Superimposed Codes and Threshold Group Testing¹

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Abstract- We will discuss superimposed codes and non-adaptive group testing designs arising from the potentialities of compressed genotyping models in molecular biology. The given paper was motivated by the 30th anniversary of D'yachkov-Rykov recurrent upper bound on the rate of superimposed codes published in 1982. We were also inspired by recent results obtained for non-adaptive threshold group testing which develop the theory of superimposed codes.

Index terms. Group testing, compressed genotyping, screening experiments, search designs, superimposed codes, rate of codes, rate of designs, bounds on the rate, shortened RC-code, threshold search designs.

1 Introduction

We consider superimposed codes and non-adaptive group testing models. These search models are also termed as combinatorial designs of screening experiments or pooling designs. Designing screening experiments (DSE) ([3], [5], [7]) can be located in applied mathematics in the border region of search and information theory [2],[6]. In many "processes" which are dependent on a large number of factors, it is natural, that one assumes a small number of "significant" factors, which really control the process, and considers the influence of the other factors as mere "experiment errors". Experiments to identify the significant factors are called screening experiments.

A typical problem from DSE theory called a symmetric model of DSE [29] or symmetric search model is the following. Among t factors there are p "significant", which need to be identified. By N tests which examine arbitrary distinct N subsets of the factors, it can be determined N values of a function depending only on the number of significant factors included in the tests. One tries to perform these experiments as economical as possible. The main criterion at this is the search duration: how many tests N are at least necessary to identify all significant factors in the most unfavorable case?

The aim of our paper is to present the principal combinatorial results for the symmetric search model. We don't discuss here the general noisy symmetric model of non-adaptive search designs which can be described using the terminology of multiple access channel (MAC) [8]. An interested reader is referred to [29]. The information theory problems for non-symmetric search model are considered in [7].

The paper is organized as follows. In Section 2, we give a brief survey of necessary definitions and bounds on the rate of superimposed codes which are the *base for studying* of non-adaptive group testing models.

In Section 3, we introduce the concept of non-adaptive group testing designs arising from the potentialities of compressed genotyping models in molecular biology and establish a universal

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upper bound on their rate. The universal bound is prescribed by D'yachkov-Rykov [9] recurrent upper bound on the rate of classical superimposed codes.

In Section 4, we remind our constructions of superimposed codes based on shortened Reed-Solomon codes (RS-codes) [18]-[22] and other ideas [27]-[28]. In these papers we essentially extended optimal and suboptimal construction of classical superimposed codes suggested in [1]. Note that we included in [18]-[22] the detailed tables with parameters of the best known superimposed codes. We don't mention other authors because, unfortunately, we don't know any papers containing relevant results, i.e., the similar or improved tables of parameters. Any extension of our tables is the important open problem.

In Section 5, the threshold group testing model is discussed. We apply the conventional terminology of superimposed code theory to refine the description of a new lower bound on the rate of threshold designs recently obtained in [37].

1.1 Notations, Definitions and Relevant Issues

Let [n] be the set of integers from 1 to n and the symbol \triangleq denote definitional equalities. For integers $N \ge 2$ and $t \ge 2$, symbols $\Omega_j \subset [N]$, $j = 1, 2, \ldots, t$, denote subsets of [N]. Subsets Ω_j , $j \in [t]$, are identified with binary columns $\mathbf{x}(j) \triangleq (x_1(j), x_2(j), \ldots, x_N(j))$ in which

$$x_i(j) \triangleq \begin{cases} 1 & \text{if } i \in \Omega_j, \\ 0 & \text{if } i \notin \Omega_j, \quad i \in [N]. \end{cases}$$

An incidence matrix $X \triangleq ||x_i(j)||$, $i \in [N]$, $j \in [t]$, is called a *code* with t codewords (columns) $\mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(t)$ of length N corresponding to a *family of subsets* $\Omega_1, \Omega_2, \ldots, \Omega_t$.

Let $P \subset [t]$ be an arbitrary fixed subset of [t] and |P| is its size, i.e.,

$$P \triangleq \{p_1, p_2, \dots, p_{|P|}\} \subset [t], \quad 1 \le p_1 < p_2 < \dots < p_{|P|} \le t.$$

Denote by $\mathcal{P}(t, \leq s)$ ($\mathcal{P}(t, = s)$) the collection of all $\sum_{i=0}^{s} {t \choose i}$ (${t \choose s}$) subsets P of size $|P| \leq s$ (|P| = s). Let $N \geq 2$ be an integer and $\mathsf{A} = \{A_1, A_1, \ldots, A_N\}, A_i \subset [t], i \in [N]$, is a fixed family of subsets of [t]. Subsets A_i are identified with binary rows $\mathbf{x}_i \triangleq (x_i(1), x_i(2), \ldots, x_i(t))$ in which

$$x_i(j) \triangleq \begin{cases} 1 & \text{if } j \in A_i, \\ 0 & \text{if } j \notin A_i, \quad i \in [N], \ j \in [t]. \end{cases}$$

We will identify the family A with its incidence matrix (code) $X = ||x_i(j)||, i \in [N], j \in [t].$

In the theory of group testing [30] (designing screening experiments [29]) the given, in advance, family $A = \{A_1, A_1, \ldots, A_N\}$ is interpreted as a non-adaptive search design consisting of N group tests (experiments) $A_i, i \in [N]$. An experimenter wants to construct group tests $A_i, i \in [N]$, to carry out the corresponding experiments and then to identify an unknown subset $P \subset [t]$ with the help of test outcomes provided that $P \subset \mathcal{P}(t, \leq s)$ or $P \subset \mathcal{P}(t, = s)$, where $s \ll t$. If for each test $A_i, i \in [N]$, its outcome depends only on the size of intersection

$$|P \cap A_i| = \sum_{m=1}^{|P|} x_i(p_m), \quad i \in [N],$$

then we will say that a symmetric model [29] of non-adaptive search design is considered.

2 Superimposed (z, u)-Codes

In this section we give a brief survey of necessary definitions and bounds on the rate of superimposed codes which are the *base for studying* of non-adaptive group testing models.

Let z and u be positive integers such that $z + u \leq t$.

Definition 1. [22]. A family of subsets $\Omega_1, \Omega_2, \ldots, \Omega_t$, where $\Omega_j \subseteq [N], j \in [t]$, is called an (z, u)-cover-free family if for any two non-intersecting subsets $Z, U \subset [t], Z \cap U = \emptyset$, such that |Z| = z, |U| = u, the following condition holds:

$$\bigcap_{j \in U} \Omega_j \not\subseteq \bigcup_{j \in Z} \Omega_j$$

An incidence matrix $X = ||x_i(j)||, i \in [N], j \in [t]$, corresponding to (z, u)-cover-free family is called a *superimposed* (z, u)-code.

The following evident necessary and sufficient condition for Definition 1 takes place.

Proposition 1. [22]. Any binary $(N \times t)$ -matrix X is a superimposed (z, u)-code if and only if for any two subsets $Z, U \subset [t]$, such that |Z| = z, |U| = u and $Z \cap U = \emptyset$ the matrix X contains a row $\mathbf{x}_i = (x_i(1), x_i(2) \dots, x_i(t), \text{ for which})$

$$x_i(j) = 1$$
 for all $j \in U$, $x_i(j) = 0$ for all $j \in Z$.

