# Shortest Repetition-Free Words Accepted by Automata 

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#### Abstract

We consider the following problem: given that a finite automaton $M$ of $N$ states accepts at least one $k$-power-free (resp., overlapfree) word, what is the length of the shortest such word accepted? We give upper and lower bounds which, unfortunately, are widely separated.


## 1 Introduction

Let $L$ be an interesting language, such as the language of primitive words, or the language of non-palindromes. We are interested in the following kind of question: given that an automaton $M$ of $N$ states accepts a member of $L$, what is a good bound on the length $\ell(N)$ of the shortest word accepted?

For example, Ito et al. [7] proved that if $L$ is the language of primitive words, then $\ell(N) \leq 3 N-3$. Horváth et al. [6 proved that if $L$ is the language of non-palindromes, then $\ell(N) \leq 3 N$. For additional results along these lines, see [1.

For an integer $k \geq 2$, a $k$-power is a nonempty word of the form $x^{k}$. A word is $k$-power-free if it has no $k$-powers as factors. A word of the form axaxa, where $a$ is a single letter, and $x$ is a (possibly empty) word, is called an overlap. A word is overlap-free if it has no factor that is an overlap.

In this paper we address two open questions left unanswered in 1, corresponding to the case where $L$ is the language of $k$-power-free (resp., overlap-free) words. For these words and a large enough alphabet we give a class of DFAs of $N$ states for which the shortest $k$-power (resp., overlap) is of length $N^{\frac{1}{4}(\log N)+O(1)}$. For overlaps over a binary alphabet we give an upper bound of $2^{O\left(N^{4 N}\right)}$.

## 2 Notation

For a finite alphabet $\Sigma$, let $\Sigma^{*}$ denote the set of finite words over $\Sigma$. Let $w=$ $a_{0} a_{1} \cdots a_{n-1} \in \Sigma^{*}$ be a word. Let $w[i]=a_{i}$, and let $w[i . . j]=a_{i} \cdots a_{j}$. By convention we have $w[i]=\epsilon$ for $i<0$ or $i \geq n$, and $w[i . . j]=\epsilon$ for $i>j$. A prefix $p$ of $w$ is a period of $w$ if $w[i+r]=w[i]$ for $0 \leq i<|w|-r$, where $r=|p|$.

For words $x, y$, let $x \preceq y$ denote that $x$ is a factor of $y$. A factor $x$ of $y$ is proper if $x \neq y$. Let $x \preceq_{p} y$ (resp., $x \preceq_{s} y$ ) denote that $x$ is a prefix (resp., suffix) of $y$. Let $x \prec_{p} y$ (resp., $x \prec_{s} y$ ) denote that $x$ is a prefix (resp., suffix) of $y$ and $x \neq y$.

A word is primitive if it is not a $k$-power for any $k \geq 2$. Two words $x, y$ are conjugate if one is a cyclic shift of the other; that is, if there exist words $u, v$ such that $x=u v$ and $y=v u$. One simple observation is that all conjugates of a $k$-power are $k$-powers.

Let $h: \Sigma^{*} \rightarrow \Sigma^{*}$ be a morphism, and suppose $h(a)=a x$ for some letter $a$. The fixed point of $h$, starting with $a \in \Sigma$, is denoted by $h^{\omega}(a)=a x h(x) h^{2}(x) \cdots$. We say that a morphism $h$ is $k$-power-free (resp., overlap-free) if $h(w)$ is $k$-powerfree (resp., overlap-free) if $w$ is.

Let $\Sigma_{m}=\{0,1, \ldots, m-1\}$. Define the morphism $\mu: \Sigma_{2}^{*} \rightarrow \Sigma_{2}^{*}$ as follows

$$
\begin{aligned}
& \mu(0)=01 \\
& \mu(1)=10 .
\end{aligned}
$$

We call $\mathbf{t}=\mu^{\omega}(0)$ the Thue-Morse word. It is easy to see that

$$
\mu(\mathbf{t}[0 . . n-1])=\mathbf{t}[0 . .2 n-1] \text { for } n \geq 0
$$

From classical results of Thue 1011 , we know that the morphism $\mu$ is overlapfree. From [2], we know that that $\mu(x)$ is $k$-power free for each $k>2$.

For a DFA $D=\left(Q, \Sigma, \delta, q_{0}, F\right)$ the set of states, input alphabet, transition function, set of final states, and initial state are denoted by $Q, \Sigma, \delta, F$, and $q_{0}$, respectively. Let $L(D)$ denote the language accepted by $D$. As usual, we have $\delta(q, w a)=\delta(\delta(q, w), a)$ for a word $w$.

We state the following basic result without proof.
Proposition 1. Let $D=\left(Q, \Sigma, \delta, q_{0}, F\right)$ be a (deterministic or nondeterministic) finite automaton. If $L(D) \neq \emptyset$, then $D$ accepts at least one word of length smaller than $|Q|$.

## 3 Lower bound

In this section, we construct an infinite family of DFAs such that the shortest $k$-power-free word accepted is rather long, as a function of the number of states. Up to now only a linear bound was known.

