# Primitive Words and Lyndon Words in Automatic and Linearly Recurrent Sequences 

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#### Abstract

We investigate questions related to the presence of primitive words and Lyndon words in automatic and linearly recurrent sequences. We show that the Lyndon factorization of a $k$-automatic sequence is itself $k$-automatic. We also show that the function counting the number of primitive factors (resp., Lyndon factors) of length $n$ in a $k$-automatic sequence is $k$-regular. Finally, we show that the number of Lyndon factors of a linearly recurrent sequence is bounded.


## 1 Introduction

We start with some basic definitions. A nonempty word $w$ is called a power if it can be written in the form $w=x^{k}$, for some integer $k \geq 2$. Otherwise $w$ is called primitive. Thus murmur is a power, but murder is primitive. A word $y$ is a factor of a word $w$ if there exist words $x, z$ such that $w=x y z$. If further $x=\epsilon$ (resp., $z=\epsilon$ ), then $y$ is a prefix (resp., suffix) of $w$. A prefix or suffix of a word $w$ is called proper if it is unequal to $w$.

Let $\Sigma$ be an ordered alphabet. We recall the usual definition of lexicographic order on the words in $\Sigma^{*}$. We write $w<x$ if either
(a) $w$ is a proper prefix of $x$; or
(b) there exist words $y, z, z^{\prime}$ and letters $a<b$ such that $w=y a z$ and $x=y b z^{\prime}$.

For example, using the usual ordering of the alphabet, we have common $<$ con $<$ conjugate. As usual, we write $w \leq x$ if $w<x$ or $w=x$.

A word $w$ is a conjugate of a word $x$ if there exist words $u, v$ such that $w=u v$ and $w=v u$. Thus, for example, enlist and listen are conjugates. A word is said to be Lyndon if it is primitive and lexicographically least among all its conjugates. Thus, for example, academy is Lyndon, while googol and googoo are not. A classical theorem is that a finite word is Lyndon if and only if it is lexicographically less than each of its proper suffixes 10 .

We now turn to (right-) infinite words. We write an infinite word in boldface, as $\mathbf{x}=a_{0} a_{1} a_{2} \cdots$ and use indexing starting at 0 . For $i \leq j+1$, we let $[i . . j]$ denote
the set $\{i, i+1, \ldots, j\}$. (If $i=j+1$ we get the empty set.) We let $\mathbf{x}[i . . j]$ denote the word $a_{i} a_{i+1} \cdots a_{j}$. Similarly, $[i . . \infty]$ denotes the infinite set $\{i, i+1, \ldots\}$ and $\mathbf{x}[i . . \infty]$ denotes the infinite word $a_{i} a_{i+1} \cdots$.

An infinite word or sequence $\mathbf{x}=a_{0} a_{1} a_{2} \cdots$ is said to be $k$-automatic if there is a deterministic finite automaton (with outputs associated with the states) that, on input $n$ expressed in base $k$, reaches a state $q$ with output $\tau(q)$ equal to $a_{n}$. For more details, see [5] or [3]. In several previous papers [14|17|19]11], we have developed a technique to show that many properties of automatic sequences are decidable. The fundamental tool is the following:

Theorem 1. Let $P(n)$ be a predicate associated with a $k$-automatic sequence $\mathbf{x}$, expressible using addition, subtraction, comparisons, logical operations, indexing into $\mathbf{x}$, and existential and universal quantifiers. Then there is a computable finite automaton accepting the base-k representations of those $n$ for which $P(n)$ holds. Furthermore, we can decide if $P(n)$ holds for at least one $n$, or for all $n$, or for infinitely many $n$.

If a predicate is constructed as in the previous theorem, we just say it is "expressible". Any expressible predicate is decidable. As an example, we prove

Theorem 2. Let $\mathbf{x}$ be a $k$-automatic sequence. The predicate $P(i, j)$ defined by " $\mathrm{x}[i . . j]$ is primitive" is expressible.

Proof. (due to Luke Schaeffer) It is easy to see that a word is a power if and only if it is equal to some cyclic shift of itself, other than the trivial shift. Thus a word is a power if and only if there is a $d, 0<d<j-i+1$, such that $x[i . . j-d]=x[i+d . . j]$ and $x[j-d+1 . . j]=x[i . . i+d-1]$. A word is primitive if there is no such $d$.

