# Shortest Repetition-Free Words Accepted by Automata

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Abstract. We consider the following problem: given that a finite automaton M of N states accepts at least one k-power-free (resp., overlap-free) word, what is the length of the shortest such word accepted? We give upper and lower bounds which, unfortunately, are widely separated.

### 1 Introduction

Let L be an interesting language, such as the language of primitive words, or the language of non-palindromes. We are interested in the following kind of question: given that an automaton M of N states accepts a member of L, what is a good bound on the length  $\ell(N)$  of the shortest word accepted?

For example, Ito et al. [7] proved that if L is the language of primitive words, then  $\ell(N) \leq 3N - 3$ . Horváth et al. [6] proved that if L is the language of non-palindromes, then  $\ell(N) \leq 3N$ . For additional results along these lines, see [1].

For an integer  $k \ge 2$ , a *k*-power is a nonempty word of the form  $x^k$ . A word is *k*-power-free if it has no *k*-powers as factors. A word of the form *axaxa*, where *a* is a single letter, and *x* is a (possibly empty) word, is called an *overlap*. A word is *overlap*-free if it has no factor that is an overlap.

In this paper we address two open questions left unanswered in [1], corresponding to the case where L is the language of k-power-free (resp., overlap-free) words. For these words and a large enough alphabet we give a class of DFAs of N states for which the shortest k-power (resp., overlap) is of length  $N^{\frac{1}{4}(\log N)+O(1)}$ . For overlaps over a binary alphabet we give an upper bound of  $2^{O(N^{4N})}$ .

# 2 Notation

For a finite alphabet  $\Sigma$ , let  $\Sigma^*$  denote the set of finite words over  $\Sigma$ . Let  $w = a_0a_1 \cdots a_{n-1} \in \Sigma^*$  be a word. Let  $w[i] = a_i$ , and let  $w[i..j] = a_i \cdots a_j$ . By convention we have  $w[i] = \epsilon$  for i < 0 or  $i \ge n$ , and  $w[i..j] = \epsilon$  for i > j. A prefix p of w is a *period* of w if w[i+r] = w[i] for  $0 \le i < |w| - r$ , where r = |p|.

For words x, y, let  $x \leq y$  denote that x is a factor of y. A factor x of y is proper if  $x \neq y$ . Let  $x \leq_p y$  (resp.,  $x \leq_s y$ ) denote that x is a prefix (resp., suffix) of y. Let  $x \prec_p y$  (resp.,  $x \prec_s y$ ) denote that x is a prefix (resp., suffix) of y and  $x \neq y$ .

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A word is *primitive* if it is not a k-power for any  $k \ge 2$ . Two words x, y are *conjugate* if one is a cyclic shift of the other; that is, if there exist words u, v such that x = uv and y = vu. One simple observation is that all conjugates of a k-power are k-powers.

Let  $h: \Sigma^* \to \Sigma^*$  be a morphism, and suppose h(a) = ax for some letter a. The *fixed point* of h, starting with  $a \in \Sigma$ , is denoted by  $h^{\omega}(a) = a x h(x) h^2(x) \cdots$ . We say that a morphism h is k-power-free (resp., overlap-free) if h(w) is k-power-free (resp., overlap-free) if w is.

Let  $\Sigma_m = \{0, 1, \dots, m-1\}$ . Define the morphism  $\mu : \Sigma_2^* \to \Sigma_2^*$  as follows

$$\mu(0) = 01$$
  
 $\mu(1) = 10.$ 

We call  $\mathbf{t} = \mu^{\omega}(0)$  the *Thue-Morse word*. It is easy to see that

$$\mu(\mathbf{t}[0..n-1]) = \mathbf{t}[0..2n-1] \text{ for } n \ge 0.$$

From classical results of Thue [10,11], we know that the morphism  $\mu$  is overlapfree. From [2], we know that that  $\mu(x)$  is k-power free for each k > 2.

For a DFA  $D = (Q, \Sigma, \delta, q_0, F)$  the set of states, input alphabet, transition function, set of final states, and initial state are denoted by  $Q, \Sigma, \delta, F$ , and  $q_0$ , respectively. Let L(D) denote the language accepted by D. As usual, we have  $\delta(q, wa) = \delta(\delta(q, w), a)$  for a word w.

We state the following basic result without proof.