For a word $w$ of length $n$ and $i \geq 1$, let

$$
\operatorname{cyc}_{i}(w)=w[i . . n-1] w[0 . . i-2]
$$

denote $w$ 's $i$ th cyclic shift to the left, followed by removing the last symbol. Also define

$$
\operatorname{cyc}_{0}(w)=w[0 . . n-2] .
$$

For example, we have

$$
\begin{aligned}
& \operatorname{cyc}_{2}(\text { recompute })=\text { computer }, \\
& \operatorname{cyc}_{4}(\text { richly })=\text { lyric }
\end{aligned}
$$

We call each $\operatorname{cyc}_{i}(w)$ a partial conjugate of $w$, which is not a reflexive, symmetric, or transitive relation.

A word $w$ is a simple $k$-power if it is a $k$-power and it contains no $k$-power as a proper factor.

We start with a few lemmas.
Lemma 2. Let $w=p^{k}$ be a simple $k$-power. Then the word $p$ has $|p|$ distinct conjugates.

Proof. By contradiction. If $p^{k}$ is a simple $k$-power, then $p$ is a primitive word. Suppose that $p=u v=x y$ such that $x \prec_{p} u$ and $y x=v u$. Without loss of generality, we can assume that $x v \neq \epsilon$. Then there exists a word $t \neq \epsilon$ such that $u=x t$ and $y=t v$. From $v u=y x$ we get

$$
v x t=t v x .
$$

Using the second theorem of Lyndon and Schützenberger [8] we get that there exists $z \neq \epsilon$ such that

$$
\begin{gathered}
v x=z^{i} \\
t=z^{j}
\end{gathered}
$$

for some positive integers $i, j$. So $y x=z^{i+j}$, and hence $p=x y$ is not primitive, a contradiction.

Lemma 3. Let $w$ be a simple $k$-power of length $n$. Then we have

$$
\begin{equation*}
\operatorname{cyc}_{i}(w)=\operatorname{cyc}_{j}(w) i f f i \equiv j\left(\bmod \frac{n}{k}\right) \tag{1}
\end{equation*}
$$

Proof. Let $w=p^{k}$. If $i \equiv i^{\prime}\left(\bmod \frac{n}{k}\right)$ and $i^{\prime}<\frac{n}{k}$, then

$$
\operatorname{cyc}_{i}(w)=\left(p\left[i^{\prime} \cdot \cdot \frac{n}{k}-1\right] p\left[0 . . i^{\prime}-1\right]\right)^{k-1} \operatorname{cyc}_{i^{\prime}}(p)
$$

Similarly, if $j \equiv j^{\prime}\left(\bmod \frac{n}{k}\right)$ and $j^{\prime}<\frac{n}{k}$, then

$$
\operatorname{cyc}_{j}(w)=\left(p\left[j^{\prime} \cdot \cdot \frac{n}{k}-1\right] p\left[0 . . j^{\prime}-1\right]\right)^{k-1} \operatorname{cyc}_{j^{\prime}}(p)
$$

If $i^{\prime}=j^{\prime}$, then clearly $\operatorname{cyc}_{i}(w)=\operatorname{cyc}_{j}(w)$. If $i^{\prime} \neq j^{\prime}$, we get that

$$
p\left[i^{\prime} \cdot . \frac{n}{k}-1\right] p\left[0 . . i^{\prime}-1\right] \neq p\left[j^{\prime} \cdot . \frac{n}{k}-1\right] p\left[0 . . j^{\prime}-1\right]
$$

using Lemma 2, and hence $\operatorname{cyc}_{i}(w) \neq \operatorname{cyc}_{j}(w)$.

Lemma 4. All conjugates of a simple $k$-power are simple $k$-powers.


Fig. 1: starting positions of the occurrences of $q$ inside $x$

Proof. By contradiction. Let $w=p^{k}$ be a simple $k$-power, and let $z \neq w$ be a conjugate of $w$. Clearly $z$ is a $k$-power. Suppose $z$ contains $q^{k}$ and $z \neq q^{k}$. Thus $|q|<|p|$. Since $w$ is simple $q^{k} \npreceq w=p^{k}$. The word $x=p^{k+1}$ contains $z$ as a factor. So $x=u q^{k} v$, for some words $u, v \preceq p$.
Note that $u$ and $v$ are nonempty and not equal to $p$ since $q^{k} \npreceq p^{k}$. Letting $e:=|p|-|q|$, and considering the starting positions of the occurrences of $q$ in $x$ (see Fig. (1), we can write

$$
x\left[\left|p^{i} u\right|-i e . .\left|p^{i} u\right|-(i-1) e-1\right]=x\left[\left|p^{j} u\right|-j e . .\left|p^{j} u\right|-(j-1) e-1\right]
$$

for every $0 \leq i, j<k$. Since $p$ is a period of $x$, we can write

$$
x[|u|-i e . .|u|-(i-1) e-1]=x[|u|-j e . .|u|-(j-1) e-1]
$$

which means $x[u-(k-1) e . . u+e-1] \preceq w$ is a $k$-power. Therefore $w$ contains a $k$-power other than itself, a contradiction.

