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Multiple Objects: Error Exponents in Hypotheses Testing and Identification

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Abstract: We survey a series of investigations of optimal testing of multiple hypotheses concerning various multiobject models.

These studies are a bright instance of application of methods and technics developed in Shannon information theory to solution of typical statistical problems.

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1. Introduction

"One can conceive of Information Theory in the broad sence as covering the theory of Gaining, Transferring, and Storing Information, where the first is usually called Statistics." [2].

Shannon information theory and mathematical statistics interaction revealed to be effective. This interplay is mutually fruitful, in some works results of probability theory and statistics were obtained with application of information-theoretical methods and there are studies where statistical results provide ground for new findings in information theory [12], [14], [16]–[19], [35], [39], [49], [54], [57]–[59].

This paper can serve an illustration of application of information-theoretical methods in statistics: on one hand this is analogy in problem formulation and on the other hand this is employment of technical tools of proof, specifically of the method of types [15], [17].

It is often necessary in statistical research to make decisions regarding the nature and parameters of stochastic model, in particular, the probability distribution of the object. Decisions can be made on the base of results of observations of the object. The vector of results is called a sample. The correspondence between samples and hypotheses can be designed based on some selected criterion. The procedure of statistical hypotheses detection is called test.

The classical problem of statistical hypothesis testing refers to two hypotheses. Based on data samples a statistician makes decision on which of the two proposed hypotheses must be accepted. Many mathematical investigations, some of which have also applied significance, were implemented in this direction [50].

The need of testing of more than two hypotheses in many scientific and applied fields has essentially increased recently. As an instance microarray analysis could be mentioned [22].

The decisions can be erroneous due to randomness of the sample. The test is considered as good if the probabilities of the errors in given conditions are as small as possible.

Frequently the problem was solved for the case of a tests sequence, where the probabilities of error decrease exponentially as 2^{-NE}, when the number of observations N tends to the infinity. We call the exponent of error probability E the *reliability*. In case of two hypotheses both reliabilities corresponding to two possible error probabilities could not be increased simultaneously, it is an accepted way to fix the value of one of the reliabilities and try to make the tests sequence get the greatest value of the remaining reliability. Such a test is called *logarithmically asymptotically optimal* (LAO). Such optimal tests were considered first by Hoeffding [48], examined later by Csiszár and Longo [18], Tusnady [57], [58] (he called such test series *exponentially rate optimal* (ERO)), Longo and Sgarro [52]. The term LAO for testing of two hypotheses was proposed by Birge [11]. Amongst papers on testing, associated with information theory, we can also note works of Natarajan [54], Gutman [25], Anantharam [8], Han [26] and of many others. Some objectives in this direction were first suggested in original introductory article by Dobrushin, Pinsker and Shiryaev [21]. The achievable region of error probability exponents was examined by Tuncel [56].

The problem has common features with the issue studied in the information theory on interrelation between the rate R of the code and the exponent E of the error probability. In information theory the relation $E(R)$ is called according to Shannon the *reliability function*, while $R(E)$ is named the *E-capacity*, or the *reliability-rate function*, as it was introduced by Haroutunian [28], [34], [43].

Simple but actual concept of not only separate but also simultaneous investigation of some number of objects of the same type, evidently, was first formulated by Ahlswede and Haroutunian [6] for reliable testing of distributions of multiple items. But simultaneous examination of properties of many similar objects may be attractive and effective in plenty of other statistical situations.

The organization of this paper is as follows. We start with the definitions and notations in the next section. In section 3 we introduce the problem of multihypotheses testing concerning one object. In section 4 we consider the reliability approach to multihypotheses testing for many independent and dependent objects. Section 5 is dedicated to the problem of statistical identification under condition of optimality. Section 6 is devoted to description of characteristics of LAO hypotheses testing with permission of rejection of decision for the model consisting of one and of more independent objects.

2. Definitions and Notations

We denote finite sets by script capitals. The cardinality of a set $\mathcal{X}$ is denoted as $|\mathcal{X}|$. Random variables (RVs), which take values in finite sets $\mathcal{X}$, $\mathcal{S}$ are denoted by X, S. Probability distributions (PDs) are denoted by Q, P, G, W, V, $Q \circ V$.

Let PD of RV X, characterizing an object, be $Q \stackrel{\triangle}{=} \{Q(x),\ x \in \mathcal{X}\}$, and conditional PD of RV X for given value of state s of the object be $V \stackrel{\triangle}{=} \{V(x|s),\ x \in \mathcal{X},\ s \in \mathcal{S}\}$.

The Shannon entropy $H_Q(X)$ of RV X with PD Q is:

$$H_Q(X) \stackrel{\triangle}{=} -\sum_{x\in\mathcal{X}} Q(x)\log Q(x).$$

The conditional entropy $H_{P,V}(X \mid S)$ of RV X for given RV S with corresponding PDs is:

$$H_{P,V}(X \mid S) \stackrel{\triangle}{=} -\sum_{x\in\mathcal{X},s\in\mathcal{S}} P(s)V(x|s)\log V(x|s).$$

The divergence (Kullback-Leibler information, or "distance") of PDs Q and G on $\mathcal{X}$ is:

$$D(Q||G) \stackrel{\triangle}{=} \sum_{x\in\mathcal{X}} Q(x)\log\frac{Q(x)}{G(x)},$$

and conditional divergence of the PD $P \circ V = \{P(s)V(x|s),\ x \in \mathcal{X},\ s \in \mathcal{S}\}$ and PD $P \circ W = \{P(s)\,W(x|s), x \in \mathcal{X},\ s \in \mathcal{S}\}$ is:

$$D(P \circ V||P \circ W) = D(V||W|P) \stackrel{\triangle}{=} \sum_{x,s} P(s)V(x|s)\log\frac{V(x|s)}{Wx|s)}.$$

For our investigations we use the method of types, one of the important technical tools in Shannon theory [17, 15]. The type $Q_{\mathbf{x}}$ of a vector $\mathbf{x} = (x_1, ..., x_N) \in \mathcal{X}^N$ is a PD (the empirical distribution)

$$Q_{\mathbf{x}} = \left\{Q_{\mathbf{x}}(x) = \frac{N(x|\mathbf{x})}{N}, x \in \mathcal{X}\right\},$$

where $N(x|\mathbf{x})$ is the number of repetitions of symbol x in vector $\mathbf{x}$.

The joint type of vectors $\mathbf{x} \in \mathcal{X}$ and $\mathbf{s} = (s_1, s_2, ..., s_N) \in \mathcal{S}^N$ is the PD

$$P_{\mathbf{x},\mathbf{s}} = \left\{\frac{N(x, s|\mathbf{x}, \mathbf{s})}{N},\ x \in \mathcal{X},\ s \in \mathcal{S}\right\},$$

where $N(x, s|\mathbf{x}, \mathbf{s})$ is the number of occurrences of symbols pair (x, s) in the pair of vectors $(\mathbf{x}, \mathbf{s})$. The conditional type of $\mathbf{x}$ for given $\mathbf{s}$ is conditional PD

$$V_{\mathbf{x}|\mathbf{s}} = \{V(x|s), x \in \mathcal{X}, s \in \mathcal{S}\},$$

defined by relation $N(x, s|\mathbf{x}, \mathbf{s}) = N(x|\mathbf{x})V_{\mathbf{x}|\mathbf{s}}(x|s)$ for all $x \in \mathcal{X},\ s \in \mathcal{S}$.

We denote by $\mathcal{Q}^N(\mathcal{X})$ the set of all types of vectors in $\mathcal{X}^N$ for given N, by $\mathcal{P}^N(\mathcal{S})$ – the set of all types of vectors $\mathbf{s}$ in $\mathcal{S}^N$ and by $\mathcal{V}^N(\mathcal{X}|\mathbf{s})$ – the set of all possible conditional types of vectors $\mathbf{x}$ in $\mathcal{X}^N$ for given $\mathbf{s} \in \mathcal{S}^N$. The set of vectors $\mathbf{x}$ of type Q is denoted by $\mathcal{T}_Q^N(X)$ and the family of vectors $\mathbf{x}$ of conditional type V for given $\mathbf{s} \in \mathcal{S}^N$ of type P by $\mathcal{T}_{P,V}^N(X \mid \mathbf{s})$. The set of all possible PDs Q on $\mathcal{X}$ and PDs P on $\mathcal{S}$ is denoted, correspondingly, by $\mathcal{Q}(X)$ and $\mathcal{P}(\mathcal{S})$.

We need the following frequently used inequalities [17]:

$$\mid \mathcal{Q}^N(\mathcal{X}) \mid \leq (N+1)^{|\mathcal{X}|}, \tag{2.1}$$

$$\mid \mathcal{V}^N(\mathcal{X}|\mathbf{s}) \mid \leq (N+1)^{|\mathcal{S}||\mathcal{X}|}, \tag{2.2}$$

for any type $Q \in \mathcal{Q}^N(\mathcal{X})$

$$(N+1)^{-|\mathcal{X}|} \exp\{NH_Q(X)\} \leq \mid \mathcal{T}_Q^N(X) \mid \leq \exp\{H_Q(X)\}, \tag{2.3}$$

and for any type $P \in \mathcal{P}^N(\mathcal{S})$ and $V \in \mathcal{V}^N(\mathcal{X}|\mathbf{s})$

$$(N+1)^{-|\mathcal{S}||\mathcal{X}|} \exp\{NH_{P,V}(X|S)\} \leq \mid \mathcal{T}_{P,V}^N(X \mid \mathbf{s}) \mid \leq \exp\{H_{P,V}(X|S)\}. \tag{2.4}$$

3. LAO Testing of Multiple Hypotheses for One Object

The problem of optimal testing of *multiple* hypotheses was proposed by Dobrushin [20], and was investigated in [29]–[33]. Fu and Shen [23] explored the case of two hypotheses when side information is absent. The problem concerning arbitrarily varying sources solved in [38] was induced by the ideas of the paper of Ahlswede [1]. The case of two hypotheses with side information about states was considered in [3]. In the same way as in [23] from result on LAO testing, the rate-reliability and the reliability-rate functions for arbitrarily varying source with side information were obtained in [38].

The problem of multiple hypotheses LAO testing for discrete stationary Markov source of observations was solved by Haroutunian [30]–[32]. In [37] Haroutunian and Grigoryan generalized results from [23], [30]–[32] for multi-hypotheses LAO testing by a non-informed statistician for arbitrarily varying Markov source.

Here for clearness we expose the results on multiple hypotheses LAO testing for the case of the most simple invariant object.

Let $\mathcal{X}$ be a finite set of values of random variable (RV) X. M possible PDs $G_m = \{G_m(x),\ x \in \mathcal{X}\}$, $m = \overline{1, M}$, of RV X characterizing the object are known.

The statistician must detect one among M alternative hypotheses G_m, using sample $\mathbf{x} = (x_1, ..., x_N)$ of results of N independent observations of the object.

The procedure of decision making is a non-randomized test $\varphi_N(\mathbf{x})$, it can be defined by division of the sample space $\mathcal{X}^N$ on M disjoint subsets $\mathcal{A}_m^N = \{\mathbf{x} :\ \varphi_N(\mathbf{x}) = m\}$, $m = \overline{1, M}$. The set $\mathcal{A}_m^N$ consists of all samples $\mathbf{x}$ for which the hypothesis G_m must be adopted. We study the probabilities $\alpha_{l|m}(\varphi_N)$ of the erroneous acceptance of hypothesis G_l provided that G_m is true

$$\alpha_{l|m}(\varphi_N) \stackrel{\triangle}{=} G_m^N(A_l^N),\ l, m = \overline{1, M},\ \ m \neq l. \tag{3.1}$$

The probability to reject hypothesis G_m, when it is true, is also considered

$$\begin{aligned}\alpha_{m|m}(\varphi_N) &\stackrel{\triangle}{=} \sum_{l\neq m}\alpha_{l|m}(\varphi_N)\\ &= G_m^N(\overline{\mathcal{A}_m^N})\\ &= (1-G_m^N(\mathcal{A}_m^N)).\end{aligned} \tag{3.2}$$

A quadratic matrix of M^2 error probabilities $\{\alpha_{l|m}(\varphi_N),\ m=\overline{1,M},\ l=\overline{1,M}\}$ is the *power* of the tests.

Error probability exponents of the infinite sequence φ of tests, which we call *reliabilities*, are defined as follows:

$$E_{l|m}(\varphi) \stackrel{\triangle}{=} \overline{\lim_{N\to\infty}}\left\{-\frac{1}{N}\log\alpha_{l|m}(\varphi_N)\right\},\ \ m,l=\overline{1,M}. \tag{3.3}$$

We see from (3.2) and (3.3) that

$$E_{m|m}(\varphi)=\min_{l\neq m}E_{l|m}(\varphi),\ \ m=\overline{1,M}. \tag{3.4}$$

The matrix

$$\mathbf{E}(\varphi)=\begin{pmatrix} E_{1|1}\ \dots\ E_{l|1}\ \dots\ E_{M|1}\\ \dots\dots\dots\dots\dots\dots\dots\\ E_{1|m}\dots E_{l|m}\dots E_{M|m},\\ \dots\dots\dots\dots\dots\dots\dots\\ E_{1|M}\dots E_{l|M}\dots E_{M|M}\end{pmatrix}$$

called the *reliabilities matrix* of the tests sequence φ is the object of our investigation.

