Sufficiency of Cut-Generating Functions*

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Abstract

This note settles an open problem about cut-generating functions, a concept that has its origin in the work of Gomory and Johnson from the 1970's and has received renewed attention in recent years.

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1 Introduction

We consider sets of the form

$$X = X(R, S) := \{x \in \mathbb{R}^n_+ : Rx \in S\}, \tag{1a}$$

$$\text{where } \begin{cases} R = [r_1 \dots r_n] \text{ is a real } q \times n \text{ matrix,} \\ S \subset \mathbb{R}^q \text{ is a nonempty closed set with } 0 \notin S. \end{cases} \tag{1b}$$

This model has been studied in [Joh81] and [CCD$^+$13]. It appears in cutting plane theory [Gom69, GJ72, ALWW07, JSRF06] where the goal is to generate inequalities that are valid for X but not for the origin. Such cutting planes are well-defined [CCD$^+$13, Lemma 2.1] and can be written as

$$c^\top x \geqslant 1. \tag{2}$$

Let $S \subset \mathbb{R}^q$ be a given nonempty closed set with $0 \notin S$. The set S is assumed to be fixed in this paragraph. [CCD$^+$13] introduce the notion of a *cut-generating function*: This is any function $\rho : \mathbb{R}^q \mapsto \mathbb{R}$ that produces coefficients $c_j := \rho(r_j)$ of a cut (2) valid for $X(R, S)$ for any choice of n and

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$R = [r_1 \dots r_n]$. It is shown in [CCD+13] that cut-generating functions enjoy significant structure. For instance, the minimal ones are sublinear and are closely related to S-free neighborhoods of the origin. We say that a closed convex set is *S-free* if it contains no point of S in its interior. For any minimal cut-generating function ρ, there exists a closed convex S-free set $V \subset \mathbb{R}^q$ such that $0 \in \operatorname{int} V$ and $V = \{r \in \mathbb{R}^q : \rho(r) \leqslant 1\}$. A cut (2) with coefficients $c_j := \rho(r_j)$ is called an *S-intersection cut*.

Now assume that both S and R are fixed. Noting $X(R, S) \subset \mathbb{R}^n_+$, we say that a cutting plane $c^\top x \geqslant 1$ *dominates* $b^\top x \geqslant 1$ if $c_j \leqslant b_j$ for $j \in [n]$. A natural question is whether every cut (2) valid for $X(R, S)$ is dominated by an S-intersection cut. [CCD+13] give an example showing that this is not always the case. However, this example has the peculiarity that S contains points that cannot be obtained as Rx for any $x \in \mathbb{R}^n_+$. [CCD+13] propose the following open problem: Assuming $S \subset \operatorname{cone} R$, is it true that every cut (2) valid for $X(R, S)$ is dominated by an S-intersection cut? Our main theorem shows that this is indeed the case. This generalizes the main result of [CCZ10] and Theorem 6.3 in [CCD+13].

Theorem 1.1. *Suppose $S \subset \operatorname{cone} R$. Then any valid inequality $c^\top x \geqslant 1$ separating the origin from X is dominated by an S-intersection cut.*

2 Proof of the Main Theorem

Our proof of Theorem 1.1 will use several lemmas. We first introduce some terminology. Given a convex cone $K \subseteq \mathbb{R}^d$, let $K^\circ := \{w \in \mathbb{R}^d : u^\top w \leqslant 0, \forall u \in K\}$ (resp. $K^* := \{w \in \mathbb{R}^d : u^\top w \geqslant 0, \forall u \in K\}$) denote the *polar* (resp. *dual*) of K. Let $\sigma_W(u) := \sup_{w \in W} u^\top w$ be the *support function* of a set $W \subseteq \mathbb{R}^d$. A function $\rho : \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\}$ is said to be *positively homogeneous* if $\rho(\lambda u) = \lambda \rho(u)$ for all $\lambda > 0$ and $u \in \mathbb{R}^d$ and *subadditive* if $\rho(u_1) + \rho(u_2) \geqslant \rho(u_1 + u_2)$ for all $u_1, u_2 \in \mathbb{R}^d$. Moreover, ρ is *sublinear* if it is both positively homogeneous and subadditive. Sublinear functions are known to be convex and it is not difficult to show that support functions are sublinear (see, e.g., [HUL04, Chapter C]). Given a closed convex neighborhood V of the origin, a *representation of V* is any sublinear function $\rho : \mathbb{R}^q \mapsto \mathbb{R}$ such that $V = \{r \in \mathbb{R}^q : \rho(r) \leqslant 1\}$. S-intersection cuts are generated via representations of closed convex S-free neighborhoods of the origin.

