The 1/4-Core of the Uniform Bin Packing Game Is Nonempty

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Abstract. A cooperative bin packing game is an N-person game, where the player set N consists of k bins of capacity 1 each and n items of sizes $a_1, \cdots, a_n$. The value of a coalition of players is defined to be the maximum total size of items in the coalition that can be packed into the bins of the coalition. We adopt the taxation model proposed by Faigle and Kern (1993) [6] and show that the 1/4-core is nonempty for all instances of the bin packing game. This strengthens the main result in [3].

1 Intorduction

Since many years, logistics and supply chain management are playing an important role in both industry and our daily life. In view of the big profit generated in this area, the question therefore arises how to "fairly" allocate profits among the "players" that are involved. Take online shopping as an example: Goods are delivered by means of carriers. Generally, shipping costs are proportional to the weight or volume of the goods, and the total cost is basically determined by the competitors. But there might be more subtle ways to compute "fair" shipping cost (and allocation between senders and receivers). It is natural to study such allocation problems in the framework of cooperative games. As a first step in this direction we analyze a simplified model with uniform packet sizes as described in more detail in section 2.

A *cooperative game* is defined by a tuple $\langle N, v\rangle$, where N is a set of *players* and $v : 2^N \to \mathbb{R}$ is a *value function* satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a *coalition* and N is called the *grand coalition.* The usual goal in cooperative games is to " fairly" allocate the total gain $v(N)$ of the grand coalition N among the individual players. A well known concept is the *core* of a cooperative game, defined by all *allocation vectors* $x \in \mathbb{R}^N$ satisfying

(i) $x(N) = v(N)$,
(ii) $x(S) \geq v(S)$ for all $S \subseteq N$.

As usual, we abbreviate $x(S) = \sum_{i\in S} x_i$.

We say a cooperative game is balanced if there exists a core allocation for any instance. Unfortunately, many games are not balanced. Players in a non-balanced game may not cooperate because no matter how the total gain is distributed,

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there will always be some coalition S with $x(S) < v(S)$, i.e., it gets paid less than it could earn on its own. For this case, one naturally seeks to relax the condition (ii) above in such a way that the modified core becomes nonempty. Faigle and Kern [6] introduced the multiplicative ϵ-core as follows. Given $\epsilon > 0$, the ϵ-core consists of all vectors $x \in \mathbb{R}^N$ satisfying condition (i) above together with

(ii') $x(S) \geq (1-\epsilon)v(S)$ for all subsets $S \subseteq N$.

We can interpret ϵ as a tax rate in the sense that coalition S is allowed to keep only $(1-\epsilon)v(S)$ on its own. Thus, the ϵ-core provides an ϵ-approximation to balancedness. If the value function v is nonnegative, the 1-core is obviously nonempty. In order to approximate the core as close as possible, one would like to have the taxation rate ϵ as small as possible while keeping the ϵ-core nonempty. Discussions of the ϵ-core allocation for other cooperative games, say, facility location games, TSP games etc. can be found in [10],[11],[8],[5].

As motivated at the beginning of this paper, we study a specific game of the following kind: There are two disjoint sets of players, say, A and B. Each player $i \in A$ possesses an item of value/size a_i, for $i = 1, \cdots, n$, and each player $j \in B$ possesses a truck /bin of capacity 1. The items produce a profit proportional to their size a_i if they are brought to the market place. The value $v(N)$ of the grand coalition thus represents the maximum profit achievable. How should $v(N)$ be allocated to the owners of the items and the owners of the trucks?

Bin packing games were first investigated by Faigle and Kern [6]. In their paper, they considered the non-uniform bin packing game (i.e., bin capacities are distinct) and showed that the 1/2-core is nonempty if any item can be packed to any bin. Later, researchers focused on the uniform case, where all bins have the same capacity. Woeginger [17] showed that the 1/3-core is always nonempty. Recently, Kern and Qiu [13] improved this result to $1/3-1/108$. Eventually, however, it turned out that the standard (unmodified) version of the set packing heuristic, combined with the matching arguments already used in [7] and yet another heuristic based on replacing complete bins as described in detail in section 3 led to the improvement $\epsilon_N \geq 1/4$. As $v'(N) \geq v(N)$, this amounts to a strengthening of the main result in [3], hopefully also providing additional insight.

Other previous results on bin packing games can be summarized as follows. In [7], Faigle and Kern show that the integrality gap, defined by the difference of the optimum fractional packing value and the optimum integral packing value (*cf.* section 2) is bounded by 1/4, if all item sizes are strictly larger than 1/3, thereby implying that the 1/7-core is nonempty in that case (which was independently shown by Kuipers [15]). Moreover, in the general case, given a fixed $\epsilon \in (0,1)$, they prove that the ϵ-core is always non-empty if the number of bins is sufficiently large ($k \geq O(\epsilon^{-5})$). Liu [16] gives complexity results on testing emptiness of the core and core membership for bin packing games, stating that both problems are NP-complete. Also, Kern and Qiu [14] studied the non-uniform bin packing game and showed that the 1/2-core is nonempty for any instance. Moreover, the problem of approximating the maximum packing value $v(N)$ is also studied in literature (called "multiple subset sum problem"): PTAS and 3/4 approximation algorithms are proposed in [2] and [3]. Other variants of the bin packing game can also be found in [1], [18], [4], [9] etc..