**Proposition 1.** Let  $D = (Q, \Sigma, \delta, q_0, F)$  be a (deterministic or nondeterministic) finite automaton. If  $L(D) \neq \emptyset$ , then D accepts at least one word of length smaller than |Q|.

# 3 Lower bound

In this section, we construct an infinite family of DFAs such that the shortest k-power-free word accepted is rather long, as a function of the number of states. Up to now only a linear bound was known.

For a word w of length n and  $i \ge 1$ , let

$$\operatorname{cyc}_{i}(w) = w[i..n-1]w[0..i-2]$$

denote w's *i*th cyclic shift to the left, followed by removing the last symbol. Also define

$$\operatorname{cyc}_{0}(w) = w[0..n-2].$$

For example, we have

 $\operatorname{cyc}_2(\operatorname{recompute}) = \operatorname{computer},$  $\operatorname{cyc}_4(\operatorname{richly}) = \operatorname{lyric}.$  We call each  $\operatorname{cyc}_i(w)$  a *partial conjugate* of w, which is not a reflexive, symmetric, or transitive relation.

A word w is a *simple k-power* if it is a *k*-power and it contains no *k*-power as a proper factor.

We start with a few lemmas.

**Lemma 2.** Let  $w = p^k$  be a simple k-power. Then the word p has |p| distinct conjugates.

*Proof.* By contradiction. If  $p^k$  is a simple k-power, then p is a primitive word. Suppose that p = uv = xy such that  $x \prec_p u$  and yx = vu. Without loss of generality, we can assume that  $xv \neq \epsilon$ . Then there exists a word  $t \neq \epsilon$  such that u = xt and y = tv. From vu = yx we get

$$vxt = tvx.$$

Using the second theorem of Lyndon and Schützenberger [8], we get that there exists  $z\neq\epsilon$  such that

 $vx = z^i$  $t = z^j$ 

for some positive integers i, j. So  $yx = z^{i+j}$ , and hence p = xy is not primitive, a contradiction.

**Lemma 3.** Let w be a simple k-power of length n. Then we have

$$\operatorname{cyc}_i(w) = \operatorname{cyc}_j(w) \text{ iff } i \equiv j \pmod{\frac{n}{k}}.$$
 (1)

*Proof.* Let  $w = p^k$ . If  $i \equiv i' \pmod{\frac{n}{k}}$  and  $i' < \frac{n}{k}$ , then

$$\operatorname{cyc}_{i}(w) = (p[i' .. \frac{n}{k} - 1] p[0..i' - 1])^{k-1} \operatorname{cyc}_{i'}(p).$$

Similarly, if  $j \equiv j' \pmod{\frac{n}{k}}$  and  $j' < \frac{n}{k}$ , then

$$\operatorname{cyc}_{j}(w) = (p[j' .. \frac{n}{k} - 1] p[0..j' - 1])^{k-1} \operatorname{cyc}_{j'}(p).$$

If i' = j', then clearly  $\operatorname{cyc}_i(w) = \operatorname{cyc}_j(w)$ . If  $i' \neq j'$ , we get that

$$p[i' .. \frac{n}{k} - 1] p[0..i' - 1] \neq p[j' .. \frac{n}{k} - 1] p[0..j' - 1]$$

using Lemma 2, and hence  $\operatorname{cyc}_i(w) \neq \operatorname{cyc}_i(w)$ .

**Lemma 4.** All conjugates of a simple k-power are simple k-powers.

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Fig. 1: starting positions of the occurrences of q inside x

*Proof.* By contradiction. Let  $w = p^k$  be a simple k-power, and let  $z \neq w$  be a conjugate of w. Clearly z is a k-power. Suppose z contains  $q^k$  and  $z \neq q^k$ . Thus |q| < |p|. Since w is simple  $q^k \not\preceq w = p^k$ . The word  $x = p^{k+1}$  contains z as a factor. So  $x = uq^k v$ , for some words  $u, v \preceq p$ .

Note that u and v are nonempty and not equal to p since  $q^k \not\preceq p^k$ . Letting e := |p| - |q|, and considering the starting positions of the occurrences of q in x (see Fig. 1), we can write

$$x[|p^{i}u| - ie..|p^{i}u| - (i-1)e - 1] = x[|p^{j}u| - je..|p^{j}u| - (j-1)e - 1]$$

for every  $0 \le i, j < k$ . Since p is a period of x, we can write

$$x[|u| - ie..|u| - (i - 1)e - 1] = x[|u| - je..|u| - (j - 1)e - 1]$$

which means  $x[u - (k - 1)e...u + e - 1] \leq w$  is a k-power. Therefore w contains a k-power other than itself, a contradiction.