Corollary 5. Partial conjugates of simple $k$-powers are $k$-power-free.
The next lemma shows that there are infinitely many simple $k$-powers over a binary alphabet for $k>2$. We also show that there are infinitely many simple squares over a ternary alphabet, using a result of Currie [4].

## Lemma 6.

(i) Let $p=\mathbf{t}\left[0 . .2^{n}-1\right]$ where $n \geq 0$. For every $k>2$, the word $p^{k}$ is a simple $k$-power.
(ii) There are infinitely many simple squares over a ternary alphabet.

## Proof.

(i) By induction on $n$. For $n=0$ we have $p^{k}=0^{k}$ which is a simple $k$-power. Suppose $n>0$. To get a contradiction, suppose that there exist words $u, v, x$ with $u v \neq \epsilon$ and $x \neq \epsilon$ such that $p^{k}=u x^{k} v$. Note that $|x|<|p|$, so $|u v| \geq k$. Without loss of generality, we can assume that $|v| \geq\left\lceil\frac{k}{2}\right\rceil \geq 2$. Let $q=\mathbf{t}\left[0 . .2^{n-1}-1\right]$. We know that

$$
p^{k}=\mu\left(q^{k}\right)
$$

We can write

$$
w=u x^{k} \preceq_{p} \mu\left(q^{k-1} q[0 \ldots|q|-2]\right) .
$$

Since $\mu$ is $k$-power-free, the word $q^{k-1} q[0 . .|q|-2]$ contains a $k$-power. Hence $q^{k}$ contains at least two $k$-powers, a contradiction.
(ii) Currie [4] proved that over a ternary alphabet, for every $n \geq 18$, there is a word $p$ of length $n$ such that all its conjugates are squarefree. Such squarefree words are called circularly squarefree words.
We claim that for every circularly squarefree word $p$, the word $p^{2}$ is a simple square. To get a contradiction, let $q^{2}$ be the smallest square in $p^{2}$. So there exist words $u, y$ with $u y \neq \epsilon$ such that $p^{2}=u q^{2} y$. We have $\left|q^{2}\right|>|p|$ since $p$ is circularly squarefree. Therefore, if we let $p=u v=x y$, then $|x|>|u|$ and $|v|>|y|$. So there exists $t$ such that $x=u t$ and $v=t y$. We can assume $|t|<|q|$, since otherwise $|t|=|q|$ and $|u y|=0$, a contradiction. Now since $q^{2}=v x=t y u t$, we get that $q$ begins and ends with $t$, which means $t^{2} \prec q^{2}$. Therefore $p^{2}$ has a smaller square than $q^{2}$, a contradiction.

Next we show how to construct arbitrarily long simple $k$-powers from smaller ones. Fix $k=2$ (resp., $k \geq 3$ ) and $m=3$ (resp., $m=2$ ). Let $w_{1} \in \Sigma_{m}^{*}$ be a simple $k$-power. Using the previous lemma, there are infinitely many choices for $w_{1}$. Let $w_{1}$ be of length $n$. Define $w_{i+1} \in \Sigma_{m+i}^{*}$ for $i \geq 1$ recursively by

$$
\begin{equation*}
w_{i+1}=\operatorname{cyc}_{0}\left(w_{i}\right) a_{i} \operatorname{cyc}_{n^{i-1}}\left(w_{i}\right) a_{i} \operatorname{cyc}_{2 n^{i-1}}\left(w_{i}\right) a_{i} \cdots \operatorname{cyc}_{(n-1) n^{i-1}}\left(w_{i}\right) a_{i} \tag{2}
\end{equation*}
$$

where $a_{i}=m+i-1$ and $w_{0}=0$. The next lemma states that $w_{i}$, for $i \geq 1$, is a simple $k$-power. Therefore, using Corollary 5 , each word $\operatorname{cyc}_{0}\left(w_{i}\right)$ is $k$-power-free. For $i \geq 1$, it is easy to see that

$$
\begin{equation*}
\left|w_{i}\right|=n\left|w_{i-1}\right|=n^{i} \tag{3}
\end{equation*}
$$

Lemma 7. For every $i \geq 1$, the word $w_{i}$ is a simple $k$-power.
Proof. By induction on $i$. The word $w_{1}$ is a simple $k$-power. Now suppose that $w_{i}$ is a simple $k$-power for some $i \geq 1$. Using Lemma 3, we have $\operatorname{cyc}_{j n^{i-1}}\left(w_{i}\right)=$ $\operatorname{cyc}_{\left(j+\frac{n}{k}\right) n^{i-1}}\left(w_{i}\right)$, since $\frac{\left|w_{i}\right|}{k}=\frac{n^{i}}{k}$.