Let t(N, z, u) be the maximal possible size of superimposed (z, u)-codes. For fixed $1 \le u < z$, define a *rate* of (z, u)-codes:

$$R(z, u) \triangleq \overline{\lim_{N \to \infty} \frac{\log_2 t(N, z, u)}{N}}$$

For the classical case u = 1, superimposed (z, 1)-codes and their applications were introduced by W.H Kautz, R.C. Singleton in [1]. Further, these codes along with new applications were investigated in [9]-[29]. The best known upper and lower bounds on the rate R(z, 1) can be found in papers [9],[14] and [22].

2.1 Recurrent Upper Bounds on R(z, 1) and R(z, u)

Let $h(\alpha) \triangleq -\alpha \log_2 \alpha - (1-\alpha) \log_2(1-\alpha), \ 0 < \alpha < 1$, be the binary entropy. To formulate an *upper bound* on the rate $R(z, 1), z \ge 1$, we introduce the function [9]

$$f_z(\alpha) \triangleq h(\alpha/z) - \alpha h(1/z), \quad z = 1, 2, \dots,$$

of argument α , $0 < \alpha < 1$.

Theorem 1. [9]-[10]. (Recurrent upper bound on R(z, 1)). If z = 1, 2, ..., then the rate $R(z, 1) \leq \overline{R}(z, 1)$, where

$$\overline{R}(1,1) = R(1,1) = 1, \quad \overline{R}(2,1) \triangleq \max_{0 < \alpha < 1} f_2(\alpha) = 0.321928$$
(1)

and sequence $\overline{R}(z,1), z = 3, 4, \dots$, is defined as the unique solution of recurrent equation

$$\overline{R}(z,1) = f_z \left(1 - \frac{\overline{R}(z,1)}{\overline{R}(z-1,1)} \right).$$
(2)

Up to now, the recurrent sequence $\overline{R}(z,1)$, z = 1, 2, ..., defined by (1)-(2) and called a *recurrent upper bound* has been the best known upper bound on the rate R(z,1). The reciprocal values of $\overline{R}(z,1)$, z = 2, 3, ..., 17, taken from [10], are given in Table 1.

z	$1/\overline{R}(z,1)$	z	$1/\overline{R}(z,1)$	z	$1/\overline{R}(z,1)$	z	$1/\overline{R}(z,1)$
2	3.1063	6	12.0482	10	24.5837	14	40.3950
3	5.0180	7	14.8578	11	28.2402	15	44.8306
4	7.1196	8	17.8876	12	32.0966	16	49.4536
5	9.4660	9	21.1313	13	36.1493	17	54.2612

Table 1.

Applying Theorem 1 and the corresponding calculus arguments, we proved

Theorem 2. [9]-[10]. (Non-recurrent upper bound on R(z, 1)). For any $z \ge 2$, the rate R(z, 1) satisfies inequality

$$R(z,1) \le \frac{2\log_2[e(z+1)/2]}{z^2}, \qquad z=2,3,\dots$$

which leads to the asymptotic inequality

$$R(z,1) \le \frac{2\log_2 z}{z^2} (1+o(1)), \quad z \to \infty.$$

Theorem 3. [26] (Recurrent inequality for R(z, u)). If $z \ge u \ge 2$, then for any $i \in [z-1]$ and $j \in [u-1]$, the rate

$$R(z,u) \leq \frac{R(z-i,u-j)}{R(z-i,u-j) + \frac{(i+j)^{i+j}}{i^{i}\cdot j^{j}}}.$$
(3)

Recurrent inequality (3) and the known numerical values of recurrent upper bound $\overline{R}(z, 1)$, $z = 1, 2, \ldots$, defined by (1)-(2), give numerical values of the best known upper bound $\overline{R}(z, u)$ on the rate R(z, u), $z \ge u \ge 2$. An asymptotic consequence from the given upper bound is presented by

Theorem 4. [23] If $z \to \infty$ and $u \ge 2$ is fixed, then

$$R(z,u) \le \overline{R}(z,u) \le \frac{(u+1)^{u+1}}{2e^{u-1}} \cdot \frac{\log_2 z}{z^{u+1}} \cdot (1+o(1)).$$

2.2 Random Coding Lower Bounds on R(z, u) and R(z, 1)

Theorem 5. [22] A random coding lower bound on the rate R(z, u) has the form:

$$R(z,u) \ge \underline{R}(z,u) \triangleq -(z+u-1)^{-1}\log_2\left(1 - \frac{z^z u^u}{(z+u)^{z+u}}\right), \quad 2 \le u < z.$$

If $u \geq 2$ is fixed and $z \to \infty$, then the asymptotic form of the given lower bound is

$$R(z,u) \ge \underline{R}(z,u) = \frac{e^{-u} \cdot u^u \cdot \log_2 e}{z^{u+1}} \cdot (1+o(1)).$$

If u = 1, then the best known random coding lower bound on the rate R(z, 1) is given by **Theorem 6.** [15] For any z = 1, 2, ..., the rate $R(z, 1) \ge \underline{R}(z, 1) \triangleq \frac{A(z)}{z}$, where

$$A(z) \triangleq \max_{0 < \alpha < 1, \ 0 < Q < 1} \left\{ -(1-Q)\log(1-\alpha^{z}) + z\left(Q\log\frac{\alpha}{Q} + (1-Q)\log\frac{1-\alpha}{1-Q}\right) \right\}.$$

If $z \to \infty$, then the rate

$$R(z,1) \ge \underline{R}(z,1) = \frac{1}{z^2 \log e} (1+o(1)) = \frac{0.693}{z^2} (1+o(1)).$$

In the first and second rows of Table 2, we give values of $\underline{R}(s,1) < 1/s$, s = 2, 3..., 8, along with the corresponding values of $\overline{R}(s,1) < 1/s$, s = 2, 3..., 8, taken from Table 1.

S	2	3	4	5	6	7	8
$\underline{\widetilde{R}}_1 (\le s) = \underline{R}(s,1)$.182	.079	.044	.028	.019	.014	.011
$\overline{\widetilde{R}}_1 (\le s) = \overline{R}(s,1)$.3219	.1993	.1405	.1056	.0830	.0673	.0559
$\underline{\widetilde{R}}_2 (\leq s) = \underline{R}(s-1,2)$	-	.0321	.0127	.0068	.0037	.0024	.0015
$\overline{\widetilde{R}}_2 (\le s) = \overline{R}(s-1,2)$	-	.1610	.0745	.0455	.0287	.0204	.0146
$\underline{\widetilde{R}}_3 (\leq s) = \underline{R}(s-2,3)$	-	-	.0127	.0046	.0020	.0010	.0001
$\overline{\widetilde{R}}_3 (\le s) = \overline{R}(s-2,3)$	_	_	.0745	.0387	.0183	.0109	.0067
$\underline{R}\left(F_{0}^{1},=s\right)$.302	.142	.082	.053	.037	.027	.021

Table 2

3 $(F^{\ell}, \leq s)$ -Designs, $(F^{\ell}, = s)$ -Designs and \mathcal{D}_s^{ℓ} -Codes

In this section we introduce the concept of non-adaptive group testing designs arising from the potentialities of compressed genotyping models in molecular biology and establish a universal upper bound on their rate. The universal bound is prescribed by our recurrent upper bound on the rate of classical superimposed codes. Using notations of Section 1, we give

