Suffix conjugates for a class of morphic subshifts

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Abstract

Let A be a finite alphabet and $f: A^* \to A^*$ be a morphism with an iterative fixed point $f^{\omega}(\alpha)$, where $\alpha \in A$. Consider the subshift (\mathcal{X}, T) , where \mathcal{X} is the shift orbit closure of $f^{\omega}(\alpha)$ and $T: \mathcal{X} \to \mathcal{X}$ is the shift map. Let S be a finite alphabet that is in bijective correspondence via a mapping c with the set of nonempty suffixes of the images f(a) for $a \in A$. Let $S \subset S^{\mathbb{N}}$ be the set of infinite words $\mathbf{s} = (s_n)_{n \geq 0}$ such that $\pi(\mathbf{s}) := c(s_0) f(c(s_1)) f^2(c(s_2)) \cdots \in \mathcal{X}$. We show that if f is primitive and f(A) is a suffix code, then there exists a mapping $H: S \to S$ such that (S, H) is a topological dynamical system and $\pi: (S, H) \to (\mathcal{X}, T)$ is a conjugacy; we call (S, H) the suffix conjugate of (\mathcal{X}, T) . In the special case when f is the Fibonacci or the Thue-Morse morphism, we show that the subshift (S, T) is sofic, that is, the language of S is regular.

1 Introduction

Let A be a finite alphabet and $f: A^* \to A^*$ a morphism with an iterative fixed point $f^{\omega}(\alpha) = \lim_{n \to \infty} f^n(\alpha)$. Consider the shift orbit closure \mathcal{X} generated by $f^{\omega}(\alpha)$. If $\mathbf{x} \in \mathcal{X}$, then there exist a letter $a \in A$ and an infinite word $\mathbf{y} \in \mathcal{X}$ such that $\mathbf{x} = sf(\mathbf{y})$, where s is a nonempty suffix of f(a) [5, Lemma 6]. This formula has been observed several times in different contexts, see [7] and the references therein. Since $\mathbf{y} \in \mathcal{X}$, this process can be iterated to generate an expansion

$$\mathbf{x} = s_0 f(s_1) f^2(s_2) \cdots f^n(s_n) \cdots, \tag{1}$$

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where each s_n is a nonempty suffix of an image of some letter in A. In general, however, not every sequence $(s_n)_{n>0}$ of suffixes gives rise to an infinite word in \mathcal{X} by means of this kind of expansion. Therefore, in this paper we introduce the set S that consists of those $(s_n)_{n\geq 0}$ whose expansion (1) is in \mathcal{X} . Our goal is then to understand the structure of \mathcal{S} . By endowing \mathcal{S} with the usual metric on infinite words, \mathcal{S} becomes a metric space. Furthermore, \mathcal{S} can be associated with a mapping $G: \mathcal{S} \to \mathcal{S}$ (see below) giving rise to a topological dynamical system (\mathcal{S}, G) that is an extension of (\mathcal{X}, f) ; see the discussion around Eq. (4) However, imposing some further restrictions on f, we obtain a much stronger result: If f is a circular morphism such that $|f^n(a)| \to \infty$ for all $a \in A$ and f(A) is a suffix code, then there exists a mapping $H: \mathcal{S} \to \mathcal{S}$ such that (\mathcal{S}, H) and (\mathcal{X}, T) , where T is the usual shift operation, are conjugates (Theorem 1). We call (S, H) the suffix conjugate of (X, T). Since primitive morphisms are circular (i.e., recognizable) by Mossé's theorem [14], primitivity of f together with the suffix code condition suffice for the existence of the suffix conjugate. In particular, both the Fibonacci morphism $\varphi \colon 0 \mapsto 01, 1 \mapsto 0$ and the Thue-Morse morphism μ : $0 \mapsto 01$, $1 \mapsto 10$ satisfy these conditions, and so the corresponding Fibonacci subshift $(\mathcal{X}_{\varphi}, T)$ and the Thue-Morse subshift (\mathcal{X}_{μ}, T) have suffix conjugates. In this paper we characterize the language of both subshifts and show that they are regular.