### Corollary 5. Partial conjugates of simple k-powers are k-power-free.

The next lemma shows that there are infinitely many simple k-powers over a binary alphabet for k > 2. We also show that there are infinitely many simple squares over a ternary alphabet, using a result of Currie [4].

#### Lemma 6.

- (i) Let  $p = \mathbf{t}[0..2^n 1]$  where  $n \ge 0$ . For every k > 2, the word  $p^k$  is a simple k-power.
- (ii) There are infinitely many simple squares over a ternary alphabet.

# Proof.

(i) By induction on *n*. For n = 0 we have  $p^k = 0^k$  which is a simple *k*-power. Suppose n > 0. To get a contradiction, suppose that there exist words u, v, x with  $uv \neq \epsilon$  and  $x \neq \epsilon$  such that  $p^k = ux^k v$ . Note that |x| < |p|, so  $|uv| \ge k$ . Without loss of generality, we can assume that  $|v| \ge \lceil \frac{k}{2} \rceil \ge 2$ . Let  $q = \mathbf{t}[0..2^{n-1} - 1]$ . We know that

$$p^k = \mu(q^k).$$

We can write

$$w = ux^k \preceq_p \mu(q^{k-1}q[0..|q|-2])$$

Since  $\mu$  is k-power-free, the word  $q^{k-1}q[0..|q|-2]$  contains a k-power. Hence  $q^k$  contains at least two k-powers, a contradiction.

(ii) Currie [4] proved that over a ternary alphabet, for every  $n \ge 18$ , there is a word p of length n such that all its conjugates are squarefree. Such squarefree words are called *circularly squarefree words*.

We claim that for every circularly squarefree word p, the word  $p^2$  is a simple square. To get a contradiction, let  $q^2$  be the smallest square in  $p^2$ . So there exist words u, y with  $uy \neq \epsilon$  such that  $p^2 = uq^2 y$ . We have  $|q^2| > |p|$  since p is circularly squarefree. Therefore, if we let p = uv = xy, then |x| > |u|and |v| > |y|. So there exists t such that x = ut and v = ty. We can assume |t| < |q|, since otherwise |t| = |q| and |uy| = 0, a contradiction. Now since  $q^2 = vx = tyut$ , we get that q begins and ends with t, which means  $t^2 \prec q^2$ . Therefore  $p^2$  has a smaller square than  $q^2$ , a contradiction.

Next we show how to construct arbitrarily long simple k-powers from smaller ones. Fix k = 2 (resp.,  $k \ge 3$ ) and m = 3 (resp., m = 2). Let  $w_1 \in \Sigma_m^*$  be a simple k-power. Using the previous lemma, there are infinitely many choices for  $w_1$ . Let  $w_1$  be of length n. Define  $w_{i+1} \in \Sigma_{m+i}^*$  for  $i \ge 1$  recursively by

$$w_{i+1} = \operatorname{cyc}_0(w_i)a_i\operatorname{cyc}_{n^{i-1}}(w_i)a_i\operatorname{cyc}_{2n^{i-1}}(w_i)a_i\cdots\operatorname{cyc}_{(n-1)n^{i-1}}(w_i)a_i \qquad (2)$$

where  $a_i = m + i - 1$  and  $w_0 = 0$ . The next lemma states that  $w_i$ , for  $i \ge 1$ , is a simple k-power. Therefore, using Corollary 5, each word  $cyc_0(w_i)$  is k-power-free. For  $i \geq 1$ , it is easy to see that

$$|w_i| = n|w_{i-1}| = n^i. (3)$$

**Lemma 7.** For every  $i \ge 1$ , the word  $w_i$  is a simple k-power.

*Proof.* By induction on *i*. The word  $w_1$  is a simple k-power. Now suppose that  $w_i$  is a simple k-power for some  $i \ge 1$ . Using Lemma 3, we have  $\operatorname{cyc}_{in^{i-1}}(w_i) =$  $\operatorname{cyc}_{(j+\frac{n}{k})n^{i-1}}(w_i)$ , since  $\frac{|w_i|}{k} = \frac{n^i}{k}$ . We now claim that  $w_{i+1}$  is a k-power and

$$w_{i+1} = (\operatorname{cyc}_0(w_i)a_i \operatorname{cyc}_{n^{i-1}}(w_i)a_i \operatorname{cyc}_{2n^{i-1}}(w_i)a_i \cdots \operatorname{cyc}_{(\frac{n}{\tau}-1)n^{i-1}}(w_i)a_i)^k.$$

To see this, suppose that  $w_{i+1}$  contains a k-power  $y^k$  such that  $w_{i+1} \neq y^k$ .