Theorem 3. Let $\mathbf{x}$ be a $k$-automatic sequence. The predicate $L L(i, j, m, n)$ defined by " $\mathbf{x}[i . . j]<\mathbf{x}[m . . n]$ " is expressible.

Proof. We have $\mathbf{x}[i . . j]<\mathbf{x}[m . . n]$ if and only if either
(a) $j-i<n-m$ and $\mathbf{x}[i . . j]=\mathbf{x}[m . . m+j-i]$; or
(b) there exists $t<\min (j-i, n-m)$ such that $\mathbf{x}[i . . i+t]=\mathbf{x}[m . . m+t]$ and $\mathbf{x}[i+t+1]<\mathbf{x}[m+t+1]$.

Theorem 4. Let $\mathbf{x}$ be a $k$-automatic sequence. The predicate $L(i, j)$ defined by " $\mathrm{x}[i . . j]$ is a Lyndon word" is expressible.

Proof. It suffices to check that $\mathbf{x}[i . . j]$ is lexicographically less than each of its proper suffixes, that is, that $L L\left(i, j, i^{\prime}, j\right)$ holds for all $i^{\prime}$ with $i<i^{\prime} \leq j$.

We can extend the definition of lexicographic order to infinite words in the obvious way. We can extend the definition of Lyndon words to (right-) infinite words as follows: an infinite word $\mathbf{x}=a_{0} a_{1} a_{2} \cdots$ is Lyndon if it is lexicographically less than all its suffixes $\mathbf{x}[j . . \infty]=a_{j} a_{j+1} \cdots$ for $j \geq 1$. Then we have the following theorems.

Theorem 5. Let $\mathbf{x}$ be a $k$-automatic sequence. The predicate $L L_{\infty}(i, j)$ defined by " $\mathbf{x}[i . . \infty]<\mathbf{x}[j . . \infty]$ is expressible.

Proof. This is equivalent to $\exists t \geq 0$ such that $\mathbf{x}[i . . i+t-1]=\mathbf{x}[j . . j+t-1]$ and $\mathbf{x}[i+t]<\mathbf{x}[j+t]$.

Theorem 6. Let $\mathbf{x}$ be a $k$-automatic sequence. The predicate $L_{\infty}(i)$ defined by " $\mathrm{x}[i . . \infty]$ is an infinite Lyndon word" is expressible.

Proof. This is equivalent to $L L_{\infty}(i, j)$ holding for all $j>i$.

## 2 Lyndon factorization

Siromoney et al. [15] proved that every infinite word $\mathbf{x}=a_{0} a_{1} a_{2} \cdots$ can be factorized uniquely in exactly one of the following two ways:
(a) as $\mathbf{x}=w_{1} w_{2} w_{3} \cdots$ where each $w_{i}$ is a finite Lyndon word and $w_{1} \geq w_{2} \geq$ $w_{3} \cdots$; or
(b) as $\mathbf{x}=w_{1} w_{2} w_{3} \cdots w_{r} \mathbf{w}$ where $w_{i}$ is a finite Lyndon word for $1 \leq i \leq r$, and $\mathbf{w}$ is an infinite Lyndon word, and $w_{1} \geq w_{2} \geq \cdots \geq w_{r} \geq \mathbf{w}$.

If (a) holds we say that the Lyndon factorization of $\mathbf{x}$ is infinite; otherwise we say it is finite.

Ido and Melançon 1413 gave an explicit description of the Lyndon factorization of the Thue-Morse word $\mathbf{t}$ and the period-doubling sequence (among other things). (Recall that the Thue-Morse word is given by $\mathbf{t}[n]=$ the number of 1's in the binary expansion of $n$, taken modulo 2.) For the Thue-Morse word, this factorization is given by

$$
\mathbf{t}=w_{1} w_{2} w_{3} w_{4} \cdots=(011)(01)(0011)(00101101) \cdots,
$$

where each term in the factorization, after the first, is double the length of the previous. Séébold 18 and Černý generalized these results to other related automatic sequences.