Throughout this section, we assume that $X \neq \emptyset$ and $c^\top x \geqslant 1$ is a valid inequality separating the origin from X.

Lemma 2.1. *If $u \in \mathbb{R}^n_+$ and $Ru = 0$, then $c^\top u \geqslant 0$, or, equivalently, $c \in \mathbb{R}^n_+ + \operatorname{Im} R^\top$.*

Proof. Let $\overline{x} \in X$. Note that $R(\overline{x} + tu) = R\overline{x} \in S$ and $\overline{x} + tu \geqslant 0$ for all $t \geqslant 0$. By the validity of c, we have $c^\top(\overline{x} + tu) \geqslant 1$ for all $t \geqslant 0$. Observing $tc^\top u \geqslant 1 - c^\top \overline{x}$ and letting $t \to +\infty$ implies $c^\top u \geqslant 0$ as desired. Because u is an arbitrary vector in $\mathbb{R}^n_+ \cap \operatorname{Ker} R$, we can write $c \in (\mathbb{R}^n_+ \cap \operatorname{Ker} R)^*$. The equality $(\mathbb{R}^n_+ \cap \operatorname{Ker} R)^* = \mathbb{R}^n_+ + \operatorname{Im} R^\top$ follows from the facts $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$, $(\operatorname{Ker} R)^* = \operatorname{Im} R^\top$ and $\mathbb{R}^n_+ + \operatorname{Im} R^\top$ is closed (see, e.g., [Roc70, Cor. 16.4.2]). ∎

Let

$$h(r) := \min \quad c^\top x \qquad \begin{aligned} & Rx = r, \\ & x \geqslant 0. \end{aligned} \tag{3}$$

Remark 2.2. *$h(r_j) \leqslant c_j$ for all $j \in [n]$.*

Lemma 2.3. *h is a piecewise-linear sublinear function on the domain $\operatorname{cone} R$.*

Proof. The domain of h must be a subset of $\operatorname{cone} R$ because (3) is infeasible for $r \notin \operatorname{cone} R$. The dual of (3) is

$$\begin{array}{ll} \max & r^\top y \\ & R^\top y \leqslant c. \end{array} \tag{4}$$

Let $P := \{y \in \mathbb{R}^q : R^\top y \leqslant c\}$. By Lemma 2.1, $c = c' + c''$ where $c' \in \mathbb{R}^n_+$ and $c'' \in \operatorname{Im} R^\top$. Because $c'' \in \operatorname{Im} R^\top$, there exists $y'' \in \mathbb{R}^q$ such that $R^\top y'' = c'' \leqslant c$. Hence $y'' \in P$ which shows that the dual LP is always feasible, strong duality holds and $h(r) = \sigma_P(r)$ for all $r \in \operatorname{cone} R$. In particular, $h(0) = 0$ and $h(r)$ is finite for all $r \in \operatorname{cone} R$. Now let W be a finite set of points for which $P = \operatorname{conv} W + \operatorname{rec} P$. Observe that $\operatorname{rec} P = (\operatorname{cone} R)^\circ$ and $r^\top u \leqslant 0$ for all $r \in \operatorname{cone} R$ and $u \in \operatorname{rec} P$. Therefore, $h(r) = \sigma_P(r) = \sigma_W(r)$ for all $r \in \operatorname{cone} R$ which implies that h is piecewise-linear and sublinear on the domain $\operatorname{cone} R$. ■

Lemma 2.4. *Theorem 1.1 holds when $\operatorname{cone} R = \mathbb{R}^q$.*

Proof. In this case, h is finite everywhere. Let $V := \{r \in \mathbb{R}^q : h(r) \leqslant 1\}$. Because the Slater condition is satisfied, we have $\operatorname{int} V = \{r \in \mathbb{R}^q : h(r) < 1\}$ (see, e.g., [HUL04, Prop. D.1.3.3]). Thus V is a closed convex neighborhood of the origin and h represents V by definition.