If y contains more than one occurrence of  $a_i$ , then  $y = ua_i \operatorname{cyc}_i(w_i)a_i v$  for some words u, v and an integer j. Since  $y^2 = ua_i \operatorname{cyc}_i(w_i)a_i v ua_i \operatorname{cyc}_i(w_i)a_i v \preceq$  $w_{i+1}$ , using (2) and Lemma 3, we get

$$|y| = \left|\operatorname{cyc}_{j}(w_{i})a_{i}vua_{i}\right| \ge \frac{n}{k}n^{i} = \frac{|w_{i+1}|}{k},$$

and hence  $y^k = w_{i+1}$ , a contradiction.

If y contains just one  $a_i$ , then  $y = ua_i v$  for some words u, v which contain no  $a_i$ . So  $y^k = u(avu)^{k-1}av$  for  $a = a_i$ . Therefore vu is a partial conjugate of  $w_i$ . However the distance between two equal partial conjugates of  $w_i$  in  $w_{i+1}$  is longer than just one letter, using (2) and Lemma 3.

Finally, if y contains no  $a_i$ , then a partial conjugate of  $w_i$  contains a k-power, which is impossible due to Lemma 4. 

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To make our formulas easier to read, we define  $a_0 = w_1[n-1]$ .

**Theorem 8.** For  $i \ge 1$ , there is a DFA  $D_i$  with  $2^{i-1}(n-1)+2$  states such that  $\operatorname{cyc}_0(w_i)$  is the shortest k-power-free word in  $L(D_i)$ .

*Proof.* Define  $D_1 = (Q_1, \Sigma_{a_1}, \delta_1, q_{1,0}, F_1)$  where

$$Q_{1} := \{q_{1,0}, q_{1,1}, q_{1,2}, \dots, q_{1,n-1}, q_{d}\},\$$
  
$$F_{1} := \{q_{1,n-1}\},\$$
  
$$\delta_{1}(q_{1,j}, w[j]) := q_{1,j+1} \text{ for } 0 \le j < n-1,$$

and the rest of the transitions go to the dead state  $q_d$ . Clearly we have  $|Q_1| = n+1$ and  $L(D_1) = {cyc_0(w_1)}$ .

We define  $D_i = (Q_i, \Sigma_{a_i}, \delta_i, q_{1,0}, F_i)$  for  $i \ge 2$  recursively. For the rest of the proof s and t denote (possibly empty) sequences of integers and j denotes a single integer (a sequence of length 1). We use integer sequences as subscripts of states in  $Q_i$ . For example,  $q_{1,0}, q_{s,j}$ , and  $q_{s,2,t}$  might denote states of  $D_i$ . For  $i \ge 1$ , define

$$Q_{i+1} := Q_i \cup \{q_{i+1,t} : q_t \in (Q_i - F_i) - \{q_d\}\},\tag{4}$$

$$F_{i+1} := \{ q_{i+1,i,t} : \delta_i(q_{i,t}, c) = q_{1,n-1} \text{ for some } c \in \Sigma_{a_i} \},$$
(5)

if 
$$q_t \in Q_i$$
 and  $c \in \Sigma_{a_i}$ , then  $\delta_{i+1}(q_t, c) := \delta_i(q_t, c)$  (6)

if 
$$q_t, q_s \in (Q_i - F_i) - \{q_d\}, c \in \Sigma_{a_i}$$
, and  $\delta_i(q_t, c) = q_s$ ,

then 
$$\delta_{i+1}(q_{i+1,t},c) := q_{i+1,s}$$
 (7)

if 
$$q_t \in F_i$$
, then  $\delta_{i+1}(q_t, a_i) := q_{1,1}$  and  $\delta_{i+1}(q_t, a_{i-1}) := q_{i+1,1,0}$  (8)

if  $i > 1, q_{i+1,t} \notin F_{i+1}$ , and  $\delta_i(q_t, a_{i-1}) = q_{1,j}$ ,

then 
$$\delta_{i+1}(q_{i+1,t}, a_i) := q_{1,j+1}$$
 (9)

and finally for the special case of i = 1,

$$\delta_2(q_{2,1,j}, a_1) := q_{1,j+2} \text{ for } 0 \le j < n-2.$$
(10)