In this section, generalizing the work of Ido, Melançon, Séébold, and Černý, we prove that the Lyndon factorization of a $k$-automatic sequence is itself $k$ automatic. Of course, we need to explain how the factorization is encoded. The easiest and most natural way to do this is to use an infinite word over $\{0,1\}$, where the 1 's indicate the positions where a new term in the factorization begins. Thus the $i$ 'th 1 , for $i \geq 0$, appears at index $\left|w_{1} w_{2} \cdots w_{i}\right|$. For example, for the Thue-Morse word, this encoding is given by

$$
100101000100000001 \cdots .
$$

If the factorization is infinite, then there are infinitely many 1's in its encoding; otherwise there are finitely many 1 's.

In order to prove the theorem, we need a number of results. We draw a distinction between a factor $f$ of $\mathbf{x}$ (which is just a word) and an occurrence of
that factor (which specifies the exact position at which $f$ occurs). For example, in the Thue-Morse word $\mathbf{t}$, the factor 0110 occurs as $\mathbf{x}[0 . .3]$ and $\mathbf{x}[11 . .15]$ and many other places. We call [0..3] and [11..15], and so forth, the occurrences of 0110. An occurrence is said to be Lyndon if the word at that position is Lyndon. We say an occurrence $O_{1}=[i . . j]$ is inside an occurrence $O_{2}=\left[i^{\prime} . . j^{\prime}\right]$ if $i^{\prime} \leq i$ and $j^{\prime} \geq j$. If, in addition, either $i^{\prime}<i$ or $j<j^{\prime}$ (or both), then we say $O_{1}$ is strictly inside $O_{2}$. These definitions are easily extended to the case where $j$ or $j^{\prime}$ are equal to $\infty$, and they correspond to the predicates $I$ (inside) and $S I$ (strictly inside) given below:

$$
\begin{aligned}
I\left(i, j, i^{\prime}, j^{\prime}\right) \text { is } & i^{\prime} \leq i \text { and } j^{\prime} \geq j \\
S I\left(i, j, i^{\prime}, j^{\prime}\right) \text { is } & I\left(i, j, i^{\prime}, j^{\prime}\right) \text { and }\left(\left(i^{\prime}<i\right) \text { or }\left(j^{\prime}>j\right)\right)
\end{aligned}
$$

An infinite Lyndon factorization

$$
\mathbf{x}=w_{1} w_{2} w_{3} \cdots
$$

then corresponds to an infinite sequence of occurrences

$$
\left[i_{1} . . j_{1}\right],\left[i_{2} . . j_{2}\right], \cdots
$$

where $w_{n}=\mathbf{x}\left[i_{n} . . j_{n}\right]$ and $i_{n+1}=j_{n}+1$ for $n \geq 1$, while a finite Lyndon factorization

$$
\mathbf{x}=w_{1} w_{2} \cdots w_{r} \mathbf{w}
$$

corresponds to a finite sequence of occurrences

$$
\left[i_{1} . . j_{1}\right],\left[i_{2} . . j_{2}\right], \ldots,\left[i_{r} . . j_{r}\right],\left[i_{r+1} . . \infty\right]
$$

where $w_{n}=\mathbf{x}\left[i_{n} . . j_{n}\right]$ and $i_{n+1}=j_{n}+1$ for $1 \leq n \leq r$.
Theorem 7. Let $\mathbf{x}$ be an infinite word. Every Lyndon occurrence in $\mathbf{x}$ appears inside a term of the Lyndon factorization of $\mathbf{x}$.

Proof. We prove the result for infinite Lyndon factorizations; the result for finite factorizations is exactly analogous.

Suppose the factorization is $\mathbf{x}=w_{1} w_{2} w_{3} \cdots$. It suffices to show that no Lyndon occurrence can span the boundary between two terms of the factorization. Suppose, contrary to what we want to prove, that $u w_{i} w_{i+1} \cdots w_{j} v$ is a Lyndon word for some $u$ that is a nonempty suffix of $w_{i-1}$ (possibly equal to $w_{i-1}$ ), and $v$ that is a nonempty prefix of $w_{j+1}$ (possibly equal to $w_{j+1}$ ), and and $i \leq j+1$. (If $i=j+1$ then there are no $w_{i}$ 's at all between $u$ and $v$.)