Claim 2.1*:* V is S-free. Suppose this is not the case. Let $\overline{r} \in S$ be a point in $\operatorname{int} V$. Then there exists $\overline{x} \geqslant 0$ such that $R\overline{x} = \overline{r} \in S$ and $c^\top \overline{x} = h(\overline{r}) < 1$. Because $\overline{x} \in X$, this contradicts the validity of $c^\top x \geqslant 1$. □

Therefore, $\sum_{j=1}^n h(r_j)x_j \geqslant 1$ is an S-intersection cut that can be obtained from the closed convex S-free neighborhood V of the origin. By Remark 2.2, $h(r_j) \leqslant c_j$ for all $j \in [n]$. This shows that $\sum_{j=1}^n h(r_j)x_j \geqslant 1$ dominates $c^\top x \geqslant 1$. ■

We now consider the case where $\operatorname{cone} R \subsetneq \mathbb{R}^q$. We want to extend the definition of h to the whole of $\mathbb{R}^q$ and show that this extension is a cut-generating function. We will first construct a function h' such that 1) h' is finite everywhere on $\operatorname{span} R$, 2) h' coincides with h on $\operatorname{cone} R$. If $\dim(R) < q$, we will further extend h' to the whole of $\mathbb{R}^q$ by letting $h'(r) = h'(r')$ for all $r \in \mathbb{R}^q$, $r' \in \operatorname{span} R$, $r'' \in (\operatorname{span} R)^\perp$ such that $r = r' + r''$. Our proof of Theorem 1.1 will show that this procedure yields a function h' that is the desired extension of h.

Let $r_0 \in -\operatorname{ri}(\operatorname{cone} R)$ where $\operatorname{ri}(\cdot)$ denotes the relative interior. Note that this guarantees $\operatorname{cone}(R \cup \{r_0\}) = \operatorname{span} R$ since there exist $\epsilon > 0$ and $d := \dim(R)$ linearly independent vectors $a_1, \dots, a_d \in \operatorname{span} R$ such that $-r_0 \pm \epsilon a_i \in \operatorname{cone} R$ for all $i \in [d]$ which implies $\pm a_i \in \operatorname{cone}(R \cup \{r_0\})$. Now we define c_0 as

$$c_0 := \sup_{r \in \operatorname{cone} R} \sup_{\alpha > 0} \frac{h(r) - h(r + \alpha(-r_0))}{\alpha}. \tag{5}$$

Lemma 2.5. *c_0 is finite.*

Proof. Any pair $\overline{r} \in \operatorname{cone} R$ and $\overline{\alpha} > 0$ yields a lower bound on c_0: Our choice of r_0 ensures $\overline{r} + \overline{\alpha}(-r_0) \in \operatorname{cone} R$ and $c_0 \geqslant \frac{h(\overline{r}) - h(\overline{r} + \overline{\alpha}(-r_0))}{\overline{\alpha}}$. To get an upper bound on c_0, consider the LPs (3) and (4). Let $\tilde{r} \in \operatorname{cone} R$ and $\tilde{\alpha} \geqslant 0$. Observe that $\tilde{r} + \tilde{\alpha}(-r_0) \in \operatorname{cone} R$ and, as in the proof of Lemma 2.3, one can show that both LPs are feasible when we plug in $\tilde{r} + \tilde{\alpha}(-r_0)$ for r. Therefore, strong duality holds and $h(\tilde{r}+\tilde{\alpha}(-r_0)) = \sigma_P(\tilde{r}+\tilde{\alpha}(-r_0))$ where $P := \{y \in \mathbb{R}^q : R^\top y \leqslant c\}$ is the feasible region of (4). Let W be a finite set of points for which $P = \operatorname{conv} W + \operatorname{rec} P$. Because $\operatorname{rec} P = (\operatorname{cone} R)^\circ$, we have $(\tilde{r}+\tilde{\alpha}(-r_0))^\top u \leqslant 0$ for all $u \in \operatorname{rec} P$. This implies $\sigma_P(\tilde{r}+\tilde{\alpha}(-r_0)) = \sigma_W(\tilde{r}+\tilde{\alpha}(-r_0))$ and we can write