The rest of the paper is organized as follows. In section 2, we give a formal definition of bin packing games and introduce fractional packings. In section 3, we prove that the 1/4-core is always nonempty. Finally, in section 4, we remark on some open problems.

2 Preliminaries

A *bin packing game* N is defined by a set A of n items $1, 2, \cdots, n$ of sizes $a_1, a_2, \cdots, a_n$, and a set B of k bins, of capacity 1 each, where we assume, w.l.o.g., $0 \le a_i \le 1$.

A *feasible packing* of an item set $A' \subseteq A$ into a set of bins $B' \subseteq B$ is an assignment of some (or all) elements in A' to the bins in B' such that the total size of items assigned to any bin does not exceed its capacity. Items that are assigned to a bin are called *packed* and items that are not assigned are called *not packed*. The *value* or *size* of a feasible packing is the total size of packed items.

The player set N consists of all items and all bins. The value $v(S)$ of a coalition $S \subseteq N$, where $S = A_S \cup B_S$ with $A_S \subseteq A$ and $B_S \subseteq B$, is the maximum value of all feasible packings of A_S into B_S. A corresponding feasible packing is called an *optimum packing*.

Let F be an item set, and denote by $a_F = \sum_{i \in F} a_i$ the *value* of F. F is called a *feasible set* if $a_F \le 1$. Denote by $\mathcal{F}$ the set of all feasible sets, w.r.t. all items of N. Then the total earning $v(N)$ of the grand coalition N equals

$$\begin{aligned}
&\max \sum_{F \in \mathcal{F}} a_F y_F \\
&\text{s.t.} \sum_{F \in \mathcal{F}} y_F \le k, \\
&\sum_{F \ni i} y_F \le 1 \quad (i = 1, 2, \cdots, n), \\
&y_F \in \{0, 1\}\,.
\end{aligned} \tag{2.1}$$

The value $v'(N)$ of an *optimum fractional packing* is defined by the relaxation of (2.1), i.e.,

$$\begin{aligned}
&\max \sum_{F \in \mathcal{F}} a_F y_F \\
&\text{s.t.} \sum_{F \in \mathcal{F}} y_F \le k, \\
&\sum_{F \ni i} y_F \le 1 \quad (i = 1, 2, \cdots, n), \\
&y_F \in [0, 1].
\end{aligned} \tag{2.2}$$

A *fractional packing* is a vector y satisfying the constraints of the linear program (2.2). We call a feasible set F *selected/packed* by a packing y' if $y'_F > 0$. Accordingly, we refer to the original "feasible packing" as *integral packing*, which meets the constraints of (2.1). To illustrate the definition of the fractional packing, we consider an example as follows.

Example. Given 2 bins and 4 items of sizes $a_i = 1/2$ for $i = 1, 2, 3$ and $a_4 = 1/2 + \epsilon$, with a small $\epsilon > 0$.

Obviously, packing items 1, 2 into the first bin and packing item 4 to the second bin results in an optimum integral packing of total value $v(N) = 3/2 + \epsilon$. Let $F_1 = \{1, 2\}$, $F_2 = \{2, 3\}$, $F_3 = \{1, 3\}$, $F_4 = \{4\}$. By solving the linear program (2.2), the optimum fractional packing $y' = (y'_F)$ selects $F_1, \cdots, F_4$ with a fraction 1/2 each, resulting in a value

$$v'(N) = \sum_{j=1}^{4} y'_{F_j} a_{F_j} = \frac{7}{4} + \frac{\epsilon}{2} > v(N).$$

An intriguing problem is to find the "minimal" taxation rate $\epsilon_{\min}$ such that the $\epsilon_{\min}$-core is nonempty for all instances of the bin packing game. Due to the following observation of Faigle and Kern in [7], this amounts to bounding the *relative gap* $(v'(N) - v(N))/v'(N)$ for any bin packing game instance N.