Definition 2. Let ℓ , s, t be integers with $1 \leq \ell < s < t$ and $F^{\ell} = F^{\ell}(n)$ be an arbitrary fixed function of integer argument $n = 0, 1, ..., \ell$ such that for any $n = 0, 1, ..., \ell - 1$, its value $F^{\ell}(n) \neq F^{\ell}(\ell)$. Define the vector

$$\mathbf{y}^{\ell}(P,\mathsf{A}) \triangleq \left(y_1^{\ell}, y_2^{\ell}, \dots, y_N^{\ell}\right), \quad y_i^{\ell} \triangleq \begin{cases} F^{\ell}(n) & \text{if } |P \cap A_i| = n, \quad n = 0, 1, \dots, \ell - 1, \\ F^{\ell}(\ell) & \text{if } |P \cap A_i| \ge \ell, \quad i \in [N], \end{cases}$$

or

$$\mathbf{y}^{\ell}(P,X) \triangleq \left(y_{1}^{\ell}, y_{2}^{\ell}, \dots, y_{N}^{\ell}\right), \quad y_{i}^{\ell} \triangleq \begin{cases} F^{\ell}(n) & \text{if } \sum_{m=1}^{|P|} x_{i}(p_{m}) = n, \quad n = 0, 1, \dots, \ell - 1, \\ F^{\ell}(\ell) & \text{if } \sum_{m=1}^{|P|} x_{i}(p_{m}) \ge \ell, \quad i \in [N]. \end{cases}$$

A code X of length N and size t is called an $(F^{\ell}, \leq s)$ -design, $((F^{\ell}, = s)$ -design), $1 \leq \ell < s < t$, for group testing model if $\mathbf{y}^{\ell}(P, X) \neq \mathbf{y}^{\ell}(P', X)$ for any

$$P \neq P', \quad P \in \mathcal{P}(t, \le s), \ P' \in \mathcal{P}(t, \le s) \quad \left(P \in \mathcal{P}(t, = s), \ P' \in \mathcal{P}(t, = s)\right).$$

Remark 1. $(F^{\ell}, \leq s)$ -design and $(F^{\ell}, = s)$ -design are examples, which can be interpreted as compressed genotyping [36] models in molecular biology.

Remark 2. In [38], a special $(F^{\ell}, \leq s)$ -design is considered. The authors introduce the ranges $(0 \triangleq r_0 < r_1 < r_2 < \ldots < r_k \triangleq p)$ and set

$$F^{\ell}(r_0 + 1) = \dots = F^{\ell}(r_1) = 1$$

$$F^{\ell}(r_1 + 1) = \dots = F^{\ell}(r_2) = 2$$

$$\vdots = \dots = \vdots$$

$$F^{\ell}(r_{k-1} + 1) = \dots = F^{\ell}(r_k) = k.$$

This model can be viewed as an adder model followed by a quantizer.

Let $1 \leq \ell < s < t$ be integers. For any set $\mathcal{S} \subset [t]$ of size $|\mathcal{S}| = s$, we denote by $\binom{\mathcal{S}}{\ell}$ the collection of all $\binom{s}{\ell}$ ℓ -subsets of the set \mathcal{S} .

Definition 3. [11]. A family of subsets $\Omega_1, \Omega_2, \ldots, \Omega_t$ is called an \mathcal{D}_s^{ℓ} -family if for any $\mathcal{S} \subset [t], |\mathcal{S}| = s$, and any $j \notin \mathcal{S}$,

$$\Omega_j \not\subseteq \bigcup_{\binom{S}{\ell}} \left\{ \bigcap_{k=1}^{\ell} \Omega_{j_k} \right\}, \text{ where } \binom{S}{\ell} \triangleq \left\{ (j_1, j_2, \dots, j_{\ell}) : j_i \in \mathcal{S}, \quad j_1 < j_2 < \dots < j_{\ell} \right\}.$$

An incidence matrix $X = ||x_i(j)||, i \in [N], j \in [t]$, corresponding to \mathcal{D}_s^{ℓ} -family is called a superimposed \mathcal{D}_s^{ℓ} -code (briefly, \mathcal{D}_s^{ℓ} -code).

One can easily check the following

Proposition 2. Any binary $(N \times t)$ -matrix X is a \mathcal{D}_s^{ℓ} -code, $1 \leq \ell < s < t$, if and only if for any collection of s + 1 integers $j_1, j_2, \ldots, j_s, j_{s+1}, j_k \neq j_m, j_k \in [t]$, there exists $i \in [N]$ such that

$$x_i(j_{s+1}) = 1, \qquad \sum_{k=1}^{\circ} x_i(j_k) \le \ell - 1.$$

For $\ell = 1$ and s = 2, 3..., the definition of \mathcal{D}_s^1 -code coincides with the definition of superimposed (s, 1)-code. In addition, if $1 \leq \ell < s - 1$, then any \mathcal{D}_s^ℓ -code is a $\mathcal{D}_s^{\ell+1}$ -code.

Remark 3. For $s > \ell \ge 2$, \mathcal{D}_s^{ℓ} -codes were suggested in [11] for the study of some communication systems with random multiple access.

3.1 Universal Upper Bound for $(F^{\ell}, \leq s)$ -Designs

Let $t(N, \mathcal{D}_s^{\ell})$, $t(N, F^{\ell}, \leq s)$ and $t(N, F^{\ell}, = s)$ be the maximal size of superimposed \mathcal{D}_s^{ℓ} -codes, $(F^{\ell}, \leq s)$ -designs and $(F^{\ell}, = s)$ -designs. For fixed $1 \leq \ell < s$, define the corresponding *rates*:

$$R\left(\mathcal{D}_{s}^{\ell}\right) \triangleq \overline{\lim_{N \to \infty}} \ \frac{\log_{2} t\left(N, \mathcal{D}_{s}^{\ell}\right)}{N}, \quad 1 \leq \ell < s,$$

0

$$R\left(F^{\ell}, \leq s\right) \triangleq \lim_{N \to \infty} \frac{\log_2 t\left(N, F^{\ell}, \leq s\right)}{N}, \qquad R\left(F^{\ell}, = s\right) \triangleq \lim_{N \to \infty} \frac{\log_2 t\left(N, F^{\ell}, = s\right)}{N}.$$

Obviously, for any $1 \le \ell < s$, the following inequalities hold:

$$t\left(N, F^{\ell}, \leq s\right) \leq t\left(N, F^{\ell}, = s\right), \quad R\left(F^{\ell}, \leq s\right) \leq R\left(F^{\ell}, = s\right) \leq \frac{\log_2(\ell+1)}{s}.$$
 (4)

Proposition 3. [11]. If $1 \leq \ell < s - 1$, then any $(F^{\ell}, \leq s)$ -design is a superimposed \mathcal{D}_{s-1}^{ℓ} -code, i.e.,

$$t\left(N, F^{\ell}, \leq s\right) \leq t\left(N, \mathcal{D}_{s-1}^{\ell}\right), \qquad R\left(F^{\ell}, \leq s\right) \leq R\left(\mathcal{D}_{s-1}^{\ell}\right), \quad 1 \leq \ell < s-1.$$

Proof. By contradiction. If a code $X = ||x_i(j)||$, $i \in [N]$, $j \in [t]$ doesn't satisfy the definition of \mathcal{D}_{s-1}^{ℓ} -code, then in virtue of Proposition 1, there exists a collection of s integers $j_1, j_2, \ldots, j_{s-1}, j_s, j_k \neq j_m, j_k \in [t]$, such that for any $i \in [N]$,

$$x_i(j_s) = 1 \implies \sum_{k=1}^{s-1} x_i(j_k) \ge \ell.$$

Hence, for (s-1)-subset $P \triangleq \{j_1, j_2, \ldots, j_{s-1}\} \subset [t]$ and s-subset $P' \triangleq \{j_1, j_2, \ldots, j_{s-1}, j_s\} \subset [t]$, the vector $\mathbf{y}^{\ell}(P, X) = \mathbf{y}^{\ell}(P', X)$. This contradicts to the definition of $(F^{\ell}, \leq s)$ -design.

