

## Flooding edge or node weighted graphs

#### Fernand Meyer

Centre de Morphologie Mathématique

22 March 2013

Fernand Meyer (Centre de Morphologie Math Flooding edge or node weighted graphs

Floodings are useful for:

- filtering images
- suppressing regional minima and filling them by lakes -> regularization and control of the watershed segmentation

Combined with the dual operator, the razings, they permit to construct powerful autodual filters (alternate sequential, flattenings and levelings) We flood two types of graphs:

- node weighted graphs with a ground level on the nodes
- edge weighted graph without ground level on the nodes The flooding assigns a flooding to each node.
- We characterize valid floodings on both types of graphs: lakes, regional minima lakes.
- We show that flooding a node weighted graph is equivalent with flooding an edge weighted graph with appropriate edge weights

We then introduce dominated floodings under a ceiling function. We present two classes of algorithms :

- shortest path algorithms for the ultrametric flooding distance
- direct construction of the flooding on the dendrogram of the closed balls of the flooding distance

#### Reminders on graphs

3

A non oriented graph G = [N, E]: N =nodes ; E =,edges ; an edge  $u \in E =$  a pair of vertices

A chain of length n is a sequence of n edges  $L = \{e_1, e_2, ..., e_n\}$ , with successive edges having a common node.

A *path* between two nodes x and y is a sequence of nodes

 $(n_1 = x, n_2, ..., n_k = y)$  with successive nodes linked by an edge.

A cocycle is the set of all edges with one extremity in a subset Y and the other in the complementary set  $\overline{Y}$ .

The subgraph spanning a set  $A \subset N$  is the graph  $G_A = [A, E_A]$ , where  $E_A$  are the edges linking two nodes of A.

The partial graph associated to the edges  $E' \subset E$  is G' = [N, E'].

In a graph G = [N, E], edges and nodes may be weighted :

- $e_{ij}$  is the weight of the edge (i, j)
- n<sub>i</sub> the weight of the node i. The weights take their value in a completely ordered lattice T.

A subgraph G' of a node weighted graph G is a flat zone, if any two nodes of G' are connected by a path where all nodes have the same altitude. A subgraph G' of a graph G is a regional minimum if G' is a flat zone and all neighboring nodes have a higher altitude

#### Distances on a graph

Case of edge weighed graphs

Distances on an edge weighted graph have chains as support :

- 1) Definition of the weight of a chain, as a measure derived from the edge weights of the chain elements (example : sum, maximum, etc.)
- 2) Comparison of two chains by their weight. The chain with the smallest weight is called the shortest.
- The distance d(x, y) between two nodes x and y of a graph is  $\infty$  if there is no chain linking these two nodes and equal to the weight of the shortest chain if such a chain exists.
- Given three nodes (x, y, z) the concatenation of the shortest chain  $\pi_{xy}$  between x and y and the shortest chain  $\pi_{yz}$  between y and z is a chain  $\pi_{xz}$  between x and z, whose weight is smaller or equal to the weight of the shortest chain between x and z. To each distance corresponds a particular triangular inequality :  $d(x, z) \leq weight(\pi_{xy} \triangleright \pi_{yz})$  where  $\pi_{xy} \triangleright \pi_{yz}$  represents the concatenation of both chains.

**Length of a chain:** The length of a chain between two nodes *x* and *y* is defined as the sum of the weights of its edges.

**Distance:** The distance d(x, y) between two nodes x and y is the minimal length of all chains between x and y. If there is no chain between them, the distance is equal to  $\infty$ .

**Triangular inequality :** For  $(x, y, z) : d(x, z) \le d(x, y) + d(y, z)$ 

The weights are assigned to the edges, and represent their altitudes. **Altitude of a chain:** The altitude of a chain is equal to the highest weight of the edges along the chain.

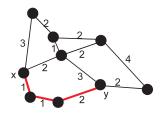
**Flooding distance between two nodes:** The flooding distance fldist(x, y) between nodes x and y is equal to the minimal altitude of all chains between x and y. During a flooding process, in which a source is placed at location x, the flood would proceed along this chain of minimal highest altitude to reach the pixel y. If there is no chain between them, the level distance is equal to  $\infty$ .

**Triangular inequality :** For  $(x, y, z) : d(x, z) \le d(x, y) \lor d(y, z) :$ ultrametric inequality An ultrametric distance verifies

- \* reflexivity : d(x, x) = 0
- \* symmetry: d(x, y) = d(y, x)

\* ultrametric inequality: for all  $x, y, z : d(x, y) \le max\{d(x, z), d(z, y)\}$ : the lowest lake containing both x and y is lower or equal than the lowest lake containing x, y and z.

## Distances on a graph : sum and maximum of the edge weights



The shortest chain (sum of weights of the edges) between x and y is a red line and has a length of 4.

The lowest chain (maximal weight of the edges) between x and y is a red line and a maximal weight of 2. A flooding between x and y would follow this chain.

Flooding a topographic surface or flooding a graph

### The region adjacency graph

We will work with "neighborhood graphs" where the nodes are the catchment basins and the edges connect neighboring bassins. The edges are weighted by a dissimilarity measure between adjacent catchment basins; the simplest being the altitude of the pass-point between two basins.

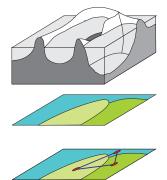


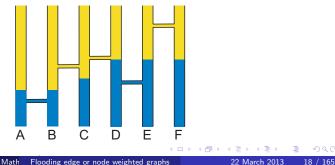
Figure: The region adjacency graph of a topographic surface

An image may be considered as a topographic surface. The altitude of each pixel corresponds to its gray level. An image may be modelled by a graph, the nodes being the pixels and the edges connecting neighboring pixels. A first weight distribution f represents the ground level. For a flooded surface, the nodes hold a second weight  $\tau \ge f$  equal to the flooding level.

The edges are not weighted.

## Representation of a flooded RAG as an edge weighted graph

A physical interpretation of a flooded RAG: the nodes are tanks with infinite height and depth, their weight represent the height of the flooding in the tank. If two nodes are connected by a weighted edge, the corresponding tanks are linked by a pipe at an altitude of the weight. The pipes allow the water to pass from tank to tank, according the laws of hydrostatics. We call such a graph tank network (TN). The level in each tank is indicated in blue.



# Flooding a topographic surface of its region adjacency graph

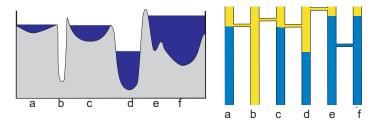


Figure: Flooding a topographic surface or flooding its region adjacency graph.

The flooding of a topographic surface is perfectly defined if one knows the flooding level in or above each catchment basin. The same flooding may be represented on the region adjacency graph by assigning to each node the flooding level in the corresponding basin.

## Modelling the laws of hydrostatics in node and edge weighted tanks

As flooding a topographic surface and flooding its RAG represent the same phenomenon, we have to find two models, one for node weighted graphs, the other for edge weighted graphs, expressing this same phenomenon

## Flooding a topographic surface or nodes weighted graph

#### Definition

A function g is a flooding of a function f if and only if  $g \ge f$  and for any couple of neighboring pixels  $(p, q) : g_p > g_q \Rightarrow g_p = f_p$ 

Fig. 4A presents a physically possible flooding. On the contrary the flooding in fig. 4B is impossible, as the lake containing the pixel p where  $g_p > f_p$  is not limited by solid ground since  $g_p > g_q$ .

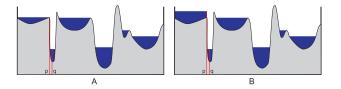


Figure: A possible and an impossible flooding

Images are particular node weighted graphs, the pixels being the nodes. Neighboring pixels are linked by an unweighted edge. We now consider arbitrary node weighted graphs. The node weights  $f_i$  indicate the ground level. The edges are not weighted. Such a topographic graph is flooded if the nodes are assigned a second family of weights indicating the level of the flooding at each node.

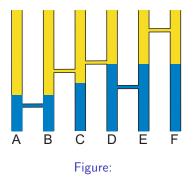
A distribution  $\tau$  of node weights will represent an effective flooding if it verifies a number of conditions of equilibrium:

- A flooding being always above the ground level:  $\tau_i \ge f_i$ .
- As there is nothing to prevent the water to flow from a higher to a lower position, an inequal level of water at two neighboring nodes p and q is impossible, except when the highest node is dry; hence τ<sub>p</sub> > τ<sub>q</sub> ⇒ τ<sub>p</sub> = f<sub>p</sub> indicating that the highest level is dry, without water.
- Consequence 1: in a lake, the level of all nodes is the same.
- Consequence 2: floodings are connected operators :  $f_p = f_q \Rightarrow \tau_p = \tau_q$

э

G = [E, N]: a node and edge weighted graph, E = edges, N = nodes. The edges are weighted: the weight  $e_{ij}$  of the edge (i, j) represents the altitude at which a flood coming from one extremity may reach the other extremity of the edge.

The nodes also are weighted;  $\tau_i$  represents the altitude of the flood at node *i*.

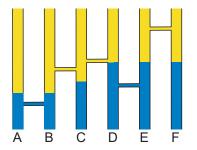


We consider the nodes as vertical tanks of infinite height and depth : there is no ground level.

The weight  $\tau_i$  represents the level of water in the tank *i*.

Two neighboring tanks *i* and *j* are linked by a pipe at an altitude  $e_{ij}$  equal to the weight of the edge.

We call such an edge weighted graph a tank network.



Laws of hydrostatics:

- if the level τ<sub>i</sub> in the tank i is higher than the pipe e<sub>ij</sub>, then the levels is the same in both tanks i and j: τ<sub>i</sub> = τ<sub>j</sub>.
- the level  $\tau_i$  in the tank *i* cannot be higher than the level  $\tau_j$ , unless  $e_{ij} \ge \tau_i$ .

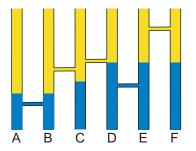


Figure: Tank and pipe network:

- A and B form a regional minimum with  $\tau_A = \tau_B = \lambda$ ;  $e_{AB} \leq \lambda$ ;  $e_{BC} > \lambda$
- B and C have unequal levels but are separated by a higher pipe.
- D and E form a full lake, reaching the level of its lowest exhaust pipe  $e_{CD}$
- E and F have the same level ; however they do not form a lake, as they are linked by a pipe which is higher

#### Definition

The distribution  $\tau$  of water in the pipes of the graph [E, N] is a flooding of this graph, i.e. is a stable distribution of fluid if it verifies the criterion: for any couple of neighboring nodes (p, q) we have:  $(\tau_p > \tau_q \Rightarrow e_{pq} \ge \tau_p)$  (criterion 1)

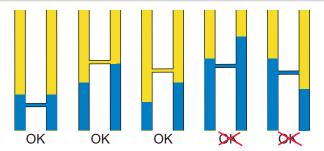


Figure: The water distribution marked OK are compatible with the laws of physics ; the others are not.

$$\begin{array}{l} (\tau_{p} > \tau_{q} \Rightarrow e_{pq} \geq \tau_{p}) \Leftrightarrow (\mathsf{not} \ (\tau_{p} > \tau_{q}) \ \mathsf{or} \ e_{pq} \geq \tau_{p}) \Leftrightarrow (\tau_{p} \leq \tau_{q} \ \mathsf{or} \\ \tau_{p} \leq e_{pq}) \Leftrightarrow \\ (\tau_{p} \leq \tau_{q} \lor e_{pq}) \quad (\textit{criterion 2}) \\ \Leftrightarrow (\tau_{p} \leq \tau_{q} \lor \bigwedge (\tau_{q} \lor e_{pq}) \quad (\textit{criterion 3}) \\ (p,q) \ \mathsf{neighbors} \end{array}$$

#### Remark

The criterion  $(\tau_p > \tau_q \Rightarrow e_{pq} \ge \tau_p)$  is equivalent with  $(e_{pq} < \tau_p \Rightarrow \tau_p \le \tau_q)$ . Hence if  $e_{pq} < \tau_p$ , we have  $\tau_p \le \tau_q$ ; so we also have  $e_{pq} < \tau_q$  implying  $\tau_q \le \tau_p$ ; finally  $\tau_p = \tau_q$ .