We recognize that a sequence φ^* of tests is LAO if for given positive values of $M-1$ diagonal, elements of matrix $\mathbf{E}(\varphi^*)$ the procedure provides maximal values for all other elements of it.

Now we form the LAO test by constructing decision sets noted $\mathcal{R}_m^{(N)}$. Given strictly positive numbers $E_{m|m}$, $m=\overline{1,M-1}$, we define the following regions:

$$\mathcal{R}_m \stackrel{\triangle}{=} \{Q:\ \ D(Q||G_m)\leq E_{m|m}\},\ \ \ m=\overline{1,M-1}, \tag{3.5}$$

$$\mathcal{R}_M \stackrel{\triangle}{=} \{Q:\ \ D(Q||G_m)> E_{m|m},\ \ \ m=\overline{1,M-1}\}, \tag{3.6}$$

$$\mathcal{R}_m^{(N)} \stackrel{\triangle}{=} \mathcal{R}_m\bigcap\mathcal{Q}^N(\mathcal{X}),\ \ m=\overline{1,M}, \tag{3.7}$$

and corresponding values:

$$E^*_{m|m}=E^*_{m|m}(E_{m|m})\stackrel{\triangle}{=} E_{m|m},\ \ \ m=\overline{1,M-1}, \tag{3.8}$$

$$E^*_{m|l}=E^*_{m|l}(E_{m|m})\stackrel{\triangle}{=}\inf_{Q\in\mathcal{R}_m}D(Q||G_l),\ \ \ l=\overline{1,M},\ \ m\neq l,\ \ m=\overline{1,M-1}, \tag{3.9}$$

$$E^*_{M|m} = E^*_{M|m}(E_{1|1}, E_{2|2}, ..., E_{M-1|M-1}) \stackrel{\triangle}{=} \inf_{P \in \mathcal{R}_M} D(Q||G_m), \quad m = \overline{1, M-1}, \tag{3.10}$$

$$E^*_{M|M} = E^*_{M|M}(E_{1|1}, E_{2|2}, ..., E_{M-1|M-1}) \stackrel{\triangle}{=} \min_{m: m=\overline{1,M-1}} E^*_{M|m}. \tag{3.11}$$

Theorem 3.1 [33]: *If for described model all conditional PDs G_m, $m = \overline{1, M}$, are different in the sense that, $D(G_l||G_m) > 0$, $l \neq m$, and the positive numbers $E_{1|1}, E_{2|2}, ..., E_{M-1|M-1}$ are such that the following $M-1$ inequalities, called compatibility conditions, hold*

$$E_{1|1} < \min_{m=\overline{2,M}} D(G_m||G_1), \tag{3.12}$$

$$E_{m|m} < \min \left[\min_{l=\overline{1,m-1}} E^*_{l|m}(E_{l|l}), \quad \min_{l=\overline{m+1,L}} D(G_l||G_m) \right], \ m = \overline{2, M-1},$$

then there exists a LAO sequence φ^* of tests, the reliabilities matrix of which $\mathbf{E}(\varphi^*) = \{E^*_{m|l}\}$ is defined in (3.8)–(3.11) and all elements of it are positive.

When one of inequalities (3.12) is violated, then at least one element of matrix $\mathbf{E}(\varphi^*)$ is equal to 0.

The proof of Theorem 3.1 is postponed to the Appendix.

It is worth to formulate the following useful property of reliabilities matrix of the LAO test.

Remark 3.1 [39]: *The diagonal elements of the reliabilities matrix of the LAO test in each row are equal only to the element of the last column:*

$$E^*_{m|m} = E^*_{M|m}, \text{ and } E^*_{m|m} < E^*_{l|m}, \ l = \overline{1, M-1}, \ l \neq m, \ m = \overline{1, M}. \tag{3.13}$$

That is the elements of the last column are equal to the diagonal elements of the same row and due to (3.4) are minimal in this row. Consequently the first $M-1$ elements of the last column also can play a part as given parameters for construction of a LAO test.

4. Reliability Approach to Multihypotheses Testing for Many Objects

In [6] Ahlswede and Haroutunian proposed a new aspect of the statistical theory – investigation of models with many objects. This work developed the ideas of papers on Information theory [1], [5], of papers on many hypotheses testing [29]-[33] and of book [9], devoted to research of sequential procedures solving decision problems such as ranking and identification. The problem of hypotheses testing for the model consisting of two independent and of two strictly dependent objects (when they cannot admit the same distribution) with two possible hypothetical distributions were solved in [6]. In [39] the specific characteristics of the model consisting of $K(\geq 2)$ objects each independently of others following one of given $M(\geq 2)$ probability distributions were explored. In [47] the model composed by stochastically related objects was investigated. The result concerning two independent Markov chains is presented in [36]. In this section we expose these results.

4.1. Multihypotheses LAO Testing for Many Independent Objects

Let us now consider the model with three independent similar objects. For brevity we solve the problem for three objects, the generalization of the problem for K independent objects will be discussed hereafter along the text.

Let X_1, X_2 and X_3 be independent RVs taking values in the same finite set $\mathcal{X}$, each of them with one of M hypothetical PDs $G_m = \{G_m(x), \;\; x \in \mathcal{X}\}$. These RVs are the characteristics of the objects. The random vector (X_1, X_2, X_3) assumes values $(x^1, x^2, x^3) \in \mathcal{X}^3$.

Let $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) \stackrel{\triangle}{=} ((x_1^1, x_1^2, x_1^3), \dots, (x_n^1, x_n^2, x_n^3), \dots, (x_N^1, x_N^2, x_N^3)),\ x_n^k \in \mathcal{X}$, $k = \overline{1,3}$, $n = \overline{1,N}$, be a vector of results of N independent observations of the family (X_1, X_2, X_3). The test has to determine unknown PDs of the objects on the base of observed data. The detection for each object should be made from the same set of hypotheses: G_m, $m = \overline{1,M}$. We call this procedure the *compound test* for three objects and denote it by Φ_N, it can be composed of three individual tests φ_N^1, φ_N^2, φ_N^3 for each of three objects. The test φ_N^i, $i = \overline{1,3}$, is a division of the space $\mathcal{X}^N$ into M disjoint subsets $\mathcal{A}_m^i$, $m = \overline{1,M}$. The set $\mathcal{A}_m^i$, $m = \overline{1,M}$, contains all vectors $\mathbf{x}_i$ for which the hypothesis G_m is adopted. Hence test Φ_N is realised by division of the space $\mathcal{X}^N \times \mathcal{X}^N \times \mathcal{X}^N$ into M^3 subsets $\mathcal{A}_{m_1,m_2,m_3} = \mathcal{A}_{m_1}^1 \times \mathcal{A}_{m_2}^2 \times \mathcal{A}_{m_3}^3$, $m_i = \overline{1,M}$, $i = \overline{1,3}$. We denote the infinite sequence of compound tests by Φ. When we have K independent objects the test Φ is composed of tests φ^1, φ^2, $\dots$, φ^K.

The probability of the falsity of acceptance of hypotheses triple $(G_{l_1}, G_{l_2}, G_{l_3})$ by the test Φ_N provided that the triple of hypotheses $(G_{m_1}, G_{m_2}, G_{m_3})$ is true, where $(m_1, m_2, m_3) \neq (l_1, l_2, l_3)$, $m_i, l_i = \overline{1,M}$, $i = \overline{1,3}$, is:

$$\begin{aligned}
\alpha_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi_N) &\stackrel{\triangle}{=} G_{m_1}^N \circ G_{m_2}^N \circ G_{m_3}^N \left(\mathcal{A}_{l_1,l_2,l_3}^N\right) \\
&\stackrel{\triangle}{=} G_{m_1}^N \left(\mathcal{A}_{l_1}^N\right) \cdot G_{m_2}^N \left(\mathcal{A}_{l_2}^N\right) \cdot G_{m_3}^N \left(\mathcal{A}_{l_3}^N\right) \\
&= \sum_{\mathbf{x}_1 \in \mathcal{A}_{l_1}^N} G_{m_1}^N(\mathbf{x}_1) \sum_{\mathbf{x}_2 \in \mathcal{A}_{l_2}^N} G_{m_2}^N(\mathbf{x}_2) \sum_{\mathbf{x}_3 \in \mathcal{A}_{l_3}^N} G_{m_3}^N(\mathbf{x}_3).
\end{aligned}$$

The probability to reject a true triple of hypotheses $(G_{m_1}, G_{m_2}, G_{m_3})$ by analogy with (3.2) is defined as follows:

$$\alpha_{m_1,m_2,m_3|m_1,m_2,m_3}(\Phi_N) \stackrel{\triangle}{=} \sum_{(l_1,l_2,l_3) \neq (m_1,m_2,m_3)} \alpha_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi_N). \tag{4.1}$$

We study corresponding reliabilities $E_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi)$ of the sequence of tests Φ,

$$E_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi) \stackrel{\triangle}{=} \overline{\lim_{N \to \infty}} \left\{ -\frac{1}{N} \log \alpha_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi_N) \right\},$$
$$m_i, l_i = \overline{1,M}, \;\; i = \overline{1,3}. \tag{4.2}$$

Definitions (4.1) and (4.2) imply (compare with (3.4)) that

$$E_{m_1,m_2,m_3|m_1,m_2,m_3}(\Phi) = \min_{(l_1,l_2,l_3)\neq(m_1,m_2,m_3)} E_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi). \tag{4.3}$$

Our aim is to analyze the reliabilities matrix $\mathbf{E}(\Phi^*) = \{E_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi^*)\}$ of LAO test sequence Φ^* for three objects. We call the test sequence LAO for the model with many objects if for given positive values of certain part of elements of reliabilities matrix the procedure provides maximal values for all other elements of it.

Let us denote by $\mathbf{E}(\varphi^i)$ the reliabilities matrices of the sequences of tests φ^i, $i = \overline{1,3}$. The following Lemma is a generalization of Lemma from [6].

Lemma 4.1: *If elements $E_{l|m}(\varphi^i)$, $m, l = \overline{1,M}$, $i = \overline{1,3}$, are strictly positive, then the following equalities hold for $\mathbf{E}(\Phi)$, $\Phi = (\varphi^1,\varphi^2,\varphi^3)$, $l_i, m_i = \overline{1,M}$:*

$$E_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi) = \sum_{i=1}^{3} E_{l_i|m_i}(\varphi^i), \quad m_i \neq l_i, \tag{4.4}$$

$$E_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi) = \sum_{i\in[[1,2,3]-k]} E_{l_i|m_i}(\varphi^i), \quad m_k = l_k,\ m_i \neq l_i,\ k = \overline{1,3}, \tag{4.5}$$

$$E_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi) = E_{l_i|m_i}(\varphi^i), \quad i = \overline{1,3},\ m_k = l_k,\ m_i \neq l_i,\ k \in [[1,2,3]-i]. \tag{4.6}$$

Equalities (4.4) are valid also if $E_{l_i|m_i}(\varphi^i) = 0$ for several pairs (m_i, l_i) and several i.

The proof of Lemma 4.1 is exposed in Appendix.

Now we shall show how we can erect the LAO test from the set of compound tests when $3(M-1)$ strictly positive elements of the reliabilities matrix $E_{M,M,M|m,M,M}$, $E_{M,M,M|M,m,M}$ and $E_{M,M,M|M,M,m}$, $m = \overline{1,M-1}$, are preliminarily given.

The following subset of tests:

$$\mathcal{D} = \{\Phi : E_{m|m}(\varphi^i) > 0, \quad m = \overline{1,M}, \ i = \overline{1,3}\}$$

is distinguished by the property that when $\Phi \in \mathcal{D}$ the elements $E_{M,M,M|m,M,M}(\Phi)$, $E_{M,M,M|M,m,M}(\Phi)$ and $E_{M,M,M|M,M,m}(\Phi)$, $m = \overline{1,M-1}$, of the reliabilities matrix are strictly positive.