We now claim that $w_{i+1}$ is a $k$-power and

$$
w_{i+1}=\left(\operatorname{cyc}_{0}\left(w_{i}\right) a_{i} \operatorname{cyc}_{n^{i-1}}\left(w_{i}\right) a_{i} \operatorname{cyc}_{2 n^{i-1}}\left(w_{i}\right) a_{i} \cdots \operatorname{cyc}_{\left(\frac{n}{k}-1\right) n^{i-1}}\left(w_{i}\right) a_{i}\right)^{k}
$$

To see this, suppose that $w_{i+1}$ contains a $k$-power $y^{k}$ such that $w_{i+1} \neq y^{k}$.
If $y$ contains more than one occurrence of $a_{i}$, then $y=u a_{i} \operatorname{cyc}_{j}\left(w_{i}\right) a_{i} v$ for some words $u, v$ and an integer $j$. Since $y^{2}=u a_{i} \operatorname{cyc}_{j}\left(w_{i}\right) a_{i} v u a_{i} \operatorname{cyc}_{j}\left(w_{i}\right) a_{i} v \preceq$ $w_{i+1}$, using (2) and Lemma 3, we get

$$
|y|=\left|\operatorname{cyc}_{j}\left(w_{i}\right) a_{i} v u a_{i}\right| \geq \frac{n}{k} n^{i}=\frac{\left|w_{i+1}\right|}{k}
$$

and hence $y^{k}=w_{i+1}$, a contradiction.
If $y$ contains just one $a_{i}$, then $y=u a_{i} v$ for some words $u, v$ which contain no $a_{i}$. So $y^{k}=u(a v u)^{k-1} a v$ for $a=a_{i}$. Therefore $v u$ is a partial conjugate of $w_{i}$. However the distance between two equal partial conjugates of $w_{i}$ in $w_{i+1}$ is longer than just one letter, using (2) and Lemma 3.

Finally, if $y$ contains no $a_{i}$, then a partial conjugate of $w_{i}$ contains a $k$-power, which is impossible due to Lemma 4

To make our formulas easier to read, we define $a_{0}=w_{1}[n-1]$.
Theorem 8. For $i \geq 1$, there is a DFA $D_{i}$ with $2^{i-1}(n-1)+2$ states such that $\operatorname{cyc}_{0}\left(w_{i}\right)$ is the shortest $k$-power-free word in $L\left(D_{i}\right)$.

Proof. Define $D_{1}=\left(Q_{1}, \Sigma_{a_{1}}, \delta_{1}, q_{1,0}, F_{1}\right)$ where

$$
\begin{aligned}
& Q_{1}:=\left\{q_{1,0}, q_{1,1}, q_{1,2}, \ldots, q_{1, n-1}, q_{d}\right\} \\
& F_{1}:=\left\{q_{1, n-1}\right\} \\
& \delta_{1}\left(q_{1, j}, w[j]\right):=q_{1, j+1} \text { for } 0 \leq j<n-1
\end{aligned}
$$

and the rest of the transitions go to the dead state $q_{d}$. Clearly we have $\left|Q_{1}\right|=n+1$ and $L\left(D_{1}\right)=\left\{\operatorname{cyc}_{0}\left(w_{1}\right)\right\}$.

We define $D_{i}=\left(Q_{i}, \Sigma_{a_{i}}, \delta_{i}, q_{1,0}, F_{i}\right)$ for $i \geq 2$ recursively. For the rest of the proof $s$ and $t$ denote (possibly empty) sequences of integers and $j$ denotes a single integer (a sequence of length 1 ). We use integer sequences as subscripts of states in $Q_{i}$. For example, $q_{1,0}, q_{s, j}$, and $q_{s, 2, t}$ might denote states of $D_{i}$. For $i \geq 1$, define

$$
\begin{align*}
& Q_{i+1}:=Q_{i} \cup\left\{q_{i+1, t}: q_{t} \in\left(Q_{i}-F_{i}\right)-\left\{q_{d}\right\}\right\}  \tag{4}\\
& F_{i+1}:=\left\{q_{i+1, i, t}: \delta_{i}\left(q_{i, t}, c\right)=q_{1, n-1} \text { for some } c \in \Sigma_{a_{i}}\right\},  \tag{5}\\
& \text { if } q_{t} \in Q_{i} \text { and } c \in \Sigma_{a_{i}} \text {, then } \delta_{i+1}\left(q_{t}, c\right):=\delta_{i}\left(q_{t}, c\right)  \tag{6}\\
& \text { if } q_{t}, q_{s} \in\left(Q_{i}-F_{i}\right)-\left\{q_{d}\right\}, c \in \Sigma_{a_{i}}, \text { and } \delta_{i}\left(q_{t}, c\right)=q_{s}, \\
& \quad \text { then } \delta_{i+1}\left(q_{i+1, t}, c\right):=q_{i+1, s}  \tag{7}\\
& \text { if } q_{t} \in F_{i} \text {, then } \delta_{i+1}\left(q_{t}, a_{i}\right):=q_{1,1} \text { and } \delta_{i+1}\left(q_{t}, a_{i-1}\right):=q_{i+1,1,0}  \tag{8}\\
& \text { if } i>1, q_{i+1, t} \notin F_{i+1}, \text { and } \delta_{i}\left(q_{t}, a_{i-1}\right)=q_{1, j}, \\
& \text { then } \delta_{i+1}\left(q_{i+1, t}, a_{i}\right):=q_{1, j+1} \tag{9}
\end{align*}
$$

and finally for the special case of $i=1$,

$$
\begin{equation*}
\delta_{2}\left(q_{2,1, j}, a_{1}\right):=q_{1, j+2} \text { for } 0 \leq j<n-2 . \tag{10}
\end{equation*}
$$

The rest of the transitions, not indicated in (6)-(10), go to the dead state $q_{d}$. Fig. 2b depicts $D_{2}$ and $D_{3}$. Using (4), we have $\left|Q_{i+1}\right|=2\left|Q_{i}\right|-2=2^{i}(n-1)+2$ by a simple induction.