Flooding a topographic graph is the same as flooding an associated edge weighted graph

 $G_n = [E, N]$ : a topographic graph. Ground level = f. The edges are not weighted.

The lowest level of flood covering two neighboring nodes p and q is equal to  $f_p \vee f_q$ .

Consider now a second graph  $G_e$  with the same structure but with edge weights  $e_{pq} = f_p \vee f_q$ .

Any flooding  $\tau \ge f$  of  $G_e$  verifies: for (p, q) neighbors

 $(\tau_p \leq \tau_q \vee e_{pq}) \Leftrightarrow (\tau_p \leq \tau_q \vee f_q \vee f_p) \Leftrightarrow (\tau_p \leq \tau_q \vee f_p) \text{ as } \tau_q \geq f_q$ But this last criterion characterizes a flooding of  $G_n$ .

#### Theorem

There is an equivalence between the floodings  $\tau \ge f$  of  $G_e$  of  $G_n$  and the floodings  $\tau \ge f$  of  $G_e$ , with edge weights  $e_{pq} = f_p \lor f_q$ .

・ロト ・聞 ト ・ 国 ト ・ 国 ト …

#### Lakes of edge weighted graphs

э

If a node p has no neighboring node q such that  $\tau_p = \tau_q$ , then p is an isolated node and isolated lake.

Consider now two neighboring nodes p and q verifying  $\tau_p = \tau_q$ . Adding a drop of water at the node p has no impact on node q, if there exists no path linking p and q with edge weights  $\leq \tau_p, \tau_q$ . We define a binary relation between neighboring pixels

 $p, q: p \sim q \Leftrightarrow \tau_p = \tau_q \text{ and } e_{pq} \leq \tau_p, \tau_q.$ 

#### Lemma

If we cut all edges which do not verify  $p \sim q$ , we get a partial graph G; the connected components of  $\widetilde{G}$  are the lakes of the graph G.

#### Lakes of node weighted graphs

э

Consider now a topographic graph  $G_n$  with a ground level f and its derived tank network  $G_e$  with edge weights  $e_{pq} = f_p \lor f_q$ . Any flooding of  $G_n$  also is a flooding of  $G_e$ . Applying the definition of lakes given above we distinguish two cases:

- p is an isolated node : it has no neighboring node q such that  $au_p = au_q$
- p is not isolated, and has at least one neighbor q such that  $\tau_p = \tau_q$ . As  $\tau \ge f$ , we have  $\tau_p = \tau_q = \tau_p \lor \tau_q \ge f_p \lor f_q = e_{pq}$ . This shows that  $\tau_p = \tau_q \Rightarrow p \sim q$ . This shows that the lakes of  $G_n$  simply are its flat zones.

#### Definition

The lakes of a TG are its flat zones, that is maximal connected components of nodes with the same altitude.

イロト イポト イヨト イヨト

A lake on a topographic graph is dry if it has a uniform altitude at the ground level. It is a wet lake, if it contains at least one pixel p for which  $\tau_p > f_p$ . The two following lemmas concern wet lakes. The first is a reinterpretation of a lemma established for TN.

#### Lemma

If two neighboring nodes p and q verify  $\tau_p > e_{pq} = f_p \vee f_q$ , then  $\tau_p = \tau_q$ .

The second derives from criterion TG-1.

#### Lemma

If two neighboring nodes p and q verify  $\tau_p > f_p$  and  $\tau_q > f_q$ , then  $\tau_p = \tau_q$ .

**Proof:**  $\{\tau_p > \tau_q \Rightarrow \tau_p \leq f_p\} \Leftrightarrow \{\tau_p > f_p \Rightarrow \tau_p \leq \tau_q\}$ . Applying the last implication to  $\tau_p > f_p$  and  $\tau_q > f_q$  yields  $\tau_p \leq \tau_q$  and  $\tau_p \geq \tau_q$ , which together gives  $\tau_p = \tau_q$ .

▲ロト ▲圖ト ▲画ト ▲画ト 三直 - のへで

## Regional minima lakes and full lakes

## Full lakes and regional minimum lakes in a tank network

What happens at the boundary of a lake X in a tank network ? Consider 2 neighboring pixels (p, q), p being inside a lake of altitude  $\lambda$  and q outside. These pixels do not verify  $p \sim q : e_{pq} > \tau_p$  and  $\tau_p \neq \tau_q$ :

- if  $e_{pq} > \tau_p$  and one has to climb for going from p to q.
- else  $e_{pq} \leq \tau_p$  implying  $\tau_q \leq \tau_p \lor e_{pq} = \tau_p$ . As  $\tau_p \neq \tau_q$  we have  $\tau_q < \tau_p$ , which implies  $\tau_p \leq e_{pq}$ . Thus  $\tau_p = e_{pq}$  and  $\tau_q < \tau_p$ , indicating the q is an exhaust node of the lake , and the lake X is a full lake.

In other terms, in a lake without exhaust edges, all outgoing edges are higher than the level of the lake. Such a lake is called regional minimum lake. A lake with one or several exhautst edges is called full lake. Adding a drop of water to a full lake provokes an overflow through the exhaust edges.

#### Definition

A regional minimum of a tank network is a lake with all outgoing edges, or cocycle edges having a higher altitude.

#### Definition

A lake of level  $\lambda$  of the flooding of a tank network is a full lake, if there exists an an exhaust edge from an inside node p to an outside node q verifying  $\tau_p = e_{pq} = \lambda > \tau_q$ .

#### Lemma

Each regional minimum of the flooding of a tank network contains a regional minimum of the tank network itself or is an isolated regional minimum.

**Proof:** Either  $X = \{p\}$  is an isolated regional minimum node p with all adjacent edges having a weight  $> \lambda$ . If X contains inside edges, and (p, q) is the edge for which  $e_{pq}$  is minimal, then the maximal connected component containing p with edge weights equal to  $e_{pq}$  is a regional minimum of the graph.

## Full lakes and regional minimum lakes in topographic graph

A couple of neighboring nodes belongs to the cocycle of a lake X,  $p \in X$ and  $q \notin X$  only if  $\tau_p \neq \tau_q$ . Either  $\tau_p > \tau_q$ . Or  $\tau_p > \tau_q$  implying  $\tau_p = f_p$ and the lake is a full lake having an exhaust node p. In the graph  $G_e$  we have  $e_{pq} = f_p \lor f_q = \tau_p$  as  $f_p = \tau_p > \tau_q \ge f_q$ . We get the two following definitions.

#### Definition

A regional minimum is a lake for which the ground level of all outside neighbors has a higher altitude.

#### Definition

A lake of the flooding of a topographic surface is a full lake of altitude  $\lambda$  if there exist two neighboring nodes p inside the lake and q outside, such that  $\tau_p < \tau_q = f_q$ .

Each regional minimum of the flooding of a topographic graph contains a regional minimum of the topographic graph itself.

## Among all possible flooidngs, chosing one

## Specifying a particular flooding among all possible ones

Many floodings of a topographic surface of of an edge weighted graph are possible. In order to specify a particular flooding we have to add other criteria. For instance the lowest flooding for which each lake is a full lake or has a surface area higher than a given threshold specifies the so-called area flooding. we are interested by the highest flooding under a ceiling function  $\omega$ .

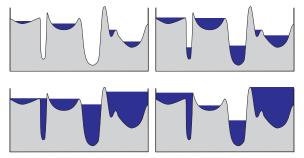


Figure: Various flood distribution on the same topographic surface

#### Lemma

If  $\tau$  and  $\nu$  are two floodings of a node or edge weighted graph G, then  $\tau \lor \nu$  and  $\tau \land \nu$  also are floodings of G.

Hence the family of floodings of the graph  $G_e$  or  $G_n$  below the function  $\omega$  is closed by supremum. This supremum is itself a flooding and is below  $\omega$ . For this reason it is the highest flooding of the graph  $G_e$  or  $G_n$  below  $\omega$ .

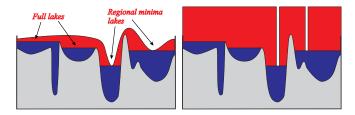


Figure: The highest flooding of a topographic surface below a ceiling function (in red). The ceiling function on the left and on the right yield the same flooding, as they constrain the level regional minima lakes at identical levels ; all other lakes being full lakes.

## Marker driven watershed segmentation

The watershed of the gradient contains the contours of the image. The regions to segment contain each a marker. A ceiling function equal to the gradient image on the markers and equal to  $\infty$  everywhere else is constructed. The highest flooding under this ceiling function has regional minima lakes at the position of the markers and full lakes everywhere else. The watershed of the flooded surface gives the result.

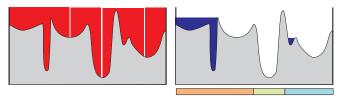


Figure: Left: a ceiling function with three minima above a topographic surface Right: the highest flooding of the topographic surface below the ceiling function. It contains three regional minima lakes. The watershed partition of this function is indicated below, each region labeled with a distinct color and contains one minimum of the ceiling function. The regional minimum lakes of dominated floodings

## Regional minima of dominated edge weighted graphs

Consider a flooding  $\tau$  of a tank network and X, one of its lakes of altitude  $\lambda$ . Each node p of X verifies  $\tau_p = \lambda$ .

If there exists a pair of nodes,  $s \in X$  and  $t \notin X$ , such that  $\tau_t < \lambda$  and

 $e_{st} = \lambda$  then X is a full lake and the edge (s, t) an exhaust edge for X.

Such a lake cannot have a higher altitude than its exhaust edges, and the constraining function plays no role in the level of the lake.

On the contrary, if all edges of the cocycle of X have an altitude above  $\lambda$ , then X is a regional minimum of the flooding  $\tau$ . If the level  $\lambda$  of the lake cannot become higher, it is because it is constrained by at least one node of the ceiling function  $\omega$  at altitude  $\lambda$ . This node cannot be any node, as stated now.

#### Theorem

Any regional minimum lake of the highest flooding of a graph  $G_e$  with edge weights e, below a ceiling function  $\omega$ , contains a regional minimum lake of the graph with edge weights  $\delta_{en}\omega \lor e$  and node weights  $\omega$ .

< //2 → < 三

#### Theorem

Any regional minimum lake of the highest flooding of a graph G with node weights f, below a ceiling function  $\omega$ , contains a regional minimum lake of the function  $\omega$ .

A regional minimum lake X on a topographic graph has a uniform flooding level  $\lambda$  and all its neighboring nodes have a flooding level  $> \lambda$ . The level of X could be higher, were it not constrained by the ceiling function  $\omega$ . There exists a node  $p \in X$  for which  $\omega_p = \tau_p = \lambda$ . The connected component Y of nodes for which  $\omega = \lambda$  contains p and is included in X, as for each outside neighbor q of X, we have  $\omega_q \ge \tau_q > \lambda$ . On X we have  $\omega \ge \tau = \lambda$ , showing that Y has no lower neighbor. Thus Y is a regional minimum of the ceiling function. The highest flooding of f under  $\omega$  if made of lakes and of dry zones, where the flooding equals the ground level. The lakes themselves are divided between full lakes and regional minima lakes. The level of the full lakes is solely determined by the altitude of the lowest pass point surounding the lakes. The level of regional minima lakes is determined by the level of the regional minima of the ceiling function. In fact, the blocking effect is the same whatever the size of this regional minimum ; a single point is sufficient.