An encoding scheme for \mathcal{X} related to ours was considered by Holton and Zamboni [7] and Canterini and Siegel [4], who studied bi-infinite primitive morphic subshifts and essentially used prefixes of images of letters where we use suffixes. Despite of this seemingly insignificant difference, though, we are not aware of any mechanism that would allow transferring results from one encoding scheme to another. See also the work by Shallit [15], who constructed a finite automaton that provides an encoding for the set of infinite overlap-free words.

2 Preliminaries and generalities

In this paper we will follow the standard notation and terminology of combinatorics on words [11, 1] and symbolic dynamics [10, 9].

Let A be a finite alphabet and $f: A^* \to A^*$ a morphism with an iterative fixed point $f^{\omega}(\alpha) = \lim_{n \to \infty} f^n(\alpha)$, where $\alpha \in A$. Let \mathcal{X} be the shift orbit closure generated by $f^{\omega}(\alpha)$. Let S' be the set of nonempty suffixes of images of letters under f. Denote $S = \{0, 1, \ldots, |S'| - 1\}$ and let $c: S \to S'$ be a bijection. We consider S as a finite alphabet.

If $s = s_0 s_1 \cdots s_n$ with $s_i \in S$, then we denote by $\pi(s)$ the word

$$\pi(s) = c(s_0)f(c(s_1))f^2(c(s_2))\cdots f^n(c(s_n)) \in A^*.$$

Then π extends to a mapping $\pi\colon S^{\mathbb{N}}\to A^{\mathbb{N}}$ in a natural way, and so we may define

$$\mathcal{S} = \{ \mathbf{s} \in S^{\mathbb{N}} \mid \pi(\mathbf{s}) \in \mathcal{X} \}.$$

Our goal in this section is to find sufficient conditions on f so that S can be endowed with dynamics that yields a conjugate to (X, T) via the mapping π .

Examples 1 and 2 below show that this task is not trivial. Such sufficient conditions are laid out in Definition 1.

If $\mathbf{x} \in \mathcal{X}$ and $\mathbf{s} \in \mathcal{S}$ such that $\pi(\mathbf{s}) = \mathbf{x}$, we say that \mathbf{x} is an expansion of \mathbf{s} .

Lemma 1 (Currie, Rampersad, and Saari [5]). For every $\mathbf{x} \in \mathcal{X}$, there exist $a \in A$, a non-empty suffix s of f(a), and an infinite word $\mathbf{y} \in \mathcal{X}$ such that $\mathbf{x} = sf(\mathbf{y})$ and $a\mathbf{y} \in \mathcal{X}$. Therefore the mapping $\pi : \mathcal{S} \to \mathcal{X}$ is surjective.

Both $A^{\mathbb{N}}$ and $S^{\mathbb{N}}$ are endowed with the usual metric

$$d((x_n)_{n\geq 0}, (y_n)_{n\geq 0}) = \frac{1}{2^n}, \text{ where } n = \inf\{n \mid x_n \neq y_n\},$$

The following lemma is obvious.

Lemma 2. The mapping $\pi: \mathcal{S} \to \mathcal{X}$ is continuous.

We denote the usual shift operation $(x_n)_{n\geq 0} \mapsto (x_{n+1})_{n\geq 0}$ in both spaces $A^{\mathbb{N}}$ and $S^{\mathbb{N}}$ by T. We have $T(\mathcal{X}) \subset \mathcal{X}$ and $f(\mathcal{X}) \subset \mathcal{X}$ by the construction of \mathcal{X} , and both T and f are clearly continuous on \mathcal{X} , so we have the topological dynamical systems (\mathcal{X},T) and (\mathcal{X},f) . Note, however, that in general $T(\mathcal{S})$ is not necessarily a subset of \mathcal{S} , as the following example shows.

Example 1. Let $f: \{\alpha, a, b\}^* \to \{\alpha, a, b\}^*$ be the morphism $\alpha \mapsto \alpha ab$, $a \mapsto a$, and $b \mapsto ab$. Then

$$f^{\omega}(\alpha) = \alpha f(b) f^{2}(b) f^{3}(b) \cdots$$
 and $T f^{\omega}(\alpha) = babaabaaab \cdots$.

Since the latter sequence is not in the shift orbit closure \mathcal{X} generated by $f^{\omega}(\alpha)$, this shows that \mathcal{S} is not closed under T for this particular morphism.