Since $u$ is a suffix of $w_{i-1}$ and $w_{i-1}$ is Lyndon, we have $u \geq w_{i-1}$. On the other hand, by the Lyndon factorization definition we have $w_{i-1} \geq w_{i} \geq \cdots \geq$ $w_{j} \geq w_{j+1}$. But $v$ is a prefix of $w_{j+1}$, so just by the definition of lexicographic ordering we have $w_{j+1} \geq v$. Putting this all together we get $u \geq v$. So $u x \geq v$ for all words $x$.

On the other hand, since $u w_{i} \cdots w_{j} v$ is Lyndon, it must be lexicographically less than any proper suffix - for instance, $v$. So $u w_{i} \cdots w_{j} v<v$. Take $x=$ $w_{i} \cdots w_{j} v$ to get a contradiction with the conclusion in the previous paragraph.

Corollary 1. The occurrence $[i . . j]$ corresponds to a term in the Lyndon factorization of $\mathbf{x}$ if and only if
(a) $[i . . j]$ is Lyndon; and
(b) $[i . . j]$ does not occur strictly inside any other Lyndon occurrence.

Proof. Suppose $[i . . j]$ corresponds to a term $w_{n}$ in the Lyndon factorization of $\mathbf{x}$. Then evidently [i..j] is Lyndon. If it occurred strictly inside some other Lyndon occurrence, say $\left[i^{\prime} . . j^{\prime}\right]$, then we know from Theorem 7 that $\left[i^{\prime} . . j^{\prime}\right]$ itself lies in inside some $w_{m}$, so $[i . . j]$ must lie strictly inside $w_{m}$, which is clearly impossible.

Now suppose [i..j] is Lyndon and does not occur strictly inside any other Lyndon occurrence. From Theorem 7 [i..j] must occur inside some term of the factorization $\left[i^{\prime} . . j^{\prime}\right]$. If $[i . . j] \neq\left[i^{\prime} . . j^{\prime}\right]$ then $[i . . j]$ lies strictly inside $\left[i^{\prime} . . j^{\prime}\right]$, a contradiction. So $[i . . j]=\left[i^{\prime} . . j^{\prime}\right]$ and hence corresponds to a term of the factorization.

Corollary 2. The predicate $L F(i, j)$ defined by " $i . . j]$ corresponds to a term of the Lyndon factorization of $\mathbf{x}$ " is expressible.
Proof. Indeed, by Corollary 1 the predicate $L F(i, j)$ can be defined by

$$
L(i, j) \text { and } \forall i^{\prime}, j^{\prime}\left(S I\left(i, j, i^{\prime}, j^{\prime}\right) \Longrightarrow \neg L\left(i^{\prime}, j^{\prime}\right)\right) .
$$

We can now prove the main result of this section.
Theorem 8. Using the encoding mentioned above, the Lyndon factorization of a $k$-automatic sequence is itself $k$-automatic.

Proof. Using the technique of [1] we can create an automaton that on input $i$ expressed in base $k$, guesses $j$ and checks if $L F(i, j)$ holds. If so, it outputs 1 and otherwise 0 . To get the last $i$ in the case that the Lyndon factorization is finite, we also accept $i$ if $L_{\infty}(i)$ holds.

We also have
Theorem 9. Let $\mathbf{x}$ be a $k$-automatic sequence. It is decidable if the Lyndon factorization of $\mathbf{x}$ is finite or infinite.

Proof. The construction given above in the proof of Theorem 8 produces an automaton that accepts finitely many distinct $i$ (expressed in base $k$ ) if and only if the Lyndon factorization of $\mathbf{x}$ is finite.

We programmed up our method and found the Lyndon factorization of the Thue-Morse sequence $\mathbf{t}$, the period-doubling sequence $\mathbf{d}$, the paperfolding sequence $\mathbf{p}$, and the Rudin-Shapiro sequence $\mathbf{r}$, and their negations. (The results for Thue-Morse and the period-doubling sequence were already given in [13], albeit in a different form.) Recall that the period-doubling sequence is defined by $\mathbf{p}[n]=|\mathbf{t}[n+1]-\mathbf{t}[n]|$. The paperfolding sequence $\mathbf{p}=0010011 \cdots$ arises from the limit of the sequence $\left(f_{n}\right)$, where $f_{0}=0$ and $f_{n+1}=f_{n} 0 \bar{f}_{n}^{R}$, where $R$ denotes reversal and $\bar{x}$ maps 0 to 1 and 1 to 0 . Finally, the Rudin-Shapiro sequence $\mathbf{r}$ is defined by $\mathbf{r}[n]=$ the number of (possibly overlapping) occurrences of 11 in the binary expansion of $n$, taken modulo 2 . The results are given in the theorem below.