Replace the ceiling function  $\omega$  by a function equal to  $\omega$  on at least one node of each regional minimum produces the same dominated flooding.

## Considering the flooding process itself

We place a source pouring water at a node  $\Omega$  of an edge weighted graph  $G_e$  and flood the graph. We are interested by the level of the flood at each other node of the graph when it reaches for the first time this node. If p is a node of the graph, the flood coming from  $\Omega$  will reach p following the easiest path: among all paths between  $\Omega$  and p, it follows the path for which the highest edge is the lowest. This value constitutes precisely the ultrametric distance  $d(\Omega, p)$  between  $\Omega$  and p.

Consider the shortest path between  $\Omega$  and p. The value  $\tau_p$  is the weight of the highest edge between  $\Omega$  and p. The last node on this shortest path is a node q. If the highest edge on the path is (q, p), then  $\tau_p = e_{pq}$ . If not  $\tau_p = \tau_q \ge e_{pq}$  as the highest edge between  $\Omega$  and p of the path  $\pi$  lies between  $\Omega$  and q. In all cases we have  $\tau_p = \tau_q \vee e_{pq}$ . Consider now any other neighboring node s of p. The path obtained by concatenating the shortest path between  $\Omega$  and p, hence  $\tau_p \le \tau_s \vee e_{ps}$ .

#### Theorem

The shortest ultrametric distance of each node p of an edge weighted graph to a particular node  $\Omega$  is a flooding of this graph.

Let *I* be the subset of nodes for which  $d(\Omega, i) = \omega_i$ . If the geodesic path between  $\Omega$  and a node *p* passes through *i*, then  $\tau_p = d(\Omega, p) = e_{\Omega\omega_i} \lor d(i, p) = \omega_i \lor d(i, p)$ .

For any node 
$$q$$
 we have  $\tau_q = \bigwedge_{i \in I} \omega_i \lor d(i, q)$ .

This shows that  $\tau$  is the highest possible flooding of  $G_e$  on all nodes  $\omega_i$ and also on all other nodes. Suppressing the node  $\Omega$  and all edges linking  $\Omega$  with another node of  $G_e$  produces a graph G' for which  $\tau$  is the highest flooding dominated by  $\omega$ .

# Inversely, each dominated flooding is produced by a flooding.

Any dominated flooding verifies  $\tau_p \leq \bigwedge_{q \text{ neighbor of } p} (\tau_q \lor e_{pq})$  and  $\tau_p \leq \omega_p : \tau_p \leq \omega_p \land \bigwedge_{q \text{ neighbor of } p} \left( \tau_q \lor e_{pq} \right).$ The highest of them verifies  $\tau_p = \omega_p \wedge \bigwedge_{q \text{ neighbor of } p} (\tau_q \vee e_{pq})$ Adding to the graph G a dummy node  $\Omega$  with a weight  $\tau_{\Omega} = 0$  linked by a dummy edge  $(\Omega, p)$  with a weight  $\omega_p$  produces a graph  $\hat{G}$ . Rewritten as  $\tau_p = (\tau_\Omega \lor e_{\Omega p}) \land \land (\tau_q \lor e_{qp}), \text{ this formula is the expression of}$ a neighbor of pthe algorithm of Berge for computing the shortest ultrametric distance of each node to  $\Omega$  in the augmented graph  $\widehat{G}$ . The algorithm of Berge expresses that the shortest path between  $\Omega$  and p is  $e_{\Omega p} = \omega_p$  if the path is simply the edge  $(\Omega, p)$  or it is equal to  $(\tau_s \vee e_{ps})$  if the path passes through the neighbor s of p, and if  $(\tau_q \vee e_{qp})$ takes its smallest value for q = s.

# Inversely, each dominated flooding is produced by a flooding.

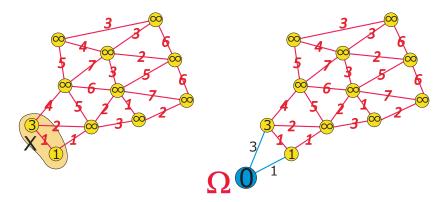


Figure: Adding a dummy node linked to each node x in X by an edge weighted by the offset at x.

# Inversely, each dominated flooding is produced by a flooding.

#### Theorem

The highest flooding of the graph G below a function  $\omega$  defined on the nodes is the shortest distance of each node to  $\Omega$  in augmented graph  $\hat{G}$ .

If the flooding is not constrained on a node p, i.e.  $\omega_p = \infty$ , then it is not necessary to link the node p with the dummy node  $\Omega$ , as the highest flooding will reach p through one of its neighboring nodes.

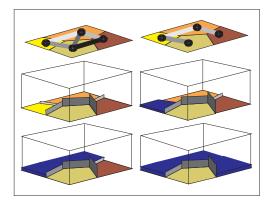
## Pruning the graph and getting the same result

Each dominated flooding results in a flood distribution verifying criterion DF-2:  $\tau_q = \bigwedge_{i \in N} \omega_i \lor d(i, q)$ .

The ultrametric distance d(i, q) is the weight of the highest edge in a path  $\pi$  of lowest sup-section between i and q. Consider now each edge (p, q) of the path  $\pi$ . If it belongs to  $\pi$  we keep it. If not we may replace it in  $\pi$  by the unique path between p and q contained in T, as all edges along this path have a weight  $\leq e_{pq}$ ; these substitutions produce paths with the same sup-section. In other words, the edges of the tree T are sufficient for computing the ultrametric distances of the graph

## The flooding always follows the union of minimum spanning trees

#### Illustration :



Any flooding  $\tau \ge f$  on a node weighted graph  $G_n$  also is a flooding of the derived edge weighted graph  $G_e$  with edge weights  $\delta_{en}e$ . The preceding results apply. Any flooding  $\tau \ge f$  of a MST of  $G_e$  also is a flooding of  $G_e$  and of  $G_n$ .

This result has interesting algorithmic implications. It is possible to compute highest flooding using a minimum spanning tree of the graph with a dramatically lower number of edges. However, one has to take in the balance the time needed for constructing the graph. It may be interesting if one has to construct several dominated flooding of the same graph. We will meet later an algorithm which combines the construction of the MST and of a particular flooding of the graph.

### Flooding with shortest distance algorithms

Many shortest distance algorithms exist. Each of the following has specific advantages. Algorithm of Berge Algorithm of Dijkstra Algorithm of Prim Core expanding algorithm Path algebra

## Algorithms for computing the highest flooding on tank networks

## The algorithm of Berge

< 一型

2

#### Initialisation

The algorithm of Berge is initialised with the function  $\omega$ . For each node  $p: \tau_p = \omega_p$ . This distribution being not a flooding, the algorithm applies until stability the relation Repeat until  $\tau_p^{(m)} = \tau_p^{(m-1)}: \tau_p^{(n)} = \omega_p \wedge \bigwedge_{\substack{q \text{ neighbor of } p}} \left( \tau_q^{(n-1)} \vee e_{pq} \right)$ 

**Convergence:**  $\tau_p^{(m)}$  decreases at each iteration. It has a lower ceiling, the smallest value of  $\omega$ , therefore it converges.

#### Improved version with less memory accesses:

As  $\tau$  can only decrease at each iteration, replacing the ceiling function  $\omega$  by the value taken by  $\tau$  at iteration (n-1) produces an equivalent algorithm with less memory accesses: the value of  $\omega$  has only to be fetched at initialization. Initialisation:  $\tau_p^{(0)} = \omega_p$ Repeat until  $\tau_p^{(m)} = \tau_p^{(m-1)} : \tau_p^{(n)} = \tau_p^{(n-1)} \wedge \bigwedge_{\substack{q \text{ neighbor of } p}} \left( \tau_q^{(n-1)} \vee e_{pq} \right)$ 

Fernand Meyer (Centre de Morphologie Math Flooding edge or node weighted graphs

# Software or hardware implementation of the algorithm of Berge

Using a local neighborhood, extremely versatile, as the nodes may be processed in any order, the algorithm of Berge t is well suited for software or hardware implementation based on a systematic scan of the graph. The algorithm is parallel or recursive:

• 
$$au_p^{(n)} = au_p^{(n-1)} \wedge \bigwedge_{q \text{ neighbor of } p} \left( au_q^{(n-1)} \vee e_{pq} \right)$$
 represents a parallel

implementation of the algorithm : the arguments for computing  $\tau_p^{(n)}$  are all those obtained during the previous scan.

• The recursive implementation separates the nodes already met during the current scanning and the nodes in the future:

$$\begin{split} \tau_{p}^{(n)} &= \tau_{p}^{(n-1)} \wedge \bigwedge_{\substack{q \text{ past neighbor of } p \\ q \text{ future neighbor of } p}} \left( \tau_{q}^{(n)} \vee e_{pq} \right) \\ & \bigwedge_{\substack{q \text{ future neighbor of } p \\ Alternating a \text{ forward scan and a backward scan for the graph permits}} \end{split}$$

an accelerated convergence of the flooding levels.

## The algorithm of Moore-Dijkstra

э

The Dijkstra algorithm is a greedy algorithm. A set S contains all nodes whose distance is known. For the outside neighbors of S, this distance is estimated: for  $p \in S$  and  $q \notin S$ ,  $\tau_q \leq \tau_p \lor e_{pq}$ . And we have  $\tau_q = \tau_p \lor e_{pq}$  if the shortest path to q follows the edge (p, q). This is the case for the node in  $\overline{S}$  with the lowest estimation. This node may be introduced into S and the estimation of the distance of its neighbors still in  $\overline{S}$  updated. The nodes introduced in S have increasing values, as the estimation of all nodes in  $\overline{S}$  is higher than the estimation of the nodes in S.

The edges linking each node with the node through which it has been flooded in the algorithm form a tree. This tree is rooted at  $\Omega$  and contains a never decreasing geodesic path between  $\Omega$  and each node.

#### Initialisation:

 $S = \{\Omega\}$  and  $\tau_{\Omega} = -\infty$ ; for each node p in  $N = \overline{S} : \tau_p = \omega_p$ Flooding:

While 
$$\overline{S} \neq \emptyset$$
 repeat:  
Select  $j \in \overline{S}$  for which  $\tau_j = \min_{i \in \overline{S}} [\tau_i]$   
 $\overline{S} = \overline{S} \setminus \{j\}$   
For any neighbor  $i$  of  $j$  in  $\overline{S}$  do  $\tau_i = \min [\tau_i, \tau_j \lor e_{ji}]$   
End While

The dummy node  $\Omega$  and the dummy edges linking  $\Omega$  with the nodes of N is useless in practice:

Initialization:

$$\begin{split} S &= \varnothing \ ; \ \overline{S} = N \ ; \ \text{for each node } p \ \text{in } N : \tau_p = \omega_p \\ \hline \textbf{Flooding:} \\ & \text{While } \overline{S} \neq \varnothing \ \text{repeat:} \\ & \quad \text{Select } j \in \overline{S} \ \text{for which } \tau_j = \min_{i \in \overline{S}} [\tau_i] \\ & \quad \overline{S} = \overline{S} \setminus \{j\} \\ & \quad \text{For any neighbor } i \ \text{of } j \ \text{in } \overline{S} \ \text{do } \tau_i = \min [\tau_i, \tau_j \lor e_{ji}] \\ & \quad \text{End While} \end{split}$$

When the node j is introduced into S, it has the highest value in S. The instruction

<For any neighbor *i* of *j* in  $\overline{S}$  do  $\tau_i = \min[\tau_i, \tau_j \lor e_{ji}] >$  may be simplified in:

<For any neighbor *i* of *j* verifying  $\tau_j \vee e_{ji} < \tau_i$  do  $\tau_i = \tau_j \vee e_{ji} >$  as a node *i* verifying  $\tau_i > \tau_j$  cannot belong to *S*. Checking whether *i* belongs to  $\overline{S}$  is not necessary, leading to the following algorithm.