If f is the morphism $0 \mapsto 01$, $1 \mapsto 0$, then f is called the *Fibonacci morphism* and we write $f = \varphi$. The unique fixed point of φ is denoted by \mathbf{f} and it is called the *Fibonacci word*. The shift orbit closure it generates is denoted by \mathcal{X}_{φ} and the pair $(\mathcal{X}_{\varphi}, T)$ is called the *Fibonacci subshift*.

Similarly, if f is $0 \mapsto 01$, $1 \mapsto 10$, then f is the *Thue-Morse morphism* and we write $f = \mu$. The fixed point $\mu^{\omega}(0)$ of μ is denoted by \mathbf{t} and it is called the *Thue-Morse word*. The shift orbit closure generated by \mathbf{t} is denoted by \mathcal{X}_{μ} , and the pair (\mathcal{X}_{μ}, T) is called the *Thue-Morse subshift*.

Example 2. Let f be the morphism $0 \mapsto 010$, $1 \mapsto 10$. The two fixed points of f generate the Fibonacci subshift. The set of suffixes of f(0) and f(1) is $S' = \{0, 10, 010\}$, and we define a bijection $c: \{0, 1, 2\} \to S'$ by c(0) = 0, c(1) = 10, and c(2) = 010. Then $\pi(01) = \pi(20) = 010010$, and therefore $\pi(01^{\omega}) = \pi(201^{\omega})$. This word equals the Fibonacci word \mathbf{f} as can be seen by observing that

$$\mathbf{f} = 010\varphi^2(10)\varphi^4(10)\varphi^6(10)\cdots$$

and $010f^n(a) = \varphi^{2n}(a)010$ for all $n \ge 0$ and $a \in \{0, 1\}$. This shows that it is possible for two distinct words in S to have the same expansions, and therefore π is not always injective.

The following lemma is a straightforward consequence of the definition of π .

Lemma 3. Let $\mathbf{s} = s_0 s_1 s_2 \cdots$, where $s_i \in S$. Then

$$f(\pi \circ T(\mathbf{s})) = T^{|c(s_0)|}\pi(\mathbf{s}).$$

and

$$\pi(\mathbf{s}) = \pi(s_0 s_1 \cdots s_{n-1}) f^n(\pi(T^n \mathbf{s})). \tag{2}$$

For finite words $x, y \in S^*$, the above reads $\pi(xy) = \pi(x) f^{|x|}(\pi(y))$.

Note that if $s \in S$ such that $c(s) \in S'$ is a letter, then $f(c(s)) \in S'$. As this connection will be frequently referred to, we define a morphism

$$\lambda \colon S_1^* \to S^*$$
 with $\lambda(s) = c^{-1}(f(c(s))),$ (3)

where $S_1 \subset S$ consists of those $s \in S$ for which |c(s)| = 1. Then in particular, $c(\lambda(s)) = f(c(s))$.

Lemma 4. Let $\mathbf{s} = s_0 s_1 \cdots \in \mathcal{S}$ with $s_i \in \mathcal{S}$, and write $\mathbf{x} = \pi(\mathbf{s}) \in \mathcal{X}$. Let $r \geq 0$ be the smallest integer, if it exists, such that $|c(s_r)| \geq 2$ and write $c(s_r) = au$, where $a \in A$ and $u \in A^+$. Then $f(\mathbf{x}) = \pi(\mathbf{t})$, where $\mathbf{t} = t_0 t_1 \cdots \in \mathcal{S}$ satisfies

- $t_i = \lambda(s_i)$ for i = 0, 1, ..., r 1,
- $t_r = c^{-1}(f(a)),$
- $t_{r+1} = c^{-1}(u)$, and
- $t_i = s_{i-1}$ for $i \ge r + 2$.