Theorem 7. (De Bonis, Vaccaro [32]). For any $1 \leq \ell < s$, the rate $R\left(\mathcal{D}_{s}^{\ell}\right)$ of superimposed \mathcal{D}_{s}^{ℓ} -codes satisfies inequality

$$R\left(\mathcal{D}_{s}^{\ell}\right) \leq R\left(\left\lfloor \frac{s}{\ell} \right\rfloor, 1\right),$$

where R(z, 1), $z \ge 1$, is the rate of classical superimposed (z, 1)-codes.

Proposition 3 and Theorem 7 lead to inequalities:

$$R\left(F^{\ell}, \leq s\right) \leq R\left(\mathcal{D}_{s-1}^{\ell}\right) \leq R\left(\left\lfloor\frac{s-1}{\ell}\right\rfloor, 1\right) \leq \overline{R}\left(\left\lfloor\frac{s-1}{\ell}\right\rfloor, 1\right), \quad 1 \leq \ell \leq s, \tag{5}$$

where $\overline{R}(z, 1)$ is the recurrent upper bound on the rate R(z, 1) presented by Theorem 1. For instance, if $(\ell = 3, s = 10)$ or $(\ell = 3, s = 13)$, then Table 2 shows that

$$\overline{R}(3,1) = .199 < .200 = 2/10$$
 or $\overline{R}(4,1) = .140 < .154 = 2/13$,

i.e., for $\ell = 3$ and s = 3k + 1, $k = 3, 4, \dots$, bound (5) improves the trivial bound (4).

From inequalities (4)-(5), it follows

Proposition 4. (Universal upper bound). For any $(F^{\ell}, \leq s)$ -design, the rate

$$R\left(F^{\ell}, \leq s\right) \leq \min\left\{\frac{\log_2(\ell+1)}{s}; \overline{R}\left(\left\lfloor\frac{s-1}{\ell}\right\rfloor, 1\right)\right\}, \quad 1 \leq \ell < s,$$

and the asymptotic inequality

$$R\left(F^{\ell}, \le s\right) \le \frac{2\ell^2 \log_2 s}{s^2} (1+o(1)), \quad \ell = 1, 2, \dots, \qquad s \to \infty,$$

holds.

4 Constructions of Superimposed (z, u)-Codes and \mathcal{D}_s^{ℓ} -Codes

4.1 Superimposed (s, 1)-Codes and \mathcal{D}_s^{ℓ} -Codes Based on Shortened Reed-Solomon Codes

Let \mathcal{Q} be the set of all primes or prime powers ≥ 2 , i.e.,

$$\mathcal{Q} \triangleq \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37, \ldots\}.$$

Let $q \in Q$ and $2 \le k \le q+1$ be fixed integers for which there exists the q-ary Reed-Solomon code (RS-code) B of size q^k , length (q+1) and the Hamming distance d = q-k+2 = (q+1)-(k-1) [4]. We will identify the code B with an $((q+1) \times q^k)$ -matrix whose columns, (i.e., (q+1)-sequences from the alphabet $\{0, 1, 2, \ldots, q-1\}$) are the codewords of B. Therefore, the maximal possible number of positions (rows) where its two codewords (columns) can coincide, called a *coincidence* of code B, is equal to k - 1.

Fix an arbitrary integer r = 0, 1, 2, ..., k - 1 and introduce the *shortened* RS-code \tilde{B} of size $t = q^{k-r}$, length n = q + 1 - r that has the same Hamming distance d = q - k + 2. Code \tilde{B} is obtained by the *shortening* of the *subcode* of B which contains 0's in the first r positions (rows) of B. Obviously, the coincidence of \tilde{B} is equal to

$$\lambda \triangleq n - d = (q + 1 - r) - d = q + 1 - r - (q - k + 2) = k - r - 1.$$
(6)

Consider the following standard transformation of the q-ary code \tilde{B} , when each symbol of the q-ary alphabet $\{0, 1, 2, \ldots, q-1\}$ is substituted for the corresponding binary column of the length q and the weight 1, namely:

$$0 \Leftrightarrow \underbrace{(1,0,0,\ldots,0)}_{q}, \quad 1 \Leftrightarrow \underbrace{(0,1,0,\ldots,0)}_{q}, \quad \ldots \quad q-1 \Leftrightarrow \underbrace{(0,0,0,\ldots,1)}_{q}.$$

As a result we have a binary constant-weight code X of size t, length N and weight w, where

$$t = q^{k-r} = q^{\lambda+1}, \quad N = n \cdot q = (q+1-r)q, \quad w = n = q+1-r.$$
 (7)

From Propositions 1-2 and (6), it follows

Proposition 5. Let integers $1 \le \ell < s$ satisfy inequalities

$$s[(k-1)-r] \le \ell (q+1-r) - 1, \quad 2 \le k \le q+1, \quad 0 \le r \le k-1.$$
(8)

Then the binary constant-weight code X with parameters (7) is a \mathcal{D}_s^{ℓ} -code if $2 \leq \ell < s$, or X is a superimposed (s, 1)-code if $\ell = 1$.

For $\ell = 1$, the detailed tables with parameters of the best known superimposed (s, 1)-codes (or \mathcal{D}_s^1 -codes) based on Proposition 5 are presented in our papers [18]-[19]. Table 3 gives an example of such table. In Table 3, we marked by the **boldface type** two *triples* of superimposed code parameters which were known from [1]. The rest triples of superimposed code parameters from Table 3 were obtained in [18]-[19].

For the general case of superimposed (z, u)-codes, $2 \le u < z$, the construction similar to Proposition 5 was developed in [22]. Another significant constructions of superimposed (z, u)codes, $2 \le u < z$, were suggested in [27]-[28]. Table 4 gives parameters of the best known superimposed (z, u)-codes if u = 2, 3 and $z = 2, 3, \ldots 9$.

4.2 Parameters of constant-weight superimposed (s, 1)-codes $2 \le s \le 8$, of weight w, length N, size $t = q^{\lambda+1}$, $2^m \le t < 2^{m+1}$, $5 \le m \le 30$, based on the q-ary shortened Reed-Solomon codes.

