b \mathrel{k} M \in w'$  by particularisation of

the supposition, and thus  $M \in cl_b^{w'}(\emptyset)$  by the definition of  $cl_b^{w'}$ . For (5), suppose that  $b \in \mathcal{C} \cup \{a\}$  and let  $w, w', w'' \in \mathcal{S}$ . Further suppose that  $w \text{ [M]}_a C_b^{\mathcal{C} \cup \{a\}} w'$  (i.e., for all  $\phi \in \mathcal{L}$ , if  $\{M\}_a ::_b^{\mathcal{C} \cup \{a\}} \phi \in w$  then  $\phi \in w'$ ) and  $w' {}_M C_a^c w''$  (i.e., for all  $\phi \in \mathcal{L}$ , if  $M ::_a^c \phi \in w'$  then  $\phi \in w''$ ). Furthermore suppose that  $(M, b) ::_a^{\mathcal{C}} \phi \in w$ . Since w is maximal,

$$((M,b)::_a^{\mathcal{C}}\phi) \to \{\!\!\{M\}\!\!\}_a::_b^{\mathcal{C}\cup\{a\}}(M::_a^{\mathcal{C}}\phi) \in w,$$

as a direct consequence of nominal peer review and then the first supposition. Hence, applying *modus ponens* consecutively,  $\{\![M]\!]_a ::_b^{\mathcal{C} \cup \{a\}} (M ::_a^{\mathcal{C}} \phi) \in w$  by the fourth supposition,  $M ::_a^{\mathcal{C}} \phi \in w'$  by particularisation of the second supposition, and finally  $\phi \in w''$  by the third supposition. 

#### Β Application examples

With the simple but powerful language of LiiP, we can concisely express otherwise difficult to formalise security requirements such as those arising in Access Control (cf. [And08, Chapter 4]) and Data-Base Privacy (cf. [And08, Chapter 9]).

#### **B.1** Access Control

According to [And08, Chapter 4]:

Access control is the traditional center of gravity of computer security. It is where security engineering meets computer science. Its function is to control which principals (persons, processes, machines...) have access to which resources in the system — which files they can read, which programs they can execute, how they share data with other principals, and so on.

"Principals" and "resources" mean "agents" in our terminology. Access rights can be specified by application-specific access-control *policies*  $\Phi$ ; and specific access is then granted when certain access-authorisation *credentials* C are presented. These credentials are examples of application-specific base data B(cf. Definition 1), whose validity typically is, first, temporary and thus nonmonotonic as in the case of one-time credentials and credentials revokable by other, so-called revocation credentials, and, second, restricted to certain agent communities  $\mathcal{C} \subseteq \mathcal{A}$ . Conceptually, an access-control policy can be understood as a set  $\Phi$  of implicational laws  $\phi$  that together with elementary access-right facts P constitutes a Horn-logical (cf. Prolog) or even an efficiently decidable

and

Datalog theory. In LiiP, we can formalise each elementary access-right fact as an application-specific atomic proposition  $P \in \mathcal{P}$ . An example of such a fact is that an agent *a* may write-access resource *r* guarded by a different agent *b* (acting thus as a *reference monitor*), which we can formalise as an atomic proposition  $P_1 := \mathsf{maywrite}(a, r, b)$ . Thus we can let  $\mathcal{C} := \{a, b\} \subset \{a, r, b\} \subseteq \mathcal{A}$ . Naturally, agent *a* may then also read resource *r* guarded by agent *b*, i.e.,  $\phi_1 := (\mathsf{maywrite}(a, r, b) \to \mathsf{mayread}(a, r, b))$ . Et cetera up to  $\phi_m$  and  $P_n$  for some natural numbers  $m, n \in \mathbb{N}$ . Now, define the resulting access-control policy as  $\Phi := \{\phi_i\}_{1 \le i \le m}$ , the resulting access-control LiiP-theory over  $\Phi$  as

$$\operatorname{LiiP}_{\Phi} := \operatorname{Cl}(\Phi)$$

(where Cl is as in Definition 2), and  $\vdash_{\text{LiiP}_{\Phi}}$  similarly to  $\vdash_{\text{LiiP}}$ . Whence the following instance of a direct consequence of nominal peer review