Really, because $E_{m|m}(\varphi^i) > 0$, $m = \overline{1,M}$, $i = \overline{1,3}$, then in view of (3.4) $E_{M|m}(\varphi^i)$ are also strictly positive. From equalities (4.4)–(4.6) we obtain that the noted elements are strictly positive for $\Phi \in \mathcal{D}$ and $m = \overline{1,M-1}$

$$E_{M,M,M|m,M,M}(\Phi) = E_{M|m}(\varphi^1), \tag{4.7}$$

$$E_{M,M,M|M,m,M}(\Phi) = E_{M|m}(\varphi^2), \tag{4.8}$$

$$E_{M,M,M|M,M,m}(\Phi) = E_{M|m}(\varphi^3). \tag{4.9}$$

For given positive elements

$$E_{M,M,M|m,M,M},\ E_{M,M,M|M,m,M},\ E_{M,M,M|M,M,m},\ m = \overline{1, M-1},$$

define the following family of decision sets of PDs:

$$\mathcal{R}_m^{(i)} \triangleq \{Q : D(Q||G_m) \leq E_{M,M,M|m_1,m_2,m_3},\ m_i = m,\ m_j = M,\ \ i \neq j, j = \overline{1,3}\}$$
$$m = \overline{1, M-1},\ \ i = \overline{1,3}, \tag{4.10}$$

$$\mathcal{R}_M^{(i)} \triangleq \{Q : D(Q||G_m) > E_{M,M,M|m_1,m_2,m_3},\ m_i = m,\ m_j = M, i \neq j, j = \overline{1,3},$$
$$m = \overline{1, M-1}\},\ \ i = \overline{1,3}. \tag{4.11}$$

Define also the elements of the reliability matrix of the compound LAO test for three objects:

$$\begin{aligned}
E^*_{M,M,M|m,M,M} &\triangleq E_{M,M,M|m,M,M},\\
E^*_{M,M,M|M,m,M} &\triangleq E_{M,M,M|M,m,M},\\
E^*_{M,M,M|M,M,m} &\triangleq E_{M,M,M|M,M,m},
\end{aligned} \tag{4.12}$$

$$E^*_{l_1,l_2,l_3|m_1,m_2,m_3} \triangleq \inf_{Q \in R^{(i)}_{l_i}} D(Q||G_{m_i}),$$
$$i = \overline{1,3},\ m_k = l_k,\ m_i \neq l_i,\ i \neq k,\ k \in [[1,2,3] - i], \tag{4.13}$$

$$E^*_{m_1,m_2,m_3|l_1,l_2,l_3} \triangleq \sum_{i \neq k} \inf_{Q \in R^{(i)}_{l_i}} D(Q||G_{m_i}),$$
$$m_k = l_k,\ m_i \neq l_i,\ k = \overline{1,3},\ i \in [[1,2,3] - k], \tag{4.14}$$

$$E^*_{l_1,l_2,l_3|m_1,m_2,m_3} \triangleq \sum_{i=1}^{3} \inf_{Q \in R^{(i)}_{l_i}} D(Q||G_{m_i}),\ m_i \neq l_i, i = \overline{1,3}. \tag{4.15}$$

The following theorem is a generalization and improvement of the corresponding theorem proved in [6] for the case $K = 2,\ M = 2$.

Theorem 4.1 [39]: *For considered model with three objects, if all distributions G_m, $m = \overline{1, M}$, are different, (and equivalently $D(G_l||G_m) > 0$, $l \neq m$, $l, m = \overline{1, M}$), then the following statements are valid:*

a) when given strictly positive elements $E_{M,M,M|m,M,M}$, $E_{M,M,M|M,m,M}$ and $E_{M,M,M|M,M,m}$, $m = \overline{1, M-1}$, meet the following conditions

$$\max(E_{M,M,M|1,M,M}, E_{M,M,M|M,1,M}, E_{M,M,M|M,M,1})$$

$$< \min_{l=\overline{2,M}} D(G_l||G_1), \tag{4.16}$$

and for $m = \overline{2, M-1}$,

$$E_{M,M,M|m,M,M} < \min\left[\min_{l=\overline{1,m-1}} E^*_{l,m,m|m,m,m}, \min_{l=\overline{m+1,M}} D(G_l||G_m)\right], \tag{4.17}$$

$$E_{M,M,M|M,m,M} < \min\left[\min_{l=\overline{1,m-1}} E^*_{m,l,m|m,m,m}, \min_{l=\overline{m+1,M}} D(G_l||G_m)\right], \tag{4.18}$$

$$E_{M,M,M|M,M,m} < \min\left[\min_{l=\overline{1,m-1}} E^*_{m,m,l|m,m,m}, \min_{l=\overline{m+1,M}} D(G_l||G_m)\right], \tag{4.19}$$

then there exists a LAO test sequence $\Phi^* \in \mathcal{D}$*, the reliability matrix of which* $\mathbf{E}(\Phi^*)$ *is defined in* (4.12)–(4.15) *and all elements of it are positive,*

b) if even one of the inequalities (4.16)–(4.19) *is violated, then there exists at least one element of the matrix* $\mathbf{E}(\Phi^*)$ *equal to* 0*.*

For the proof of Theorem 4.1 see Appendix.

When we consider the model with K independent objects the generalization of Lemma 4.1 will take the following form.

Lemma 4.2: *If elements* $E_{l_i|m_i}(\varphi^i)$*,* $m_i, l_i = \overline{1,M}$*,* $i = \overline{1,K}$*, are strictly positive, then the following equalities hold for* $\Phi = (\varphi^1, \varphi^2, ..., \varphi^K)$:

$$E_{l_1,l_2,\dots,l_K|m_1,m_2,\dots,m_K}(\Phi) = \sum_{i=1}^{K} E_{l_i|m_i}(\varphi^i), \quad m_i \neq l_i,\ i = \overline{1,K},$$

$$E_{l_1,l_2,\dots,l_K|m_1,m_2,\dots,m_K}(\Phi) = \sum_{i:\ m_i \neq l_i} E_{l_i|m_i}(\varphi^i).$$

For given $K(M-1)$ strictly positive elements $E_{M,M,\dots,M|m,M,\dots,M}$, $E_{M,M,\dots,M|M,m,\dots,M}$, ... , $E_{M,\dots,M,M,|M,M\dots,m}$, $m = \overline{1, M-1}$, for K independent objects we can find the LAO test Φ^* in a way similar to case of three independent objects.

Comment 4.1: Idea to renumber K-distributions from 1 to M^K and consider them as PDs of one complex object offers an alternative way of testing for models with K objects. We can give $M^K - 1$ diagonal elements of corresponding large matrix $\mathbf{E}(\Phi)$ and apply Theorem 3.2 concerning one composite object. In this direct algorithm the number of the preliminarily given elements of the matrix $\mathbf{E}(\Phi)$ would be greater (because $M^K - 1 > K(M-1)$, $M \geq 2, K \geq 2$) but the procedure of calculations would be longer than in our algorithm presented above in this section. Our approach to the problem gives also the possibility to define the LAO tests for each of the separate objects, but the approach with renumbering of K-vectors of hypotheses does not have this opportunity. In the same time in the case of direct algorithm there is opportunity for the investigator to define preliminarily the greater number of elements of the matrix $\mathbf{E}(\Phi)$.

In applications one of two approaches may be used in conformity with preferences of the investigator.

4.1.1. Example

Some illustrations of exposed results are in an example concerning two objects. The set $\mathcal{X} = \{0, 1\}$ contains two elements and the following PDs are given on $\mathcal{X}$: $G_1 = \{0,10; 0,90\}$, $G_2 = \{0,85; 0,15\}$, $G_3 = \{0,23; 0,77\}$. As it follows from relations (4.12)–(4.15), several elements of the reliability matrix are functions of one of given elements, there are also elements which are functions of two, or three given elements. For example, in Fig. 1 and Fig. 2 the results of calculations of functions $E_{1,2|2,1}(E_{3,3|1,3}, E_{3,3|3,2})$ and $E_{1,2|2,2}(E_{3,3|1,3})$ are presented. For these distributions we have $\min(D(G_2||G_1), D(G_3||G_1)) \approx 2,2$ and $\min(E_{2,2|2,1}, D(G_3||G_2)) \approx 1,4$. We see that, when the inequalities (4.16) or (4.19) are violated, $E_{1,2|2,1} = 0$ and $E_{1,2|2,2} = 0$.

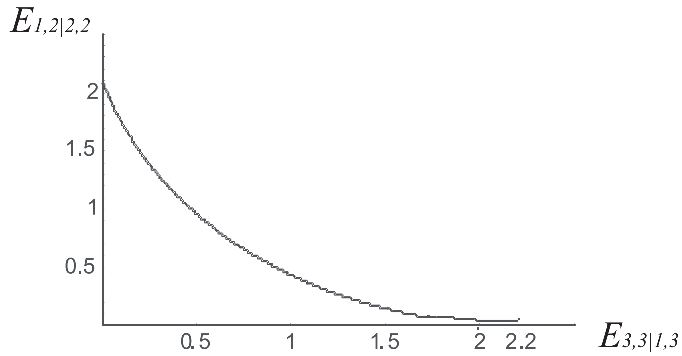

Fig 1.

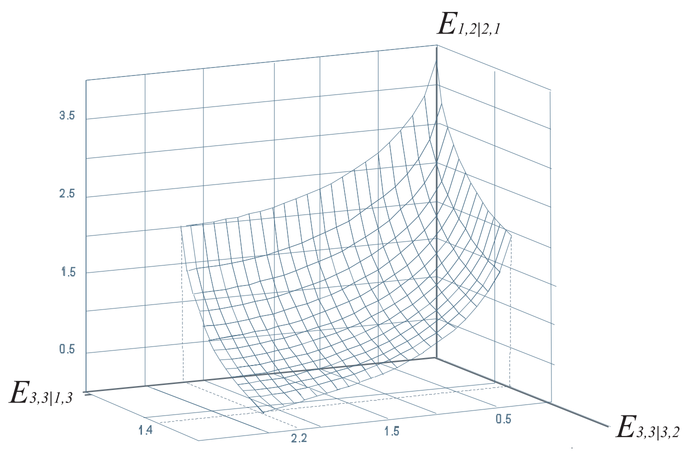

Fig 2.

4.2. Multihypotheses LAO Testing for Two Dependent Objects

We consider characteristics of procedures of LAO testing of probability distributions of two related (*stochastically*, *statistically* and *strictly dependent*) objects. We use these terms for different kinds of dependence of two objects.

Let X_1 and X_2 be RVs taking values in the same finite set $\mathcal{X}$ and $\mathcal{P}(\mathcal{X})$ be the space of all possible distributions on $\mathcal{X}$.

Let $(\mathbf{x_1}, \mathbf{x_2}) = ((x_1^1, x_1^2), (x_2^1, x_2^2), ...(x_N^1, x_N^2))$ be a sequence of results of N independent observations of the pair of objects.

First we consider the model, which consists of two *stochastically* related objects. We name so the following more general dependence. There are given M_1 PDs

$$G_{m_1} = \{G_{m_1}(x^1),\ x^1 \in \mathcal{X}\},\ \ m_1 = \overline{1, M_1}.$$

The first object is characterized by RV X_1 which has one of these M_1 possible PDs and the second object is dependent on the first and is characterized by RV X_2 which can have one of $M_1 \times M_2$ conditional PDs

$$G_{m_2|m_1} = \{G_{m_2|m_1}(x^2|x^1),\ x^1, x^2 \in \mathcal{X}\},\ m_1 = \overline{1, M_1},\ m_2 = \overline{1, M_2}.$$

Joint PD of the pair of objects is

$$G_{m_1,m_2} = G_{m_1} \circ G_{m_2|m_1} = \{G_{m_1,m_2}(x^1, x^2),\ x^1, x^2 \in \mathcal{X}\},$$

where

$$G_{m_1,m_2}(x^1, x^2) = G_{m_1}(x^1)G_{m_2|m_1}(x^2|x^1),\ \ m_1 = \overline{1, M_1},\ m_2 = \overline{1, M_2}.$$

The probability $G^N_{m_1,m_2}(\mathbf{x_1}, \mathbf{x_2})$ of N-vector $(\mathbf{x_1}, \mathbf{x_2})$ is the following product:

$$\begin{aligned} G^N_{m_1,m_2}(\mathbf{x_1}, \mathbf{x_2}) &\stackrel{\triangle}{=} G^N_{m_1}(\mathbf{x_1})G^N_{m_2|m_1}(\mathbf{x_2}|\mathbf{x_1}) \\ &\stackrel{\triangle}{=} \prod_{n=1}^{N} G_{m_1}(x^1_n)G_{m_2|m_1}(x^2_n|x^1_n), \end{aligned}$$

with $G^N_{m_1}(\mathbf{x_1}) = \prod\limits_{n=1}^{N} G_{m_1}(x^1_n)$ and $G^N_{m_2 m_1}(\mathbf{x_2}|\mathbf{x_1}) = \prod\limits_{n=1}^{N} G_{m_2|m_1}(x^2_n|x^1_n)$.

In somewhat particular case, when X_1 and X_2 are related *statistically* [44], [60], the second object depends on index of PD of the first object but not depends on value x^1 taken by the first object. The second object is characterized by RV X_2 which can have one of $M_1 \times M_2$ conditional PDs $G_{m_2|m_1} = \{G_{m_2|m_1}(x^2),\ x^2 \in \mathcal{X}\}$, $m_1 = \overline{1, M_1}$, $m_2 = \overline{1, M_2}$.

In the third case of *strict* dependence, the objects X_1 and X_2 can have only different distributions from the same given family of M PDs G_1, G_2, ..., G_M.

Discussed in Comment 4.1 the *direct approach* for LAO testing of PDs for two related objects, consisting in consideration of the pair of objects as one composite object and then use of Theorem 3.1, is applicable for first two cases [45]. But now we consider also another approach.