Initialization:

 $\begin{array}{l} S = \varnothing \ ; \ \overline{S} = N \ ; \ \text{for each node } p \ \text{in } N : \tau_p = \omega_p \\ \text{Flooding:} \\ \text{While } \overline{S} \neq \varnothing \ \text{repeat:} \\ & \quad \text{Select } j \in \overline{S} \ \text{for which } \tau_j = \min_{i \in \overline{S}} \left[\tau_i\right] \\ & \quad \overline{S} = \overline{S} \backslash \{j\} \\ & \quad \text{For any neighbor } i \ \text{of } j \ \text{verifying } \tau_j \lor e_{ji} < \tau_i \ \text{do } \tau_i = \tau_j \lor e_{ji} \\ \text{End While} \end{array}$ 

Fetching the node with the smallest value  $\tau_j$  within  $\overline{S}$  is made easy by using an adequate data structure such as a "funnel" structure: nodes with any order of priority may be stored in the funnel; but only one of the nodes with the smallest priority is extracted at any time. Possible implementation: an ordered bucket structure. A node is introduced in the bucket corresponding to its priority. Each extracted node is chosen among the nodes in the bucket with highest priority. If the buckets have the structure of a queue, we speak about hierarchical queues. Before being introduced into S, the distance of each node has to be estimated anew every time one of its neighbor is introduced into S. If we use a "singel occupancy funnel", a node occupies only one location and its priority is updated if needed. In a multiple occupancy funnel (MOF), a node may occupy more than one location, with distinct priority. When a node is extracted for the first time, its value is correct. When it is extracted another time one has to discard it.

# The algorithm of Dijkstra with a funnel structure

Consider a flooding of the following graph. Initially the node p is in  $\Phi$  with a value 3 and a yellow colour. The two other nodes are not in  $\Phi$ . When p is extracted from  $\Phi$ , both its neighbors are introduced in  $\Phi$  with estimates equal to 5 and 9 (yellow colour). The node q is then extracted from  $\Phi$  and its neighbor r is introduced in  $\Phi$  with a new estimate 7.; at the same time, the node r of the graph gets this same value. When r is extracted from  $\Phi$  for the first time, there is an identity between its priority and the flooding level of r in the graph. The second time it is extracted, its priority is 9, higher than the value in the graph and has to be discarded from further processing.

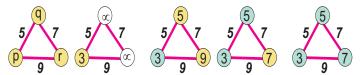


Figure: Propagation of the flooding on an edge weighted graph using the Dijkstra algorithm.

22 March 2013

78 / 165

#### Initialization:

```
create \Phi, a multi occupancy funnel ; \tau = \infty
for each node p verifying \omega_p < \infty
introduce p into \Phi with a priority \omega_p ; \tau_p = \omega_p
```

## Flooding

While  $\Phi$  is not empty repeat:

Extract from  $\Phi$  the node j with the lowest prioriy  $\lambda$  If  $\tau_j=\lambda$ 

For any neighbor *i* of *j* such that  $\tau_i \vee e_{ii} < \tau_i$ 

$$au_i = au_j \lor e_{ji}$$

introduce *i* into  $\Phi$  with the priority  $\tau_i$ 

End While

In the case of marker based segmentation a number of nodes are markers. The aim is to produce a Voronoi tessellation of the graph: each node is assigned to the marker which is closest for the ultrametric distance. Several solutions exist for breaking the ties, if a node is at the same distance of distinct markers.

The preceding algorithm may be used, the reduced set  $\widehat{S}$  of ceiling minima, being the set of nodes belonging to the markers. In the context of segmentation, a distinct label is assigned to each marker. This label will be assigned to the total region flooded through this marker.

#### Initialization:

create a HQ  $\Phi$ 

create an image  $\boldsymbol{\zeta}$  which will hold the labels

assign to each marker node p a distinct label  $\zeta_p$ , a flooding value

$$au_p = 0$$
 and introduce  $p$  into  $\Phi$  with a priority 0

for all other nodes  $au=\infty$ 

#### Flooding

While  $\Phi$  is not empty repeat:

Extract from  $\Phi$  the node *j* with the lowest prioriy  $\lambda$ If  $\tau_i = \lambda$ 

For any neighbor *i* of *j* such that  $\tau_i \vee e_{ii} < \tau_i$ 

$$\begin{aligned} \tau_i &= \tau_j \lor e_{ji} \\ \zeta_i &= \zeta_j \\ \text{introduce } i \text{ into } \Phi \text{ with the priority } \tau_i \end{aligned}$$

End While

A hierarchical queue are governed by a double hierarchical order. A hierarchical queue is a series of first first out queues, having each a priority. A node is introduced in the queue corresponding to its priority. The node which is extracted at any time of the HQ is the node which has been introduced first in the queue with the highest priority. The priorities among the queues organize that the flooding progresses in an order of increasing altitudes. The FIFO structure of the queue ensures that a node inside a plateau is flooded in an order proportional to its distance to the lower border of the plateau.

## The algorithm of Prim

- 一司

2

On a graph, from a node p to a node q, the flood always follows the paths of lowest sup-section linking p and q. All such paths belong to the MST of the graph. Hence, it is possible to combine the flooding with the construction of the MST. The algorithm of PRIM constructs the MST rooted in  $\Omega$ .

#### Initialisation

Initially, the tree T spans only the node  $\Omega$ .

## Expansion

As long as the tree does not contain all nodes of the graph:

Chose the lowest edge (q, s) in the cocycle of T, such that  $q \in T$ and  $s \notin T$ . Append the node s to the tree:  $T = T \cup \{s\}$  Flooding with the algorithm of PRIM

The result of the preceding algorithm is a tree rooted at  $\Omega$ . Each other node p is linked with  $\Omega$  through a unique path. The flood coming from  $\Omega$ necessarily follows this path. The flooding of the nodes and the construction of the tree may be done simultaneously.

#### Initialisation

Initially, the tree T spans only the node  $\Omega$  :  $T = \{\Omega\}$ .  $\tau_{\Omega} = 0$ . Expansion

As long the tree does not contain all nodes of the graph:

Chose the lowest edge (q, s) in the cocycle of T, such that  $q \in T$  and  $s \notin T$ .

Append the edge (q, s) and the node s to the tree:  $T = T \cup \{s\}$  $\tau_s = \tau_q \lor e_{qs}$ 

#### Analysis of the algorithm

The nodes are introduced with a never decreasing flood level. A node with a flood level  $\lambda$  first floods its neighbors appended through an edge which is lower or equal to the current flooding level: these neighbors get the current flood level  $\lambda$  and are appended to the tree. If this is not possible anymode, the smallest edge in the cocycle of the tree with a weight  $> \lambda$  is followed, introducing the first node with a weight  $> \lambda$  into the tree. The PRIM algorithm is a particular avatar of Dijkstra's algorithm. Among all neighboring nodes of T for which the estimated flooding level is the smallest, the algorithm of PRIM first considers those linked with the tree through the lowest edge.

# Scheduling with a HQ

## Initialization:

create  $\Phi$ , a multi occupancy funnel.

 $\tau = \infty$ 

for each node p verifying  $\omega_p < \infty$ , introduce p into  $\Phi$  with a priority

 $\omega_p$ 

 $\lambda = -\infty$ 

#### Flooding

While  $\Phi$  is not empty repeat:

Extract from  $\Phi$  the node *p* with the lowest prioriy  $\mu$  if  $\tau_p = \infty$ 

$$\tau_p = \infty$$

If 
$$\mu > \lambda : \lambda = \mu$$

$$\tau_p = \lambda$$

For any neighbor q of p such that  $au_q = \infty$ 

introduce q into  $\Phi$  with the priority  $e_{pq}$ 

22 March 2013

87 / 165

**Remark:** Replacing the last instruction with <introduce *i* into the funnel with the priority  $\lambda \lor e_{ji} >$  produces the algorithm of Dijkstra.

#### Initialisation

 $T = \{\emptyset\}$ <br/>For each marker p :

 $\tau_p = 0$   $T = T \cup \{p\}$ assign a new label  $\zeta_p$ 

#### Expansion

As long the tree does not contain all nodes of the graph:

Chose the lowest edge (q, s) in the cocycle of T, such that  $q \in T$ and  $s \notin T$ .

Append the edge (q, s) and the node s to the tree:  $T = T \cup \{s\}$  $\tau_s = \tau_q \lor e_{qs}$  $\zeta_s = \zeta_q$  If we are not interested by the flooding level but only by the Voronoi partition associated to the markers:

#### Initialisation

For each marker p, assign a new label  $\zeta_p$ 

For all other nodes  $q: \zeta_q = -\infty$ 

#### Expansion

As long as there are nodes with a label  $\zeta = -\infty$ 

Chose the lowest edge (q,s) verifying  $\zeta_q>-\infty$  and  $\zeta_s=-\infty$   $\zeta_s=\zeta_q$ 

Shortest path algorithms on node weighted graphs

Any flooding  $\tau$  of a node weighted graph  $G_n$ , above the ground level f also is a flooding on an edge weighted graph  $G_e$  with edge weights  $e_{pq} = f_p \lor f_q$ : -> all results and algorithms established for  $G_e$  are applicable to r  $G_n$  simply by replacing  $e_{pq}$  by  $f_p \lor f_q$  and remembering that  $\tau \ge f$ , they get simpler.

## The algorithm of Berge

< 一型

2

## The algorithm of Berge

Initialisation: 
$$\tau_p^{(0)} = \omega_p$$
  
Repeat until  $\tau_p^{(m)} = \tau_p^{(m-1)} : \tau_p^{(n)} =$   
 $\omega_p \wedge \bigwedge_{\substack{q \text{ neighbor of } p}} \left( \tau_q^{(n-1)} \vee f_p \vee f_q \right) = \omega_p \wedge \bigwedge_{\substack{q \text{ neighbor of } p}} \left( \tau_q^{(n-1)} \vee f_p \right)$ 

#### A variant of the algorithm

Replacing the ceiling function  $\omega$  by the value taken by  $\tau$  at iteration (n-1) since  $\tau$  can only decrease at each iteration. The algorithm becomes:

Initialisation:  $\tau_p^{(0)} = \omega_p$ Repeat until  $\tau_p^{(m)} = \tau_p^{(m-1)} : \tau_p^{(n)} = \tau_p^{(n-1)} \wedge \bigwedge_{\substack{q \text{ neighbor of } p}} \left( \tau_q^{(n-1)} \vee f_p \right)$ We recognize the classical algorithm. Repeat until stability  $\tau = f \vee \varepsilon \tau$ 

## The algorithm of Prim

- 一司

2

The algorithm of PRIM remains exactly the same as for pipe networks. By replacing  $e_{qs}$  by its value  $f_q \vee f_s$ , and since  $\tau_q \ge f_q$ , we get  $\tau_q \vee f_q \vee f_s = \tau_q \vee f_s$ . The flooding of the nodes and the construction of the tree may be done simultaneously.

#### Initialisation

Initially, the tree T has only the node  $\Omega$  and no edge:

$$T = \{\Omega, \varnothing\}. \ \tau_{\Omega} = 0.$$

#### Expansion

As long the tree does not contain all nodes of the graph:

Chose the edge (q, s) with the lowest weight  $f_q \vee f_s$  in the cocycle of T, such that  $q \in T$  and  $s \notin T$ .