 $\vdash_{\mathrm{LiiP}_{\Phi}} ((C, b) ::_{a}^{\mathcal{C}} \mathsf{maywrite}(a, r, b)) \to \{\!\![C]\!\!\}_{a} ::_{b}^{\mathcal{C}} \mathsf{maywrite}(a, r, b)$ 

and the following instance of epistemic truthfulness

 $\vdash_{\mathrm{LiiP}_{\Phi}} (\{\!\![C]\!\!]_a ::_b^{\mathcal{C}} \mathsf{maywrite}(a, r, b)) \to (b \, \mathsf{k} \, \{\!\![C]\!\!]_a \to \mathsf{maywrite}(a, r, b)) \,.$ 

Hence by transitivity of logical implication,

 $\vdash_{\mathrm{LiiP}_{\Phi}} ((C, b) ::_{a}^{\mathcal{C}} \mathsf{maywrite}(a, r, b)) \to (b \, \mathsf{k} \, \{\!\![C]\!\!\}_{a} \to \mathsf{maywrite}(a, r, b)) \, .$ 

This means that if it is commonly accepted in C that (C, b) can prove to (and thus inform) a that a may write-access r guarded by b, then if further b knows  $\{\![C]\!]_a$  (through a presenting  $\{\![C]\!]_a$  to b, since only a can generate her own signature), then indeed a may write-access r—and the guard b knows that (due to Fact 1) and thus will grant a the requested access. Actually b will also grant a read-access since according to the policy  $\Phi$ , write access implies read access:

$$\begin{split} \vdash_{\mathrm{LiiP}_{\Phi}} & ((C,b) ::_{a}^{\mathcal{C}} \mathrm{maywrite}(a,r,b)) \to (b \, \mathsf{k} \, \{\!\!\{C\}\!\!\}_{a} \to \\ & (\mathrm{maywrite}(a,r,b) \wedge \mathrm{mayread}(a,r,b))) \, . \end{split}$$

Note that we could refine our arguably rough policy  $\Phi$  with respect to *agent* roles and thus specify a refined policy  $\Phi'$ . For example, we could specify that  $\Phi \subseteq \Phi'$  and that for all  $x, y \in C$ ,  $guest(x) \in \mathcal{P}$  and  $host(y) \in \mathcal{P}$  as well as  $((guest(x) \land host(y)) \rightarrow mayread(x, r, y)) \in \Phi'$ ,  $(host(y) \rightarrow maywrite(y, r, y)) \in \Phi'$ . Et cetera. Orthogonally to agent roles, we could refine  $\Phi$  with respect to agent clearances and corresponding resource classifications (cf. Information Flow Control [And08, Section 8.3.1–2]) and thus specify a refined policy  $\Phi''$ . For example we could specify that  $\Phi \subseteq \Phi''$  and that for all  $a \in \mathcal{A}$  (and thus for all resources r), topsecret(a), secret(a), confidential(a), unclassified(a)  $\in \mathcal{P}$  as well as (topsecret(a)  $\rightarrow$  secret(a))  $\in \Phi''$ , (secret(a)  $\rightarrow$  confidential(a))  $\in \Phi''$ , ((topsecret(a))

 $\wedge$  topsecret(r))  $\rightarrow$  maywrite(a, r, b))  $\in \Phi''$ , and ((topsecret $(a) \wedge (\text{secret}(r) \lor \text{confidential}(r) \lor \text{unclassified}(r))) \rightarrow \text{mayread}(a, r, b)) \in \Phi''$ . Et cetera for other, so-called *no-read-up* and *no-write-down/up* requirements.

### **B.2** Data-Base Privacy

An important example of a resource is a relational data-base, say a medical data-base d, over application-specific atomic pieces of content data B (cf. Definition 1). Note that d typically evolves, whence the point of non-monotonicity. Then, each unary relation in the data-base d can be understood as a finite (sub)set of content data B, each binary relation as a finite set of ordered pairs (B, B') of data B and B', each relation of higher finite arity as a finite set of such pairs of pairs, and the content of d as a finite set of such relations (finite sets). Finally, finite sets can be coded as data pairs and thus the entire content  $\mathcal{D}$  of d can be understood as a subset of  $\mathcal{M}$  over the atomic data B. Now, data-base privacy with respect to the data-base d means that certain agents a must not be able to infer certain facts  $\phi$  from d (cf. Inference Control [And08, Section 9.3]). In order to meet this privacy requirement, certain atomic data B in  $\mathcal{D}$  are blinded (e.g., replaced by some dummy datum), resulting in a new, partially blinded content  $\mathcal{D}' \subseteq \mathcal{M}'$ . The privacy requirement can now be formalised in the language of LiiP by simply stipulating that for all  $M \in \mathcal{M}'$ ,

 $\neg (M ::_a^{\emptyset} \phi).$ 

The requirement could be proved by induction over the well-structured data.