$$
\begin{aligned}
c_0 &= \sup_{r \in \operatorname{cone} R} \sup_{\alpha > 0} \frac{\sigma_W(r) - \sigma_W(r + \alpha(-r_0))}{\alpha} \\
&\leqslant \sup_{r \in \operatorname{cone} R} \sup_{\alpha > 0} \frac{\sigma_W(\alpha r_0)}{\alpha} \\
&= \sigma_W(r_0)
\end{aligned}
$$

where we have used the sublinearity of σ_W in the inequality and the second equality. The conclusion follows now from the fact that W is a finite set. ■

We define a sublinear function h' over $\operatorname{span} R$:

$$
\begin{aligned}
h'(r) := \min \quad & c_0 x_0 + c^\top x \\
& r_0 x_0 + Rx = r, \\
& x_0 \geqslant 0,\ x \geqslant 0.
\end{aligned} \tag{6}
$$

Lemma 2.6. *The function h' coincides with h on $\operatorname{cone} R$. Furthermore, for any $r \in \operatorname{cone} R$, (6) admits an optimal solution of the form $(0, x) \in \mathbb{R} \times \mathbb{R}^n$.*

Proof. It is clear that $h' \leqslant h$. Let $\bar{r} \in \operatorname{cone} R$ and suppose $h'(\bar{r}) < h(\bar{r})$. Then there exists $(\bar{x}_0, \bar{x})$ satisfying $r_0\bar{x}_0 + R\bar{x} = \bar{r}$, $\bar{x} \geqslant 0$, $\bar{x}_0 > 0$ and $c_0\bar{x}_0 + c^\top \bar{x} < h(\bar{r})$. Rearranging the terms and using Remark 2.2, we obtain

$$
c_0 < \frac{h(\bar{r}) - c^\top \bar{x}}{\bar{x}_0} \leqslant \frac{h(\bar{r}) - \sum_{j=1}^n h(r_j)\bar{x}_j}{\bar{x}_0}.
$$

Finally, the sublinearity of h and the observation that $R\bar{x} = \bar{r} - r_0\bar{x}_0$ give

$$
c_0 < \frac{h(\bar{r}) - \sum_{j=1}^n h(\bar{x}_j r_j)}{\bar{x}_0} \leqslant \frac{h(\bar{r}) - h(R\bar{x})}{\bar{x}_0} = \frac{h(\bar{r}) - h(\bar{r} - r_0\bar{x}_0)}{\bar{x}_0}.
$$

This contradicts the definition of c_0 and proves the first claim. Now let $\tilde{x}$ be an optimal solution to (3) for $r = \bar{r}$. We have $c^\top \tilde{x} = h(\bar{r}) = h'(\bar{r})$ and $(0, \tilde{x})$ is feasible to (6). This shows that $(0, \tilde{x})$ is an optimal solution to (6). ■

If $\dim(R) < q$, we extend the function h' defined in (6) to the whole of $\mathbb{R}^q$ by letting

$$
h'(r) = h'(r') \text{ for all } r \in \mathbb{R}^q,\ r' \in \operatorname{span} R,\ r'' \in (\operatorname{span} R)^\perp \text{ such that } r = r' + r''. \tag{7}
$$

Proof of Theorem 1.1. Let h' be defined as in (6) and (7) and let $V' := \{r \in \mathbb{R}^q : h'(r) \leqslant 1\}$. Observe that V' is a closed convex neighborhood of the origin because h' is sublinear and finite everywhere. Furthermore, $\operatorname{int}(V') = \{r \in \mathbb{R}^q : h'(r) < 1\}$ by the Slater property.