Lemma 1 ([7]). *Given $\epsilon \in (0, 1)$ and a bin packing game N, the ϵ-core $\neq \emptyset$ if and only if $\epsilon \geq 1 - v(N)/v'(N)$.*

We let ϵ_N denote the relative gap of N. Trivially, if all items are packed in a feasible integral packing, we get $v(N) = v'(N)$, implying that $\epsilon_N = 0$, so the core is nonempty. Thus let us assume in what follows that no feasible integral packing packs all items. Clearly, any feasible integral packing y with corresponding packed sets $F_1, ..., F_k$ yields a lower bound $v(N) \geq w(y) = \sum_{i=1}^{k} a_{F_i}$. In view of Lemma 1 we are particularly interested in integral packings y of value $w(y) \geq (1 - \epsilon)v'(N)$ for certain $\epsilon > 0$. For $\epsilon = 1/2$, such integral packings are easy to find by means of a simple *greedy packing* heuristic, that constructs a feasible set F_j to be packed into bin j in the following way: First order the available (yet unpacked) items non-increasingly, say, $a_1 \geq a_2 \geq ....$ Then, starting with $F_j = \emptyset$, keep adding items from the list as long as possible (i.e., $a_{F_j} + a_i \leq 1 \Rightarrow F_j \leftarrow F_j \cup \{i\}$, else proceed to $i + 1$). Clearly, this eventually yields a feasible F_j of size $> \frac{1}{2}$. Indeed - unless all items get packed - the final F_j has size $> 1 - a$, where a is the minimum size of an unpacked item. Applying greedy packing to all bins will exhibit an integral packing y with $a_{F_j} > \frac{1}{2}$ for all j, so $w(y) > k/2 \geq v'(N)/2$, thus proving non-emptiness of the $\frac{1}{2}$-core by Lemma 1.

A bit more work is required to exhibit integral packings with $w(y) \geq \frac{2}{3}v'(N)$ (*c.f.* [17], [13]). We end this section by mentioning two trivial cases, namely $k = 1$ and $k = 2$: Indeed, in case $k = 1$, we obviously have $v(N) = v'(N)$, thus $\epsilon_N = 0$ and in case $k = 2$, the bound $\epsilon_N \leq \frac{1}{4}$ can be seen as follows: Let y' be an optimal fractional solution and let F be a largest (most valuable) feasible set in the support of y'. Then the integral packing y that packs F into the first bin and fills the second bin greedily to at least 1/2 (as explained above) is easily seen to yield a value $w(y) \geq \frac{3}{4}w(y')$, so that $\epsilon_N \leq \frac{1}{4}$.

3 Proof of Non-emptiness of the 1/4-Core

Throughout this section, we assume that N is a smallest counterexample, i.e., a game with 1/4-core(N)= $\emptyset$ and $n + k$, the number of players is as small as possible. We start with a simplification, similar to that in [17] for the case $\epsilon = 1/3$. The following result is a special case of Lemma 2.3 in [13], but we include a proof for convenience:

Lemma 2. *All items have size $a_i > 1/4$.*

Proof. Assume to the contrary that some item, say, item n has size $a_n \leq 1/4$. Let $\bar{N}$ denote the game obtained from N by removing this item. Thus 1/4-core($\bar{N}$)$\neq \emptyset$ or, equivalently, $v(\bar{N}) \geq \frac{3}{4}v'(\bar{N})$. Adding item n to $\bar{N}$ can clearly not increase v' by more than a_n, i.e.,

$$v'(N) \leq v'(\bar{N}) + a_n. \tag{3.1}$$

Let $\bar{y}$ be any optimal integer packing for $\bar{N}$, i.e., $w(\bar{y}) = v(\bar{N})$. Then either item n can be packed "on top of $\bar{y}$" (namely when some bin is filled up to $\leq 1-a_n$) – or not. In the latter case, the packing y fills each bin to more than $1-a_n \geq 3/4$, thus $w(\bar{y}) \geq \frac{3}{4}k \geq \frac{3}{4}v'(N)$, contradicting the assumption that $\frac{1}{4}$-core(N)= $\emptyset$. Hence, item n can indeed be packed on top of $\bar{y}$, yielding $v(N) \geq w(\bar{y})+a_n = v(\bar{N})+a_n$. Together with (3.1) this yields $v'(N) - v(N) \leq v'(\bar{N}) - v(\bar{N})$ and hence

$$\epsilon_N = \frac{v'(N) - v(N)}{v'(N)} \leq \frac{v'(\bar{N}) - v(\bar{N})}{v'(\bar{N})} = \epsilon_{\bar{N}} \leq \frac{1}{4},$$

– a contradiction again. □

Note that this property implies each feasible set contains at most 3 items. In the following we reconsider (slight variants of) two packing heuristics that have been introduced earlier in [2] resp. [13]. The first one, which we call Item Packing, seeks to first pack large items, then small ones on top of these, without regarding the optimum fractional solution. The second one, which we call Set Packing, starts out from the optimum fractional solution y' and seeks to extract an integer packing based on the feasible sets that are (fractionally) packed by y'.