The rest of the transitions, not indicated in (6)–(10), go to the dead state  $q_d$ . Fig. 2b depicts  $D_2$  and  $D_3$ . Using (4), we have  $|Q_{i+1}| = 2|Q_i| - 2 = 2^i(n-1) + 2$  by a simple induction.

An easy induction on *i* proves that  $|F_i| = 1$ . So let  $f_i$  be the appropriate integer sequence for which  $F_i = \{q_{f_i}\}$ . Using (6)–(10), we get that for every  $1 \leq j < n$ , there exists exactly one state  $q_t \in Q_i$  for which  $\delta_i(q_t, a_{i-1}) = q_{1,j}$ .

By induction on *i*, we prove that for  $i \ge 2$  if  $\delta_i(q_t, a_{i-1}) = q_{1,j}$ , then

$$x_1 = \operatorname{cyc}_{(j-1)n^{i-2}}(w_{i-1}), \tag{11}$$

$$x_2 = w_i [0..jn^{i-1} - 2], \tag{12}$$

$$x_3 = w_i[(j-1)n^{i-1}..n^i - 2].$$
(13)

are the shortest k-power-free words for which

$$\delta_i(q_{1,j-1}, x_1) = q_t, \tag{14}$$

$$\delta_i(q_{1,0}, x_2) = q_t, \tag{15}$$

$$\delta_i(q_{1,j-1}, x_3) = q_{f_i}.$$
(16)

In particular, from (13) and (16), for j = 1, we get that  $cyc_0(w_i)$  is the shortest k-power-free word in  $L(D_i)$ .

The fact that our choices of  $x_1, x_2$ , and  $x_3$  are k-power-free follows from the fact that proper factors of simple k-powers are k-power-free. For i = 2 the proofs of (14)-(16) are easy and left to the readers.

Suppose that (14)–(16) hold for some  $i \ge 2$ . Let us prove (14)–(16) for i + 1. Suppose that

$$\delta_{i+1}(q_t, a_i) = q_{1,j}.$$
(17)

First we prove that the shortest k-power-free word x for which

$$\delta_{i+1}(q_{1,j-1},x) = q_t,$$

is  $x = \operatorname{cyc}_{(j-1)n^{i-1}}(w_i)$ . If  $q_t \in Q_i$ , from (8) and (17), we have

$$q_t = q_{f_i}, \text{ and}$$
  
 $\delta_{i+1}(q_t, a_i) = q_{1,1}$ 

By induction hypothesis, the  $cyc_0(w_i)$  is the shortest k-power-free word in  $L(D_i)$ . In other words, we have  $\delta_i(q_{1,0}, \operatorname{cyc}_0(w_i)) = q_{f_i} = q_t$ , which can be rewritten using (6) as  $\delta_{i+1}(q_{1,0}, \operatorname{cyc}_0(w_i)) = q_t$ .

Now suppose  $q_t \notin Q_i$ . Then by (9) and (17), we get that there exists t' such that

$$t = i + 1, t';$$
  

$$\delta_i(q_{t'}, a_{i-1}) = q_{1,j-1}$$

From the induction hypothesis, i.e., (15) and (16), we can write

$$\delta_i(q_{1.0}, w_i[0..(j-1)n^{i-1} - 2]) = q_{t'}, \tag{18}$$

$$\delta_i(q_{1,j-1}, w_i[(j-1)n^{i-1}..n^i - 2]) = q_{f_i}.$$
(19)

In addition  $w_i[0..(j-1)n^{i-1}-2]$  and  $w_i[(j-1)n^{i-1}..n^i-2]$  are the shortest k-power-free transitions from  $q_{1,0}$  to  $q_{t'}$  and from  $q_{1,j-1}$  to  $q_{f_i}$  respectively. Using (6), we can rewrite (18) and (19) for  $\delta_{i+1}$  as follows:

$$\delta_{i+1}(q_{1,0}, w_i[0..(j-1)n^{i-1}-2]) = q_{t'}, \tag{20}$$

$$\delta_{i+1}(q_{1,j-1}, w_i[(j-1)n^{i-1}..n^i - 2]) = q_{f_i}.$$
(21)