Let us remark that test Φ^N can be composed of a pair of tests φ^N_1 and φ^N_2 for the separate objects: $\Phi^N = (\varphi^N_1, \varphi^N_2)$. Denote by φ^1, φ^2 and Φ the infinite sequences of tests for the first, for the second and for the pair of objects, respectively.

Let X_1 and X_2 be related *stochastically*. For the object characterized by X_1 the non-randomized test $\varphi^1_N(\mathbf{x_1})$ can be determined by partition of the sample space $\mathcal{X}^N$ on M_1 disjoint subsets $\mathcal{A}^N_{l_1} = \{\mathbf{x_1} : \varphi^1_N(\mathbf{x_1}) = l_1\}$, $l_1 = \overline{1, M_1}$, i.e. the set $\mathcal{A}^N_{l_1}$ consists of vectors $\mathbf{x_1}$ for which the PD G_{l_1} is adopted. The probability $\alpha_{l_1|m_1}(\varphi^1_N)$ of the erroneous acceptance of PD G_{l_1} provided that G_{m_1} is true, $l_1, m_1 = \overline{1, M_1}$, $m_1 \neq l_1$, is defined by the set $\mathcal{A}^N_{l_1}$

$$\alpha_{l_1|m_1}(\varphi^1_N) \stackrel{\triangle}{=} G^N_{m_1}(\mathcal{A}^N_{l_1}).$$

We define the probability to reject G_{m_1}, when it is true, as follows

$$\alpha_{m_1|m_1}(\varphi_N^1) \stackrel{\triangle}{=} \sum_{l_1: l_1 \neq m_1} \alpha_{l_1|m_1}(\varphi_N^1) = G_{m_1}^N(\overline{\mathcal{A}_{m_1}^N}). \tag{4.20}$$

Corresponding error probability exponents are:

$$E_{l_1|m_1}(\varphi^1) \stackrel{\triangle}{=} \overline{\lim_{N\to\infty}} \left\{ -\frac{1}{N} \log \alpha_{l_1|m_1}(\varphi_N^1) \right\}, \ \ m_1, l_1 = \overline{1, M_1}. \tag{4.21}$$

It follows from (4.20) and (4.21) that

$$E_{m_1|m_1}(\varphi^1) = \min_{l_1: l_1 \neq m_1} E_{l_1|m_1}(\varphi^1), \ \ l_1, m_1 = \overline{1, M_1}.$$

For construction of the LAO test we assume given strictly positive numbers $E_{m_1|m_1}$, $m = \overline{1, M_1 - 1}$ and we define regions $\mathcal{R}_{l_1}$, $l = \overline{1, M_1}$ as in (3.5)–(3.6).

For the second object characterized by RV X_2 depending on X_1 the non-randomized test $\varphi_N^2(\mathbf{x_2}, \mathbf{x_1}, l_1)$, based on vectors $(\mathbf{x_1}, \mathbf{x_2})$ and on the index of the hypothesis l_1 adopted for X_1, can be given for each l_1 and $\mathbf{x}_1$ by division of the sample space $\mathcal{X}^N$ on M_2 disjoint subsets

$$\mathcal{A}_{l_2|l_1}^N(\mathbf{x_1}) \stackrel{\triangle}{=} \{\mathbf{x_2} : \ \varphi_2^N(\mathbf{x_2}, \mathbf{x_1}, l_1) = l_2\}, \ l_1 = \overline{1, M_1}, \ l_2 = \overline{1, M_2}. \tag{4.22}$$

Let

$$\mathcal{A}_{l_1,l_2}^N \stackrel{\triangle}{=} \{(\mathbf{x_1}, \mathbf{x_2}) : \ \mathbf{x_1} \in A_{l_1}^N, \ \ \mathbf{x_2} \in \mathcal{A}_{l_2|l_1}^N(\mathbf{x_1})\}. \tag{4.23}$$

The probabilities of the erroneous acceptance for $(l_1, l_2) \neq (m_1, m_2)$ are

$$\alpha_{l_1,l_2|m_1,m_2} \stackrel{\triangle}{=} G_{m_1,m_2}^N(\mathcal{A}_{l_1,l_2}^N).$$

Corresponding reliabilites are denoted $E_{l_1,l_2|m_1,m_2}$ and are defined as in (4.2). We can upper estimate the probabilities of the erroneous acceptance for $(l_1, l_2) \neq (m_1, m_2)$

$$\begin{aligned}
G_{m_1,m_2}^N(\mathcal{A}_{l_1,l_2}^N) &= \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\mathcal{A}_{l_1,l_2}^N} G_{m_1}^N(\mathbf{x}_1) G_{m_2|m_1}^N(\mathbf{x}_2|\mathbf{x_1}) \\
&= \sum_{\mathbf{x}_1 \in \mathcal{A}_{l_1}^N} G_{m_1}^N(\mathbf{x}_1) G_{m_2|m_1}^N(\mathcal{A}_{l_2|l_1}^N(\mathbf{x}_1)|\mathbf{x_1}) \\
&\leq \max_{\mathbf{x}_1 \in \mathcal{A}_{l_1}^N} G_{m_2|m_1}^N(\mathcal{A}_{l_2|l_1}^N(\mathbf{x}_1)|\mathbf{x_1}) \sum_{\mathbf{x}_1 \in \mathcal{A}_{l_1}^N} G_{m_1}^N(\mathbf{x}_1) \\
&= G_{m_1}^N(\mathcal{A}_{l_1}^N) \max_{\mathbf{x}_1 \in \mathcal{A}_{l_1}^N} G_{m_2|m_1}^N(\mathcal{A}_{l_2|l_1}^N(\mathbf{x}_1)|\mathbf{x_1}).
\end{aligned}$$

These upper estimates of $\alpha_{l_1,l_2|m_1,m_2}(\Phi_N)$ for each $(l_1, l_2) \neq (m_1, m_2)$ we denote by

$$\beta_{l_1,l_2|m_1,m_2}(\Phi_N) \stackrel{\triangle}{=} G_{m_1}^N(\mathcal{A}_{l_1}^N) \max_{\mathbf{x}_1 \in \mathcal{A}_{l_1}^N} G_{m_2|m_1}^N(\mathcal{A}_{l_2|l_1}^N(\mathbf{x}_1)|\mathbf{x_1}).$$

Consequently we can deduce that for $l_1, m_1 = \overline{1, M_1},\ l_2, m_2 = \overline{1, M_2}$, new parameters

$$F_{l_1,l_2|m_1,m_2}(\Phi) \stackrel{\triangle}{=} \overline{\lim_{N\to\infty}}\{-\frac{1}{N}\log \beta^N_{l_1,l_2|m_1,m_2}(\Phi_N)\},$$

are lower estimates for reliabilities $E_{l_1,l_2|m_1,m_2}(\Phi)$.
We can introduce for $l_1, m_1 = \overline{1, M_1},\ l_2, m_2 = \overline{1, M_2},\ m_2 \neq l_2$,

$$\beta_{l_2|l_1,m_1,m_2}(\varphi^2_N) \stackrel{\triangle}{=} \max_{\mathbf{x}_1 \in \mathcal{A}^N_{l_1}} G^N_{m_2|m_1}(\mathcal{A}^N_{l_2|l_1}(\mathbf{x}_1)|\mathbf{x_1}),$$

and also consider

$$\beta_{m_2|l_1,m_1,m_2}(\varphi^2_N) \stackrel{\triangle}{=} \max_{\mathbf{x}_1 \in \mathcal{A}^N_{l_1}} G^N_{m_2|m_1}(\overline{\mathcal{A}^N_{m_2|l_1}(\mathbf{x}_1)}|\mathbf{x_1})$$

$$= \sum_{l_2\neq m_2} \beta_{l_2|l_1,m_1,m_2}(\varphi^2_N). \quad (4.24)$$

The corresponding estimates of the reliabilities of test φ^2_N, are the following

$$F_{l_2|l_1,m_1,m_2}(\varphi^2) \stackrel{\triangle}{=} \overline{\lim_{N\to\infty}}\left\{-\frac{1}{N}\log \beta_{l_2|l_1,m_1,m_2}(\varphi^2_N)\right\},$$

$$l_1, m_1 = \overline{1, M_1},\ l_2, m_2 = \overline{1, M_2},\ m_2 \neq l_2. \quad (4.25)$$

It is clear from (4.24) and (4.25) that for every $l_1, m_1 = \overline{1, M_1},\ l_2, m_2 = \overline{1, M_2}$

$$F_{m_2|l_1,m_1,m_2}(\varphi^2) = \min_{l_2 : l_2 \neq m_2} F_{l_2|l_1,m_1,m_2}(\varphi^2). \quad (4.26)$$

For given positive numbers $F_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, M_2 - 1}$, for $Q \in \mathcal{R}_{l_1}$ and for each pair $l_1, m_1 = \overline{1, M_1}$ let us define the following regions and values:

$$\mathcal{R}_{l_2|l_1}(Q) \stackrel{\triangle}{=} \{V : \ D(V||G_{l_2|l_1}|Q) \leq F_{l_2|l_1,m_1,l_2}\},\ l_2 = \overline{1, M_2 - 1}, \quad (4.27)$$

$$\mathcal{R}_{M_2|l_1}(Q) \stackrel{\triangle}{=} \left\{V : \ D(V||G_{l_2|l_1}|Q) > F_{l_2|l_1,m_1,l_2},\ l_2 = \overline{1, M_2 - 1}\right\}, \quad (4.28)$$

$$F^*_{l_2|l_1,m_1,l_2}(F_{l_2|l_1,m_1,l_2}) \triangleq F_{l_2|l_1,m_1,l_2},\ l_2 = \overline{1, M_2-1}, \tag{4.29}$$

$$F^*_{l_2|l_1,m_1,m_2}(F_{l_2|l_1,m_1,l_2}) \triangleq \inf_{Q\in\mathcal{R}_{l_1}} \inf_{V\in\mathcal{R}_{l_2|l_1}(Q)} D(V||G_{m_2|m_1}|Q),$$
$$m_2 = \overline{1, M_2},\ , m_2 \neq l_2,\ l_2 = \overline{1, M_2-1}, \tag{4.30}$$

$$F^*_{M_2|l_1,m_1,m_2}(F_{1|l_1,m_1,1}, ..., F_{M_2-1|l_1,m_1,M_2-1})$$
$$\triangleq \inf_{Q\in\mathcal{R}_{l_1}} \inf_{V\in\mathcal{R}_{M_2|l_1}(Q)} D(V||G_{m_2|m_1}|Q),$$
$$m_2 = \overline{1, M_2-1}, \tag{4.31}$$

$$F^*_{M_2|l_1,m_1,M_2}(F_{1|l_1,m_1,1}, ..., F_{M_2-1|l_1,m_1,M_2-1})$$
$$\triangleq \min_{l_2=\overline{1,M_2-1}} F^*_{l_2|l_1,m_1,M_2}. \tag{4.32}$$

We denote by $\mathbf{F}(\varphi_2)$ the matrix of lower estimates for elements of matrix $\mathbf{E}(\varphi_2)$.

Theorem 4.2 [47]**:** *If for given $m_1, l_1 = \overline{1, M_1}$, all conditional PDs $G_{l_2|l_1}$, $l_2 = \overline{1, M_2}$, are different in the sense that $D(G_{l_2|l_1}||G_{m_2|m_1}|Q) > 0$, for all $Q \in \mathcal{R}_{l_1}$, $l_2 \neq m_2$, $m_2 = \overline{1, M_2}$, when the strictly positive numbers $F_{1|l_1,m_1,1}$, $F_{2|l_1,m_1,2}$,...,$F_{M_2-1|l_1,m_1,M_2-1}$ are such that the following compatibility conditions hold*

$$F_{1|l_1,m_1,1} < \min_{l_2=\overline{2,M_2}} \inf_{Q\in\mathcal{R}_{l_1}} D(G_{l_2|l_1}||G_{1|m_1}|Q), \tag{4.33}$$

$$F_{m_2|l_1,m_1,m_2} < \min\Bigg(\min_{l_2=\overline{m_2+1,M_2}} \inf_{Q\in\mathcal{R}_{l_1}} D(G_{l_2|l_1}||G_{m_2|m_1}|Q),$$
$$\min_{l_2=\overline{1,m_2-1}} F^*_{l_2|l_1,m_1,m_2}(F_{l_2|l_1,m_1,l_2})\Bigg),\ m_2 = \overline{1, M_2-1}, \tag{4.34}$$

then there exists a sequence of tests $\varphi^{2,*}$, such that the lower estimates are defined in (4.29)– (4.32) and are strictly positive.

Inequalities (4.33), (4.34) are necessary for existence of test sequence with matrix $\mathbf{F}(\varphi^{2,*})$ of positive lower estimates having given elements $F_{l_2|l_1,m_1,l_2}$, $l_2 = \overline{1, M_2-1}$ in diagonal.