We first deal with Item Packing. Call an item i *large* if $a_i > 1/3$ and *small* otherwise. Let L and S denote the set of large resp. small items. If no misunderstanding is possible, we will also interpret L as the game N restricted to the large items (and k bins). Note that at most two large items fit into a bin. Thus packing L is basically a matching problem. This is why we can easily solve it to optimality and why the gap is fairly small (just like in the example from section 2). More precisely, Theorem 3.2 from [7] can be stated as

Lemma 3 ([7]). *$gap(L) = v'(L) - v(L) \leq \frac{1}{4}$.*

This motivates the following Item Packing heuristic:

Item Packing

· Compute an optimum integral packing y for L.
· Try to add as many small items on top of y as possible.

There is no need to specify how exactly small items are added in step 2. A straightforward way would be to sort the small items non-increasingly and apply some first or best fit heuristic. Let $\hat{F}_1, \cdots, \hat{F}_k$ denote the feasible sets in the integral packing $\hat{y}$ produced by Item Packing, i.e.,

$$\text{Output Item Packing: } \hat{y} \mathrel{\widehat{=}} (\hat{F}_1, \cdots, \hat{F}_k). \tag{3.2}$$

Note that, due to Lemma 3, Item Packing performs quite well w.r.t. the large items. Thus we expect Item Packing to perform rather well in case there are only a few small items or, more generally, if Item Packing leaves only a few small items unpacked. Let $S_1 \subseteq S$ denote the set of small items that get packed in step 2 and let $S_2 := S \backslash S_1$ be the set of unpacked items. We can then prove the following bounds:

Lemma 4. *$a_{\hat{F}_j} > \frac{2}{3}$ for all $j = 1, ..., k$. Hence, $v(N) > \frac{2}{3}k$ and $v'(N) > \frac{8}{9}k$.*

Proof. By definition, the first step of Item Packing produces an optimum integral packing of L of value $v(L)$. Thus the final outcome $\hat{y}$ has value $w(\hat{y}) = v(L) + a_{S_1}$. Hence $v(N) \geq v(L) + a_{S_1}$. The fractional value, on the other hand, is clearly bounded by $v'(N) \leq v'(L) + a_S$. Thus in case $S_2 = \emptyset$ (i.e., $S_1 = S$) we find

$$v'(N) - v(N) \leq v'(L) - v(L) \leq 1/4,$$

implying that

$$\epsilon_N = \frac{v'(N) - v(N)}{v'(N)} \leq \frac{1/4}{k/2} \leq \frac{1}{4} \quad \text{for } k \geq 2,$$

contradicting our assumption that 1/4-core(N)=$\emptyset$. (Note that for $k = 1$ we always have $v = v'$, i.e., $\epsilon_N = 0$, as remarked earlier in section 2.)

Thus we conclude that $S_2 \neq \emptyset$. But this means that the packing $\hat{y}$ produced by Item Packing fills each bin to more than 2/3, i.e., $a_{\hat{F}_j} > 2/3$ for all j (otherwise, if $a_{\hat{F}_j} \leq 2/3$, any item in S could be packed on top of $\hat{F}_j$). Hence $v(N) \geq w(\hat{y}) = \sum_j a_{\hat{F}_j} > \frac{2}{3}k$. Furthermore, due to our assumption that $\epsilon_N > 1/4$, we know that $v'(N) > \frac{4}{3}v(N) > \frac{8}{9}k$. □

As we have seen in the proof of Lemma 4, we may assume $S_2 \neq \emptyset$. The following result strengthens this observation:

Lemma 5. $|S_2| \geq \frac{2}{3}k - \frac{3}{4}$.

Proof. As in the proof of Lemma 4 we find

$$v'(N) \leq v'(L) + a_{S_1} + a_{S_2},$$
$$v(N) \geq v(L) + a_{S_1}.$$

Thus, using Lemma 3, we get

$$v'(N) - v(N) \leq a_{S_2} + \frac{1}{4} \leq \frac{|S_2|}{3} + \frac{1}{4}. \tag{3.3}$$

On the other hand, $\epsilon_N > 1/4$ and $v'(N) > \frac{8}{9}k$ (Lemma 4) imply

$$v'(N) - v(N) = \epsilon_N v'(N) > \frac{2}{9}k. \tag{3.4}$$

Combining (3.3) and (3.4), the bound on $|S_2|$ follows. □

Thus we are left to deal with instances where Item Packing leaves a considerable amount $|S_2|$ of small items unpacked. As it turns out, a completely different kind of packing heuristic, so-called Greedy Selection, first analyzed in [13] is helpful in this case. The basic idea is simple: We start with an optimum fractional packing y' of value $w(y') = v'(N)$. Let $F'_1, \cdots, F'_m$ denote the feasible sets in the support of y' (i.e., those that are fractionally packed) and assume that $a_{F'_1} \geq \cdots \geq a_{F'_m}$. Greedy Selection starts with $F_1 := F'_1$ and then, after having selected $F_1, \cdots, F_{j-1}$, defines F_j to be the first set in the sequence $F'_1, \cdots, F'_m$ that is disjoint from $F_1 \cup \cdots \cup F_{j-1}$. It is not difficult to see (*cf.* below) that we can select disjoint feasible sets $F_1, \cdots, F_r$ with $r = \lceil k/3 \rceil$ in this way.