If each of $c(s_i)$ is a letter, then $f(\mathbf{x}) = \pi(\mathbf{t})$, where

$$\mathbf{t} = \lambda(s_0)\lambda(s_1)\cdots\lambda(s_n)\cdots$$

Proof. Suppose r exists. The identity $\mathbf{x} = \pi(\mathbf{s})$ says that

$$\mathbf{x} = c(s_0) f(c(s_1)) \cdots f^{r-1}(c(s_{r-1})) f^r(c(s_r)) f^{r+1}(c(s_{r+1})) \cdots$$

Therefore, by denoting $f(c(s_i)) = \hat{s}_i \in S'$ for i = 0, 1, ..., r - 1, we see that

$$f(\mathbf{x}) = f(c(s_0)) f^2(c(s_1)) \cdots f^r(c(s_{r-1})) f^{r+1}(c(s_r)) f^{r+2}(c(s_{r+1})) \cdots$$

$$= \hat{s}_0 f(\hat{s}_1) \cdots f^{r-1}(\hat{s}_{r-1}) f^{r+1}(au) f^{r+2}(c(s_{r+1})) \cdots$$

$$= \hat{s}_0 f(\hat{s}_1) \cdots f^{r-1}(\hat{s}_{r-1}) f^r(f(a)) f^{r+1}(u) f^{r+2}(c(s_{r+1})) \cdots$$

$$= c(t_0) f(c(t_1)) f^2(c(t_2)) \cdots,$$

where the t_i 's are as in the statement of the lemma. The case when r does not exist is a special case of the above.

Let $\mathbf{s} \in \mathcal{S}$ and $\mathbf{t} \in \mathcal{S}$ be defined as in the previous lemma. This defines a mapping $G \colon \mathcal{S} \to \mathcal{S}$ for which $G(\mathbf{s}) = \mathbf{t}$, which is obviously continuous. Thus we have a topological dynamical system (\mathcal{S}, G) . Furthermore, by the definition of G, we have

$$f \circ \pi = \pi \circ G. \tag{4}$$

Therefore $\pi: (\mathcal{S}, G) \to (\mathcal{X}, f)$ is a factor map because π is surjective by Lemma 1 and continuous by Lemma 2. We can get a more concise definition for G if we extend the domain of λ defined in (3) to S as follows. If $s \in S \setminus S_1$, then f(c(s)) = au with $a \in A$ and $u \in A^+$, and we define

$$\lambda(s) = c^{-1}(f(a))c^{-1}(u). \tag{5}$$

Then we have, for all $\mathbf{s} \in \mathcal{S}$,

$$G(\mathbf{s}) = \begin{cases} \lambda(ps)\mathbf{t} & \text{if } \mathbf{s} = ps\mathbf{t} \text{ with } p \in S_1^* \text{ and } s \in S \setminus S_1 \\ \lambda(\mathbf{s}) & \text{if } \mathbf{s} \in S_1^{\mathbb{N}}. \end{cases}$$
 (6)

We got this far without imposing any restrictions on f, but now we have to introduce some further concepts.

If \mathcal{Y} is the shift orbit closure of some infinite word \mathbf{x} , then the set of finite factors of \mathbf{x} is called the *language* of \mathcal{Y} or \mathbf{x} and denoted by $\mathcal{L}(\mathcal{Y})$ or by $\mathcal{L}(\mathbf{x})$.

If x is a finite word and y a finite or infinite word and x is a factor of y, we will express this by writing $x \subset y$. This handy notation has been used before at least in [6].

A key property we would like our morphism f to have is called *circularity*, which has various formulations and is also called *recognizability*. We use the formulation of Cassaigne [3] and Klouda [8]; see also [12, 9]. The morphism f whose fixed point generates the shift orbit closure \mathcal{X} is called *circular on* $\mathcal{L}(\mathcal{X})$ if f is injective on $\mathcal{L}(\mathcal{X})$ and there exists a *synchronization delay* $\ell \geq 1$ such that if $w \in \mathcal{L}(\mathcal{X})$ and $|w| \geq \ell$, then it has a *synchronizing point* (w_1, w_2) satisfying the following two conditions: First, $w = w_1 w_2$. Second,

$$\forall v_1, v_2 \in A^* \left[v_1 w v_2 \in f \left(\mathcal{L}(\mathcal{X}) \right) \Longrightarrow v_1 w_1 \in f \left(\mathcal{L}(\mathcal{X}) \right) \quad \text{and} \quad w_2 v_2 \in f \left(\mathcal{L}(\mathcal{X}) \right) \right].$$

A well-known result due to Mossé [14] (see also [9]) says that a primitive morphism with an aperiodic fixed point is circular (or *recognizable*).