Note that from (7) and (20), we get

$$\delta_{i+1}(q_{i+1,1,0}, w_i[0..(j-1)n^{i-1}-2]) = q_{i+1,t'} = q_t.$$
(22)

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We also have  $\delta_{i+1}(q_{f_i}, a_i) = q_{i+1,1,0}$ , using (8). So together with (21) and (22), we get

$$\delta_{i+1}(q_{1,j-1}, \operatorname{cyc}_{(j-1)n^{i-1}}(w_i)) = q_t$$

and  $\operatorname{cyc}_{(j-1)n^{i-1}}(w_i)$  is the shortest k-power-free transition from  $q_{1,j-1}$  to  $q_t$ . The proofs of (15) and (16) are similar.

In what follows, all logarithms are to the base 2.

**Corollary 9.** For infinitely many N, there exists a DFA with N states such that the shortest k-power-free word accepted is of length  $N^{\frac{1}{4}\log N+O(1)}$ .

*Proof.* Let  $i = \lfloor \log n \rfloor$  in Theorem 8. Then  $D = D_i$  has

$$N = 2^{\lfloor \log n \rfloor - 1} (n - 1) + 2 = \Omega(n^2)$$

states. In addition, the shortest k-power-free word in L(D) is  $\operatorname{cyc}_0(w_{\lfloor \log n \rfloor})$ . Now, using (3) we can write

$$\left|\operatorname{cyc}_{0}(w_{\lfloor \log n \rfloor})\right| = n^{\lfloor \log n \rfloor} - 1.$$

Suppose  $2^t \leq n < 2^{t+1} - 1$ , so that  $t = \lfloor \log n \rfloor$  and Then  $\log N = 2t + O(1)$ , so  $\frac{1}{4}(\log N)^2 = t^2 + O(t)$ . On the other hand  $\log |w| = \lfloor \log n \rfloor(\log n) = t(t+O(1)) = t^2 + O(t)$ . Now  $2^{O(t)} = n^{O(1)} = N^{O(1)}$ , and the result follows.  $\Box$ 

Remark 10. The same bound holds for overlap-free words. To do so, we define a simple overlap as a word of the form axaxa where axax is a simple square. In our construction of the DFAs, we use complete conjugates of  $(ax)^2$  instead of partial conjugates.

Remark 11. The  $D_i$  in Theorem 8 are defined over the growing alphabet  $\Sigma_{m+i-1}$ . However, we can fix the alphabet to be  $\Sigma_{m+1}$ . For this purpose, we introduce  $w'_i$  which is quite similar to  $w_i$ :

$$w'_{1} = w_{1},$$
  

$$w'_{i+1} = \operatorname{cyc}_{0}(w'_{i})b_{i}\operatorname{cyc}_{n^{i-1}}(w'_{i})b_{i}\operatorname{cyc}_{2n^{i-1}}(w'_{i})b_{i}\cdots\operatorname{cyc}_{(n-1)n^{i-1}}(w'_{i})b_{i},$$

where  $b_i = mc_i m$  such that  $c_i$  is (any of) the shortest nonempty k-power-free word over  $\Sigma_m$  not equal to  $c_1, \ldots, c_{i-1}$ . Clearly we have  $|b_i| \leq |b_{i-1}| + 1 = O(i)$ , and hence  $w'_i = \Theta(n^i)$ .

One can then prove Lemma 7 and Theorem 8 for  $w'_i$  with minor modifications of the argument above. In particular, we construct DFA  $D'_i$  that accepts  $\operatorname{cyc}_0(w'_i)$ as the shortest k-power-free word accepted, and a  $D'_i$  that is quite similar to  $D_i$ . In particular, they have asymptotically the same number of states.



Fig. 2: transition diagrams

 $q_{2,1,n-2}$ 

(b) transition diagram of  $D_3$ 

 $q_{2,1,n-2}$ 

(a) transition diagram of

 $D_2$ 

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### 4 Upper bound for overlap-free words

In this section, we prove an upper bound on the length of the shortest overlapfree word accepted by a DFA D over a binary alphabet.