Let us define the following subsets of $\mathcal{P}(\mathcal{X})$ for given strictly positive elements

$E_{M_1,l_2|l_1,l_2}$, $F_{l_1,M_2|l_1,l_2}$, $l_1=\overline{1,M_1-1}$, $l_2=\overline{1,M_2-1}$:

$$\mathcal{R}_{l_1} \stackrel{\triangle}{=} \{Q : D(Q||G_{l_1}) \leq E_{M_1,l_2|l_1,l_2}\},\ l_1=\overline{1,M_1-1},\ l_2=\overline{1,M_2-1}, \tag{4.35}$$

$$\mathcal{R}_{l_2|l_1}(Q) \stackrel{\triangle}{=} \{V : D(V||G_{l_2|l_1}|Q) \leq F_{l_1,M_2|l_1,l_2}\},\ l_1=\overline{1,M_1-1},\ l_2=\overline{1,M_2-1}, \tag{4.36}$$

$$\mathcal{R}_{M_1} \stackrel{\triangle}{=} \{Q : D(Q||G_{l_1}) > E_{M_1,l_2|l_1,l_2},\ l_1=\overline{1,M_1-1},\ l_2=\overline{1,M_2-1}\}, \tag{4.37}$$

$$\mathcal{R}_{M_2|l_1}(Q) \stackrel{\triangle}{=} \{V : D(V||G_{l_2|l_1}|Q) > F_{l_1,M_2|l_1,l_2},\ l_1=\overline{1,M_1-1},\ l_2=\overline{1,M_2-1}\}. \tag{4.38}$$

Assume also that

$$F^*_{l_1,M_2|l_1,l_2} \stackrel{\triangle}{=} F_{l_1,M_2|l_1,l_2}, \tag{4.39}$$

$$E^*_{M_1,l_2|l_1,l_2} \stackrel{\triangle}{=} E_{M_1,l_2|l_1,l_2},\ l_1=\overline{1,M_1-1},\ l_2=\overline{1,M_2-1}, \tag{4.40}$$

$$E^*_{l_1,l_2|m_1,l_2} \stackrel{\triangle}{=} \inf_{Q:Q\in R_{l_1}} D(Q||G_{m_1}),\quad m_1\neq l_1, \tag{4.41}$$

$$F^*_{l_1,l_2|l_1,m_2} \stackrel{\triangle}{=} \inf_{Q\in\mathcal{R}_{l_1}} \inf_{V:V\in R_{l_2/l_1}(Q)} D(V||G_{m_2/m_1}|Q),\ m_2\neq l_2, \tag{4.42}$$

$$F^*_{l_1,l_2|m_1,m_2} \stackrel{\triangle}{=} F^*_{m_1,l_2|m_1,m_2} + E^*_{l_1,m_2|m_1,m_2},\quad m_i\neq l_i,\ i=1,2, \tag{4.43}$$

$$F^*_{m_1,m_2|m_1,m_2} \stackrel{\triangle}{=} \min_{(l_1,l_2)\neq(m_1,m_2)} F^*_{l_1,l_2|m_1,m_2}. \tag{4.44}$$

Theorem 4.3 [44]**:** *If all PDs G_{m_1}, $m_1=\overline{1,M_1}$, are different, that is $D(G_{l_1}||G_{m_1})>0$, $l_1\neq m_1$, $l_1,m_1=\overline{1,M_1}$, and all conditional PDs $G_{l_2|l_1}$, $l_2=\overline{1,M_2}$, are also different for all $l_1=\overline{1,M_1}$, in the sense that $D(G_{l_2|l_1}||G_{m_2|m_1}|Q)>0$, $l_2\neq m_2$, then the following statements are valid.*

When given strictly positive elements $E_{M_1,l_2|m_1,l_2}$ and $F_{l_1,M_2|l_1,m_2}$, $m_1=\overline{1,M_1-1}$, $m_2=\overline{1,M_2-1}$, meet the following conditions

$$E_{M_1,l_2|1,l_2} < \min_{l_1=\overline{2,M_1}} D(G_{l_1}||G_1), \tag{4.45}$$

$$F_{l_1,M_2|l_1,1} < \min_{l_2=\overline{2,M_2}} \inf_{Q\in\mathcal{R}_{l_1}} D(G_{l_2|l_1}||G_{1|m_1}|Q), \tag{4.46}$$

$$E_{M_1,l_2|m_1,l_2} < \min[\min_{l_1=\overline{1,m_1-1}} E^*_{l_1,l_2|m_1,l_2},\ \min_{l_1=\overline{m_1+1,M_1}} D(G_{l_1}||G_{m_1})],$$
$$m_1=\overline{2,M_1-1}, \tag{4.47}$$

$$F_{l_1,M_2|l_1,m_2} < \min[\min_{l_2=\overline{1,m_2-1}} F^*_{l_1,l_2|l_1,m_2},\ \min_{l_2=\overline{m_2+1,M_2}} \inf_{Q\in\mathcal{R}_{l_1}} D(G_{l_2|l_1}||G_{m_2|m_1}|Q)],$$
$$m_2=\overline{2,M_2-1}, \tag{4.48}$$

then there exists a LAO test sequence Φ^*, the matrix of lower estimates of which $\mathbf{F}(\Phi^*) = \{F_{l_1,l_2|m_1,m_2}(\Phi^*)\}$ is defined in (4.39)-(4.44) and all elements of it are positive.

When even one of the inequalities (4.45)-(4.48) is violated, then at least one element of the lower estimate matrix $\mathbf{F}(\Phi^*)$ is equal to 0.

When X_1 and X_2 are related statistically [44], [60] we will have instead of (4.24), (4.25) $\mathcal{A}^N_{l_2|l_1} = \{\mathbf{x_2} : \varphi_2^N(\mathbf{x_2}, l_1) = l_2\}$, $l_1 = \overline{1, M_1}$, $l_2 = \overline{1, M_2}$, and $\mathcal{A}^N_{l_1,l_2} \stackrel{\triangle}{=} \{(\mathbf{x}_1, \mathbf{x}_2) : \ \mathbf{x}_1 \in \mathcal{A}^N_{l_1}, \ \mathbf{x}_2 \in \mathcal{A}^N_{l_2|l_1}(\mathbf{x}_1)\}$. In that case we have error probabilities

$$\begin{aligned}
G^N_{m_1,m_2}(\mathcal{A}^N_{l_1,l_2}) &\stackrel{\triangle}{=} \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\mathcal{A}^N_{l_1,l_2}} G^N_{m_1}(\mathbf{x}_1) G^N_{m_2|m_1}(\mathbf{x}_2) \\
&= \sum_{\mathbf{x}_1\in\mathcal{A}^N_{l_1}} G^N_{m_1}(\mathbf{x}_1) \sum_{\mathbf{x}_2\in\mathcal{A}^N_{l_2|l_1}} G^N_{m_2|m_1}(\mathbf{x}_2) \\
&= G^N_{m_2|m_1}(\mathcal{A}^N_{l_2|l_1}) G_{m_1}(\mathcal{A}^N_{l_1}), \ (l_1, l_2) \neq (m_1, m_2).
\end{aligned}$$

For the second object the conditional probabilities of the erroneous acceptance of PD $G_{l_2|l_1}$ provided that $G_{m_2|m_1}$ is true, for $l_1, m_1 = \overline{1, M_1}$, $l_2, m_2 = \overline{1, M_2}$, are the following

$$\alpha^N_{l_2|l_1,m_1,m_2}(\varphi^2_N) \stackrel{\triangle}{=} G^N_{m_2|m_1}(\mathcal{A}^N_{l_2|l_1}),\ l_2 \neq m_2.$$

The probability to reject $G_{m_2|m_1}$, when it is true is denoted as follows

$$\alpha^N_{m_2|l_1,m_1,m_2}(\varphi^2_N) \stackrel{\triangle}{=} G^N_{m_2|m_1}(\overline{\mathcal{A}_{m_2|l_1}}) = \sum_{l_2\neq m_2} \alpha^N_{l_2|l_1,m_1,m_2}(\varphi^2_N).$$

Thus in the conditions and in the results of Theorems 4.2 and Theorems 4.3 instead of conditional divergences $\inf\limits_{Q\in\mathcal{R}_{l_1}} D(G_{l_2|l_1}||G_{m_2|m_1}|Q)$, $\inf\limits_{Q\in\mathcal{R}_{l_1}} D(V||G_{m_2|m_1}|Q)$ we will have just divergences $D(G_{l_2|l_1}||G_{m_2|m_1})$, $D(V||G_{m_2|m_1})$ and in place of $F_{l_2|l_1,m_1,m_2}(\Phi)$, $F_{l_1,l_2|m_1,m_2}(\Phi)$, $l_1, m_1 = \overline{1, M_1}$, $l_2, m_2 = \overline{1, M_2}$, will be $E_{l_2|l_1,m_1,m_2}(\Phi)$, $E_{l_1,l_2|m_1,m_2}(\Phi)$, $l_1, m_1 = \overline{1, M_1}$, $l_2, m_2 = \overline{1, M_2}$. And in that case regions defined in (4.27), (4.28) will be changed as follows:

$$\begin{aligned}
\mathcal{R}_{l_2|l_1} &\stackrel{\triangle}{=} \{V : \ D(V||G_{l_2|l_1}) \leq E_{l_2|l_1,m_1,l_2}\}, \quad l_2 = \overline{1, M_2 - 1}, \\
\mathcal{R}_{M_2|l_1} &\stackrel{\triangle}{=} \{V : \ D(V||G_{l_2|l_1}) > E_{l_2|l_1,m_1,l_2}, \quad l_2 = \overline{1, M_2 - 1}\},
\end{aligned}$$

In case of two statistically dependent objects the corresponding regions will be

$$\mathcal{R}_{l_1} \stackrel{\triangle}{=} \{Q : D(Q||G_{l_1}) \leq E_{M_1,l_2|l_1,l_2}\},\ \ l_1 = \overline{1, M_1 - 1},\ l_2 = \overline{1, M_2 - 1},$$

$$\mathcal{R}_{l_2|l_1} \stackrel{\triangle}{=} \{V : D(V||G_{l_2|l_1}) \leq E_{l_1,M_2|l_1,l_2}\},\ \ l_1 = \overline{1, M_1 - 1},\ l_2 = \overline{1, M_2 - 1},$$

$$\mathcal{R}_{M_1} \stackrel{\triangle}{=} \{Q : D(Q||G_{l_1}) > E_{M_1,l_2|l_1,l_2},\ l_1 = \overline{1, M_1 - 1},\ l_2 = \overline{1, M_2 - 1}\},$$

$$\mathcal{R}_{M_2|l_1} \stackrel{\triangle}{=} \{V : D(V||G_{l_2|l_1}) > E_{l_1,M_2|l_1,l_2},\ l_1 = \overline{1, M_1 - 1},\ l_2 = \overline{1, M_2 - 1}\}.$$

So in this case the matrix of reliabilities $\mathbf{E}(\Phi*) = \{E*_{l_1,l_2|m_1,m_2},\ l_1, m_1 = \overline{1, M_1},\ l_1, m_1 = \overline{1, M_1}\}$, will have the following elements:

$$\begin{aligned}
E^*_{l_1,M_2|l_1,l_2} &\stackrel{\triangle}{=} E_{l_1,M_2|l_1,l_2},\\
E^*_{M_1,l_2|l_1,l_2} &\stackrel{\triangle}{=} E_{M_1,l_2|l_1,l_2},\\
&\quad l_1 = \overline{1, M_1 - 1},\ l_2 = \overline{1, L_2 - 1},\\
E^*_{l_1,l_2|m_1,l_2} &\stackrel{\triangle}{=} \inf_{Q:Q\in R_{l_1}} D(Q||G_{m_1}),\ \ m_1 \neq l_1,\\
E^*_{l_1,l_2|l_1,m_2} &\stackrel{\triangle}{=} \inf_{V:V\in R_{l_2|l_1}} D(V||G_{m_2|m_1}),\quad m_2 \neq l_2,\\
E^*_{l_1,l_2|m_1,m_2} &\stackrel{\triangle}{=} E^*_{m_1,l_2|m_1,m_2} + E^*_{l_1,m_2|m_1,m_2},\ \ m_i \neq l_i,\ \ i = 1, 2,\\
E^*_{m_1,m_2|m_1,m_2} &\stackrel{\triangle}{=} \min_{(l_1,l_2)\neq(m_1,m_2)} E^*_{l_1,l_2|m_1,m_2}.
\end{aligned}$$

Theorem 4.4: [44] *If all PDs G_{m_1}, $m_1 = \overline{1, M_1}$, are different, that is $D(G_{l_1}||G_{m_1}) > 0$, $l_1 \neq m_1$, $l_1, m_1 = \overline{1, M_1}$, and all conditional PDs $G_{l_2|l_1}$, $l_2 = \overline{1, M_2}$, are also different for all $l_1 = \overline{1, M_1}$, in the sense that $D(G_{l_2|l_1}||G_{m_2|m_1}) > 0$, $l_2 \neq m_2$, then the following statements are valid.*

When given strictly positive elements $E_{M_1,l_2|l_1,l_2}$ and $E_{l_1,M_2|l_1,l_2}$, $l_1 = \overline{1, M_1 - 1}$, $l_2 = \overline{1, M_2 - 1}$, meet the following compatibility conditions

$$\begin{aligned}
E_{M_1,l_2|1,l_2} &< \min_{l_1=\overline{2,M_1}} D(G_{l_1}||G_1),\\
E_{l_1,M_2|l_1,1} &< \min_{l_2=\overline{2,M_2}} D(G_{l_2|l_1}||G_{1|m_1}),\\
E_{M_1,l_2|m_1,l_2} &< \min\left[\min_{l_1=\overline{1,m_1-1}} E^*_{l_1,l_2|m_1,l_2},\ \min_{l_1=\overline{m_1+1,M_1}} D(G_{l_1}||G_{m_1})\right],\\
&\quad m_1 = \overline{2, M_1 - 1},\\
E_{l_1,M_2|l_1,m_2} &< \min\left[\min_{l_2=\overline{1,m_2-1}} E^*_{l_1,l_2|l_1,m_2},\ \min_{l_2=\overline{m_2+1,M_2}} D(G_{l_2|l_1}||G_{m_2|m_1})\right],\\
&\quad m_2 = \overline{2, M_2 - 1},
\end{aligned}$$

then there exists a LAO test sequence Φ^*, the matrix of which $\mathbf{E}(\Phi^*)$ is stated above and all elements of it are positive.