Theorem 10. The occurrences corresponding to the Lyndon factorization of each word is as follows:

- the Thue-Morse sequence $\mathbf{t}:[0 . .2],[3 . .4],[5 . .8],[9 . .16],[17 . .32], \ldots,\left[2^{i}+1 . .2^{i+1}\right], \ldots$;
- the negated Thue-Morse sequence $\overline{\mathbf{t}}:[0 . .0],[1 . . \infty]$;
- the Rudin-Shapiro sequence $\mathbf{r}:[0 . .6],[7 . .14],[15 . .30], \ldots,\left[2^{i}-1 . .2^{i+1}-2\right], \ldots$;
- the negated Rudin-Shapiro sequence $\overline{\mathbf{r}}:[0 . .0],[1 . .1],[2 . .2],[3 . .10],[11 . .42],[43 . .46],[47 . .174], \ldots,\left[4^{i}-\right.$ $\left.4^{i-1}-4^{i-2}-1 . .4^{i}-4^{i-1}-2\right],\left[4^{i}-4^{i-1}-1 . .4^{i+1}-4^{i}-4^{i-1}-1\right], \ldots ;$
- the paperfolding sequence $\mathbf{p}:[0 . .6],[7 . .14],[15 . .30], \ldots,\left[2^{i}-1 . .2^{i+1}-2\right], \ldots$;
- the negated paperfolding sequence $\overline{\mathbf{p}}:[0 . .0],[1 . .1],[2 . .4],[5 . .9],[10 . .20],[21 . .84],[85 . .340], \ldots,\left[\left(4^{i}-\right.\right.$ 1) $\left./ 3 . .4\left(4^{i}-1\right) / 3\right], \ldots$;
- the period-doubling sequence $\mathbf{d}:[0 . .0],[1 . .4],[5 . .20],[21 . .84], \ldots,\left[\left(4^{i}-1\right) / 3 . .4\left(4^{i}-\right.\right.$ 1)/3] , ..;
- the negated period-doubling sequence $\overline{\mathbf{d}}:[0 . .1],[2 . .9],[10 . .41],[42 . .169], \ldots,\left[2\left(4^{i}-\right.\right.$ 1) $\left./ 3 . .2\left(4^{i+1}-1\right) / 3-1\right], \ldots$


## 3 Enumeration

There is a useful generalization of $k$-automatic sequences to sequences over $\mathbb{N}$, the non-negative integers. A sequence $\left(a_{n}\right)_{n \geq 0}$ over $\mathbb{N}$ is called $k$-regular if there exist vectors $u$ and $v$ and a matrix-valued morphism $\mu$ such that $a_{n}=u \mu(w) v$, where $w$ is the base- $k$ representation of $n$. For more details, see [2].

The subword complexity function $\rho(n)$ of an infinite sequence $\mathbf{x}$ counts the number of distinct length- $n$ factors of $\mathbf{x}$. There are also many variations, such as counting the number of palindromic factors or unbordered factors. If $\mathbf{x}$ is $k$-automatic, then all three of these are $k$-regular sequences [1]. We now show that the same result holds for the number $\rho_{\mathbf{x}}^{P}(n)$ of primitive factors and for the number $\rho_{\mathbf{x}}^{L}$ of Lyndon factors. We refer to these two quantities as the "primitive complexity" and "Lyndon complexity", respectively.

Theorem 11. The function counting the number of length-n primitive (resp., Lyndon) factors of a $k$-automatic sequence $\mathbf{x}$ is $k$-regular.

Proof. By the results of [4, it suffices to show that there is an automaton accepting the base- $k$ representations of pairs $(n, i)$ such that the number of $i$ 's associated with each $n$ equals the number of primitive (resp., Lyndon) factors of length $n$.