Assign the node *s* to the tree:  $T = T \cup \{s\}$ 

$$\tau_s = \tau_q \vee f_q \vee f_s = \tau_q \vee f_s$$

## The algorithm of Dijkstra

2

Initialization:

 $\begin{array}{l} S = \varnothing \ ; \mbox{ for each node } p \ \mbox{ in } N : \tau_p = \omega_p \\ \mbox{Flooding:} \\ \mbox{While } S \neq N \ \mbox{repeat:} \\ & \mbox{Select } j \in \overline{S} \ \mbox{for which } \tau_j = \min_{i \in \overline{S}} \left[ \tau_i \right] \\ & \mbox{Select } j \in \overline{S} \ \mbox{for which } \tau_j = \min_{i \in \overline{S}} \left[ \tau_i \right] \\ & \mbox{Select } S \cup \{j\} \\ & \mbox{For any neighbor } i \ \mbox{of } j \ \mbox{verifying } \tau_i > \tau_j \lor f_i \ \mbox{do } \tau_i = \tau_j \lor f_i \\ \mbox{End While } \end{array}$ 

*j* is the smallest neighbor of *i*. If *i* is flooded through one of its neighbors, this neighbor can only be *j* and as soon the value  $\tau_i$  is computed once, this value is correct and final :  $\tau_i = \tau_j \vee f_i$ . For this reason, we may use an image of binary flags  $\zeta$ , in order to flag all nodes for which the flooding value is known

Initialization:

 $S = \varnothing$  ; for each node p in  $N : \tau_p = \omega_p$  and  $\zeta_p = 0$ Flooding:

While  $S \neq N$  repeat: Select  $j \in \overline{S}$  for which  $\tau_j = \min_{i \in \overline{S}} [\tau_i]$   $S = S \cup \{j\}$ For any neighbor i of j such that  $\zeta_i = 0$  and  $\tau_i > \tau_j \lor f_i$   $\zeta_i = 1$  $\tau_i = \tau_j \lor f_i$ 

End While

## The regional minima of the ceiling function are sufficient, and any overset of them...

Each regional minimum of a flooding contains a regional minimum of the ceiling function  $\omega$ . If it were not constrained at this level, the level of the flooding would be higher in this regional minimum. Outside the regional minima, the level of the flooding not constrained by  $\omega$ . Ideally, the algorithm of Dijkstra should be initialized using one node and only one node in each regional minimum of  $\omega$ . As the regional minima are costly to compute, a cheap overset of these nodes offers a better compromise.

The regional minima of  $\omega$  are plateaus of uniform altitude, without lower neighbors. During a forward raster scan of the image, the pixels are detected which have only higher neighbors in the past and no lower neighbor in the future. The set X is obtined in a forward scan through the image (for a 2D image, from left to rights and from top to bottom).  $X = \left\{ p \mid \tau_p < \bigwedge_{q \in \text{past}(p)} \tau_q \right\} \land \left\{ p \mid \tau_p \leq \bigwedge_{q \in \text{future}(p)} \tau_q \right\}$ This algorithm finds the entry points in the regional minima and in a certain number of plateaus. We may reduce the plateaus as follows.

# Reducing the number of candidates

A classical algorithm for constructing regional minima. The flooding of  $\omega$  with  $\omega + 1$  as ceiling function produces a new function  $\hat{\omega}$ . The regional minima of  $\omega$  are all nodes verifying  $\hat{\omega}$ . For the sake of economy only a partial flooding is done, suppressing a number of plateaus which are not regional minima:

• using geodesic erosions defined as:

$$\varepsilon_{\omega}(g) = \varepsilon g \lor \omega$$
  
 $\varepsilon_{\omega}^{(n)}(g) = \varepsilon_{\omega}(\varepsilon_{\omega}^{(n)}(g))$ 

The set of nodes Y verifying  $\varepsilon_{\omega}^{(n)}(g) > \omega$  is an overset of the regional minima, decreasing with the number of iterations *n*.

 $\bullet$  Using one pass of a recursive geodesic erosion of  $\omega$  above g with a backward scanning order

$$\overleftarrow{\varepsilon}_{\omega}(g)(p) = \omega_p \vee \bigwedge_{q \in \text{past}(p)} g_q,$$

The set Z verifying  $\overleftarrow{\varepsilon}_{\omega}(g) > \omega$  is an overset of the regional minima

CEILING\_MINIMA is one of the following sets :  $X_{\alpha} X \land Y_{\alpha}$  or  $X \land Z_{\alpha}$ 

# The Dijkstra algorithm with a reduced initialisation set of ceiling minima.

A binary tag  $\zeta : \zeta = 0$  for nodes with an unknown flooding level;  $\zeta = 1$  after the first time the flooding level is computed.

#### Initialization:

create a HQ  $\Phi$ ;  $\tau = \infty$ ;  $\zeta = 0$ for each node p belonging to  $\hat{S} =$ "ceiling minima", introduce p into  $\Phi$  with a priority  $\omega_p$ 

 $\tau_p = \omega_p$ 

#### Flooding

While  $\Phi$  is not empty repeat:

Extract from  $\Phi$  the node j with the lowest prioriy  $\lambda$ If  $\tau_i = \lambda$ 

For any neighbor *i* of *j* such that  $\zeta_i = 0$  and  $\tau_j \vee e_{ji} < \tau_i$  $\tau_i = \tau_i \vee e_{ji}$ ;  $\zeta_i = 1$ ; introduce *i* into  $\Phi$  with the

priority  $\tau_i$ End While The following algorithm is equivalent and uses a 3 state flag: ("unknown", "final", "in S") = ("u", "f', "s") = (0, 1, 2) with the following meanings: a) "unknown"="u" = 0 are the nodes whose flooding value has not been computed yet; b) "final"="f"=1 computed nodes, not yet in S; c) "in S"= "s" = 2 for pixels introduced into S

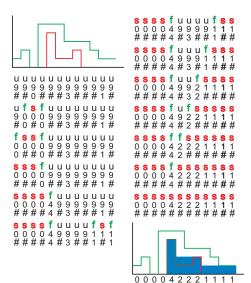
#### Flooding algorithm Initialization:

create a HQ  $\Phi$  ;  $\tau = \infty$  ;  $\zeta = 0$ for each node p belonging to  $\widehat{S} =$  "ceiling minima", introduce p into  $\Phi$  with a priority  $\omega_p$  $\tau_p = \omega_p$ Flooding While  $\Phi$  is not empty repeat: Extract from  $\Phi$  the node *j* with the lowest prioriy if  $\zeta_i < 2$  $\zeta_i = 2$ For any neighbor *i* of *j* such that  $\zeta_i = 0$ if  $\tau_i \vee e_{ii} < \tau_i$  then  $\tau_i = \tau_i \vee e_{ii}$  $\zeta_i = 1$ ; introduce *i* into  $\Phi$  with the priority  $\tau_i$ 

End While

The figure presents in red a topographic surface and in green a ceiling function. The successive lines below the figure present the evolution of the algorithm. The algorithm is illustrated step by step. In each group of 3 lines

- the first represent the status of the node,  $"u"\!=\!\!unknow,\,"f'$  = flooded, "s" in S
- the second represent the current estimated flooding value of the nodes
- the third represent the nodes in  $\boldsymbol{\Phi}$



Fernand Meyer (Centre de Morphologie Math Flooding edge or node weighted graphs

(日) (同) (三) (三)

2

In the Dijkstra algorithm, S contains all nodes with a known flooding ; at each iteration, the node with the smallest estimated flooding value is introduced into S. One node is introduced at each iteration of the algorithm. Faster floodings are possible if one remarks:

- if  $f_p = f_q$ , then  $\tau_p = \tau_q$ , as the flooding is a connected operator.
- If  $f_p = \omega_p$ , then ground and ceiling levels are identical, hence  $\tau_p = f_p = \omega_p$
- if f<sub>p</sub> ∨ f<sub>q</sub> < τ<sub>p</sub> then τ<sub>q</sub> = f<sub>q</sub>. The proof is the following. {f<sub>p</sub> ∨ f<sub>q</sub> < τ<sub>p</sub>} ⇔ {f<sub>p</sub> < τ<sub>p</sub> and f<sub>q</sub> < τ<sub>p</sub>}. On the other hand the criterion for floodings {τ<sub>p</sub> > τ<sub>q</sub>} ⇒ {f<sub>p</sub> = τ<sub>p</sub>} is equivalent with {f<sub>p</sub> < τ<sub>p</sub>} ⇒ {τ<sub>p</sub> ≤ τ<sub>q</sub>}. On the other hand τ<sub>q</sub> ≤ τ<sub>p</sub> ∨ f<sub>q</sub> = τ<sub>p</sub> ∨ f<sub>q</sub> = τ<sub>p</sub> ∨ f<sub>q</sub> = τ<sub>p</sub> which shows that τ<sub>p</sub> = τ<sub>q</sub>
  if f<sub>q</sub> < τ<sub>p</sub> then τ<sub>q</sub> ≤ f<sub>q</sub> ∨ τ<sub>p</sub> = τ<sub>p</sub>.

э.

- 4 週 ト - 4 国 ト - 4 国 ト

The two following rules permit to speed up the algorithm of Dijkstra.

- if f<sub>q</sub> ≥ τ<sub>p</sub> then τ<sub>q</sub> = f<sub>q</sub>. The proof is the following. For any neighboring nodes p and q the flooding levels verify τ<sub>q</sub> ≤ τ<sub>p</sub> ∨ f<sub>q</sub> = f<sub>q</sub>. But as τ<sub>q</sub> ≥ f<sub>q</sub> we get τ<sub>q</sub> = f<sub>q</sub>.
- Suppose that (p, q) are neighbors and p is the node of ∂<sup>-</sup>S for which the flooding level τ<sub>p</sub> is the lowest. If f<sub>q</sub> < τ<sub>p</sub>, and if q is to be flooded by a node in S, this node necessarily is p and the estimated flooding level of q is τ<sub>q</sub> = τ<sub>p</sub> ∨ f<sub>q</sub> = τ<sub>p</sub>.

3

We may now derive a fast algorithm from these remarks. Suppose that during the flooding, the set S represents all flooded nodes and p is the node of  $\partial^- S$  for which the flooding level  $\tau_p$  is the lowest. If  $f_q \ge \tau_p$  then  $\tau_q = f_q$ . If on the contrary  $f_q < \tau_p$ , we apply Dijkstra's algorithm and introduce into S the node with the smallest estimate. If there exists in  $\overline{S}$  a node j with a smaller ceiling value  $\omega_j$  as  $\tau_p : S = S \cup \{j\}$ . If not,  $\tau_q = \tau_p$  is the flooding estimation of a node in  $\overline{S}$  which is the lowest and  $S = S \cup \{q\}$ .

This shows that in the case where  $\omega_j \ge \tau_p$ , all neighbors of p may be introduced at once into the set S, yielding the following algorithm:

## Initialization:

 $S = \emptyset$ ;

## Flooding:

While  $S \neq N$  repeat: Set  $\lambda = \min_{i \in \overline{S}} [\omega_i]$ ; if j does not exist  $\lambda = \infty$ Set  $\mu = \tau_p$  for  $p \in \partial^- S$  for which  $\tau_p = \min_{i \in \partial^- S} [\tau_i]$ if  $\lambda < \mu$ :  $S = S \cup \{j\}$  and  $\tau_j = \omega_j$ else For each neighbor q of p in  $\overline{S}$  do:  $\tau_q = \tau_p \lor f_q$  $S = S \cup \{q\}$ 

End While

At each iteration, the algorithm has to fetch the node p with the smallest flooding value in  $\partial^- S$  which is easily done with a HQ; the node j with the smallest ceiling value  $\omega_{j_{.}}$  can be easily done if the nodes are ordered with increasing values in a FIFO.

The topographic surface is in red and the ceiling function in green. At initialisation, the nodes with a label "i" are the nodes of the ceiling minima. The smallest of them is introduced into S and immediately expanded, introducing both its neighbors into S. In line 5, another ceiling minima is introduced into S and immediately expanded.