An easy induction on $i$ proves that $\left|F_{i}\right|=1$. So let $f_{i}$ be the appropriate integer sequence for which $F_{i}=\left\{q_{f_{i}}\right\}$. Using (6)-(10), we get that for every $1 \leq j<n$, there exists exactly one state $q_{t} \in Q_{i}$ for which $\delta_{i}\left(q_{t}, a_{i-1}\right)=q_{1, j}$.

By induction on $i$, we prove that for $i \geq 2$ if $\delta_{i}\left(q_{t}, a_{i-1}\right)=q_{1, j}$, then

$$
\begin{align*}
x_{1} & =\operatorname{cyc}_{(j-1) n^{i-2}}\left(w_{i-1}\right),  \tag{11}\\
x_{2} & =w_{i}\left[0 . . j n^{i-1}-2\right],  \tag{12}\\
x_{3} & =w_{i}\left[(j-1) n^{i-1} . . n^{i}-2\right] . \tag{13}
\end{align*}
$$

are the shortest $k$-power-free words for which

$$
\begin{align*}
& \delta_{i}\left(q_{1, j-1}, x_{1}\right)=q_{t}  \tag{14}\\
& \delta_{i}\left(q_{1,0}, x_{2}\right)=q_{t}  \tag{15}\\
& \delta_{i}\left(q_{1, j-1}, x_{3}\right)=q_{f_{i}} . \tag{16}
\end{align*}
$$

In particular, from (13) and (16), for $j=1$, we get that $\operatorname{cyc}_{0}\left(w_{i}\right)$ is the shortest $k$-power-free word in $L\left(D_{i}\right)$.

The fact that our choices of $x_{1}, x_{2}$, and $x_{3}$ are $k$-power-free follows from the fact that proper factors of simple $k$-powers are $k$-power-free. For $i=2$ the proofs of (14)-(16) are easy and left to the readers.

Suppose that (14)-(16) hold for some $i \geq 2$. Let us prove (14)-(16) for $i+1$. Suppose that

$$
\begin{equation*}
\delta_{i+1}\left(q_{t}, a_{i}\right)=q_{1, j} . \tag{17}
\end{equation*}
$$

First we prove that the shortest $k$-power-free word $x$ for which

$$
\delta_{i+1}\left(q_{1, j-1}, x\right)=q_{t},
$$

is $x=\operatorname{cyc}_{(j-1) n^{i-1}}\left(w_{i}\right)$.
If $q_{t} \in Q_{i}$, from (8) and (17), we have

$$
\begin{aligned}
& q_{t}=q_{f_{i}}, \text { and } \\
& \delta_{i+1}\left(q_{t}, a_{i}\right)=q_{1,1} .
\end{aligned}
$$

By induction hypothesis, the $\operatorname{cyc}_{0}\left(w_{i}\right)$ is the shortest $k$-power-free word in $L\left(D_{i}\right)$. In other words, we have $\delta_{i}\left(q_{1,0}, \operatorname{cyc}_{0}\left(w_{i}\right)\right)=q_{f_{i}}=q_{t}$, which can be rewritten using (6) as $\delta_{i+1}\left(q_{1,0}, \operatorname{cyc}_{0}\left(w_{i}\right)\right)=q_{t}$.

Now suppose $q_{t} \notin Q_{i}$. Then by (9) and (17), we get that there exists $t^{\prime}$ such that

$$
\begin{aligned}
& t=i+1, t^{\prime} \\
& \delta_{i}\left(q_{t^{\prime}}, a_{i-1}\right)=q_{1, j-1}
\end{aligned}
$$

From the induction hypothesis, i.e., (15) and (16), we can write

$$
\begin{align*}
& \delta_{i}\left(q_{1,0}, w_{i}\left[0 . .(j-1) n^{i-1}-2\right]\right)=q_{t^{\prime}},  \tag{18}\\
& \delta_{i}\left(q_{1, j-1}, w_{i}\left[(j-1) n^{i-1} . . n^{i}-2\right]\right)=q_{f_{i}} . \tag{19}
\end{align*}
$$

In addition $w_{i}\left[0 . .(j-1) n^{i-1}-2\right]$ and $w_{i}\left[(j-1) n^{i-1} . . n^{i}-2\right]$ are the shortest $k$-power-free transitions from $q_{1,0}$ to $q_{t^{\prime}}$ and from $q_{1, j-1}$ to $q_{f_{i}}$ respectively. Using (6), we can rewrite (18) and (19) for $\delta_{i+1}$ as follows:

$$
\begin{align*}
& \delta_{i+1}\left(q_{1,0}, w_{i}\left[0 . .(j-1) n^{i-1}-2\right]\right)=q_{t^{\prime}},  \tag{20}\\
& \delta_{i+1}\left(q_{1, j-1}, w_{i}\left[(j-1) n^{i-1} . . n^{i}-2\right]\right)=q_{f_{i}} . \tag{21}
\end{align*}
$$