s	2	3	4	5	6	7	8
$\underline{R}(s,1)$.182	.079	.044	.028	.019	.014	.011
$\overline{R}(s,1)$.322	.199	.140	.106	.083	.067	.056
m	q, λ, N	q,λ,N	q, λ, N				
5	—	7, 1, 28	7, 1, 35	7, 1, 42	7, 1, 49	_	—
6	4, 2, 20	8, 1, 32	8, 1, 40	8, 1, 48	8, 1, 56	9, 1, 72	11, 1, 99
7	—	—	13, 1, 65	13, 1, 78	13, 1, 91	13, 1, 104	13, 1, 117
8	7, 2, 35	7, 2, 49	_	16, 1, 96	16, 1, 112	16, 1, 128	16, 1, 144
9	8, 2, 40	8, 2, 56	8, 2, 72	_	23, 1, 161	23, 1, 184	23, 1, 207
10	—	11, 2, 77	11, 2, 99	11, 2, 121	—	_	—
11	7, 3, 49	_	13, 2, 117	13, 2, 143	13, 2, 169	_	—
12	8, 3, 56	9, 3, 90	16, 2, 144	16, 2, 176	16, 2, 208	16, 2, 240	16,2,272
$\frac{12}{N}$.214	.133	.083	.068	.058	.050	.044
13	_	11, 3, 110		23, 2, 253	23, 2, 299	23, 2, 345	23, 2, 391
14	_	13, 3, 130	13, 3, 169	_	27, 2, 351	27, 2, 405	27, 2, 459
15	8, 4, 72	_	_	_	—	32, 2, 480	32, 2, 544
16	_	16, 3, 160	16, 3, 208	16, 3, 256	19, 3, 361		—
17	11, 4, 99			_	—		—
18	13, 4, 117	13, 4, 169		23, 3, 368	23, 3, 437	23, 3, 506	25, 3, 625
19	_			27, 3, 432	27, 3, 513	27, 3, 594	27, 3, 675
20	11, 5, 121	16, 4, 208	16,4,272	—	32, 3, 608	32, 3, 704	32, 3, 800
$\frac{20}{N}$.165	.096	.074	-	.034	.028	.025
21	—	_	19, 4, 323	—	—	_	41, 3, 1025
22	13, 5, 143	_	23, 4, 391	23, 4, 483	_	_	—
23	—	_	25, 4, 425	25, 4, 525	25, 4, 625	_	—
24	—	16, 5, 256	_	27, 4, 609	29, 4, 725	29, 4, 841	—
25	13, 6, 169	19, 5, 304	—	—	32, 4, 800	32, 4, 928	32, 4, 1056
$\frac{25}{N}$.148	.082	-	-	.031	.027	.024
26	—	—	_	—	37, 4, 925	37, 4, 1073	37, 4, 1221
27	_	_	23, 5, 483	_	_	43, 4, 1247	43, 4, 1419
28	16, 6, 208	_	27, 5, 702	25, 5, 650	_		49, 4, 1617
29	—	19, 6, 361	29, 5, 609	29, 5, 754	31, 5, 961	—	—
$\frac{29}{N}$	—	.080	.048	.038	.030	_	—
30	—	—	—	32, 5, 832	32, 5, 992		—

Table 3

Table 3 also contains numerical values of the rate for several obtained codes, namely: the values of fraction $\frac{m}{N}$, m = 12, 20, 25, 29. The comparison with lower $\underline{R}(s, 1)$ and upper $\overline{R}(s, 1)$ bounds from Table 2 (their values are included in Table 3 as well) yields the following conclusions:

- if s = 2 and $m \le 15$, then the values $\frac{m}{N}$ exceed the random coding rate $\underline{R}(2, 1) = .182$;
- if $s \ge 3$ and $m \le 30$, then the values $\frac{m}{N}$ exceed the random coding rate $\underline{R}(s, 1)$.

(2,2)	(3,2)	(4,2)	(5,2)	(6,2)	(7,2)	(8,2)	(9,2)
t, N	t, N	t, N	t, N	t, N	t, N	t, N	t, N
8, 14	7, 21	11, 55	11, 55	20, 190	26, 260	16, 120	38, 703
9, 18	8, 28	13,65	16, 120	25, 210	50,350	32, 496	82, 738
10, 20	10, 30	17,68	26, 130	49, 294	64, 448	65, 520	120, 1090
12, 22	16, 42	22, 77	48, 246	63, 385	80,568	81, 648	166, 1562
16, 26	21, 56	25,100	62, 330	79, 497	118, 882	119, 981	250, 2531
18, 30	24, 76	47, 205	78, 434	117, 792	164, 1308	165, 1430	282, 2933
22, 34	49, 147	64, 252	$121,\ 605$	169, 1014	256, 1800	256, 2040	361, 3249
24, 37	_	_	_	_	_	_	—
32, 43	(3,3)	(4,3)	(5,3)	(6,3)	(7,3)	(8,3)	(9,3)
40, 50	t, N	t, N	t, N	t, N	t, N	t, N	t, N
48, 59	7, 35	12, 220	16, 560	17,680	19,969	20, 1140	22, 1540
56, 65	8, 54	13, 253	19,612	20, 816	21, 1071	21, 1330	23,1771
64, 68	11,66	23, 253	$25,\ 700$	26, 910	27, 1170	22, 1386	45, 14190
80, 76	16, 112	24, 532	31, 3951	32, 4683	52, 11313	53, 12757	54, 14352
112, 96	22, 176	169, 3289	50, 8830	51,10008	529, 25740	729, 73125	729, 81900
128, 100	23, 399	—	256, 8960	361, 15504	—	_	—
144, 109	121, 660	_	—	—	—	—	—
512, 126	—	_	—	—	—	—	—

4.3 Size t and Length N of Superimposed (z, u)-Codes, u = 2, 3 and $z = 2, 3, \ldots 9$

Table 4

4.4 Examples of \mathcal{D}_s^{ℓ} -Codes

Example 1. If q = 5, then for the pair ($\ell = 2, s = 3$), inequalities (8) are fulfilled at k = 5 and r = 2. Therefore, the construction of Proposition 4 yields a binary constant-weight \mathcal{D}_3^2 -code X with parameters

$$t = q^{k-r} = 5^3 = 125, \quad N = n \cdot q = (q+1-r)q = 4 \cdot 5 = 20, \quad w = n = q+1-r = 4.$$
(9)

Parameters (9) give the following lower bound on the maximal size: $t(20, \mathcal{D}_3^2) \ge 125$.

Example 2. If q = 7, then for the pair ($\ell = 2, s = 4$), inequalities (8) are fulfilled at k = 6 and r = 3. Therefore, the construction of Proposition 4 yields a binary constant-weight \mathcal{D}_4^2 -code X with parameters

$$t = q^{k-r} = 7^3 = 343, \quad N = n \cdot q = (q+1-r)q = 5 \cdot 7 = 35, \quad w = n = q+1-r = 5.$$
 (10)

Parameters (10) give the following lower bound on the maximal size: $t(35, \mathcal{D}_4^2) \geq 343$.

Example 3. If q = 8, then for two pairs of integers ($\ell = 2, s = 6$) and ($\ell = 3, s = 10$), inequalities (8) are fulfilled at k = 5 and r = 2. Therefore, the construction of Proposition 4 yields a binary constant-weight \mathcal{D}_6^2 -code X and a binary constant-weight \mathcal{D}_{10}^3 -code X with parameters

$$t = q^{k-r} = 8^3 = 512, \quad N = n \cdot q = (q+1-r)q = 7 \cdot 8 = 56, \quad w = n = q+1-r = 7.$$
 (11)

Parameters (11) give the following lower bounds on the maximal size $t(N, \mathcal{D}_s^{\ell})$ of \mathcal{D}_s^{ℓ} -codes:

$$t(56, \mathcal{D}_6^2) \ge 512, \quad t(56, \mathcal{D}_{10}^3) \ge 512.$$

For comparison, if (u = 1, z = 6) and N = 56, then the best known lower bound on the size of optimal superimposed (6, 1)-codes, calculated in [18], is $t(56, 6, 1) \ge 64$. In addition, this example shows that for $\ell = 3$, the parameter s = 10 of \mathcal{D}_{10}^3 -code X can exceed the corresponding code weight w = 7.

5 Threshold Group Testing Model

5.1 Superimposed (z, u)-Codes and $(F_0^{\ell}, \leq s)$ -Designs

Let the function $F^{\ell} \triangleq F_0^{\ell} = F_0^{\ell}(n), 1 \leq \ell < s$, takes binary values, namely:

$$F_0^{\ell}(n) \triangleq \begin{cases} 0 & \text{if} \quad n = 0, 1, \dots, \ell - 1, \\ 1 & \text{if} \quad n = \ell. \end{cases}$$

If $\ell \geq 2$, then the given particular case is called a *threshold group testing model* [33]. For the non-adaptive threshold group testing model which is the principal model for applications [36], a *refined form* of Definition 2 can be written as follows.