Let L = L(D) and let R be the set of overlap-free words over  $\Sigma_2^*$ . Carpi [3] defined a certain operation  $\Psi$  on binary languages, and proved that  $\Psi(R)$  is regular. We prove that  $\Psi(L)$  is also regular, and hence  $\Psi(L) \cap \Psi(R)$  is regular. The next step is to apply Proposition 1 to get an upper bound on the length of the shortest word in  $\Psi(L) \cap \Psi(R)$ . This bound then gives us an upper bound on the length of the shortest overlap-free word in L.

Let  $H = \{\epsilon, 0, 1, 00, 11\}$ . Carpi defines maps

$$\Phi_l, \Phi_r: \Sigma_{25} \to H$$

such that for every pair  $h, h' \in H$ , one has

$$h = \Phi_l(a), h' = \Phi_r(a)$$

for exactly one letter  $a \in \Sigma_{25}$ .

For every word  $w \in \Sigma_{25}^*$ , define  $\Phi(w) \in \Sigma_2^*$  inductively by

$$\Phi(\epsilon) = \epsilon, \Phi(aw) = \Phi_l(a)\mu(\Phi(w))\Phi_r(a) \qquad (w \in \Sigma_{25}^*, a \in \Sigma_{25}).$$
(23)

Expanding (23) for  $w = a_0 a_1 \cdots a_{n-1}$ , we get

$$\Phi_l(a_0)\mu(\Phi_l(a_1))\cdots\mu^{n-1}(\Phi_l(a_{n-1}))\mu^{n-1}(\Phi_r(a_{n-1}))\cdots\mu(\Phi_r(a_1))\Phi_r(a_0).$$
 (24)

For  $L \subseteq \Sigma_2^*$  define  $\Psi(L) = \bigcup_{x \in L} \Phi^{-1}(x)$ . Based on the decomposition of Restivo and Salemi [9] for finite overlap-free words, the language  $\Psi(x)$  is always nonempty for an overlap-free word  $x \in \Sigma_2^*$ . The next theorem is due to Carpi [3].

**Theorem 12.**  $\Psi(R)$  is regular.

Carpi constructed a DFA A with less than 400 states that accepts  $\Psi(R)$ . We prove that  $\Psi$  preserves regular languages.

**Theorem 13.** Let  $D = (Q, \Sigma_2, \delta, q_0, F)$  be a DFA with N states, and let L = L(D). Then  $\Psi(L)$  is regular and is accepted by a DFA with at most  $N^{4N}$  states.

*Proof.* Let  $\iota: Q \to Q$  denote the identity function, and define  $\eta_0, \eta_1: Q \to Q$  as follows

$$\eta_i(q) = \delta(q, i) \text{ for } i = 0, 1.$$
(25)

For functions  $\zeta_0, \zeta_1 : Q \to Q$ , and a word  $x = b_0 b_1 \cdots b_{n-1} \in \Sigma_2^*$ , define  $\zeta_x = \zeta_{b_{n-1}} \circ \cdots \circ \zeta_{b_1} \circ \zeta_{b_0}$ . Therefore we have  $\zeta_y \circ \zeta_x = \zeta_{xy}$ . Also by convention  $\zeta_{\epsilon} = \iota$ . So for example  $x \in L(D)$  if and only if  $\eta_x(q_0) \in F$ . We create DFA  $D' = (Q', \Sigma_{25}, \delta', q'_0, F')$  where

$$Q' = \{ [\kappa, \lambda, \zeta_0, \zeta_1] : \kappa, \lambda, \zeta_0, \zeta_1 : Q \to Q \}, \delta'([\kappa, \lambda, \zeta_0, \zeta_1], a) = [\zeta_{\Phi_l(a)} \circ \kappa, \lambda \circ \zeta_{\Phi_r(a)}, \zeta_1 \circ \zeta_0, \zeta_0 \circ \zeta_1].$$

Also let

$$q'_{0} = [\iota, \iota, \eta_{0}, \eta_{1}],$$
  

$$F' = \{[\kappa, \lambda, \zeta_{0}, \zeta_{1}] : \lambda \circ \kappa(q_{0}) \in F\}.$$
(26)

We can see that  $|Q'| = N^{4N}$ . We claim that D' accepts  $\Psi(L)$ . Indeed, on input w, the DFA D' simulates the behavior of D on  $\Phi(w)$ .