Claim 2.2: V' is S-free. Suppose this is not the case. Let $\bar{r} \in S$ be a point in $\operatorname{int}(V')$. By Lemma 2.6, there exists $\bar{x} \geqslant 0$ such that $R\bar{x} = \bar{r} \in S$ and $c^\top \bar{x} = h'(\bar{r}) < 1$. Because $\bar{x} \in X$, this contradicts the validity of $c^\top x \geqslant 1$. □

Now, by Remark 2.2 and Lemma 2.6, $h'(r_j) = h(r_j) \leqslant c_j$ for all $j \in [n]$. This shows that the S-intersection cut $\sum_{j=1}^n h'(r_j)x_j \geqslant 1$ dominates $c^\top x \geqslant 1$. ■

3 Constructing the S-Free Convex Neighborhood of the Origin

We now give a geometric interpretation for the proof of Theorem 1.1. Again let $c^\top x \geqslant 1$ be a valid inequality separating the origin from X. Assume without any loss of generality that the vectors $r_1, \ldots, r_n$ have been normalized so that $c_j \in \{0, \pm 1\}$ for all $j \in [n]$. Define the sets $J_+ := \{j \in [n] : c_j = +1\}$, $J_- := \{j \in [n] : c_j = -1\}$ and $J_0 := \{j \in [n] : c_j = 0\}$. Let $C := \operatorname{conv}(\{0\} \cup \{r_j : j \in J_+\})$ and $K := \operatorname{cone}(\{r_j : j \in J_0 \cup J_-\} \cup \{r_j + r_i : j \in J_+, i \in J_-\})$. Let $Q := C + K$ and h be defined as in (3). One can show $Q = \{r \in \mathbb{R}^q : h(r) \leqslant 1\}$. However, when $\operatorname{cone} R \neq \mathbb{R}^q$, the origin lies on the boundary of Q. In the proof of Theorem 1.1, we overcame this difficulty by extending h into a function h' which is defined on the whole of $\mathbb{R}^q$ and coincides with h on $\operatorname{cone} R$. We can also follow a similar approach here. Let $r_0 \in -\operatorname{ri}(\operatorname{cone} R)$ and let c_0 be as defined in (5). When $c_0 \neq 0$, scale r_0 so that $c_0 \in \{\pm 1\}$. Introduce r_0 into the relevant subset of $[n]$ according to the sign of c_0: If $c_0 = +1$, let $J'_+ := J_+ \cup \{0\}$, $J'_0 := J_0$ and $J'_- := J_-$; otherwise, if $c_0 = 0$, let $J'_+ := J_+$, $J'_0 := J_0 \cup \{0\}$ and $J'_- := J_-$; otherwise ($c_0 = -1$), let $J'_+ := J_+$, $J'_0 := J_0$ and $J'_- := J_- \cup \{0\}$. Finally, let $C' := \operatorname{conv}(\{0\} \cup \{r_j : j \in J'_+\})$, $K' := \operatorname{cone}(\{r_j : j \in J'_0 \cup J'_-\} \cup \{r_j + r_i : j \in J'_+, i \in J'_-\})$ and $Q' := C' + K' + (\operatorname{span} R)^\perp$. The following proposition shows that h' represents Q' and Q' can be used to generate an S-intersection cut that dominates $c^\top x \geqslant 1$.

Proposition 3.1. *$Q' = \{r \in \mathbb{R}^q : h'(r) \leqslant 1\}$ where h' is defined as in* (6) *and* (7).

Proof. Let $V' := \{r \in \mathbb{R}^q : h'(r) \leqslant 1\}$. Note that V' is convex by the sublinearity of h'. We have $h'(r_j) \leqslant c_j = 1$ for all $j \in J'_+$, $h'(r_j) \leqslant c_j \leqslant 0$ for all $j \in J'_0 \cup J'_-$ and $h'(r_j + r_i) \leqslant h'(r_j) + h'(r_i) \leqslant c_j + c_i = 0$ for all $j \in J'_+$ and $i \in J'_-$. Moreover, $h'(r) = h'(r + r')$ for all $r \in \mathbb{R}^q$ and $r' \in (\operatorname{span} R)^\perp$ by the definition of h'. Hence $C' \subseteq V'$, $K' \subseteq \operatorname{rec}(V')$ and $(\operatorname{span} R)^\perp \subseteq \operatorname{lin}(V')$ which together give us $Q' = C' + K' + (\operatorname{span} R)^\perp \subseteq V'$.