Note that from (7) and (20), we get

$$
\begin{equation*}
\delta_{i+1}\left(q_{i+1,1,0}, w_{i}\left[0 . .(j-1) n^{i-1}-2\right]\right)=q_{i+1, t^{\prime}}=q_{t} . \tag{22}
\end{equation*}
$$

We also have $\delta_{i+1}\left(q_{f_{i}}, a_{i}\right)=q_{i+1,1,0}$, using (8). So together with (21) and (22), we get

$$
\delta_{i+1}\left(q_{1, j-1}, \operatorname{cyc}_{(j-1) n^{i-1}}\left(w_{i}\right)\right)=q_{t}
$$

and $\operatorname{cyc}_{(j-1) n^{i-1}}\left(w_{i}\right)$ is the shortest $k$-power-free transition from $q_{1, j-1}$ to $q_{t}$.
The proofs of (15) and (16) are similar.

In what follows, all logarithms are to the base 2 .

Corollary 9. For infinitely many $N$, there exists a DFA with $N$ states such that the shortest $k$-power-free word accepted is of length $N^{\frac{1}{4} \log N+O(1)}$.

Proof. Let $i=\lfloor\log n\rfloor$ in Theorem 8. Then $D=D_{i}$ has

$$
N=2^{\lfloor\log n\rfloor-1}(n-1)+2=\Omega\left(n^{2}\right)
$$

states. In addition, the shortest $k$-power-free word in $L(D)$ is $\operatorname{cyc}_{0}\left(w_{\lfloor\log n\rfloor}\right)$. Now, using (3) we can write

$$
\left|\operatorname{cyc}_{0}\left(w_{\lfloor\log n\rfloor}\right)\right|=n^{\lfloor\log n\rfloor}-1
$$

Suppose $2^{t} \leq n<2^{t+1}-1$, so that $t=\lfloor\log n\rfloor$ and Then $\log N=2 t+O(1)$, so $\frac{1}{4}(\log N)^{2}=t^{2}+O(t)$. On the other hand $\log |w|=\lfloor\log n\rfloor(\log n)=t(t+O(1))=$ $t^{2}+O(t)$. Now $2^{O(t)}=n^{O(1)}=N^{O(1)}$, and the result follows.

Remark 10. The same bound holds for overlap-free words. To do so, we define a simple overlap as a word of the form axaxa where axax is a simple square. In our construction of the DFAs, we use complete conjugates of $(a x)^{2}$ instead of partial conjugates.

Remark 11. The $D_{i}$ in Theorem8 are defined over the growing alphabet $\Sigma_{m+i-1}$. However, we can fix the alphabet to be $\Sigma_{m+1}$. For this purpose, we introduce $w_{i}^{\prime}$ which is quite similar to $w_{i}$ :

$$
\begin{aligned}
& w_{1}^{\prime}=w_{1} \\
& w_{i+1}^{\prime}=\operatorname{cyc}_{0}\left(w_{i}^{\prime}\right) b_{i} \operatorname{cyc}_{n^{i-1}}\left(w_{i}^{\prime}\right) b_{i} \operatorname{cyc}_{2 n^{i-1}}\left(w_{i}^{\prime}\right) b_{i} \cdots \operatorname{cyc}_{(n-1) n^{i-1}}\left(w_{i}^{\prime}\right) b_{i}
\end{aligned}
$$

where $b_{i}=m c_{i} m$ such that $c_{i}$ is (any of) the shortest nonempty $k$-power-free word over $\Sigma_{m}$ not equal to $c_{1}, \ldots, c_{i-1}$. Clearly we have $\left|b_{i}\right| \leq\left|b_{i-1}\right|+1=O(i)$, and hence $w_{i}^{\prime}=\Theta\left(n^{i}\right)$.

One can then prove Lemma 7 and Theorem 8 for $w_{i}^{\prime}$ with minor modifications of the argument above. In particular, we construct DFA $D_{i}^{\prime}$ that accepts $\operatorname{cyc}_{0}\left(w_{i}^{\prime}\right)$ as the shortest $k$-power-free word accepted, and a $D_{i}^{\prime}$ that is quite similar to $D_{i}$. In particular, they have asymptotically the same number of states.


Fig. 2: transition diagrams

## 4 Upper bound for overlap-free words

In this section, we prove an upper bound on the length of the shortest overlapfree word accepted by a DFA $D$ over a binary alphabet.

Let $L=L(D)$ and let $R$ be the set of overlap-free words over $\Sigma_{2}^{*}$. Carpi [3] defined a certain operation $\Psi$ on binary languages, and proved that $\Psi(R)$ is regular. We prove that $\Psi(L)$ is also regular, and hence $\Psi(L) \cap \Psi(R)$ is regular. The next step is to apply Proposition 1 to get an upper bound on the length of the shortest word in $\Psi(L) \cap \Psi(R)$. This bound then gives us an upper bound on the length of the shortest overlap-free word in $L$.