Definition 4. Let ℓ , $1 \leq \ell < s < t$ be integers. For code $X = ||x_i(k)||, k \in [t], i \in [N]$, and a subset $P \in \mathcal{P}(t, \leq s)$, define the *i*-th outcome of non-adaptive threshold group testing

$$y_i^{\ell}(P,X) \triangleq \begin{cases} 0 & \text{if } \sum_{k \in P} x_i(k) \le \ell - 1, \\ 1 & \text{if } \sum_{k \in P} x_i(k) \ge \ell, \quad i \in [N]. \end{cases}$$

A code X is called a $(F_0^{\ell}, \leq s)$ -design, $((F_0^{\ell}, = s)$ -design) if for any $P, P' \in \mathcal{P}(t, \leq s), P \neq P'$, and such that $P, P' \in \mathcal{P}(t, \leq s) \setminus \mathcal{P}(t, \leq \ell - 1)$ $(P \in \mathcal{P}(t, = s), P' \in \mathcal{P}(t, = s))$, there exists an index $i \in [N]$, where $y_i^{\ell}(P, X) \neq y_i^{\ell}(P', X)$.

An important connection between $(F_0^{\ell}, \leq s)$ -designs and superimposed $(s - \ell + 1, \ell)$ -codes is described by

Proposition 6. ([34], [35]). If $1 \le \ell < s$, then any superimposed $(s - \ell + 1, \ell)$ -code is a $(F_0^{\ell}, \le s)$ -design, i.e.

$$t\left(N,s-\ell+1,\ell\right) \leq t\left(N,F_0^\ell,\leq s\right), \qquad R(s-\ell+1,\ell) \leq R\left(F_0^\ell,\leq s\right).$$

The lower bound of Theorem 5 and Propositions 6 lead to the following lower bound on the rate of $(F_0^{\ell}, \leq s)$ -designs.

Proposition 7. (Random coding bound). For any $1 \le \ell < s$, the rate

$$R\left(F_0^{\ell}, \le s\right) \ge R(s - \ell + 1, \ell) \ge -\frac{1}{s} \log_2\left[1 - \frac{(s - \ell + 1)^{s - \ell + 1} \cdot \ell^{\ell}}{(s + 1)^{s + 1}}\right], \quad 1 \le \ell < s.$$
(12)

If $\ell \geq 1$ is fixed and $s \to \infty$, then the asymptotic form of the given lower bound is

$$R\left(F_0^{\ell}, \le s\right) \ge \frac{e^{-\ell} \cdot \ell^\ell \cdot \log_2 e}{s^{\ell+1}} \cdot (1+o(1)).$$

$$\tag{13}$$

5.2 Bounds on the Rate of $(F_0^1, \leq s)$ and $(F_0^1, = s)$ -Designs

If $\ell = 1$ and $s \ge 2$, then the universal upper bound of Proposition 4 lead to inequalities :

$$R(F_0^1, \le s) \le \min\{1/s; \overline{R}(s-1,1)\}, \qquad s=2,3,\dots,$$

where $\overline{R}(z, 1)$, z = 1, 2, ..., is the recurrent upper bound from Theorem 1. Hence, the asymptotic upper bound

$$R\left(F_0^1, \le s\right) \le \overline{R}(s-1, 1) = \frac{2 \cdot \log_2 s}{s^2} \cdot (1+o(1)), \quad s \to \infty,$$

holds.

In [14]-[15] (see, also [29]), we obtained the best known asymptotic random coding lower bounds on $R(F_0^1, \leq s)$ and $R(F_0^1, = s)$ along with the best known upper bound on $R(F_0^1, = s)$. These bounds have the form:

$$R\left(F_0^1, \le s\right) \ge \underline{R}(s, 1) = \frac{1}{s^2 \cdot \log_2 e} \cdot (1 + o(1)) = \frac{0.693}{s^2} \cdot (1 + o(1)), \quad s \to \infty,$$
(14)

$$R\left(F_{0}^{1},=s\right) \geq \underline{R}\left(F_{0}^{1},=s\right) = \frac{2}{s^{2} \cdot \log_{2} e} \cdot (1+o(1)) = \frac{1.386}{s^{2}} \cdot (1+o(1)), \quad s \to \infty,$$
(15)

$$R\left(F_0^1, = s\right) \le \overline{R}\left(F_0^1, = s\right) = \frac{4 \cdot \log_2 s}{s^2} \cdot (1 + o(1)), \quad s \to \infty.$$

$$\tag{16}$$

Lower bound (14), i.e., function $\underline{R}(s,1)$ is defined in Theorem 6. For the particular case $\ell = 1$, bound (14) is better than the lower bound (13) of Proposition 7. The numerical values of lower bound (15), i.e., numbers $\underline{R}(F_0^1, = s)$, $s = 2, 3, \ldots, 8$, are given in Table 2.

In addition, applying the corresponding non-asymptotic results [29], one can calculate numerical values of upper bound (16), i.e., numbers $\overline{R}(F_0^1, = s)$, $s \ge 1$, which lead to inequalities: $R(F_0^1, = s) < 1/s$ if $s \ge 11$. For s = 2, the nontrivial inequality $R(F_0^1, = 2) < 0.4998 < 1/2$ was proved in [31]. For $3 \le s \le 10$, the inequality $R(F_0^1, = s) < 1/s$ can be considered as our conjecture.

5.3 Lower Bound on the Rate of $(F_0^{\ell}, \leq s)$ -Designs

For $(F_0^{\ell}, \leq s)$ -designs, $\ell \geq 2$, the lower bound (12) of Proposition 7 can be improved [37]. An improvement is obtained with the help of the following auxiliary concepts.

Definition 5. [37]. Let ℓ , $1 \leq \ell < s < t/2$ be integers. For code $X = ||x_i(k)||$, $k \in [t]$, $i \in [N]$, and a subset $P \in \mathcal{P}(t, \leq s)$, define the *i*-th outcome of non-adaptive threshold group testing

$$y_i^{\ell}(P, X) \triangleq \begin{cases} 0 & \text{if } \sum_{k \in P} x_i(k) \leq \ell - 1, \\ 1 & \text{if } \sum_{k \in P} x_i(k) \geq \ell, \quad i \in [N] \end{cases}$$

A code X is called a *threshold* $(\ell, \leq s)$ -design of length N and size t if for any $P, P' \in \mathcal{P}(t, \leq s)$, $P \neq P'$, and such that

$$|P| \ge |P'|$$
 $P, P' \in \mathcal{P}(t, \le s) \setminus \mathcal{P}(t, \le \ell - 1)$

there exists an index $i \in [N]$, where the *i*-th outcome of non-adaptive threshold group testing is

$$y_i^{\ell}(P, X) = 1$$
 and $y_i^{\ell}(P', X) = 0$.