Greedy Selection

(Input: Opt. fractional y' with supp $y' \widehat{=} \{F'_1, \cdots, F'_m\}$, $a_{F'_1} \geq \cdots \geq a_{F'_m}$)
Initiate: $\mathcal{F}'_1 := \{F'_1, \cdots, F'_m\}$
FOR $j = 1$ to $r = \lceil k/3 \rceil$ DO
 · $F_j \leftarrow$ first set in the list $\mathcal{F}'_j$
 · $\mathcal{F}'_{j+1} \leftarrow \mathcal{F}'_j \setminus \{F'_t \mid F'_t \cap F_j \neq \emptyset\}$
End

Note that Greedy Selection starts with $\mathcal{F}' = $ supp y', a system of feasible sets of total *length* $y'_{\mathcal{F}'} := \sum_{F' \in \mathcal{F}'} y'_{F'} = k$. In each step, since $|F_j| \leq 3$, we remove feasible sets F'_t of total length $\sum_{F'_t \in \mathcal{F}'_j, F'_t \cap F_j \neq \emptyset} y'_{F'_t}$ bounded by

$$\sum_{\substack{F'_t \in \mathcal{F}'_j \\ F'_t \cap F_j \neq \emptyset}} y'_{F'_t} \leq \sum_{i \in F_j} \sum_{\substack{F'_t \in \mathcal{F}' \\ F'_t \ni i}} y_{F'_t} \leq \sum_{i \in F_j} 1 \leq 3.$$

Actually, the upper bound ≤ 3 is strict since $F'_t = F_j$ is counted $|F_j|$ times (once for each $i \in F_j$).

Thus, in each step we remove feasible sets of total length at most 3, so we certainly can continue the construction for $k = \lceil k/3 \rceil$ steps.

Lemma 6. *Greedy Selection constructs feasible sets* $F_1, \cdots, F_r \in \mathcal{F}'$ *with* $r = \lceil k/3 \rceil = \frac{1}{3}(k+s)$, $s \equiv -k \bmod 3$, *of total value* $a_{F_1} + \cdots + a_{F_r} \geq \frac{1}{3}(v'(N) + \frac{2}{3}s)$.

Proof. The first part has been argued already above. (Note that if we write $r = \frac{1}{3}(k+s)$, then $s = 3r - k$, so $s \equiv -k \mod 3$.) To prove the lower bound, we first prove

Claim 1: $a_{F_r} > \frac{2}{3}$.

Proof of Claim 1. Assume to the contrary that $a_{F_r} \leq 2/3$. In the constructive process of Greedy Selection, when we have selected $F_1, \cdots, F_{r-1}$, the remaining feasible set system F'_r has still length $\geq k - 3(r-1) = 3 - s \geq 1$. As F_r has maximum size (value) among all sets in $\mathcal{F}'_r$, we know that each set in $\mathcal{F}'_r$ has size $\leq 2/3$. Decrease y' on $\mathcal{F}'_r$ arbitrarily such that the resulting fractional packing $\tilde{y}'$ has total length $\sum_F \tilde{y}'_F = k - 1$. By construction, the corresponding game $\tilde{N}$ with one bin removed has fractional value

$$v'(\tilde{N}) \geq w(\tilde{y}') \geq w(y') - \frac{2}{3} = v'(N) - \frac{2}{3}.$$

By minimality of our counterexample N, the game $\tilde{N}$ admits an integral packing $\tilde{y}$ of value

$$v(\tilde{N}) = w(\tilde{y}) \geq \frac{3}{4}v'(\tilde{N}) \geq \frac{3}{4}(v'(N) - \frac{2}{3}) = \frac{3}{4}v'(N) - \frac{1}{2}.$$

Extending this packing $\tilde{y}$ by filling the k^{th} bin to at least $1/2$ in a simple greedy manner (order not yet packed items non-increasingly in size and try to pack them into bin k in this order) yields an integral packing y for N of value

$$w(y) \geq w(\tilde{y}) + \frac{1}{2} \geq \frac{3}{4}v'(N),$$

contrary to our assumption on N. This finishes the proof of Claim 1.

To prove the bound on $a_{F_1} + \cdots + a_{F_r}$ in Lemma 6, let $\mathcal{R}'_j \subseteq \mathcal{F}'_j$ denote those feasible sets that are removed from $\mathcal{F}'_j$ in step j, i.e., $\mathcal{R}'_j = \mathcal{F}'_j \backslash \mathcal{F}'_{j+1}$. As we have seen, $\mathcal{R}'_j$ has total length $y'_{\mathcal{R}'_j} = \sum_{F' \in \mathcal{R}'_j} y'_{F'} \leq 3$. Assume that we, in addition, also decrease the y'-value on the least valuable sets in $\mathcal{F}'_r$ by a total of $3 - y'_{\mathcal{R}'_j}$ in each step. Thus we actually decrease the length of y' by exactly 3 in each step without any further impact on the construction. The total amount of value removed this way in step j is bounded by $3a_{F_j}$. If $k \equiv 0 \mod 3$, we remove in all $r = k/3$ rounds exactly $v'(N)$. Thus

$$v'(N) \leq \sum_{j=1}^{r} 3a_{F_j},$$

as claimed.