When even one of the compatibility conditions is violated, then at least one element of the matrix $\mathbf{E}(\Phi^*)$ is equal to 0.

5. Identification of Distribution for One and for Many Objects

In [9] Bechhofer, Kiefer, and Sobel presented investigations on sequential multiple-decision procedures. This book concerns principally with a particular class of problems referred to as ranking problems. Chapter 10 of the book by Ahlswede and Wegener [7] is devoted to statistical identification and ranking. Problems of distribution identification and distributions ranking for one object applying the concept of optimality developed in [11], [48], [29]–[32] were solved in [6]. In papers [40], [46], [47] and [53] identification problems for models composed with two independent, or strictly dependent objects were investigated.

In [6], [40], [47] and [53] models considered in [9] and [7] and variations of these models inspired by the pioneering paper by Ahlswede and Dueck [5], applying the concept of optimality developed in [11], [29]–[32], [48], were studied.

First we formulate the concept of the identification for one object, which was considered in [6]. There are known $M \geq 2$ possible PDs, related with the object in consideration. Identification gives the answer to the question whether r-th PD occured, or not. This answer can be given on the base of a sample $\mathbf{x}$ and by a test $\varphi_N^*(\mathbf{x})$. More precisely, identification can be considered as an answer to the question: is result l of testing algorithm equal to r (that is $l = r$), or not equal l (that is $l \neq r$).

There are two types of error probabilities of identification for each $r = \overline{1, M}$: the probability $\alpha_{l\neq r|m=r}(\varphi_N)$ to accept l different from r, when r is in reality, and the probability $\alpha_{l=r|m\neq r}(\varphi_N)$ that r is accepted by test φ_N, when r is not correct.

The probability $\alpha_{l\neq r|m=r}(\varphi_N)$ coincides with the error probability of testing $\alpha_{r|r}(\varphi_N)$ (see (6)) which is equal to $\sum\limits_{l:l\neq r} \alpha_{l|r}(\varphi_N)$. The corresponding reliability $E_{l\neq r|m=r}(\varphi)$ is equal to $E_{r|r}(\varphi)$ which satisfies the equality (3.4).

And what is the reliability approach to identification? It is necessary to determine the dependence of optimal reliability $E^*_{l=r|m\neq r}$ upon given $E^*_{l\neq r|m=r} = E^*_{r|r}$, which can be assigned a value satisfying conditions analogical to (3.12).

The result from paper [6] is:

Theorem 5.1: *In the case of distinct hypothetical PDs $G_1, G_2, ..., G_M$, for a given sample $\mathbf{x}$ we define its type Q, and when $Q \in \mathcal{R}_l^{(N)}$ (see (3.5)–(3.7)) we accept the hypothesis l. Under condition that the a priori probabilities of all M hypotheses are positive the reliability of such identification $E_{l=r|m\neq r}$ for given $E_{l\neq r|m=r} = E_{r|r}$ is the following:*

$$E_{l=r|m\neq r}(E_{r|r}) = \min_{m:m\neq r} \inf_{Q:D(Q\|G_r)\leq E_{r|r}} D(Q\|G_m), \quad r = \overline{1, M}.$$

We can accept the supposition of positivity of a priory probabilities of all hypotheses with loss of generality, because the PD which is known to have probability 0, that is being impossible, must not be included in the studied family.

Now let us consider the model consisting of two independent objects. Let hypothetical characteristics of objects X_1 and X_2 be independent RVs taking values in the same finite set $\mathcal{X}$ with one of M PDs. Identification means that the statistician has to answer the question whether the pair of distributions (r_1, r_2) occurred or not. Now the procedure of testing for two objects can be used. Let us study two types of error probabilities for each pair (r_1, r_2), $r_1, r_2 = \overline{1, M}$. We denote by $\alpha^{(N)}_{(l_1,l_2)\neq(r_1,r_2)|(m_1,m_2)=(r_1,r_2)}$ the probability, that pair (r_1, r_2) is true, but it is rejected. Note that this probability is equal to $\alpha_{r_1,r_2|r_1,r_2}(\Phi_N)$. Let $\alpha^{(N)}_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ be the probability that (r_1, r_2) is identified, when it is not correct. The corresponding reliabilities are $E_{(l_1,l_2)\neq(r_1,r_2)|(m_1,m_2)=(r_1,r_2)} = E_{r_1,r_2|r_1,r_2}$ and $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$. Our aim is to determine the dependence of $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ on given $E_{r_1,r_2|r_1,r_2}(\Phi_N)$.

Let us define for each r, $r = \overline{1, M}$, the following expression:

$$A(r) = \min\left[\min_{l=\overline{1,r-1}} D(G_l||G_r), \min_{l=\overline{r+1,M}} D(G_l||G_r)\right].$$

Theorem 5.2 [40]:*For the model consisting of two independent objects if the distributions G_m, $m = \overline{1, M}$, are different and the given strictly positive number $E_{r_1,r_2|r_1,r_2}$ satisfy condition*

$$E_{r_1,r_2|r_1,r_2} < \min\left[A(r_1),\ A(r_2)\right],$$

then the reliability $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ is defined as follows:

$$E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}\left(E_{r_1,r_2|r_1,r_2}\right)$$

$$= \min_{m_1\neq r_1, m_2\neq r_2}\left[E_{m_1|r_1}(E_{r_1,r_2|r_1,r_2}), E_{m_2|r_2}(E_{r_1,r_2|r_1,r_2})\right],$$

where $E_{m_1|r_1}(E_{r_1,r_2|r_1,r_2})$ and $E_{m_2|r_2}(E_{r_1,r_2|r_1,r_2})$ are determined by (3.9).

Now we will present the lower estimates of the reliabilities for LAO identification for the *dependent* object which can be then applied for deducing the lower estimates of the reliabilities for LAO identification of two *related* objects. There exist two error probabilities for each $r_2 = \overline{1, M_2}$: the probability $\alpha_{l_2\neq r_2|l_1,m_1,m_2=r_2}(\varphi^2_N)$ to accept l_2 different from r_2, when r_2 is in reality, and the probability
$\alpha_{l_2=r_2|l_1,m_1,m_2\neq r_2}(\varphi^2_N)$ that r_2 is accepted, when it is not correct.

The upper estimate $\beta_{l_2\neq r_2|l_1,m_1,m_2=r_2}(\varphi^2_N)$ for $\alpha_{l_2\neq r_2|l_1,m_1,m_2=r_2}(\varphi^2_N)$ is already known, it coincides with $\beta_{r_2|l_1,m_1,r_2}(\varphi^2_N)$ which is equal to
$\sum\limits_{l_2:l_2\neq r_2} \beta_{l_2|l_1,m_1,r_2}(\varphi^2_N)$. The corresponding $F_{l_2\neq r_2|l_1,m_1,m_2=r_2}(\varphi^2)$ is equal to
$F_{r_2|l_1,m_1,r_2}(\varphi^2)$, which satisfies the equality (4.3).

We determine the optimal dependence of $F^*_{l_2=r_2|l_1,m_1,m_2\neq r_2}$ upon given $F^*_{l_2\neq r_2|l_1,m_1,m_2=r_2}$.

Theorem 5.3 [47]: *In case of distinct PDs $G_{1|l_1}, G_{2|l_1}, ..., G_{M_2|l_1}$, under condition that a priori probabilities of all M_2 hypotheses are strictly positive, for* *each* $r_2 = \overline{1, M_2}$ *the estimate of* $F_{l_2=r_2|l_1,m_1,m_2\neq r_2}$ *for given* $F_{l_2\neq r_2|l_1,m_1,m_2=r_2} = F_{r_2|l_1,m_1,r_2}$ *is the following:*

$$F_{l_2=r_2|l_1,m_1,m_2\neq r_2}(F_{r_2|l_1,m_1,r_2}) =$$

$$\min_{m_2:m_2\neq r_2} \inf_{Q\in\mathcal{R}_{l_1}} \inf_{V:D(V\|G_{r_2|l_1}|Q)\leq F_{r_2|l_1,m_1,r_2}} D(V\|G_{m_2|m_1}|Q).$$

The result of the reliability approach to the problem of identification of the probability distributions for two *related* objects is the following.

Theorem 5.4: *If the distributions* G_{m_1} *and* $G_{m_2|m_1}$, $m_1 = \overline{1, M_1}$, $m_2 = \overline{1, M_2}$, *are different and the given strictly positive number* $F_{r_1,r_2|r_1,r_2}$ *satisfies the condition*

$$E_{r_1|r_1} < \min\left[\min_{l=\overline{1,r_1-1}} D(G_{r_1}||G_{l_1}), \min_{l_1=\overline{r_1+1,M_1}} D(G_{l_1}||G_{r_1})\right],$$

or

$$F_{r_2|l_1,m_1,r_2} < \min\left[\inf_{Q\in\mathcal{R}_{l_1}} \min_{l_2=\overline{1,r_2-1}} D(G_{r_2|m_1}||G_{l_2|l_1}|Q),\right.$$
$$\left.\inf_{Q\in\mathcal{R}_{l_1}} \min_{l_2=\overline{r_2+1,M_2}} D(G_{l_2|l_1}||G_{r_2|m_1}|Q)\right],$$

then the lower estimate $F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ *of the reliability* $E_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}$ *can be calculated as follows*

$$F_{(l_1,l_2)=(r_1,r_2)|(m_1,m_2)\neq(r_1,r_2)}\left(F_{r_1,r_2|r_1,r_2}\right)$$
$$= \min_{m_1\neq r_1, m_2\neq r_2}\left[E_{r_1|m_1}(F_{r_1,r_2|r_1,r_2}), F_{r_2|l_1,m_1,m_2}(F_{r_1,r_2|r_1,r_2})\right],$$

where $E_{r_1|m_1}(F_{r_1,r_2|r_1,r_2})$ *and* $F_{r_2|l_1,m_1,m_2}(F_{r_1,r_2|r_1,r_2})$ *are determined respectively by (3.9) and (4.30).*

The particular case, when X_1 and X_2 are related *statistically*, was studied in [44], [60].

6. Multihypotheses Testing With Possibility of Rejection of Decision

This section is devoted to description of characteristics of LAO hypotheses testing with permission of decision rejection for the model consisting of one or more objects. The multiple hypotheses testing problem with possibility of rejection of decision for arbitrarily varying object with side information and for the model of two or more independent objects was examined by Haroutunian, Hakobyan and Yessayan [41], [42]. These works ware induced by the paper of Nikulin [55] concerning two hypotheses testing with refusal to take decision. An asymptotically optimal classification, in particular hypotheses testing problem with rejection of decision ware considered by Gutman [25].

6.1. Many Hypothesis Testing With Rejection of Decision by Informed Statistician for Arbitrarily Varying Object

In this section we consider multiple statistical hypotheses testing with possibility of rejecting to make choice between hypotheses concerning distribution of a discrete arbitrarily varying object. The arbitrarily varying object is a generalized model of the discrete memoryless one. Let $\mathcal{X}$ be a finite set of values of RV X, and $\mathcal{S}$ is an alphabet of states of the object.

M possible conditional PDs of the characteristic X of the object depending on values s of states, are given:

$$W_m \stackrel{\triangle}{=} \{W_m(x|s),\ \ x \in \mathcal{X},\ \ \ s \in \mathcal{S}\},\ \ \ m = \overline{1, M}, ...|\mathcal{S} \geq 1|,$$

but it is not known which of these alternative hypotheses W_m, $m = \overline{1, M}$, is real PD of the object. The statistician must select one among M hypotheses, or he can withdraw any judgement. It is possible for instance when it is supposed that real PD is not in the family of M given PDs. An answer must be given using the vector of results of N independent experiments $\mathbf{x} \stackrel{\triangle}{=} (x_1, x_2, ...x_N)$ and the vector of states of the object $\mathbf{s} \stackrel{\triangle}{=} (s_1, s_2, ..., s_N)$, $s_n \in \mathcal{S}$, $n = \overline{1, N}$.