Figure: expanding algorithm

< (17) × < 3

## The lakes of an ultrametric distance function form a dendrogram

The weights are assigned to the edges, and represent their altitudes. **Altitude of a chain:** The altitude of a chain is equal to the highest weight of the edges along the chain.

**Flooding distance between two nodes:** The flooding distance fldist(x, y) between nodes x and y is equal to the minimal altitude of all chains between x and y. During a flooding process, in which a source is placed at location x, the flood would proceed along this chain of minimal highest altitude to reach the pixel y. If there is no chain between them, the level distance is equal to  $\infty$ .

**Triangular inequality :** For  $(x, y, z) : d(x, z) \le d(x, y) \lor d(y, z) :$  ultrametric inequality

An ultrametric distance verifies

- \* reflexivity : d(x, x) = 0
- \* symmetry: d(x, y) = d(y, x)

\* ultrametric inequality: for all  $x, y, z : d(x, y) \le max\{d(x, z), d(z, y)\}$ : the lowest lake containing both x and y is lower or equal than the lowest lake containing x, y and z.

## The balls of an ultrametric distance

For  $p \in E$  the closed ball of centre p and radius  $\rho$  is defined by Ball $(p, \rho) = \{q \in E \mid d(p, q) \leq \rho\}$ . The open ball of centre p and radius  $\rho$  is defined by Ball $(p, \rho) = \{q \in E \mid d(p, q) < \rho\}$ . Every triangle is isosceles. Let us consider three distinct points p, q, r and suppose that the largest edge of this triangle is pq. Then  $d(p, q) \leq d(p, r) \lor d(r, q)$ , showing that the two larges edges of the triangle have the same length.

#### Lemma

Each element of a closed ball  $Ball(p, \rho)$  is centre of this ball

**Proof:** Suppose that *q* is an element of  $Ball(p, \rho)$ . Let us show that then *q* also is centre of this ball. If  $r \in Ball(p, \rho)$ :  $d(q, r) \le \max \{d(q, p), d(p, r)\} = \rho$ , hence  $r \in Ball(q, \rho)$ , showing that  $Ball(p, \rho) \subset Ball(q, \rho)$ . Exchanging the roles of *p* and *q* shows that  $Ball(p, \rho) = Ball(q, \rho)$ 

#### Lemma

Two closed balls  ${\rm Ball}(p,\rho)$  and  ${\rm Ball}(q,\rho)$  with the same radius are either disjoint or identical.

**Proof:** If  $\text{Ball}(p, \rho)$  and  $\text{Ball}(q, \rho)$  are not disjoint, then they contain at least one common point *r*. According to the preceding lemma, *r* is then centre of both balls  $\text{Ball}(p, \rho)$  and  $\text{Ball}(q, \rho)$ , showing that they are identical.

#### Lemma

The radius of a ball is equal to its diameter.

**Proof:** Let Ball $(p, \rho)$  be a ball of radius  $\rho$ . Let q and r be two nodes with the largest distance in Ball $(p, \rho)$ . This distance  $\lambda$  is called diameter of the ball and verifies :  $\lambda = d(q, r) \le d(q, p) \lor d(p, r) \le \rho$ . Hence  $\lambda \le \rho$ . If there exists two nodes in Ball $(p, \rho)$  with a distance equal to  $\rho$ , then  $\lambda \ge \rho$ . In this case  $\lambda = \rho$ .

## Reminders on dendrograms

 $\begin{array}{l} {\it E}: \mbox{ a domain with a finite number of elements called points} \\ {\cal X}: \mbox{ a subset of } {\cal P}({\it E}), \mbox{ with the order relation } \subset \\ {\it supp}({\cal X}): \mbox{ union of all sets belonging to } {\cal X} \mbox{ is called support of } {\cal X}: \\ {\it supp}({\cal X}). \end{array}$ 

The subsets of  $\mathcal X$  may be structured into:

- the summits :  $Sum(\mathcal{X}) = \{A \in \mathcal{X} \mid \forall B \in \mathcal{X} : A \subset B \Rightarrow A = B\}$
- the leaves : Leav $(\mathcal{X}) = \{A \in \mathcal{X} \mid \forall B \in \mathcal{X} : B \subset A \Rightarrow A = B\}$
- the predecessors :  $Pred(A) = \{B \in \mathcal{X} \mid A \subset B\}$
- the immediate predecessors :  $ImPred(A) = \{B \in \mathcal{X} \mid \{U \mid U \in \mathcal{X}, A \subset U \text{ and } U \subset B\} = (A, B)\}$
- the successors :  $Succ(A) = \{B \in \mathcal{X} \mid B \subset A\}$
- the immediate successors :  $ImSucc(A) = \{B \in \mathcal{X} \mid \{U \mid U \in \mathcal{X}, B \subset U \text{ and } U \subset A\} = (A, B)\}$
- the uncles : uncle(A) = { $B \in \mathcal{X} \mid \text{ImPred}(B) \in \text{Pred}(A), B \notin \text{Pred}(A), \text{ImPred}(B) \neq \text{ImPred}(A)$ }
- the brothers : brother(A) = { $B \in \mathcal{X} \mid \text{ImPred}(B) \in \text{Pred}(A)$ ,  $B \notin \text{Pred}(A)$ , ImPred(B) = ImPred(A)}

< 🗗 🕨 🔸

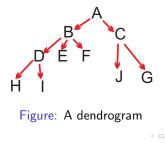
## Dendrograms

### Definition

 $\mathcal{X}$  is a dendrogram if and only if the set Pred(A) of the predecessors of A, with the order relation induced by  $\subset$  is a total order.

The maximal element of this family is a summit, which is the unique summit containing A. The smallest element is ImPred(A), the father of A, which is unique.

The arcs point from each element of the dendrogram to its immediate successor.



The following properties are equivalent:  
1)
$$\mathcal{X}$$
 is a dendrogram  
2)  $U, V, A \in \mathcal{X} : A \subset U$  and  $A \subset V \Rightarrow U \subset V$  or  $V \subset U$   
3)  $U, V \in \mathcal{X} : U \nsubseteq V$  and  $V \oiint U \Rightarrow U \cap V = \emptyset$ 

æ

Image: Image:

Due to their particular properties, the closed balls of an ultrametric distance function form a dendrogram. Consider a particular closed ball  $A = \text{Ball}(p, \rho)$ . We have to show that Pred(A) is completely ordered for  $\subset$ . Consider two predecessors of A, a ball  $B = \text{Ball}(q, \lambda)$  and a ball  $C = \text{Ball}(s, \mu)$ . As the node p belongs to both balls B and C, it is also center of these balls. Thus B and C are two balls with the same center p, and  $\text{Ball}(p, \lambda \land \mu) \subset \text{Ball}(p, \lambda \lor \mu)$ .

## Creation of a dendrogram of lakes

э

If the shortest path between  $\Omega$  and p passes through q we have  $\tau_p = \tau_q \lor d(p,q)$  if not  $\tau_p < \tau_q \lor d(p,q)$ .

#### Lemma

Any two nodes of an edge weighted graph verify  $\tau_p \leq \tau_q \lor d(p,q)$ .

Suppose that if  $d(p,q) < \tau_p$ . Then  $d(p,q) < \tau_p \leq \tau_q \lor d(p,q)$  implies  $\tau_p \leq \tau_q$ . So  $d(p,q) < \tau_p \leq \tau_q$  which similarly implies  $\tau_q \leq \tau_p$ . Hence  $\tau_p = \tau_q$  which is compatible with the laws of hydrostatics.

#### Lemma

If two nodes p and q of an edge weighted graph verify  $d(p,q) < \tau_p$  or  $d(p,q) < \tau_q$ , then  $\tau_p = \tau_q$ .

▲ロト ▲圖ト ▲画ト ▲画ト 三直 - のへで

#### Lemma

If an open ball  $Ball(p, \lambda)$  has one node with a flooding level  $\mu \ge \lambda$ , then its flooding level is uniform and equal to  $\mu$ .

**Proof:** Suppose that the node *s* in  $\operatorname{Ball}(p, \lambda)$  verifies  $\tau_s \geq \lambda$ . The node *s* as any node of an open ball is center of this ball. If  $q \in \operatorname{Ball}(s, \lambda)$ , we have  $d(s, q) < \lambda \leq \tau_s$ . Applying the preceding lemma yields  $\tau_s = \tau_q$ . As this is true for each node of  $\operatorname{Ball}(p, \lambda)$ , we have shown that the flooding level is constant and equal to  $\tau_s$  on the entire ball  $\operatorname{Ball}(p, \lambda)$ . In particular if  $\tau_p = \lambda$ , then the flooding level in  $\operatorname{Ball}(p, \lambda)$  is equal to  $\lambda$ .

Define  $\varepsilon_e(X)$ : the lowest edge of the cocycle of X. For  $\tau_p = \lambda$ , the flooding level on  $\operatorname{Ball}(p, \lambda)$  is constant and equal to  $\lambda$ . As long as  $\tau_p < \varepsilon_e(\operatorname{Ball}(p, \lambda))$ , the extension of  $\operatorname{Ball}(p, \tau_p)$  remains the same, the flooding is uniform and equal to  $\tau_p$ ;  $\operatorname{Ball}(p, \tau_p)$  is a regional minimum, as all its nodes have the same weight, its the distance between its nodes is smaller than  $\tau_p$ , and its cocycle edges are higher than  $\tau_p$ .

For  $\tau_p = \varepsilon_e(\text{Ball}(p, \lambda)) = \mu$  there exists an edge in the cocycle of Ball $(p, \lambda)$  with a weight  $\mu$ . The closed ball Ball $(p, \mu)$  strictly contains Ball $(p, \lambda)$ . As  $\tau_p = \mu$ , then for any other node s in Ball $(p, \mu)$  we have  $\tau_s \leq \tau_p \lor d(p, s) \leq \mu$ . However on Ball $(p, \mu) \subset$  Ball $(p, \mu)$  the flood is constant and equal to  $\tau_p = \mu$ . For any value  $\sigma > \mu$ , we have  $\text{Ball}(p, \mu) \subset \overset{\circ}{\text{Ball}}(p, \sigma)$ ; if  $\tau_p = \sigma$  the flood is constant on  $\overset{\circ}{\text{Ball}}(p, \sigma)$  and as  $\text{Ball}(p, \mu) \subseteq \overset{\circ}{\text{Ball}}(p, \sigma)$  the flood also is constant on  $\text{Ball}(p, \mu)$ .

In fact as long as  $\sigma < \varepsilon_e(\text{Ball}(p, \mu))$ ,  $\text{Ball}(p, \mu) = \check{\text{Ball}}(p, \sigma)$ .

#### Lemma

If there is at least one node with a weight  $\mu$  in a closed ball  $Ball(p, \mu)$  of level  $\mu$ , all other nodes in this ball have a flooding level  $\leq \mu$ .

The diameter of Y is  $\lambda$ . Such a closed ball is called lake zone of level  $\lambda$ , as the level of flooding inside is  $\leq \lambda$ .

# The growing extension of a lake containing a particular node

The extension of the lakes containing a node p as its flooding level  $\eta$  increases is:

- for  $\eta < \varepsilon_{ne}p$ , the lake  $X_0 = \{p\}$  is a regional minimum lake.

- for  $\eta = \varepsilon_{ne}p$ , the lake  $X_1 = \text{Ball}(p, \varepsilon_{ne}p)$  is a lake zone. The flood level is equal to  $\eta$  on  $X_0$  and  $\leq \eta$  everywhere else on  $X_1$ . We have  $\text{diam}(X_1) = \varepsilon_{ne}p = \varepsilon_e X_0$ .