When $k \not\equiv 0 \mod 3$, the same procedure yields a reduced length of $\mathcal{F}'_r$ after $r-1$ steps, namely $y'_{\mathcal{F}'_r} = k - 3(r-1) = 3 - s$. So in the last step we remove a set

$\mathcal{R}'_r$ of value at most $(3-s)a_{F_r} \geq \frac{2}{3}(3-s) = 2-\frac{2}{3}s$ in the last step. Summarizing, the total value removed is

$$\begin{aligned} v'(N) &\leq 3a_{F_1} + \cdots + 3a_{F_{r-1}} + (3-s)a_{F_r} \\ &= 3(a_{F_1} + \cdots + a_{F_r}) - sa_{F_r} \\ &\leq 3(a_{F_1} + \cdots + a_{F_r}) - \frac{2}{3}s, \end{aligned}$$

proving the claim. □

A natural idea is to extend the greedy selection $F_1, \cdots, F_r$ in a straightforward manner to an integral packing $y \mathrel{\widehat{=}} (F_1, \cdots, F_k)$, where $F_{r+1}, \cdots, F_k$ are determined by applying Item Packing to the remaining items and remaining $k-r$ bins. However, this does not necessarily yield a sufficiently high packing value $w(y)$: Indeed, it may happen that the remaining $k-r \approx \frac{2}{3}k$ bins get only filled to roughly $1/2$, so the total packing value is approximately $w(y) \approx \frac{1}{3}v'(N) + \frac{1}{2}\cdot\frac{2}{3}k$, which equals $\frac{2}{3}v'(N)$ in case $v'(N) \approx k$.

However, the estimate $a_{F_{r+1}} + \cdots + a_{F_k} \approx \frac{1}{2}(k-r)$ is rather pessimistic. In particular, if we assume that the packing y (obtained by Greedy Selection plus Item Packing the remaining $k-r$ bins) leaves some small items unpacked, then each of the $k-r$ bins must necessarily be filled to at least $2/3$ (otherwise any small item could be added on top of y). This would yield

$$w(y) \approx \frac{1}{3}v'(N) + \frac{2}{3}k\cdot\frac{2}{3} \geq (\frac{1}{3}+\frac{4}{9})k \geq \frac{7}{9}k \geq \frac{7}{9}v'(N),$$

which is certainly sufficient for our purposes. (We do not use this estimate later on: It is only meant to guide our intuition.) Thus the crucial instances are those where y packs all small items – and hence does not pack all large items. Thus, when Item Packing is performed on the $k-r$ remaining bins, the first phase will pack large items into these $k-r$ bins until each bin is at least filled to $1/2$. Packing small items on top of any such bin would result in a bin filled to at least $3/4$. Thus, again, the worst case appears to happen when all small items (and there are quite a few of these, *cf.* Lemma 5) are already contained in $F_1 \cup \cdots \cup F_r$. Assume for a moment that each of $F_1, \cdots, F_r$ $(r \approx k/3)$ contains two of the small items in S_2. (Recall that $|S_2| \gtrapprox \frac{2}{3}k$.) Then each of $F_1, \cdots, F_r$ can contain only one other item in addition. Thus about $\frac{2}{3}k$ of the sets $\hat{F}_j$ computed by Item Packing must be disjoint from $F_1 \cup \cdots \cup F_r$ (as no $\hat{F}_j$ contains any item from S_2). Thus we could extend $F_1, \cdots, F_r$ by roughly $\frac{2}{3}k \approx k-r$ sets, say, $\hat{F}_{r+1}, \cdots, \hat{F}_k$ from Item Packing to obtain a packing $(F_1, \cdots, F_r, \hat{F}_{r+1}, \cdots, \hat{F}_k)$ of value $\geq \frac{1}{3}v'(N) + \frac{2}{3}k\cdot\frac{2}{3}$ (as each $\hat{F}_j$ has size $\geq \frac{2}{3}$), which is again enough.