Let $t_{\ell}(N, \leq s)$, denote the maximal possible size of threshold $(\ell, \leq s)$ -designs. For fixed $1 \leq \ell < s$, define the corresponding *rate*:

$$R_{\ell} (\leq s) \triangleq \lim_{N \to \infty} \frac{\log_2 t_{\ell}(N, \leq s)}{N}$$

Obviously, any threshold $(\ell, \leq s)$ -designs is a $(F_0^{\ell}, \leq s)$ -design and the rate

$$R\left(F_0^{\ell}, \le s\right) \ge R_{\ell} (\le s), \quad 1 \le \ell < s.$$
(17)

Definition 6. [37]. Let $\ell, 1 \leq \ell < s < t/2$ be integers. A binary $(N \times t)$ -matrix X is called a *superimposed* \mathcal{M}_s^{ℓ} -code (briefly, \mathcal{M}_s^{ℓ} -code) if for any two non-intersecting subsets $Z, U \in \mathcal{P}(t, \leq s), Z \cap U = \emptyset$, such that $\ell \leq |U| \leq s, |Z| \leq |U|$ and for any element $j \in U$, the matrix X contains a row $\mathbf{x}_i = (x_i(1), x_i(2) \dots, x_i(t)), i \in [N]$, for which

$$x_i(j) = 1$$
, $\sum_{k \in U} x_i(k) = \ell$ and $x_i(k) = 0$ for all $k \in Z$.

Let $t(N, \mathcal{M}_s^{\ell})$ denote the maximal size of \mathcal{M}_s^{ℓ} -codes. For fixed $1 \leq \ell < s$, introduce

$$R\left(\mathcal{M}_{s}^{\ell}\right) \triangleq \overline{\lim_{N \to \infty}} \frac{\log_{2} t\left(N, \mathcal{M}_{s}^{\ell}\right)}{N}, \qquad 1 \leq \ell < s.$$

called a *rate* of \mathcal{M}_{s}^{ℓ} -codes. The evident connection between \mathcal{M}_{s}^{ℓ} -codes and superimposed $(2s - \ell, 1)$ -codes is given by

Proposition 8. [37]. **1.** Let $2 \le s < t/2$. If $\ell = 1$, then any \mathcal{M}_s^1 -code X of size t is a superimposed (2s-1,1)-code and, vice versa, any superimposed (2s-1,1)-code X of size t is a \mathcal{M}_s^1 -code, i.e., the rate $R(\mathcal{M}_s^1) = R(2s-1,1)$. **2.** If $2 \le \ell < s < t/2$, then any \mathcal{M}_s^ℓ -code X of size t is a superimposed $(2s-\ell,1)$ -code, i.e., the rate $R(\mathcal{M}_s^\ell) \le R(2s-\ell,1)$.

Proposition 9. [37]. If $1 \le \ell < s < t/2$, then any \mathcal{M}_s^{ℓ} -code X of size t is a threshold $(\ell, \le s)$ -design, i.e. the rate $R(\mathcal{M}_s^{\ell}) \le R_{\ell} (\le s)$.

Proof of Proposition 9. Let $X = ||x_i(k)||, k \in [t], i \in [N]$, be an arbitrary \mathcal{M}_s^{ℓ} -code. Consider arbitrary subsets: $P, P' \in \mathcal{P}(t, \leq s), P \neq P'$, and such that

$$|P| \ge |P'|, \quad P, P' \in \mathcal{P}(t, \le s) \setminus \mathcal{P}(t, \le \ell - 1), \quad \ell \le |P| \le s, \quad \ell \le |P'| \le |P|.$$

Fix an arbitrary $j \in P \setminus P'$, $j \notin P'$ and define non-intersecting subsets $U \triangleq P$ and $Z \triangleq P' \setminus P$. We have

$$\ell \le |U| \le s, \quad j \in U, \quad U \cap Z = \emptyset, \quad Z \subset P', \quad P' \setminus Z \subset U, \quad |Z| \le |P'| \le |P| = |U|.$$

Definition 6 of \mathcal{M}_s^{ℓ} -code implies that there exists an index $i \in [N]$ such that

$$\left(\sum_{k\in U} x_i(k) = \ell, \sum_{k\in Z} x_i(k) = 0, x_i(j) = 1, \sum_{k\in P'\setminus Z} x_i(k) \le \ell - 1\right) \Rightarrow$$
$$\Rightarrow \left(\sum_{k\in P} x_i(k) = \ell, \sum_{k\in P'} x_i(k) \le \ell - 1\right) \Rightarrow \left(y_i(P, X) = 1, y_i(P', X) = 0\right),$$

i.e., code X is a threshold $(\ell, \leq s)$ -design.

Proposition 9 is proved.

If $\beta \triangleq \Pr\{x_i(k) = 1\}$ and $1 - \beta \triangleq \Pr\{x_i(k) = 0\}$, then one can easily check that for any $j \in [t]$, the probability

$$\Pr\left\{\mathbf{x}(j) \text{ is } \mathcal{M}_{s}^{\ell} - \text{bad}\right\} \leq \sum_{u=\ell}^{s} \sum_{z=0}^{u} \binom{t-1}{u+z-1} \binom{u+z-1}{u-1} \times \left[1 - \binom{u-1}{\ell-1} \beta^{\ell} (1-\beta)^{u+z-\ell}\right]^{N}.$$

The given inequality leads to the following random coding lower bound on the rate of \mathcal{M}_s^{ℓ} -codes:

Proposition 10. For any β , $0 < \beta < 1$, the rate $R(\mathcal{M}_s^{\ell})$ satisfies inequality

$$R\left(\mathcal{M}_{s}^{\ell}\right) \geq \min_{\ell \leq u \leq s; \ 0 \leq z \leq u} \left\{ \frac{-\log_{2}\left[1 - \binom{u-1}{\ell-1}\beta^{\ell}\left(1-\beta\right)^{u+z-\ell}\right]}{u+z-1} \right\} \geq \min_{\ell \leq u \leq s} L_{\ell}(\beta, u),$$

where

$$L_{\ell}(\beta, u) \triangleq \left\{ \frac{-\log_2\left[1 - \binom{u-1}{\ell-1}\beta^{\ell} (1-\beta)^{2u-\ell}\right]}{2u-1} \right\}, \quad \ell \le u \le s, \quad 0 < \beta < 1.$$
(18)

From (17) and Propositions 9-10 it follows a lower bound on the rate of $(F_0^{\ell}, \leq s)$ -designs :

$$R\left(F_{0}^{\ell}, \leq s\right) \geq \underline{R}\left(F_{0}^{\ell}, \leq s\right) \triangleq \max_{0 < \beta < 1} \min_{\ell \leq u \leq s} L_{\ell}(\beta, u) =$$
$$= \max_{0 < \beta < 1} \min_{\ell \leq u \leq s} \left\{ \frac{-\log_{2}\left[1 - \binom{u-1}{\ell-1}\beta^{\ell}(1-\beta)^{2u-\ell}\right]}{2u-1} \right\}, \quad 1 \leq \ell < s.$$
(19)

The calculation of numerical values for lower bound (19) is an open problem.

5.4 Comments on Definitions 4 and 5

Let $\ell,\, 1 \leq \ell < s < t/2,$ be integers. For a comparison of Definitions 4 and 5 , introduce

Definition $\tilde{\mathbf{5}}$. A code X is called a *threshold* $\overline{(\ell, \leq s)}$ -design, of length N and size t if for any $P, P' \in \mathcal{P}(t, \leq s), P \neq P'$, and such that

$$P \setminus P' \neq \emptyset, \quad P, P' \in \mathcal{P}(t, \le s) \setminus \mathcal{P}(t, \le \ell - 1),$$

there exists an index $i \in [N]$, where the *i*-th outcome of non-adaptive threshold group testing is

$$y_i^{\ell}(P, X) = 1$$
 and $y_i^{\ell}(P', X) = 0.$

Let $\tilde{t}_{\ell}(N, \leq s)$, be the maximal size of threshold $\overline{(\ell, \leq s)}$ -designs. For fixed $1 \leq \ell < s$, define the corresponding *rate*

$$\widetilde{R}_{\ell} \, (\leq s) \triangleq \varlimsup_{N \to \infty} \frac{\log_2 \overline{t_{\ell}}(N, \leq s)}{N}.$$

The following important property is given by

Proposition 11. If $1 \leq \ell < s < t/2$, then (1) any superimposed $(s - \ell + 1, \ell)$ -code X of size t is a threshold $(\ell, \leq s)$ -design and, vice versa, (2) any threshold $(\ell, \leq s)$ -design X of size t is a superimposed $(s - \ell + 1, \ell)$ -code, i.e., the rate $\tilde{R}_{\ell} (\leq s) = R(s - \ell + 1, \ell)$.