- for diam( $X_1$ )  $< \eta < \varepsilon_e X_1$ , the lake  $X_2 = \text{Ball}(p, \eta)$  is a regional minimum lake with the extension  $X_1$ .

- for  $\eta = \varepsilon_e X_1$ , the lake  $X_2 = \text{Ball}(p, \varepsilon_e X_1)$  is a lake zone. The flood level is equal to  $\eta$  on  $X_1$  and  $\leq \eta$  everywhere else on  $X_2$ . We have  $\text{diam}(X_2) = \varepsilon_e X_1$ 

- the alternating series of regional minima lakes and lake zones goes on until all nodes of N are flooded.

- ...

We now define the level of the dominated flooding under  $\omega$  on the dendrogram of the lakes. Define  $\omega(X)$  the smallest value taken by the ceiling function  $\omega$  on X. The lakes containing the node p form an increasing series of nested sets  $\kappa^{(n)}\{p\}$ , the smallest being  $\{p\}$ , the largest being the root  $\kappa^{(m)}\{p\}$  of the dendrogram.

The operator  $\omega(X)$  is decreasing and the operator diam(X) increasing with X. As the series  $\kappa^{(n)}\{p\}$  is increasing with n, we get a series of decreasing values  $\omega(\kappa^{(n)}\{p\})$  and a series of increasing values diam $(\kappa^{(n)}\{p\})$ :

a) as the set  $\{p\}$  has no inside edge, we have diam $(\kappa^{(0)}\{p\}) = \text{diam}\{p\} = -\infty$ . Hence  $\omega\{p\} > \text{diam}\{p\} = -\infty$ 

b) if  $\kappa^{(m)}\{p\}$  is the root of the dendrogram and at the root we still have  $\omega(\kappa^{(m)}\{p\}) > \operatorname{diam}(\kappa^{(m)}\{p\})$ , i.e. the ceiling of p is higher than the root of the dendrogram, then  $\tau_p = \omega(\kappa^{(m)}\{p\})$ 

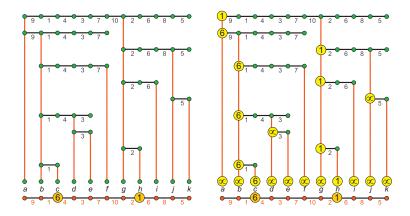
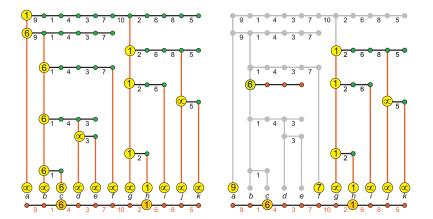


Figure: Left: Dendrogram associated to a MST. All nodes have a ceiling function equal to  $\infty$  excepting the nodes *c* and *h*, with values 6 and 1. Right: Each node of the dendrogram is assigned a ceiling value equal to the minimum ceiling value of all leaves it contains c) In the other cases, there exists an index k < m such that:  $\omega(\kappa^{(m)}\{p\}) \leq \operatorname{diam}(\kappa^{(m)}\{p\})$ , let  $k \leq m$  be the smallest index for which  $\omega(\kappa^{(k)}\{p\}) \leq \operatorname{diam}(\kappa^{(k)}\{p\})$  (rel. 1)  $\operatorname{diam}(\kappa^{(k-1)}\{p\}) < \omega(\kappa^{(k-1)}\{p\}) \leq \operatorname{diam}(\kappa^{(k)}\{p\}) = \varepsilon_e(\kappa^{(k-1)}\{p\})$ The previous relation implies that  $\tau_{\kappa^{(k-1)}\{p\}} = \omega(\kappa^{(k-1)}\{p\})$  and on  $\kappa^{(k)}\{p\}$  the maximal flooding level is  $\operatorname{diam}(\kappa^{(k)}\{p\})$ . In particular if Y is a brother of  $\kappa^{(k-1)}\{p\}$  then Y is the root of a sub-dendrogram which may be processed independently, with  $\omega(Y) = \operatorname{diam}(\kappa^{(k)}\{p\} \land \omega(Y))$  Each uncle  $Y_i$  of  $\kappa^{(k)}\{p\}$  with a father  $\kappa^{(l)}\{p\}$ , l > k becomes the root of sub-dendrogram which may be processed independently, and with a ceiling level  $\omega(Y_i) = \operatorname{diam}(\kappa^{(l)}\{p\} \land \omega(Y_i))$ . This process cuts the upstream of  $\kappa^{(k)}\{p\}$  in a number of sub-dendrograms which may then be processed independently one from another.



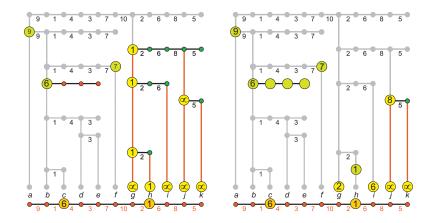
3

Image: A matrix and a matrix

2

135 / 165

What about the lake containing the node c ?The smallest index for which  $\omega(\kappa^{(k)}\{c\}) \leq \operatorname{diam}(\kappa^{(k)}\{c\})$ , is k = 3, with  $\kappa^{(3)}\{c\} = [b, c, d, e, f]$  having a diameter 7, whereas  $\omega(\kappa^{(3)}\{c\}) = 6$ . For k = 2, we get  $\kappa^{(2)}\{c\} = [b, c, d, e]$  having a diameter 4, whereas  $\omega(\kappa^{(2)}\{c\}) = 6$ . Hence:  $\kappa^{(2)}\{c\} = [b, c, d, e]$  is  $\tau_c = \tau_{\kappa^{(2)}\{c\}} = \omega(\kappa^{(2)}\{c\}) = 6$ 

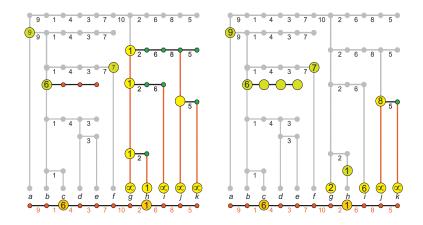


<ロ> (日) (日) (日) (日) (日)

All ancestors of  $\kappa^{(2)}\{c\}$  may be pruned. For k > 2,  $\kappa^{(k)}\{c\}$  is an ancestor of c, the flooding level of all its immediate successors which are not ancestors of c, that is, brothers of  $\kappa^{(k-1)}\{c\}$  is lower or equal than diam $(\kappa^{(k)}\{c\})$ . The edge linking each brother Y of  $\kappa^{(k-1)}\{c\}$  with its father  $\kappa^{(k)}\{c\}$  is cut; like that Y becomes the root of a sub-dendrogram; as its flooding level is lower or equal than diam $(\kappa^{(k)}\{c\})$ , one sets  $\omega(Y) = \omega(Y) \wedge \text{diam}(\kappa^{(k)}\{c\})$ . On the same time all ancestors of  $\kappa^{(2)}\{c\}$  and the edges linking them are suppressed. The set  $\kappa^{(2)}{c} = [b, c, d, e]$  got its flooding level 6 and its upstream is pruned:

-  $\kappa^{(3)}{c} = [b, c, d, e, f]$  is suppressed and the node  ${f}$  becomes the root of sub-dendrogram, with a ceiling value  $\omega({f}) = \omega({f}) \wedge \operatorname{diam}(\kappa^{(3)}{c}) = 7$ . As the sub-dendrogram is reduced to a node, its ceiling value is its flooding value, 7.  $-\kappa^{(4)}\{c\} = [a, b, c, d, e, f]$  is suppressed and the node  $\{a\}$  becomes the root of sub-dendrogram, with a ceiling value  $\omega(\{a\}) = \omega(\{a\}) \wedge \operatorname{diam}(\kappa^{(4)}\{c\}) = 9$ . As the sub-dendrogram is reduced to a node, its ceiling value is its flooding value, 9. -  $\kappa^{(5)}{c} = N$  is suppressed and the node [g, h, i, j, k] becomes the root of sub-dendrogram, with a ceiling value  $\omega([g, h, i, j, k]) = \omega([g, h, i, j, k]) \wedge \operatorname{diam}(\kappa^{(5)}\{c\}) = 1$ . The ceiling value  $\omega([g, h, i, j, k])$  of the root being known, flooding this subdendrogram becomes completely independent from the rest of the processing.

イロト 不得 とくほ とくほう しゅ



э.

<ロ> (日) (日) (日) (日) (日)

The smallest index for which  $\omega(\kappa^{(k)}\{h\}) \leq \operatorname{diam}(\kappa^{(k)}\{h\})$ , is k = 1, with  $\kappa^{(1)}{h} = [g, h]$ . The flooding level of  $\kappa^{(0)}{h} = [h]$  is  $\tau_c = \tau_{\kappa^{(0)}\{c\}} = \omega(\kappa^{(0)}\{h\}) = 1$  and the upstream of h can be pruned. -  $\kappa^{(1)}{h} = [g, h]$  is suppressed and the node  $\{g\}$  becomes the root of sub-dendrogram, with a ceiling value  $\omega(\lbrace g \rbrace) = \omega(\lbrace g \rbrace) \wedge \operatorname{diam}(\kappa^{(1)}\lbrace h \rbrace) = 1$ . As the sub-dendrogram is reduced to a node, its ceiling value is its flooding value: 1. -  $\kappa^{(2)}{h} = [g, h, i]$  is suppressed and the node  $\{i\}$  becomes the root of sub-dendrogram, with a ceiling value  $\omega(\{i\}) = \omega(\{i\}) \wedge \operatorname{diam}(\kappa^{(2)}\{h\}) = 6$ . As the sub-dendrogram is reduced to a node, its ceiling value is its flooding value, 6.  $-\kappa^{(3)}{h} = [g, h, i, j, k]$  is suppressed and the node [j, k] becomes the root of sub-dendrogram, with a ceiling value  $\omega([i, k]) = \omega([i, k]) \wedge \operatorname{diam}(\kappa^{(3)}{h}) = 8$ . The node [j, k] being the root of a dendrogram with a ceiling value higher than its diameter gets flooded at the level of the ceiling value : 8. This achieves the process since there are no more sub-dendrograms to process.

・ロン ・聞と ・ 聞と ・ 聞と

## Contraction/expansion of flat zones and dendrogram flooding

- $G_n$ : a node weighted graph with a ground level f and a ceiling function  $\omega$ . Consider an edge (p, q) such that  $f_p = f_q$ . Contracting this edge = - suppressing the edge (p, q)
- merging both nodes into a new node s, with a ground value  $f_s=f_p=f_q$  and a ceiling value  $\omega_s=\omega_p\wedge\omega_q$
- suppress the edge linking a node t with p or q and replace it with an edge (t, s).
- After contraction we get a new graph  $G_n'$  with a ground level f' and a ceiling function  $\omega'$

# Contracting inside edges of flat zones does not modify the dominated flooding

- au : the highest flooding of  $G_n$  under  $\omega$
- au' : the highest flooding of  $G_n$  under  $\omega'$
- We show that on all common nodes au= au' and that  $au_p= au_q= au_s$
- We first remark that replacing  $\omega_p$  and  $\omega_q$  by  $\omega_p \wedge \omega_q$  does not change  $\tau$ . The shortest path linking *s* in  $G'_n$  with  $\Omega$  is the same as the shortest path linking  $\Omega$  with *p* or *q* in  $G'_n$ .

### Corollary

All inside edges of flat zones may be contracted and produce each a unique node without changing the highest flooding of the graph.