The procedure of decision making is a non-randomized test $\varphi_N(\mathbf{x}, \mathbf{s})$, it can be defined by division of the sample space $\mathcal{X}^N$ for each $\mathbf{s}$ on $M+1$ disjoint subsets $\mathcal{A}_m^N(\mathbf{s}) = \{\mathbf{x} :\ \varphi_N(\mathbf{x}, \mathbf{s}) = m\}$, $m = \overline{1, M+1}$. The set $\mathcal{A}_l^N(\mathbf{s})$, $l = \overline{1, M}$, consists of vectors $\mathbf{x}$ for which the hypothesis W_l is adopted, and $\mathcal{A}_{M+1}^N(\mathbf{s})$ includes vectors for which the statistician refuses to take a certain answer.

We study the probabilities of the erroneous acceptance of hypothesis W_l provided that W_m is true

$$\alpha_{l|m}(\varphi_N) \stackrel{\triangle}{=} \max_{\mathbf{s} \in \mathcal{S}^N} W_m^N \left(\mathcal{A}_l^N(\mathbf{s})|\mathbf{s}\right),\ m, l = \overline{1, M},\ \ m \neq l. \tag{6.1}$$

When decision is declined, but hypothesis W_m is true, we consider the following probability of error:

$$\alpha_{M+1|m}(\varphi_N) \stackrel{\triangle}{=} \max_{\mathbf{s} \in \mathcal{S}^N} W_m^N \left(\mathcal{A}_{M+1}^N(\mathbf{s})|\mathbf{s}\right).$$

If the hypothesis W_m is true, but it is not accepted, or equivalently while the statistician accepted one of hypotheses W_l, $l = \overline{1, M}$, $l \neq m$, or refused to make decision, then the probability of error is the following:

$$\alpha_{m|m}(\varphi_N) \stackrel{\triangle}{=} \sum_{l:\, l \neq m} \alpha_{l|m}(\varphi_N) = \max_{\mathbf{s} \in \mathcal{S}^N} W_m^N \left(\overline{\mathcal{A}_m^N(\mathbf{s})}|\mathbf{s}\right),\ \ \ m = \overline{1, M}. \tag{6.2}$$

Corresponding reliabilities are defined similarly by to (3.2):

$$E_{l|m}(\varphi) \stackrel{\triangle}{=} \overline{\lim_{N \to \infty}} \left\{-\frac{1}{N} \log \alpha_{l|m}(\varphi_N)\right\},\ m = \overline{1, M},\ \ \ l = \overline{1, M+1}. \tag{6.3}$$

It also follows that for every test φ

$$E_{m|m}(\varphi) = \min_{l=\overline{1,M+1},\ l\neq m} E_{l|m}(\varphi), \ \ m = \overline{1,M}. \tag{6.4}$$

The matrix

$$\mathbf{E}(\varphi) = \begin{pmatrix} E_{1|1} \ \dots \ E_{l|1} \ \dots \ E_{M|1}, \ E_{M+1|1} \\ \dots\dots\dots\dots\dots\dots\dots \\ E_{1|m} \dots E_{l|m} \dots E_{M|m}, \ E_{M+1|m} \\ \dots\dots\dots\dots\dots\dots\dots \\ E_{1|M} \dots E_{l|M} \dots E_{M|M} \ E_{M+1|M} \end{pmatrix}$$

is the reliabilities matrix of the tests sequence φ for the described model.

We call the test LAO for this model if for given positive values of certain M elements of the matrix $\mathbf{E}(\varphi)$ the procedure provides maximal values for other elements of it.

For construction of LAO test positive elements $E_{1|1}$, ..., $E_{M|M}$ are supposed to be given preliminarily. The optimal dependence of error exponents was determined in [41]. This result can be easily generalized for the case of an arbitrarily varying Markov source.

6.2. Multiple Hypotheses LAO Testing With Rejection of Decision for Many Independent Objects

For brevity we consider the problem for two objects, the generalization of the problem for K independent objects will be discussed along the text.

Let X_1 and X_2 be independent RVs taking values in the same finite set $\mathcal{X}$ with one of M PDs $G_m \in \mathcal{P}(\mathcal{X})$, $m = \overline{1,M}$. These RVs are the characteristics of the corresponding independent objects. The random vector (X_1, X_2) assumes values $(x^1, x^2) \in \mathcal{X} \times \mathcal{X}$. Let $(\mathbf{x^1}, \mathbf{x^2}) \stackrel{\triangle}{=} \big((x_1^1, x_1^2), ..., (x_n^1, x_n^2), ..., (x_N^1, x_N^2)\big)$, $x_n^k \in \mathcal{X}$, $k = \overline{1,2}, n = \overline{1,N}$, be a vector of results of N independent observations of the pair of RVs (X_1, X_2). On the base of observed data the test has to determine unknown PDs of the objects or withdraw any judgement. The selection for each object should be made from the same set of hypotheses: G_m, $m = \overline{1,M}$. We call this procedure the compound test for two objects and denote it by Φ_N, it can be composed of two individual tests φ_N^1, φ_N^2 for corresponding objects. The test φ_N^i, $i = \overline{1,2}$, can be defined by division of the space $\mathcal{X}^N$ into $M+1$ disjoint subsets $\mathcal{A}_m^i$, $m = \overline{1,M+1}$. The set $\mathcal{A}_m^i$, $m = \overline{1,M}$ contains all vectors $\mathbf{x}^i$ for which the hypothesis G_m is adopted and $\mathcal{A}_{M+1}^i$ includes all vectors for which the test refuses to take a certain answer. Hence Φ_N is division of the space $\mathcal{X}^N \times \mathcal{X}^N$ into $(M+1)^2$ subsets $\mathcal{A}_{m_1,m_2} = \mathcal{A}_{m_1}^1 \times \mathcal{A}_{m_2}^2$, $m_i = \overline{1,M+1}$. We again denote the infinite sequences of tests by Φ, φ^1, φ^2.

Let $\alpha_{l_1,l_2|m_1,m_2}(\Phi_N)$ be the probability of the erroneous acceptance of the pair of hypotheses (G_{l_1}, G_{l_2}) by the test Φ_N provided that the pair of hypotheses (G_{m_1}, G_{m_2}) is true, where $(m_1, m_2) \neq (l_1, l_2)$, $m_i = \overline{1,M}$, $l_i = \overline{1,M}$, $i = \overline{1,2}$:

$$\begin{aligned} \alpha_{l_1,l_2|m_1,m_2}(\Phi_N) &= G_{m_1} \circ G_{m_2}\left(\mathcal{A}_{l_1,l_2}\right) \\ &= G_{m_1}^N(\mathcal{A}_{l_1}) \cdot G_{m_2}^N(\mathcal{A}_{l_2}). \end{aligned}$$

When the pair of hypotheses (G_{m_1}, G_{m_2}), $m_1, m_2 = \overline{1, M}$ is true, but we decline the decision the corresponding probabilities of errors are:

$$\begin{aligned}\alpha_{M+1,M+1|m_1,m_2}(\Phi_N) &= G_{m_1} \circ G_{m_2}(\mathcal{A}_{M+1,M+1}) \\ &= G^N_{m_1}(\mathcal{A}^1_{M+1}) \cdot G^N_{m_2}(\mathcal{A}^2_{M+1}).\end{aligned}$$

or

$$\alpha_{M+1,l_2|m_1,m_2}(\Phi_N) = G^N_{m_1}(\mathcal{A}^1_{M+1}) \cdot G^N_{m_2}(\mathcal{A}^2_{l_2})$$

or

$$\alpha_{l_1,M+1|m_1,m_2}(\Phi_N) = G^N_{m_1}(\mathcal{A}^1_{l_1}) \cdot G^N_{m_2}(\mathcal{A}^2_{M+1}).$$

If the pair of hypotheses (G_{m_1}, G_{m_2}) is true, but it is not accepted, or equivalently while the statistician accepted one of hypotheses (G_{l_1}, G_{l_2}), or refused to make decision, then the probability of error is the following:

$$\begin{aligned}\alpha_{m_1,m_2|m_1,m_2}(\Phi_N) = \sum_{(l_1,l_2)\neq(m_1,m_2)} \alpha_{l_1,l_2|m_1,m_2}(\Phi_N), \quad (6.5)\\ l_i = \overline{1, M+1},\ m_i = \overline{1, M},\ i = \overline{1, 2}.\end{aligned}$$

We study reliabilities $E_{l_1,l_2|m_1,m_2}(\Phi)$ of the sequence of tests Φ,

$$\begin{aligned}E_{l_1,l_2|m_1,m_2}(\Phi) \stackrel{\triangle}{=} \overline{\lim_{N\to\infty}} -\frac{1}{N} \log \alpha_{l_1,l_2|m_1,m_2}(\Phi_N), \quad (6.6)\\ m_i, = \overline{1, M},\ l_i = \overline{1, M+1}, \quad i = \overline{1, 2}.\end{aligned}$$

Definitions (6.5) and (6.6) imply that

$$E_{m_1,m_2|m_1,m_2}(\Phi) = \min_{(l_1,l_2)\neq(m_1,m_2)} E_{l_1,l_2|m_1,m_2}(\Phi). \quad (6.7)$$

We can erect the LAO test from the set of compound tests when $2M$ strictly positive elements of the reliability matrix $E_{M+1,m|m,m}$ and $E_{m,M+1|m,m}$, $m = \overline{1, M}$, are preliminarily given (see [41]).

Remark 6.1: It is necessary to note that the problem of reliabilities investigation for LAO testing of many hypotheses with possibility of rejection of decision for the model consisting of two or more independent objects can not be solved by the direct method of renumbering.

7. Conclusion and Open Problems

"A broad class of statistical problems arises in the framework of hypothesis testing in the spirit of identification for different kinds of sources, with complete or partial side information or without it. Paper [6] is a start." [2].

In this paper, we exposed solutions of a part of possible problems concerning algorithms of distributions optimal testing for certain classes of one, or multiple objects. For the same models PD optimal identification is discussed again in the spirit of error probability exponents optimal dependence. But these investigations can be continued in plenty directions.

Some problems formulated in [6] and [43], particularly, concerning the remote statistical inference formulated by Berger [10], examined in part by Ahlwede and Csiszár [4] and Han and Amari [27] still rest open.

All our results concern with discrete distribution, it is necessary to study many objects with general distributions as in [26]. For multiple objects multistage and sequential testing [13] can be also considered. Problems for many objects are present in statistics with fuzzy data [24], bayessian detection of multiple hypotheses testing [51] and geometric interpretations of tests [61].

8. Appendix

Proof of Theorem 3.1: Probability $G_m^N(\mathbf{x})$ for $\mathbf{x} \in \mathcal{T}_Q^N(X)$ can be presented as follows:

$$\begin{aligned} G_m^N(\mathbf{x}) &= \prod_{n=1}^{N} G_m(x_n) \\ &= \prod_{x} G_m(x)^{N(x|\mathbf{x})} \\ &= \prod_{x} G_m(x)^{NQ(x))} \\ &= \exp\left\{ N \sum_{x} \left(-Q(x) \log \frac{Q(x)}{G_m(x)} + Q(x) \log Q(x) \right) \right\} \\ &= \exp\left\{ -N[D(Q \parallel G_m) + H_Q(X)] \right\}. \end{aligned} \tag{8.1}$$

Let us consider the sequence of tests $\varphi_N^*(\mathbf{x})$ defined by the sets

$$\mathcal{B}_m^{(N)} \triangleq \bigcup_{P \in \mathcal{R}_m^{(N)}} \mathcal{T}_Q^N(X), \ \ m = \overline{1, M}. \tag{8.2}$$

Each $\mathbf{x}$ is in one and only in one of $\mathcal{B}_m^{(N)}$, that is

$$\mathcal{B}_l^{(N)} \bigcap \mathcal{B}_m^{(N)} = \emptyset, \ \ l \neq m, \quad \text{and} \quad \bigcup_{m=1}^{M} \mathcal{B}_m^{(N)} = \mathcal{X}^N.$$

Really, for $l = \overline{1, M-2}$, $m = \overline{2, M-1}$, for each $l < m$ let us consider arbitrary $\mathbf{x} \in \mathcal{B}_l^{(N)}$. It follows from (3.5) and (3.7) that there exists type $Q \in \mathcal{Q}^N(\mathcal{X})$ such that $D(Q||G_l) \leq E_{l|l}$ and $\mathbf{x} \in \mathcal{T}_Q^N(X)$. From (3.12) and (3.9) we have $E_{m|m} < E^*_{l|m}(E_{l|l}) < D(Q||G_m)$. From definition of $\mathcal{B}_m^{(N)}$ we see that $\mathbf{x} \notin \mathcal{B}_m^{(N)}$. Definitions (3.10), (3.12) and (3.7) show also that

$$\mathcal{B}_M^{(N)} \bigcap \mathcal{B}_m^{(N)} = \emptyset, \ \ m = \overline{1, M-1}.$$

Now, let us remark that for $m = \overline{1, M-1}$, using (2.1), (2.3), (3.1)–(3.3) and (8.1) we can estimate $\alpha^{(N)}_{m|m}(\varphi^*)$ as follows:

$$
\begin{aligned}
\alpha_{m|m}(\varphi^*_N) &= G_m^N\left(\overline{\mathcal{B}_m^{(N)}}\right) \\
&= G_m^N\left(\bigcup_{Q:D(Q||G_m)>E_{m|m}} \mathcal{T}_Q^N(X)\right) \\
&\leq (N+1)^{|\mathcal{X}|} \sup_{Q:D(Q||G_m)>E_{m|m}} G_m(\mathcal{T}_Q^N(X)) \\
&\leq (N+1)^{|\mathcal{X}|} \sup_{Q:D(Q||G_m)>E_{m|m}} \exp\{-ND(Q||G_m)\} \\
&\leq \exp\left\{-N[\inf_{Q:D(Q||G_m)>E_{m|m}} D(Q||G_m) - o_N(1)]\right\} \\
&\leq \exp\left\{-N[E_{m|m} - o_N(1)]\right\},
\end{aligned}
$$

where $o_N(1) \to 0$ with $N \to \infty$.