To do so, it suffices to show that the predicate $P(n, i)$ defined by "the factor of length $n$ beginning at position $i$ is primitive (resp., Lyndon) and is the first occurrence of that factor in $\mathbf{x "}$ is expressible. This is just

$$
P(i, i+n-1) \quad \text { and } \quad \forall j<i \mathbf{x}[i . . i+n-1] \neq \mathbf{x}[j . . j+n-1],
$$

(resp.,

$$
L(i, i+n-1) \quad \text { and } \quad \forall j<i \mathbf{x}[i . . i+n-1] \neq \mathbf{x}[j . . j+n-1]) .
$$

We used our method to compute these sequences for the Thue-Morse sequence, and the results are given below.

Theorem 12. Let $\rho_{\mathbf{t}}^{L}(n)$ denote the number of Lyndon factors of length $n$ of the Thue-Morse sequence. Then

$$
\rho_{\mathbf{t}}^{L}(n)= \begin{cases}1, & \text { if } n=2^{k} \text { or } 5 \cdot 2^{k} \text { for } k \geq 1 \\ 2, & \text { if } n=1 \text { or } n=5 \text { or } n=3 \cdot 2^{k} \text { for } k \geq 0 \\ 0, & \text { otherwise } .\end{cases}
$$

Theorem 13. Let $\rho_{\mathbf{t}}^{P}(n)$ denote the number of primitive factors of length $n$ of the Thue-Morse sequence. Then

$$
\rho_{\mathbf{t}}^{P}(n)= \begin{cases}3 \cdot 2^{t}-4, & \text { if } n=2^{t} \\ 4 n-2^{t}-4, & \text { if } 2^{t}+1 \leq n<3 \cdot 2^{t-1} \\ 5 \cdot 2^{t}-6, & \text { if } n=3 \cdot 2^{t-1} \\ 2 n+2^{t+1}-2, & \text { if } 3 \cdot 2^{t-1}<n<2^{t+1}\end{cases}
$$

We can also state a similar result for the Rudin-Shapiro sequence.
Theorem 14. Let $\rho_{\mathbf{r}}^{L}(n)$ denote the Lyndon complexity of the Rudin-Shapiro sequence. Then $\rho_{\mathbf{r}}^{L}(n) \leq 8$ for all $n$. This sequence is 2 -automatic and there is an automaton of 2444 states that generates it.

Proof. The proof was carried out by machine computation, and we briefly summarize how it was done.

First, we created an automaton $A$ to accept all pairs of integers $(n, i)$, represented in base 2 , such that the factor of length $n$ in $\mathbf{r}$, starting at position $i$, is a Lyndon factor, and is the first occurrence of that factor in $\mathbf{r}$. Thus, the number of distinct integers $i$ associated with each $n$ is $\rho_{\mathbf{r}}^{L}(n)$. The automaton $A$ has 102 states.

Using the techniques in [4], we then used $A$ to create matrices $M_{0}$ and $M_{1}$ of dimension $102 \times 102$, and vectors $v, w$ such that $v M_{x} w=\rho_{\mathbf{r}}^{L}(n)$, if $x$ is the base- 2 representation of $n$. Here if $x=a_{1} a_{2} \cdots a_{i}$, then by $M_{x}$ we mean the product $M_{a_{1}} M_{a_{2}} \cdots M_{a_{i}}$.

From this we then created a new automaton $A^{\prime}$ where the states are products of the form $v M_{x}$ for binary strings $x$ and the transitions are on 0 and 1 . This automaton was built using a breadth-first approach, using a queue to hold states whose targets on 0 and 1 are not yet known. From Theorem 18 in the next section, we know that $\rho_{\mathbf{r}}^{L}(n)$ is bounded, so that this approach must terminate. It did so at 2444 states, and the product of the $v M_{x}$ corresponding to each state with $w$ gives an integer less than or equal to 8 , thus proving the desired result and also providing an automaton to compute $\rho_{\mathbf{r}}^{L}(n)$.

Remark 1. Note that the Lyndon complexity functions in Theorems 12 and 14 are bounded. This will follow more generally from Theorem 18 below.

## 4 Finite factorizations

Of course, the original Lyndon factorization was for finite words: every finite nonempty word $x$ can be factored uniquely as a nonincreasing product $w_{1} w_{2} \cdots w_{m}$ of Lyndon words. We can apply this theorem to all prefixes of a $k$-automatic sequence. It is then natural to wonder if a single automaton can encode all the Lyndon factorizations of all finite prefixes. The answer is yes, as the following result shows.