## Combining contractions and flooding on a dendrogram

In the following figure A we want to construct the highest flooding of the red function under the green one. The contraction of the flat zones produces in fig.B a graph G' with 5 nodes (a, b, c, d, e) with ground levels (0, 4, 1, 2, 0) and ceiling levels (0, 5, 3, 3, 1). The edges are then weighted with  $\delta_{en}n$ , yielding the weights indicated in blue in fig.B. Fig.C presents the associated dendrogram. We first flood the node e.  $\kappa^{(0)}(e) = \{e\}$ . As  $\operatorname{diam}(\kappa^{(0)}\{e\}) < \omega(\kappa^{(0)}\{e\}) \leq \operatorname{diam}(\kappa^{(1)}\{e\})$ , the flooding level of e is  $\omega(\kappa^{(0)}\{e\} = 1$ .

e has two brothers, the nodes c and d, roots of subdendrograms reduced to 1 node. Having the same ceiling value, we have

$$\tau_d = \tau_c = \operatorname{diam}(\kappa^{(1)} \{ e \} \land \omega(c) = 2 \land 3 = 2.$$

e has two uncles, the nodes a and b, roots of subdendrograms reduced to 1 node. Their flooding value is  $\tau_a = \text{diam}(\kappa^{(1)}\{a\} \wedge \omega(a) = 4 \wedge 0 = 0$  and  $\tau_b = \text{diam}(\kappa^{(1)}\{b\} \wedge \omega(b) = 4 \wedge 5 = 4$ 

During the expansion, each node of the graph G' is replaced by the flat zone of the graph G it represents, with identical flooding values, as illustrated in fig.F.

### Illustration

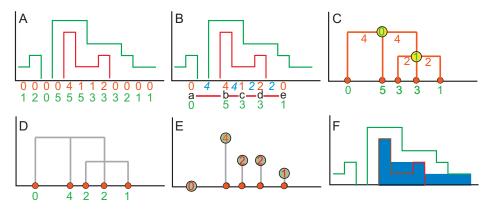


Figure: Creation of flat zones and flooding of the edge associated graph

22 March 2013 146 / 165

Given a node weighted graph  $G_n$  we assign to the edges weights equal to  $\delta_{en}n$ . It is beneficial to combine the construction of the MST of this and simultaneously contract the edges.

#### Initialisation

Create a tree with one node p of the graph.

#### Expansion

As long as the tree does not contain all nodes of the graph:

Chose the lowest edge (q, s) in the cocycle of T, such that  $q \in T$  and  $s \notin T$ .

If  $f_q=f_s$  : contract the edge (q,s) on q and link q with the neighbors of s not yet in  ${\cal T}$ 

Else : append the node s to the tree:  $T = T \cup \{s\}$ 

## Relations between floodings on edge and node weighted graphs.

# Flooding a node weighted graph = flooding an edge weighted graph

 $G_e$  = edge weighted graph,  $G^n$  = node weighted graph,  $G_e^n$  = node and edge weighted graph

For  $G_e: \eta_p = (\varepsilon_{ne}e)_p$ , i.e. the weight of the lowest edge adjacent to the node p.

#### Theorem

Consider  $G_n$ , a node weighted graph, and  $G_e$ , the derived edge weighted graph with edge weights  $e = \delta_{en}n$ . We then have the following equivalences:  $\{\tau \ge n \text{ e-flooding of } G_e\} \Leftrightarrow \{\tau \text{ n-flooding of } G_n\}(eq-1)$ 

This theorem has important algorithmic consequences. For constructing the highest flooding on the node weighted graph under a ceiling function  $\omega$  we may construct the highest flooding of the edge weighted graph  $G_e$  under  $\omega$ .

▲□▶ ▲□▶ ▲□▶ ▲□▶ = ののの

Consider an edge weighted graph  $G_e$ . The waterfall flooding consists in assigning to each node of the graph a flooding level equal to the lowest adjacent edge:  $\eta = \varepsilon_{ne}e$ . The function  $\eta$  is a particular flooding of  $G_e$ .

#### Lemma

In an edge weighted graph  $G_e$ , a function  $\tau$  on the nodes is a valid flooding if and only if  $\tau \lor \eta$  is a valid flooding: { $\tau$  e-flooding of  $G_e$ }  $\Leftrightarrow$  { $\tau \lor \eta$  e-flooding of  $G_n$ }

This equivalence has the following consequence:

- by replacing  $au_{
  ho} < \eta_{
  ho}$  by  $\eta_{
  ho}, \, au$  remains an e-flooding of  $G_e$
- by replacing  $au_p = \eta_p$  by  $au'_p < \eta_p$ , au' remains an e-flooding of  $G_e$

Given  $G_e$ : an edge weighted graph verifying  $e = \gamma_e e = \delta_{en} \varepsilon_{ne} e$ . We create a node weighted graph  $G^{\eta}$  with node weights  $\eta = \varepsilon_{ne} e$ . As the edge weights verify  $e = \delta_{en} \varepsilon_{ne} e = \delta_{en} \eta$  we have the equivalence 1:  $\{\tau \lor \eta \ge \eta \text{ e-flooding of } G_e\} \Leftrightarrow \{\tau \lor \eta \text{ n-flooding of } G^{\eta}\}$ . Equivalence eq-2:  $\{\tau \text{ e-flooding of } G_e\} \Leftrightarrow \{\tau \lor \eta \text{ e-flooding of } G_e\}$ . Thus:

#### Theorem

Consider  $G_e$ , an edge weighted graph where each edge is the lowest edge of one of its extremities (invariant by  $\gamma_e$ ), and  $G^{\eta}$ , the derived node weighted graph with node weights  $\eta = \varepsilon_{ne}e$ , we then have the following equivalences: { $\tau$  e-flooding of  $G_e$ }  $\Leftrightarrow$  { $\tau \lor \eta$  e-flooding of  $G_e$ }  $\Leftrightarrow$  { $\tau \lor \eta$ *n*-flooding of  $G^{\eta}$ }.

3

Consider now  $\tau' \ge \eta$  an e-flooding of  $G_e$  which is also a n-flooding of  $G_{\eta}$ . Consider a subset of nodes A of N. We define a new node distribution as follows:

- on  $A: \tau \leq \eta$
- on  $N/A: \tau = \tau'$

This distribution verifies  $\tau' = \tau \lor \eta$ . Hence, as stated in the preceding theorem,  $\tau$  and  $\tau'$  are both e-floodings of  $G_e$ .

Given a node weighted graph  $G^n$ , we define  $\varphi_n n = \varepsilon_{ne} \delta_{en} n = \eta$ . On the other hand we assign to the edges the weights  $e = \delta_{en} n = \delta_{en} \varepsilon_{ne} \delta_{en} n = \delta_{en} \varphi_n n = \delta_{en} \eta$ . The graph  $G^{\eta}$  is a flooding graph as  $\eta = \varepsilon_{ne} e$  and  $e = \delta_{en} \eta$ . The preceding results apply to the graphs  $G_e$ ,  $G^{\eta}$  and  $G^e$ .  $\{\tau \text{ n-flooding of } G^n\} \Leftrightarrow \{\tau \ge n \text{ e-flooding of } G_e\} \Leftrightarrow \{\tau \lor \eta \text{ e-flooding of } G^{\eta}\}$ .

#### Theorem

A flooding  $\tau$  is the highest flooding of an edge weighted graph  $G_e$  under a ceiling function  $\omega$  if and only if  $\tau \lor \eta$  is the largest flooding of  $G_e$  under the ceiling function  $\omega \lor \eta$ .

#### Theorem

If  $\chi$  is the largest flooding of  $G_e$  under the ceiling function  $\omega \lor \eta$ , then  $\chi \land \omega$  is the largest flooding of  $G_e$  under the ceiling function  $\omega$ .

- 4 伺 ト 4 ヨ ト 4 ヨ ト

Combining contraction.expansion of flat zones and closing of the isolated regional minima  $G^n$ : a node weighted graph with ground level f and ceiling level  $\omega$ . Contracting its flat zones produces a graph G' = KG with ground level f and ceiling level  $\omega_1$ . Each regional minimum of f becomes an isolated regional minimum.

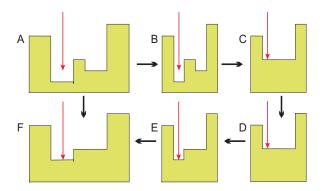
The closing  $\varphi_n$  in the graph G' fills each regional minimum to the level of its lowest neighbor.

 $\{\tau \text{ n-flooding of } G'^n \text{ dominated by } \omega_1\} \Leftrightarrow \{\tau_1 = \tau \lor \eta \text{ n-flooding of } G'^\eta \text{ dominated by } \omega_1 \lor \eta\}.$ 

If we expand the nodes of G', replacing each node by the flat zone it represents, with the flood distribution  $\tau_1 \wedge \omega$ , we get the flooding of G dominated by  $\omega$ .

#### A commutative diagram illustrates the process.

## Contraction and closing of the isolated regional minima : illustration



- Fig.A: presents a topographic surface f and a ceiling function equal to  $\infty$  everywhere except at the position of the red arrow.
- Fig.B: is obtained by contracting the flat zones with the associated ceiling function  $\omega_1$ , giving a graph KG
- Fig.C: is obtained by the closing  $\varphi_n$ , which closes the isolated regional minima with a new ceiling function  $\varphi_n f \lor \omega$
- Fig.D: is the flooding  $\tau$  of  $\varphi_n f$  under  $\varphi_n f \lor \omega$
- Fig.E: the minimum  $\tau_1 \wedge \omega_1$  (which is also the flooding of the function in fig.B under the ceiling  $\omega_1$ )
- Fig.F: the expansion of fig.E yields the flooding of f under  $\omega$ .

The preceding sequence of transformations constructs a dominated flooding of a complex function thanks to the flooding of a simpler function. The same sequence may be applied for flooding this simpler function. And so on, producing a sequence of simpler and simpler functions to flood.

## Constructing a local flooding on a topographic graph

Using all results established above we propose an algorithm for local floodings.

 $G_n$  a topographic graph, f the ground level. We want to know the flooding at a given node p for the dominated flooding of f under  $\omega$ . Assigning to the edges the weights  $\delta_{en}f$  produces a flooding graph: the lowest adjacent edge of each node has the same level than this node.

If p belongs to an isolated regional minimum and  $\omega_p \leq (\varphi_n f)_p$ , then  $\tau_p = \omega_p$ 

If not, we flood the function  $\varphi_n f$  under the ceiling function  $\omega \lor \varphi_n f$  and get a flooding  $\tau'$ . The desired flooding is  $\tau = \tau' \land \omega$ .

For  $X = \text{Ball}(p, f_p)$ : if  $\omega(X) \le f_p$ , then p is in the upstream of a lake and is dry:  $\tau_p = f_p$ If  $\omega(X) > f_p$ , we search the lake containing p: Until diam $(X) < \omega(X) \le \varepsilon_e(X)$  do  $X = \kappa(X) = \text{Ball}(p, f_p)$ 

The lake containing p is X at a level  $\omega(X)$ .

If X is lake of the flooding, its uphill up to a level  $\mu$  is constructed with the procedure "up\_hill(X,  $\mu$ )"

While  $\varepsilon_e(X) \leq \mu$   $\lambda = \varepsilon_e(X)$   $Y = \text{Ball}(X, \varepsilon_e(X))$ For each connected component  $Z_i$  of Y/X for which  $\lambda > f_{Z_i}$ : if  $\omega(Z_i) \geq \lambda : \tau_{Z_i} = \lambda$ else if  $\omega$  is minimal at q in  $Z_i$ : apply "up-hill(Ball $(q, \omega_q), \lambda$ )" For each  $p \in Y/X$  verifying  $f_p = \lambda : \tau_p = \lambda$  X = YEnd While The comparison between floodings on edge weighted graphs and on node weighted graphs has given a better insight in both of them.

New algorithms have been derived allowing to chose the one is best suited for each application (type or processor, hardware, parallel processing, etc.)