Let $H=\{\epsilon, 0,1,00,11\}$. Carpi defines maps

$$
\Phi_{l}, \Phi_{r}: \Sigma_{25} \rightarrow H
$$

such that for every pair $h, h^{\prime} \in H$, one has

$$
h=\Phi_{l}(a), h^{\prime}=\Phi_{r}(a)
$$

for exactly one letter $a \in \Sigma_{25}$.
For every word $w \in \Sigma_{25}^{*}$, define $\Phi(w) \in \Sigma_{2}^{*}$ inductively by

$$
\begin{equation*}
\Phi(\epsilon)=\epsilon, \Phi(a w)=\Phi_{l}(a) \mu(\Phi(w)) \Phi_{r}(a) \quad\left(w \in \Sigma_{25}^{*}, a \in \Sigma_{25}\right) \tag{23}
\end{equation*}
$$

Expanding (23) for $w=a_{0} a_{1} \cdots a_{n-1}$, we get

$$
\begin{equation*}
\Phi_{l}\left(a_{0}\right) \mu\left(\Phi_{l}\left(a_{1}\right)\right) \cdots \mu^{n-1}\left(\Phi_{l}\left(a_{n-1}\right)\right) \mu^{n-1}\left(\Phi_{r}\left(a_{n-1}\right)\right) \cdots \mu\left(\Phi_{r}\left(a_{1}\right)\right) \Phi_{r}\left(a_{0}\right) \tag{24}
\end{equation*}
$$

For $L \subseteq \Sigma_{2}^{*}$ define $\Psi(L)=\bigcup_{x \in L} \Phi^{-1}(x)$. Based on the decomposition of Restivo and Salemi [9] for finite overlap-free words, the language $\Psi(x)$ is always nonempty for an overlap-free word $x \in \Sigma_{2}^{*}$. The next theorem is due to Carpi [3].

Theorem 12. $\Psi(R)$ is regular.
Carpi constructed a DFA $A$ with less than 400 states that accepts $\Psi(R)$. We prove that $\Psi$ preserves regular languages.

Theorem 13. Let $D=\left(Q, \Sigma_{2}, \delta, q_{0}, F\right)$ be a DFA with $N$ states, and let $L=$ $L(D)$. Then $\Psi(L)$ is regular and is accepted by a DFA with at most $N^{4 N}$ states.

Proof. Let $\iota: Q \rightarrow Q$ denote the identity function, and define $\eta_{0}, \eta_{1}: Q \rightarrow Q$ as follows

$$
\begin{equation*}
\eta_{i}(q)=\delta(q, i) \text { for } i=0,1 \tag{25}
\end{equation*}
$$

For functions $\zeta_{0}, \zeta_{1}: Q \rightarrow Q$, and a word $x=b_{0} b_{1} \cdots b_{n-1} \in \Sigma_{2}^{*}$, define $\zeta_{x}=$ $\zeta_{b_{n-1}} \circ \cdots \circ \zeta_{b_{1}} \circ \zeta_{b_{0}}$. Therefore we have $\zeta_{y} \circ \zeta_{x}=\zeta_{x y}$. Also by convention $\zeta_{\epsilon}=\iota$. So for example $x \in L(D)$ if and only if $\eta_{x}\left(q_{0}\right) \in F$.

We create DFA $D^{\prime}=\left(Q^{\prime}, \Sigma_{25}, \delta^{\prime}, q_{0}^{\prime}, F^{\prime}\right)$ where

$$
\begin{aligned}
& Q^{\prime}=\left\{\left[\kappa, \lambda, \zeta_{0}, \zeta_{1}\right]: \kappa, \lambda, \zeta_{0}, \zeta_{1}: Q \rightarrow Q\right\}, \\
& \delta^{\prime}\left(\left[\kappa, \lambda, \zeta_{0}, \zeta_{1}\right], a\right)=\left[\zeta_{\Phi_{l}(a)} \circ \kappa, \lambda \circ \zeta_{\Phi_{r}(a)}, \zeta_{1} \circ \zeta_{0}, \zeta_{0} \circ \zeta_{1}\right] .
\end{aligned}
$$

Also let

$$
\begin{align*}
& q_{0}^{\prime}=\left[\iota, \iota, \eta_{0}, \eta_{1}\right] \\
& F^{\prime}=\left\{\left[\kappa, \lambda, \zeta_{0}, \zeta_{1}\right]: \lambda \circ \kappa\left(q_{0}\right) \in F\right\} . \tag{26}
\end{align*}
$$

We can see that $\left|Q^{\prime}\right|=N^{4 N}$. We claim that $D^{\prime}$ accepts $\Psi(L)$. Indeed, on input $w$, the DFA $D^{\prime}$ simulates the behavior of $D$ on $\Phi(w)$.