Definition 1. We write $f \in \mathcal{N}$ to indicate that $f: A^* \to A^*$ with an iterative fixed point $f^{\omega}(\alpha)$ has the following properties.

- (i) f is circular on the language of $f^{\omega}(\alpha)$:
- (ii) the set f(A) is a suffix code; i.e., no image of a letter is a suffix of another;
- (iii) each letter $a \in A$ is growing; i.e., $|f^n(a)| \to \infty$ as $n \to \infty$.

In particular, if f is primitive and $f^{\omega}(\alpha)$ aperiodic, then f is circular by Mossé's theorem, and if in addition f(A) is a suffix code, then $f \in \mathcal{N}$. Therefore both the Fibonacci morphism φ and the Thue-Morse morphism μ are in \mathcal{N} .

In Example 1 we saw that, in general, S is not necessarily closed under the shift map T for a general morphism f. The next lemma shows, however, that if $f \in \mathcal{N}$, this problem does not arise.

Lemma 5. If $f \in \mathcal{N}$, then $T(\mathcal{S}) \subseteq \mathcal{S}$. Thus (\mathcal{S}, T) is a subshift.

Proof. Let $\mathbf{s} = s_0 s_1 \cdots \in \mathcal{S}$; then $\pi(\mathbf{s}) \in \mathcal{X}$. Equation (2) says that $\pi(\mathbf{s}) = c(s_0) f(\pi(T\mathbf{s}))$, and so $f(\pi(T\mathbf{s})) \in \mathcal{X}$. Suppose that $\pi(T\mathbf{s}) \notin \mathcal{X}$.

Since $f \in \mathcal{N}$, it is circular. Let $\ell \geq 1$ be a synchronization delay for f. Note that $f^{n-1}(s_n)$ occurs both in $\pi(T\mathbf{s})$ and in $f^{\omega}(\alpha)$ for every $n \geq 1$. Since also $|f^{n-1}(s_n)| \to \infty$ as $n \to \infty$ because $f \in \mathcal{N}$, it follows that there are arbitrarily long words in $\mathcal{L}(\mathcal{X})$ that occur in infinitely many positions in $\pi(T\mathbf{s})$. Therefore there exists a word $zy \subset \pi(T\mathbf{s})$ such that z is not in $\mathcal{L}(\mathcal{X})$, $y \in \mathcal{L}(\mathcal{X})$, and $|y| \geq \ell$.

Next, consider the word $f(zy) \subset f(\pi(T\mathbf{s}))$. Since $f(y) \in \mathcal{L}(\mathcal{X})$ and $|f(y)| \geq \ell$, the word f(y) has a synchronizing point (w_1, w_2) . In particular, since $y \in \mathcal{L}(\mathcal{X})$, there exists y_1, y_2 for which $y = y_1y_2$, $f(y_1) = w_1$, and $f(y_2) = w_2$. On the other hand, $f(zy) \in \mathcal{L}(\mathcal{X})$ implies that we can write $f^{\omega}(\alpha) = put\mathbf{x}$ such that $f(zy) \subset f(ut)$ and $f(y) \subset f(t)$. Thus there exists t_1, t_2 such that $t = t_1t_2$, the word w_1 is a suffix of $f(t_1)$, and w_2 is a prefix of $f(t_2)$. Thus $f(y_1)$ is a suffix of $f(t_1)$. Since f(A) is a suffix code and $f(t_1)$ is a suffix of $f(t_2)$. But then $f(t_1)$ is a suffix of $f(t_2)$. Therefore $f(t_2)$ and so $f(t_2)$. $f(t_2)$ contradicting the choice of $f(t_2)$. Therefore $f(t_2)$ and so $f(t_2)$.

Lemma 6. If $f \in \mathcal{N}$, then the mapping $\pi : \mathcal{S} \to \mathcal{X}$ is injective.