For $l = \overline{1, M-1}$, $m = \overline{1, M}$, $l \neq m$, using (2.1), (2.3), (3.1)–(3.3) and (8.1), we can obtain similar estimates:

$$
\begin{aligned}
\alpha_{l|m}(\varphi^*_N) &= G_m^N\left(\mathcal{B}_l^{(N)}\right) \\
&= G_m^N\left(\bigcup_{Q\,:D(Q||G_l)E_{l|l}} \mathcal{T}_Q^N(X)\right) \\
&\leq (N+1)^{|\mathcal{X}|} \sup_{Q:D(Q||G_m)\leq E_{l|l}} G_m^N(\mathcal{T}_Q^N(X) \\
&\leq (N+1)^{|\mathcal{X}|} \sup_{Q:D(Q||G_m)\leq E_{l|l}} \exp\{-ND(Q||G_m)\} \\
&= \exp\left\{-N\left(\inf_{Q:D(Q||G_m)\leq E_{l|l}} D(Q||G_m) - o_N(1)\right)\right\}.
\end{aligned}
\tag{8.3}
$$

Now let us prove the inverse inequality:

$$\begin{aligned}
\alpha_{l|m}(\varphi_N^*) &= G_m^N\left(\mathcal{B}_l^{(N)}\right) \\
&= G_m^N\left(\bigcup_{Q:D(Q||G_l)\leq E_{l|l}} \mathcal{T}_Q^N(X)\right) \\
&\geq \sup_{Q:D(Q||G_l)\leq E_{l|l}} G_m^N(\mathcal{T}_Q(X) \\
&\geq (N+1)^{-|\mathcal{X}|} \sup_{Q:D(Q||G_l)\leq E_{l|l}} \exp\left\{-ND\left(Q||G_m\right)\right\} \\
&= \exp\left\{-N\left(\inf_{Q:D(Q||G_l)\leq E_{l|l}} D(Q||G_m)+o_N(1)\right)\right\}. \qquad (8.4)
\end{aligned}$$

According to the definition (3.3) $E_{l|m}(\varphi^*) = \overline{\lim_{N\to\infty}}\left\{-N^{-1}\log\alpha_{l|m}(\varphi_N^*)\right\}$, taking into account (8.3), (8.4) and the continuity of the functional $D(Q||G_l)$ we obtain that
$\lim_{N\to\infty}\left\{-N^{-1}\log\alpha_{m|l}(\varphi_N^*)\right\}$ exists and in correspondence with (3.9) equals to $E^*_{m|l}$. Thus $E_{l|m}(\varphi^*) = E^*_{l|m}$, $m=\overline{1,M}$, $l=\overline{1,M-1}$, $l\neq m$. Similarly we can obtain upper and lower bounds for $\alpha_{M|m}(\varphi_N^*)$, $m=\overline{1,M}$. Applying the same resonnement we get that the reliability $E_{M|m}(\varphi^*) = E^*_{M|m}$. By the definition (3.4) $E_{M|M}(\varphi^*) = E^*_{M|M}$. The proof of the first part of the theorem will be accomplished if we demonstrate that the sequence of tests φ^* is LAO, that is for given $E_{1|1},...,E_{M-1|M-1}$ and every sequence of tests φ for all $l,m\in\overline{1,M}$, $E_{m|l}(\varphi)\leq E^*_{m|l}$.

Let us consider any other sequence φ^{**} of tests which are defined by the sets $\mathcal{D}_1^{(N)},...,\mathcal{D}_M^{(N)}$ such that

$$E_{l|m}(\varphi^{**})\geq E^*_{l|m},\ \ m,l=\overline{1,M}. \qquad (8.5)$$

These conditions are equivalent for N large enough to the inequalities

$$\alpha_{l|m}(\varphi_N^{**})\leq\alpha_{l|m}(\varphi_N^*),\ m,l=\overline{1,M}. \qquad (8.6)$$

Let us examine the sets $D_m^{(N)}\bigcap\mathcal{B}_m^{(N)}$, $m=\overline{1,M}$. This intersection cannot be empty, because in that case

$$\begin{aligned}
\alpha_{m|m}(\varphi_N^{**}) &= G_m^N\left(\overline{\mathcal{D}}_m^{(N)}\right) \\
&\geq G_m^N\left(\mathcal{B}_m^{(N)}\right) \\
&\geq \max_{Q:D(Q||G_m)\leq E_{m|m}} G_m^N\left(\mathcal{T}_Q^N(X)\right) \\
&\geq \exp\{-N(E_{m|m}+o_N(1))\}.
\end{aligned}$$

Let us show that $\mathcal{D}_l^{(N)} \bigcap \mathcal{B}_m^{(N)} = \emptyset$, $m, l = \overline{1, M-1}$, $m \neq l$. If there exists Q such that $D(Q||G_m) \leq E_{m|m}$ and $\mathcal{T}_Q^N(X) \in \mathcal{D}_l^{(N)}$, then

$$\begin{aligned}\alpha_{l|m}(\varphi_N^{**}) &= G_m^N\left(\mathcal{D}_l^{(N)}\right)\\ &> G_m^N\left(\mathcal{T}_Q^N(X)\right)\\ &\geq \exp\{-N(E_{m|m} + o_N(1))\}.\end{aligned}$$

When $\emptyset \neq \mathcal{D}_l^{(N)} \bigcap \mathcal{T}_Q^N(X) \neq \mathcal{T}_Q^N(X)$, we also obtain that

$$\begin{aligned}\alpha_{l|m}(\varphi_N^{**}) &= G_m^N(\mathcal{D}_l^{(N)}\\ &> G_m^N(\mathcal{D}_l^{(N)} \bigcap \mathcal{T}_Q^N(X)\\ &\geq \exp\{-N(E_{m|m} + o_N(1))\}.\end{aligned}$$

Thus it follows that $E_{l|m}(\varphi^{**}) \leq E_{m|m}$, which in turn according to (3.4) provides that $E_{l|m}(\varphi^{**}) = E_{m|m}$. From condition (3.12) it follows that $E_{m|m} < E^*_{l|m}$, for all $l = \overline{1, m-1}$, hence $E_{l|m}(\varphi^{**}) < E^*_{l|m}$ for all $l = \overline{1, m-1}$, which contradicts to (8.5). Hence we obtain that $\mathcal{D}_m^{(N)} \bigcap \mathcal{B}_m^{(N)} = \mathcal{B}_m^{(N)}$ for $m = \overline{1, M-1}$. The intersection $D_m^{(N)} \bigcap \mathcal{B}_M^{(N)}$ is empty too, because otherwise

$$\alpha_{M|m}(\varphi_N^{**}) \geq \alpha_{M|m}(\varphi_N^{*}),$$

which contradicts to (8.6), hence $\mathcal{D}_m^{(N)} = B_m^{(N)}$, $m = \overline{1, M}$.

The proof of the second part of the Theorem 3.1 is simple. If one of the conditions (3.12) is violated, then from (3.9)–(3.11) it follows that at least one of the elements $E_{m|l}$ is equal to 0. For example, let in (3.12) the m-th condition be violated. It means that $E_{m|m} \geq \min\limits_{l=\overline{m+1,M}} D(G_l||G_m)$, then there exists $l^* \in \overline{m+1, M}$ such that $E_{m|m} \geq D(G_l^*||G_m)$. From latter and (3.9) we obtain that $E^*_{m|l} = 0$.

The theorem is proved.

Proof of Lemma 4.1: It follows from the independence of the objects that

$$\alpha_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi_N) = \prod_{i=1}^{3} \alpha_{l_i|m_i}(\varphi_N^i), \text{ if } m_i \neq l_i, \tag{8.7}$$

$$\alpha_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi_N) = \left(1 - \alpha_{l_k|m_k}(\varphi_N^k)\right) \prod_{i\in[[1,2,3]-k]} \alpha_{l_i|m_i}(\varphi_N^i),$$
$$m_k = l_k, \; m_i \neq l_i, \; k = \overline{1,3}, \; i \neq k, \tag{8.8}$$

$$\alpha_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi_N) = \alpha_{l_i|m_i}(\varphi_N^i) \prod_{k\in[[1,2,3]-i]} \left(1 - \alpha_{l_k|m_k}(\varphi_N^i)\right),$$
$$m_k = l_k, \; m_i \neq l_i, \; i = \overline{1,3}. \tag{8.9}$$

Remark that here we consider also the probabilities of right (not erroneous) decisions. Because $E_{l|m}(\varphi^i)$ are strictly positive then the error probability $\alpha_{l|m}(\varphi^i_N)$ tends to zero, when $N \longrightarrow \infty$. According this fact we have

$$\begin{aligned} \overline{\lim_{N\to\infty}} \left\{ -\frac{1}{N} \log \left(1-\alpha_{l|m}(\varphi^i_N)\right) \right\} &= \overline{\lim_{N\to\infty}} \frac{\alpha_{l|m}\left(\varphi^i_N\right)}{N} \frac{\log\left(1-\alpha_{l|m}(\varphi^i_N)\right)}{-\alpha_{l|m}(\varphi^i_N)} \\ &= 0. \end{aligned} \tag{8.10}$$

From definitions (4.2), equalities (8.7)–(8.9), applying (8.10) we obtain relations (4.4)–(4.6).

The Lemma is proved.

Proof of Theorem 4.1: The test $\Phi^* = (\varphi^{1,*}, \varphi^{2,*}, \varphi^{3,*})$, where $\varphi^{i,*}$, $i = \overline{1,3}$ are LAO tests of objects X_i, belongs to the set $\mathcal{D}$. Our aim is to prove that such Φ^* is a compound LAO test. Conditions (4.16)–(4.19) imply that inequalities analogous to (3.12) hold simultaneously for tests for three separate objects.

Let the test $\Phi \in \mathcal{D}$ be such that $E_{M,M,M|m,M,M}(\Phi) = E_{M,M,M|m,M,M}$ $E_{M,M,M|M,m,M}(\Phi) = E_{M,M,M|M,m,M}$, and $E_{M,M,M|m,M,M}(\Phi) = E_{M,M,M|M,M,m}$, $m = \overline{1, M-1}$.

Taking into account (4.7)–(4.9) we can see that conditions (4.16)–(4.19) for every $m = \overline{1, M-1}$ may be replaced by the following inequalities:

$$E_{M|m}(\varphi^i) < \min \left[\min_{l=\overline{1,m-1}} \inf_{Q:D(Q||G_m)\le E_{M|m}(\varphi^i)} D(Q||G_l), \min_{l=\overline{m+1,M}} D(G_l||G_m) \right]. \tag{8.11}$$

According to Remark 3.1 for LAO test $\varphi^{i,*}$, $i = \overline{1,3}$, we obtain that (8.11) meets conditions (3.12) of Theorem 3.1 for each test $\Phi \in \mathcal{D}$, $E_{m|m}(\varphi^i) > 0$, $i = \overline{1,3}$, hence it follows from (3.3) that $E_{m|l}(\varphi^i)$ are also strictly positive. Thus for a test $\Phi \in \mathcal{D}$ conditions of Lemma 4.1 are fulfilled and the elements of the reliability matrix $\mathbf{E}(\Phi)$ coincide with elements of matrix $\mathbf{E}(\varphi^i)$, $i = \overline{1,3}$, or sums of them. Then from definition of LAO test it follows that $E_{l|m}(\varphi^i) \le E_{l|m}(\varphi^{i,*})$, then $E_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi) \le E_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi^*)$. Consequently Φ^* is a LAO test and $E_{l_1,l_2,l_3|m_1,m_2,m_3}(\Phi^*)$ verify (4.12)–(4.15).

b) When even one of the inequalities (4.16)–(4.19) is violated, then at least one of inequalities (8.11) is violated. Then from Theorem 3.2 one of elements $E_{m|l}(\varphi^{i,*})$ is equal to zero. Suppose $E_{3|2}(\varphi^{1,*}) = 0$, then the elements $E_{3,m,l|2,m,l}(\Phi^*) = E_{3|2}(\varphi^{1,*}) = 0$.

The Theorem is proved.

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