In general, the Greedy Selection $F_1, \cdots, F_r$ will contain some – but not all – of S_2 and we will have to work out a trade-off between the two extreme cases considered above: Let

$$\gamma := |S_2 \cap (F_1 \cup \cdots \cup F_r)|\,.$$

Thus $F_1 \cup \cdots \cup F_r$ contains at most $3r-\gamma$ items that are not in S_2 – and hence possibly contained in some $\hat{F}_j$. So there are at most $3r-\gamma$ different $\hat{F}_j$ that intersect $F_1 \cup \cdots \cup F_r$ – and hence at least $k-(3r-\gamma)$ different $\hat{F}_j$ that do not intersect

$F_1 \cup \cdots \cup F_r$. We add these $k-3r+\gamma = \gamma - s$ different $\hat{F}_j$, say, $\hat{F}_{r+1}, \cdots, \hat{F}_{r+\gamma-s}$ to $F_1, \cdots, F_r$ to obtain a "partial" packing $F_1, \cdots, F_r, \hat{F}_{r+1}, \cdots, \hat{F}_{r+\gamma-s}$. Here we may assume that $r+\gamma-s < k$. Otherwise, we finish with a complete packing $\bar{y}$=$F_1, \cdots, F_r, \hat{F}_{r+1}, \cdots, \hat{F}_k$ of value (use Lemma 6 and $a_{\hat{F}_j} \geq 2/3$).

$$\begin{aligned}
w(\bar{y}) &\geq \frac{1}{3}(v'(N) + \frac{2}{3}s) + (k-r) \cdot \frac{2}{3} \\
&= \frac{1}{3}v'(N) + \frac{2}{9}s + (\frac{2}{3}k - \frac{1}{3}s)\frac{2}{3} \\
&= \frac{1}{3}v'(N) + \frac{4}{9}k \\
&\geq (\frac{1}{3} + \frac{4}{9})v'(N) \\
&> \frac{3}{4}v'(N),
\end{aligned}$$

contrary to our assumption that $v(N) < \frac{3}{4}v'(N)$.

Thus $r+\gamma-s < k$ holds indeed and therefore we may complete our partial packing $F_1, \cdots, F_r, \hat{F}_{r+1}, \cdots, \hat{F}_{r+\gamma-s}$ by Item Packing the remaining items to the remaining $k-(r+\gamma-s)$ bins, resulting in an integral packing

$$\bar{y} \mathrel{\widehat{=}} F_1, \cdots, F_r, \hat{F}_{r+1}, \cdots, \hat{F}_{r+\gamma-s}, F_{r+\gamma-s+1}, \cdots, F_k.$$

This completes the description of our heuristic

Set Packing

· Get $F_1, \cdots, F_r$ from Greedy Selection and let $\gamma := |S_2 \cap (F_1 \cup \cdots \cup F_r)|$
· Extend with sets $\hat{F}_j$ from Item Packing to $F_1, \cdots, F_r, \hat{F}_{r+1}, \cdots, \hat{F}_{r+\gamma-s}$
· Complete by Item Packing to $F_1, \cdots, F_r, \hat{F}_{r+1}, \cdots, \hat{F}_{r+\gamma-s}, F_{r+\gamma-s+1}, \cdots, F_k$.

Lemma 7. *Set Packing packs all of S (and hence not all of L).*

Proof. If Set Packing leaves some small item unpacked, then necessarily $a_{F_j} > 2/3$ for $j = r+\gamma-s+1, \cdots, k$. Thus the same computation as above yields

$$w(\bar{y}) \geq \frac{1}{3}(v'(N) + \frac{2}{3}s) + (k-r)\frac{2}{3} > \frac{3}{4}v'(N),$$

a contradiction. Thus all of S gets packed. If, in addition, all of L would get packed, then the value of the resulting packing $\bar{y}$ were $w(\bar{y}) \geq a_S + a_L \geq v'(N)$, a contradiction again. □

Thus, in phase 3 of Set Packing, when we apply Item Packing to the last $k-(r+\gamma-s)$ bins, each bin gets first filled to at least $1/2$ by large items, and then (possibly among other small items), the remaining $|S_2| - \gamma$ items from S_2 get packed on top of (some of) the last $k-(r+\gamma-s)$ bins. So these last $k-(r+\gamma-s)$ bins contribute at least

$$\frac{1}{2}(k-(r+\gamma-s)) + \frac{1}{4}(|S_2| - \gamma)$$

to the total value of $\bar{y}$.

Summarizing, $w(\bar{y})$ can be estimated as

$$\begin{aligned} w(\bar{y}) \geq \frac{1}{3}(v'(N) + \frac{2}{3}s) + (\gamma - s) \cdot \frac{2}{3} \\ + \frac{1}{2}(k - (r + \gamma - s)) + \frac{1}{4}(|S_2| - \gamma). \end{aligned} \tag{3.5}$$

In case $k \equiv 0 \mod 3$, we have $s = 0$, $r = k/3$ and $|S_2| \geq \frac{2}{3}k$ (by Lemma 5). So (3.5) simplifies to

$$w(\bar{y}) \geq \frac{1}{3}v'(N) + \frac{1}{2} \cdot \frac{2}{3}k + \frac{1}{4} \cdot \frac{2}{3}k - \frac{1}{12}\gamma \geq \frac{3}{4}v'(N) + \frac{1}{12}k - \frac{1}{12}\gamma,$$

proving that $w(\bar{y}) \geq \frac{3}{4}v'(N)$ if $\gamma \leq k$. But this is true: Indeed, as we have already argued above, we may even assume that $r + \gamma - s = r + \gamma < k$. (Recall that we consider the case $s = 0$ here.) In case $k \not\equiv 0 \mod 3$, by similar computation, we can show that $w(\bar{y}) \geq \frac{3}{4}v'(N)$ still holds.