Theorem 15. Suppose $\mathbf{x}$ is a $k$-automatic sequence. Then there is an automaton A accepting

$$
\begin{aligned}
& \left\{(n, i)_{k}: \text { the Lyndon factorization of } \mathbf{x}[0 . . n-1] \text { is } w_{1} w_{2} \cdots w_{m}\right. \\
& \left.\quad \text { with } w_{m}=\mathbf{x}[i . . n-1]\right\}
\end{aligned}
$$

Proof. As is well-known [10], if $w_{1} w_{2} \cdots w_{m}$ is the Lyndon factorization of $x$, then $w_{m}$ is the lexicographically least suffix of $x$. So to accept $(n, i)_{k}$ we find $i$ such that $\mathbf{x}[i . . n-1]<\mathbf{x}[j . . n-1]$ for $0 \leq j<n$ and $i \neq j$.

Given $A$, we can find the complete factorization of any prefix $\mathbf{x}[0 . . n-1]$ by using this automaton to find the appropriate $i$ (as described in [12]) and then replacing $n$ with $i$.

We carried out this construction for the Thue-Morse sequence, and the result is shown below in Figure 4

In a similar manner, there is an automaton that encodes the factorization of every factor of a $k$-automatic sequence:

Theorem 16. Suppose $\mathbf{x}$ is a $k$-automatic sequence. Then there is an automaton $A^{\prime}$ accepting

$$
\begin{aligned}
& \left\{(i, j, l)_{k}: \text { the Lyndon factorization of } \mathbf{x}[i . . j-1] \text { is } w_{1} w_{2} \cdots w_{m}\right. \\
& \left.\quad \text { with } w_{m}=\mathbf{x}[l . . n-1]\right\}
\end{aligned}
$$

We calculated $A^{\prime}$ for the Thue-Morse sequence using our method. It is a 34-state machine and is displayed in Figure 4.


Fig. 1. A finite automaton accepting the base-2 representation of $(n, i)$ such that the Lyndon factorization of $\mathbf{t}[0 . . n-1]$ ends in the term $\mathbf{t}[i . . n-1]$


Fig. 2. A finite automaton accepting the base-2 representation of $(i, j, l)$ such that the Lyndon factorization of $\mathbf{t}[i . . j-1]$ ends in the term $\mathbf{t}[l . . j-1]$

Another quantity of interest is the number of terms in the Lyndon factorization of each prefix.

Theorem 17. Let $x$ be a $k$-automatic sequence. Then the sequence $(f(n))_{n \geq 0}$ defined by

$$
f(n)=\text { the number of terms in the Lyndon factorization of } \mathbf{x}[0 . . n]
$$

is $k$-regular.
Proof. We construct an automaton to accept
$\left\{(n, i): \exists j \leq n\right.$ such that $L(i, j)$ and if $S I\left(i, j, i^{\prime}, j^{\prime}\right)$ and $0 \leq i^{\prime} \leq j^{\prime} \leq n$ then $\left.\neg L\left(i^{\prime}, j^{\prime}\right)\right\}$.
For the Thue-Morse sequence the corresponding sequence satisfies the relations

$$
\begin{aligned}
f(4 n+1) & =-f(2 n)+f(2 n+1)+f(4 n) \\
f(8 n+2) & =-f(2 n)+f(4 n)+f(4 n+2) \\
f(8 n+3) & =-f(2 n)+f(4 n)+f(4 n+3) \\
f(8 n+6) & =-f(2 n)-f(4 n+2)+3 f(4 n+3) \\
f(8 n+7) & =-f(2 n)+2 f(4 n+3) \\
f(16 n) & =-f(2 n)+f(4 n)+f(8 n) \\
f(16 n+4) & =-f(2 n)+f(4 n)+f(8 n+4) \\
f(16 n+8) & =-f(2 n)+f(4 n+3)+f(8 n+4) \\
f(16 n+12) & =-f(2 n)-2 f(4 n+2)+3 f(4 n+3)+f(8 n+4)
\end{aligned}
$$

for $n \geq 1$, which allows efficient calculation of this quantity.