Evidently, any threshold $\overline{(\ell, \leq s)}$ -design is a threshold $(\ell, \leq s)$ -design. Therefore, in virtue of Proposition 11, the rate

$$\widetilde{R}_{\ell} (\leq s) = R(s - \ell + 1, \ell) \leq R_{\ell} (\leq s) \leq R\left(F_0^{\ell}, \leq s\right).$$

Denote by $\underline{R}(z, u)$, $1 \le u \le z$, the lower bound on R(z, u) formulated in Theorems 5 and 6. Let $\overline{R}(z, u)$ be the upper bound on R(z, u) given by Theorem 3. For parameters $\ell = 1, 2, 3$ and $s = \ell + 1, \ell + 2, \ldots, 8$, numerical values of lower bound $\underline{\widetilde{R}}_{\ell} (\le s) \triangleq \underline{R}(s - \ell + 1, \ell)$ and upper bound $\underline{\widetilde{R}}_{\ell} (\le s) \triangleq \overline{R}(s - \ell + 1, \ell)$ on the rate $\widetilde{R}_{\ell} (\le s) = R(s - \ell + 1, \ell)$ are presented in Table 2.

Proof of Proposition 11. (1) Let $X = ||x_i(k)||, k \in [t], i \in [N]$, be a superimposed $(s - \ell + 1, \ell)$ -code. Consider arbitrary subsets: $P, P' \in \mathcal{P}(t, \leq s), P \neq P'$, and such that

$$P \setminus P' \neq \emptyset, \quad P, P' \in \mathcal{P}(t, \le s) \setminus \mathcal{P}(t, \le \ell - 1), \quad \ell \le |P| \le s, \quad \ell \le |P'| \le s$$

Fix an arbitrary subset $U \subset P$ such that $|U| = \ell$, and $U \setminus P' \neq \emptyset$. Note that the size of intersection $|P' \cap U| \leq \ell - 1$.

Consider the set $P' \setminus (P' \cap U)$. Introduce a set $Z, Z \subset [t]$, of size $|Z| = s - (\ell - 1)$, where the intersection $Z \cap U = \emptyset$, as follows.

1. If $|P' \setminus (P' \cap U)| \ge s - (\ell - 1)$, then we choose the set $Z, Z \subseteq P' \setminus (P' \cap U), Z \cap U = \emptyset$, as an arbitrary fixed subset of size $|Z| = s - (\ell - 1)$. Let a row $i, i \in [N]$ corresponds to the pair (U, Z) in Definition 1 of superimposed $(s - \ell + 1, \ell)$ -code X. One can easily see that

$$\sum_{k \in P} x_i(k) \ge \sum_{k \in U} x_i(k) = \ell, \ \sum_{k \in P'} x_i(k) \le |P'| - |Z| \le s - [s - (\ell - 1)] = \ell - 1$$

Hence, $(y_i(P, X) = 1, y_i(P', X) = 0).$

2. If $|P' \setminus (P' \cap U)| < s - (\ell - 1)$, then we choose the set $Z, Z \supset P' \setminus (P' \cap U)$, as an arbitrary fixed superset of size $|Z| = s - (\ell - 1)$. Let a row $i, i \in [N]$ corresponds to the pair (U, Z) in Definition 1 of superimposed $(s - \ell + 1, \ell)$ -code X. One can easily see that

$$\sum_{k \in P} x_i(k) \ge \sum_{k \in U} x_i(k) = \ell, \ \sum_{k \in P'} x_i(k) = |P' \cap U| \le \ell - 1.$$

Hence, $(y_i(P, X) = 1, y_i(P', X) = 0).$

Arguments 1. and 2. imply that code X is a threshold $(\ell, \leq s)$ -design. Therefore, the statement (1) of Proposition 11 is proved.

(2) Let $X = ||x_i(k)||, k \in [t], i \in [N]$, be a threshold $\overline{(\ell, \leq s)}$ -design. Consider two arbitrary non-intersecting sets U and Z, where

$$U \subset [t], \quad |U| = \ell, \quad Z \subset [t], \quad |Z| = s - (\ell - 1), \quad U \cap Z = \emptyset,$$

and fix an element $j \in U$. Introduce subsets $P, P' \in \mathcal{P}(t, \leq s) \setminus \mathcal{P}(t, \leq \ell - 1)$ as follows:

$$P \triangleq U, \quad P' \triangleq (U \setminus j) \cup Z, \quad P \setminus P' \neq \emptyset, \quad |P| = \ell, \quad |P'| = (\ell - 1) + s - (\ell - 1) = s.$$

Definition $\widetilde{\mathbf{5}}$ of threshold $(\ell, \leq s)$ -design means that there exists an index $i \in [N]$ such that

$$\left(y_i(P,X) = 1, \ y_i(P',X) = 0 \right) \Rightarrow \left(\sum_{k \in P} x_i(k) \ge \ell, \ \sum_{k \in P'} x_i(k) \le \ell - 1 \right) \Rightarrow$$
$$\Rightarrow \left(\sum_{k \in U} x_i(k) \ge \ell, \ \sum_{k \in U \setminus j} x_i(k) + \sum_{k \in Z} x_i(k) \le \ell - 1 \right) \Rightarrow$$
$$\Rightarrow x_i(k) = 1, \ k \in U, \ |U| = \ell; \qquad x_i(k) = 0, \ k \in Z, \ |Z| = s - (\ell - 1).$$

Hence, code X is a superimposed $(s - \ell + 1, \ell)$ -code, i.e., statement (2) is established. Proposition 11 is proved.

6 Concluding Remarks

In this Section, we would like to distinguish the principal achievements for the theory of non-adaptive group testing models and superimposed codes obtained in the last decade.

- 1. In 2003, Vladimir Lebedev [26] proved Theorem 3 which established a recurrent inequality for the rate R(z, u) of superimposed (z, u)-codes. This inequality and the best known numerical values [9, 22] of upper bound on the rate R(z, 1) gave the best known numerical values of upper bound on the rate $R(z, u), z \ge u \ge 2$.
- 2. In 2004, Vladimir Lebedev and Hyun Kim [28] presented the best known and optimal constructions (see, Table 4) of superimposed (z, u)-codes, $z \ge u \ge 2$.
- 3. In 2004, Annalisa De Bonis and Ugo Vaccaro [32] proved Theorem 7 which established an upper bound on the rate of superimposed \mathcal{D}_s^{ℓ} -codes via the rate R(z, 1) of superimposed (z, 1)-codes. The result leads to the universal upper bound (Proposition 4) on the rate of group testing designs motivated by compressed genotyping models in molecular biology.
- 4. In 2010, Mahdi Cheraghchi [37] introduced the concepts of threshold $(\ell, \leq s)$ -designs and superimposed \mathcal{M}_s^{ℓ} -codes and proved Proposition 9 which actually established an improved lower bound (19) on the rate of non-adaptive threshold group testing model.

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