To prove the converse, let $\overline{r} \in \mathbb{R}^q$ be such that $h'(\overline{r}) \leqslant 1$. We consider two distinct cases: $h'(\overline{r}) \leqslant 0$ and $0 < h'(\overline{r}) \leqslant 1$. First, let us suppose $h'(\overline{r}) \leqslant 0$. Then the definition of h' implies that there exist $(\overline{x}_0, \overline{x}) \in \mathbb{R} \times \mathbb{R}^n$ and $\overline{r}' \in (\operatorname{span} R)^\perp$ such that $(\overline{x}_0, \overline{x}) \geqslant 0$, $\sum_{j \in J'_+} \overline{x}_j - \sum_{i \in J'_-} \overline{x}_i \leqslant 0$ and $r_0 \overline{x}_0 + R\overline{x} = \overline{r} - \overline{r}'$. It can be verified by inspection that the first two sets of inequalities define a cone generated by the rays $\{e_j : j \in J'_0 \cup J'_-\} \cup \{e_j + e_i : j \in J'_+, i \in J'_-\}$. This shows $\overline{r} \in K' + (\operatorname{span} R)^\perp \subseteq Q'$. Now suppose $0 < h'(\overline{r}) \leqslant 1$. Then there exist $(\overline{x}_0, \overline{x}) \in \mathbb{R} \times \mathbb{R}^n$ and $\overline{r}' \in (\operatorname{span} R)^\perp$ such that $(\overline{x}_0, \overline{x}) \geqslant 0$, $0 < \sum_{j \in J'_+} \overline{x}_j - \sum_{i \in J'_-} \overline{x}_i \leqslant 1$ and $r_0 \overline{x}_0 + R\overline{x} = \overline{r} - \overline{r}'$. Define $\overline{x}_i^j := \overline{x}_i \frac{\overline{x}_j}{\sum_{j \in J'_+} \overline{x}_j}$ for all $i \in J'_-$ and $j \in J'_+$. These values are well-defined since $0 \leqslant \sum_{i \in J'_-} \overline{x}_i < \sum_{j \in J'_+} \overline{x}_j$. Observe that $\sum_{j \in J'_+} \overline{x}_i^j = \overline{x}_i$ and $r_0 \overline{x}_0 + R\overline{x} = \sum_{j \in J'_+} (\overline{x}_j - \sum_{i \in J'_-} \overline{x}_i^j) r_j + \sum_{i \in J'_-} \sum_{j \in J'_+} \overline{x}_i^j (r_i + r_j) + \sum_{j \in J'_0} \overline{x}_j r_j$. We have $\sum_{j \in J'_+} (\overline{x}_j - \sum_{i \in J'_-} \overline{x}_i^j) = \sum_{j \in J'_+} \overline{x}_j - \sum_{i \in J'_-} \overline{x}_i \leqslant 1$ together with $\overline{x}_j - \sum_{i \in J'_-} \overline{x}_i^j > 0$ which is true for all $j \in J'_+$ because $\sum_{i \in J'_-} \overline{x}_i^j = \overline{x}_j \frac{\sum_{i \in J'_-} \overline{x}_i}{\sum_{j \in J'_+} \overline{x}_j} < \overline{x}_j$. Hence $\sum_{j \in J'_+} (\overline{x}_j - \sum_{i \in J'_-} \overline{x}_i^j) r_j \in C'$. Moreover, $\sum_{i \in J'_-} \sum_{j \in J'_+} \overline{x}_i^j (r_i + r_j) + \sum_{j \in J'_0} \overline{x}_j r_j \in K'$. These yield $\overline{r} \in C' + K' + (\operatorname{span} R)^\perp = Q'$. ■

The proof of Theorem 1.1 shows that $V' := \{r \in \mathbb{R}^q : h'(r) \leqslant 1\}$ is a closed convex S-free neighborhood of the origin. Proposition 3.1 shows that $Q' = V'$. Therefore, $\sum_{j=1}^n h'(r_j) x_j \geqslant 1$ is an S-intersection cut obtained from Q'.

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