## 5 Linearly recurrent sequences

Definition 1. A recurrent infinite word $\mathbf{x}=a_{0} a_{1} a_{2} \cdots$, where each $a_{i}$ is $a$ letter, is called linearly recurrent with constant $L>0$ if, for every factor $u$ and its two consecutive occurrences beginning at positions $i$ and $j$ in $\mathbf{x}$ with $i<j$, we have $j-i<L|u|$. The word $a_{i} a_{i+1} \cdots a_{j-1}$ is called a return word of $u$. Thus linear recurrence can be defined from the condition that every return word $w$ of every factor $u$ of $\mathbf{x}$ satisfy $|w|<L|u|$. Let $\mathcal{R}_{u}$ denote the set of return words of $u$ in $\mathbf{x}$.

Remark 2. Linear recurrence implies that every length- $k$ factor appears at least once in every factor of length $(L+1) k-1$.

Lemma 1 (Durand, Host, and Skau [8]). Let $\mathbf{x}$ be an aperiodic linearly recurrent word with constant $L$.
(i) If $u$ is a factor of $\mathbf{x}$ and $w$ its return word, then $|w|>|u| / L$.
(ii) The number of return words of any given factor $u$ of $\mathbf{x}$ is $\# \mathcal{R}_{u} \leq L(L+1)^{2}$.

Theorem 18. The Lyndon complexity of any linearly recurrent sequence is bounded.

Proof. Let $\mathbf{x}$ be a linearly recurrent sequence with constant $L$. If $\mathbf{x}$ is ultimately periodic, it is purely periodic because it is recurrent, and thus its Lyndon complexity is bounded. Therefore assume that $\mathbf{x}$ is aperiodic, and let $n \geq L$. Denote $k=\lfloor(n+1) /(L+1)\rfloor$, so that

$$
\begin{equation*}
(L+1) k-1 \leq n<(L+1)(k+1)-1 . \tag{1}
\end{equation*}
$$

The left-hand side inequality in (11) and Remark 2 together imply that all factors in $\mathbf{x}$ of length $k$ occur in all factors of length $n$. Therefore if $u$ is the lexicographically smallest factor of length $k$, then every Lyndon factor of $\mathbf{x}$ of length $n$ must begin with $u$. Since every suffix of $\mathbf{x}$ that begins with $u$ can be factorized over $\mathcal{R}_{u}$, we conclude further that every length- $n$ Lyndon factor of $\mathbf{x}$ is a prefix of a word in $\mathcal{R}_{u}^{*}$.

The return words of $u$ have length at least $k / L$ by Lemma 1. Furthermore, the right-hand side inequality in (11) gives

$$
\frac{n}{k / L}<\frac{(L+1)(k+1)-1}{k / L}<\frac{L(L+1)(k+1)}{k} \leq 2 L(L+1)
$$

Therefore any Lyndon factor of length $n$ is a prefix of a word in $\mathcal{R}_{u}^{2 L(L+1)}$. Since $\# \mathcal{R}_{u} \leq L(L+1)^{2}$ by Lemma we conclude that

$$
\rho_{\mathbf{x}}^{L}(n) \leq \max \left\{\rho_{\mathbf{x}}^{L}(1), \rho_{\mathbf{x}}^{L}(2), \ldots, \rho_{\mathbf{x}}^{L}(L-1), L(L+1)^{4 L(L+1)}\right\}
$$

so that the Lyndon complexity of $\mathbf{x}$ is bounded.
Definition 2. Let $h: \mathcal{A}^{*} \rightarrow \mathcal{A}^{*}$ be a primitive morphism, and let $\tau: \mathcal{A} \rightarrow \mathcal{B}$ be a letter-to-letter morphism. If $h$ is prolongable, so that the limit $h^{\omega}(a):=$ $\lim _{n \rightarrow \infty} h^{n}(a)$ exists for some letter $a \in \mathcal{A}$, then the sequence $\tau\left(h^{\omega}(a)\right)$ is called primitive morphic.

Lemma 2 (Durand [7,8]). Primitive morphic sequences are linearly recurrent.
Corollary 3. The Lyndon complexity of any primitive morphic sequence is bounded.
Proof. Follows from Lemma 2 and Theorem 18 ,
Corollary 4. If $\mathbf{x}$ is $k$-automatic and primitive morphic, then its Lyndon complexity is $k$-automatic.

Proof. Follows from Corollary 3 and Theorem 11, because a $k$-regular sequence over a finite alphabet is $k$-automatic [2].

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