Let $w=a_{0} a_{1} \cdots a_{n-1} \in \Sigma_{25}^{*}$, and define

$$
\begin{aligned}
& \Phi_{1}(w)=\Phi_{l}\left(a_{a_{0}}\right) \mu\left(\Phi_{l}\left(a_{1}\right)\right) \cdots \mu^{n-1}\left(\Phi_{l}\left(a_{n-1}\right)\right), \\
& \Phi_{2}(w)=\mu^{n-1}\left(\Phi_{r}\left(a_{n-1}\right)\right) \cdots \mu\left(\Phi_{r}\left(a_{1}\right)\right) \Phi_{r}\left(a_{0}\right) .
\end{aligned}
$$

Using (24), we can write

$$
\Phi(w)=\Phi_{1}(w) \Phi_{2}(w)
$$

We prove by induction on $n$ that

$$
\begin{equation*}
\delta^{\prime}\left(q_{0}^{\prime}, w\right)=\left[\eta_{\Phi_{1}(w)}, \eta_{\Phi_{2}(w)}, \eta_{\mu^{n}(0)}, \eta_{\mu^{n}(1)}\right] . \tag{27}
\end{equation*}
$$

For $n=0$, we have $\Phi(w)=\Phi_{1}(w)=\Phi_{2}(w)=\epsilon$. So

$$
\delta^{\prime}\left(q_{0}^{\prime}, \epsilon\right)=q_{0}^{\prime}=\left[\iota, \iota, \eta_{0}, \eta_{1}\right]=\left[\eta_{\Phi_{1}(w)}, \eta_{\Phi_{2}(w)}, \eta_{\mu^{0}(0)}, \eta_{\mu^{0}(1)}\right] .
$$

So we can assume (27) holds for some $n \geq 0$. Now suppose $w=a_{0} a_{1} \cdots a_{n}$ and write

$$
\begin{align*}
& \delta^{\prime}\left(q_{0}^{\prime}, a_{0} a_{1} \cdots a_{n}\right) \\
& =\delta^{\prime}\left(\delta^{\prime}\left(q_{0}^{\prime}, a_{0} a_{1} \cdots a_{n-1}\right), a_{n}\right) \\
& =\delta^{\prime}\left(\left[\eta_{\Phi_{1}(w[0 . . n-1])}, \eta_{\Phi_{2}(w[0 . . n-1])}, \eta_{\mu^{n}(0)}, \eta_{\mu^{n}(1)}\right], a_{n}\right) \\
& =\left[\eta_{\left.\mu^{n}\left(\phi_{l}\left(a_{n}\right)\right) \circ \eta_{\Phi_{1}(w[0 . . n-1])}, \eta_{\Phi_{2}(w[0 . . n-1])} \circ \eta_{\mu^{n}\left(\phi_{r}\left(a_{n}\right)\right)}, \eta_{\mu^{n}(1)} \circ \eta_{\mu^{n}(0)}, \eta_{\mu^{n}(0)} \circ \eta_{\mu^{n}(1)}\right]}^{=\left[\eta_{\Phi_{1}(w)}, \eta_{\Phi_{2}(w)}, \eta_{\mu^{n+1}(0)}, \eta_{\mu^{n+1}(1)}\right]}\right.
\end{align*}
$$

and equality (28) holds because

$$
\begin{aligned}
& \Phi_{1}(w[0 . . n-1]) \mu^{n}\left(\phi_{l}\left(a_{n}\right)\right)=\Phi_{1}(w) \\
& \mu^{n}\left(\phi_{r}\left(a_{n}\right)\right) \Phi_{2}(w[0 . . n-1])=\Phi_{2}(w) \\
& \mu^{n}(0) \mu^{n}(1)=\mu^{n}(01)=\mu^{n}(\mu(0))=\mu^{n+1}(0), \text { and similarly } \\
& \mu^{n}(1) \mu^{n}(0)=\mu^{n+1}(1) .
\end{aligned}
$$

Finally, using (26), we have

$$
\begin{aligned}
w \in L\left(D^{\prime}\right) & \Longleftrightarrow \delta^{\prime}\left(q_{0}^{\prime}, w\right)=\left[\eta_{\Phi_{1}(w)}, \eta_{\Phi_{2}(w)}, \zeta_{0}, \zeta_{1}\right] \in F^{\prime} \\
& \Longleftrightarrow \eta_{\Phi_{1}(w)} \circ \eta_{\Phi_{2}(w)}\left(q_{0}\right) \in F \\
& \Longleftrightarrow \Phi(w)=\Phi_{1}(w) \Phi_{2}(w) \in L(D) .
\end{aligned}
$$

Theorem 14. Let $D=\left(Q, \Sigma_{2}, \delta, q_{0}, F\right)$ be a DFA with $N$ states. If $D$ accepts at least one overlap-free word, then the length of the shortest overlap-free word accepted is $2^{O\left(N^{4 N}\right)}$.

Proof. Let $L=L(D)$. Using Theorem 13, there exists a DFA $D^{\prime}$ with $N^{4 N}$ states that accepts the language $\Psi(L)$.

Since $\Psi(R)$ is regular and is accepted by a DFA with at most 400 states, we see that

$$
K=\Psi(L) \cap \Psi(R)
$$

is regular and is accepted by a DFA with $O\left(N^{4 N}\right)$ states.
Since $L$ accepts an overlap-free word, the language $K$ is nonempty. Using Proposition 1, we see that $K$ contains a word $w$ of length $O\left(N^{4 N}\right)$.

Therefore $\Phi(w)$ is an overlap-free word in $L$. By induction, one can easily prove that $|\Phi(w)|=O\left(2^{|w|}\right)$. Hence we have $|\Phi(w)|=2^{O\left(N^{4 N}\right)}$.

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