Summarizing, both cases ($k \equiv 0$ and $k \not\equiv 0 \mod 3$) yield that N cannot be a counterexample, so we have proved

Theorem 8 $\epsilon_N \leq 1/4$ *for all instances of the bin packing game.*

4 Remarks and Open Problems

We like to note that – even though our proof in section 3 is indirect – it can easily be turned into a constructive proof. Indeed, we implicitly show that either Item Packing or Set Packing yields an integral packing y of value $w(y) \geq \frac{3}{4}v'(N)$. Also note that an optimum fractional packing y' (as input to Greedy Selection) is efficiently computable: Indeed, as any feasible set may contain at most 3 items, the total number of feasible sets is $O(n^3)$, so the LP for computing y' has polynomial size. Thus, in particular, our approach also yields a strengthening of the result in [3] (efficient 3/4 approximation).

In [13] it was conjectured that $\epsilon_N \leq 1/7$ is true for all instances. (This bound would be tight as can be seen from the small example presented in section 2.) We do not expect that our arguments provide any clue about how to approach 1/7.

A probably even more challenging conjecture due to Woeginger states that the integrality gap

$$gap(N) = v'(N) - v(N)$$

is bounded by a constant. Until now, this has only been verified for item sizes $> 1/3$ (*cf.* Lemma 3). It would be interesting to investigate the case of item sizes $a_i > 1/4$. The largest gap found (*cf.* [12]) so far is $gap(N) \approx 1/3$, for a game with 6 bins and 18 items.

References

1. Bilò, V.: On the packing of selfish items. In: Proceedings of the 20th International Conference on Parallel and Distributed Processing, IPDPS 2006, p. 45. IEEE Computer Society, Washington, DC (2006)
2. Caprara, A., Kellerer, H., Pferschy, U.: The multiple subset sum problem. SIAM Journal of Optimization 11, 308–319 (1998)
3. Caprara, A., Kellerer, H., Pferschy, U.: A 3/4-approximation algorithm for multiple subset sum. Journal of Heuristics 9(2), 99–111 (2003)
4. Epstein, L., Kleiman, E.: Selfish bin packing. Algorithmica 60, 368–394 (2011)
5. Faigle, U., Fekete, S., Hochstättler, W., Kern, W.: On approximately fair cost allocation in euclidean tsp games. Operations Research Spektrum 20, 29–37 (1998)
6. Faigle, U., Kern, W.: On some approximately balanced combinatorial cooperative games. Methods and Models of Operation Research 38, 141–152 (1993)
7. Faigle, U., Kern, W.: Approximate core allocation for binpacking games. SIAM J. Discrete Math. 11, 387–399 (1998)
8. Goemans, M.X., Skutella, M.: Cooperative facility location games. Journal of Algorithms 50(2), 194–214 (2004)
9. Han, X., Chin, F.Y.L., Ting, H.-F., Zhang, G., Zhang, Y.: A new upper bound 2.5545 on 2d online bin packing. ACM Trans. Algorithms 7(4), 50:1–50:18 (2011)
10. Jain, K., Mahdian, M., Markakis, E., Saberi, A., Vazirani, V.V.: Greedy facility location algorithms analyzed using dual fitting with factor-revealing LP. J. ACM 50(6), 795–824 (2003)
11. Jain, K., Vazirani, V.: Applications of approximation algorithms to cooperative games. In: Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing, STOC 2001, pp. 364–372. ACM, New York (2001)
12. Joosten, B.: Relaxation of 3-partition instances. Master's thesis, Radboud University (December 2011)
13. Kern, W., Qiu, X.: Integrality gap analysis for bin packing games. Operations Research Letters 40(5), 360–363 (2012)
14. Kern, W., Qiu, X.: Note on non-uniform bin packing games. Discrete Applied Mathematics (2012) (in press)
15. Kuipers, J.: Bin packing games. Mathematical Methods of Operations Research 47, 499–510 (1998)
16. Liu, Z.: Complexity of core allocation for the bin packing game. Operations Research Letters 37(4), 225–229 (2009)
17. Woeginger, G.J.: On the rate of taxation in a cooperative bin packing game. Mathematical Methods of Operations Research 42, 313–324 (1995)
18. Yu, G., Zhang, G.: Bin packing of selfish items. In: Papadimitriou, C., Zhang, S. (eds.) WINE 2008. LNCS, vol. 5385, pp. 446–453. Springer, Heidelberg (2008)