Let  $w = a_0 a_1 \cdots a_{n-1} \in \Sigma_{25}^*$ , and define

$$\Phi_1(w) = \Phi_l(a_{a_0})\mu(\Phi_l(a_1))\cdots\mu^{n-1}(\Phi_l(a_{n-1})),$$
  
$$\Phi_2(w) = \mu^{n-1}(\Phi_r(a_{n-1}))\cdots\mu(\Phi_r(a_1))\Phi_r(a_0).$$

Using (24), we can write

$$\Phi(w) = \Phi_1(w)\Phi_2(w).$$

We prove by induction on n that

$$\delta'(q'_0, w) = \left[\eta_{\Phi_1(w)}, \eta_{\Phi_2(w)}, \eta_{\mu^n(0)}, \eta_{\mu^n(1)}\right].$$
(27)

For n = 0, we have  $\Phi(w) = \Phi_1(w) = \Phi_2(w) = \epsilon$ . So

$$\delta'(q'_0, \epsilon) = q'_0 = [\iota, \iota, \eta_0, \eta_1] = [\eta_{\Phi_1(w)}, \eta_{\Phi_2(w)}, \eta_{\mu^0(0)}, \eta_{\mu^0(1)}]$$

So we can assume (27) holds for some  $n \ge 0$ . Now suppose  $w = a_0 a_1 \cdots a_n$ and write

$$\begin{aligned} \delta'(q'_0, a_0 a_1 \cdots a_n) \\ &= \delta'(\delta'(q'_0, a_0 a_1 \cdots a_{n-1}), a_n) \\ &= \delta'\left(\left[\eta_{\Phi_1(w[0..n-1])}, \eta_{\Phi_2(w[0..n-1])}, \eta_{\mu^n(0)}, \eta_{\mu^n(1)}\right], a_n\right) \\ &= \left[\eta_{\mu^n(\phi_l(a_n))} \circ \eta_{\Phi_1(w[0..n-1])}, \eta_{\Phi_2(w[0..n-1])} \circ \eta_{\mu^n(\phi_r(a_n))}, \eta_{\mu^n(1)} \circ \eta_{\mu^n(0)}, \eta_{\mu^n(0)} \circ \eta_{\mu^n(1)}\right] \\ &= \left[\eta_{\Phi_1(w)}, \eta_{\Phi_2(w)}, \eta_{\mu^{n+1}(0)}, \eta_{\mu^{n+1}(1)}\right], \end{aligned}$$
(28)

and equality (28) holds because

$$\begin{split} & \varPhi_1(w[0..n-1])\mu^n(\phi_l(a_n)) = \varPhi_1(w), \\ & \mu^n(\phi_r(a_n))\varPhi_2(w[0..n-1]) = \varPhi_2(w), \\ & \mu^n(0)\mu^n(1) = \mu^n(01) = \mu^n(\mu(0)) = \mu^{n+1}(0), \text{ and similarly} \\ & \mu^n(1)\mu^n(0) = \mu^{n+1}(1). \end{split}$$

Finally, using (26), we have

$$w \in L(D') \iff \delta'(q'_0, w) = \left[\eta_{\Phi_1(w)}, \eta_{\Phi_2(w)}, \zeta_0, \zeta_1\right] \in F'$$
$$\iff \eta_{\Phi_1(w)} \circ \eta_{\Phi_2(w)}(q_0) \in F$$
$$\iff \Phi(w) = \Phi_1(w)\Phi_2(w) \in L(D).$$

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**Theorem 14.** Let  $D = (Q, \Sigma_2, \delta, q_0, F)$  be a DFA with N states. If D accepts at least one overlap-free word, then the length of the shortest overlap-free word accepted is  $2^{O(N^{4N})}$ .

*Proof.* Let L = L(D). Using Theorem 13, there exists a DFA D' with  $N^{4N}$  states that accepts the language  $\Psi(L)$ .

Since  $\varPsi(R)$  is regular and is accepted by a DFA with at most 400 states, we see that

$$K = \Psi(L) \cap \Psi(R)$$

is regular and is accepted by a DFA with  $O(N^{4N})$  states.

Since L accepts an overlap-free word, the language K is nonempty. Using Proposition 1, we see that K contains a word w of length  $O(N^{4N})$ .

Therefore  $\Phi(w)$  is an overlap-free word in *L*. By induction, one can easily prove that  $|\Phi(w)| = O(2^{|w|})$ . Hence we have  $|\Phi(w)| = 2^{O(N^{4N})}$ .

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