Proof. For every $u, v \in A^*$ and $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we have that $uf(\mathbf{x}) = vf(\mathbf{y})$ implies u = v and $\mathbf{x} = \mathbf{y}$. This follows from the circularity and suffix code property of f. (See also the proof of Lemma 5.) Therefore if $\mathbf{s}, \mathbf{s}' \in \mathcal{S}$ and $\pi(\mathbf{s}) = \pi(\mathbf{s}')$, then Lemma 3 gives

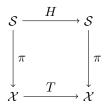
$$c(s_0)f(\pi(T\mathbf{s})) = c(s_0')f(\pi(T\mathbf{s}')),$$

so that $c(s_0) = c(s'_0)$ and $\pi(T\mathbf{s}) = \pi(T\mathbf{s}')$. Thus $s_0 = s'_0$, and since $T\mathbf{s}, T\mathbf{s}' \in \mathcal{S}$ by Lemma 5, we can repeat the argument obtaining $s_1 = s'_1, s_2 = s'_2, \ldots$ Therefore $\mathbf{s} = \mathbf{s}'$.

Remark 1. In Lemma 6 above, the assumption that f is circular is crucial: If $f: a^* \to a^*$ is defined by f(a) = aa, then $S = S^{\mathbb{N}}$ while $\mathcal{X} = \{a^{\omega}\}$, so π is anything but injective! Nevertheless, f satisfies all conditions of \mathcal{N} , except circularity.

Now we are ready to define the desired dynamics on S.

Theorem 1. Suppose that $f \in \mathcal{N}$. Let $H: \mathcal{S} \to \mathcal{S}$ be the mapping given by $H = T \circ G$. Then $\pi \circ H = T \circ \pi$ and so $\pi: (\mathcal{S}, H) \to (\mathcal{X}, T)$ is a conjugacy.



Proof. Observe first that $H(S) \subset S$ by Lemma 5, so the definition of H is sound. The mapping π is surjective by Lemma 1 and injective by Lemma 6, so it is a bijection. Furthermore π is continuous by Lemma 2. Finally, let us verify $\pi \circ H = T \circ \pi$. Let $\mathbf{s} = s_0 s_1 \cdots \in S$ with $s_i \in S$. If $|c(s_0)| \geq 2$, then we leave it to the reader to check that, by denoting $c(s_0) = au$ with $a \in A$, we have

$$\pi \circ H(\mathbf{s}) = \pi \circ T \circ G(\mathbf{s}) = uf(c(s_1))f^2(c(s_2)) \cdots = T \circ \pi(\mathbf{s}).$$

If $|c(s_0)| = 1$, then it is readily seen that $T \circ G(\mathbf{s}) = G \circ T(\mathbf{s})$. Using this, Equation (4), and Lemma 3 in this order gives

$$\pi \circ H(\mathbf{s}) = \pi \circ T \circ G(\mathbf{s}) = \pi \circ G \circ T(\mathbf{s}) = f \circ \pi \circ T(\mathbf{s}) = T^{|c_0|} \circ \pi(\mathbf{s}) = T \circ \pi(\mathbf{s}),$$

and the proof is complete.

The rest of this section is devoted to developing a few results for understanding the language of \mathcal{S} . They will be needed in the next sections that deal with the suffix conjugates of the Fibonacci and the Thue-Morse subshifts.

If u is a finite nonempty word, we denote by u^{\flat} and ${}^{\flat}u$ the words obtained from u by deleting its last and first letter, respectively.

If a finite word u is not in $\mathcal{L}(\mathcal{X})$, then u is called a forbidden word of \mathcal{X} . If both ^{d}u and u^{\flat} are in $\mathcal{L}(\mathcal{X})$, then u is a minimal forbidden word of \mathcal{X} . There is a connection between the minimal forbidden words and the so-called bispecial factors of an infinite word. See a precise formulation of this in [13] and examples in Sections 3 and 4.

We say that a word $u \in S^*$ is a *cover* of a word $v \in A^*$ if $v \subset \pi(u)$. Furthermore, we say that the cover u is *minimal* if $v \not\subset \pi(u^{\flat})$ and $v \not\subset f(\pi({}^{\flat}u))$. The latter expression comes from the identity $\pi(u) = c(u_0)f(\pi({}^{\flat}u))$, where u_0 is the first letter of u, given by Lemma 3.

Let \mathcal{C} be the set of minimal covers of the minimal forbidden factors of \mathcal{X} .

Lemma 7. Suppose $f \in \mathcal{N}$. Let $\mathbf{s} \in S^{\mathbb{N}}$. Then $\mathbf{s} \notin \mathcal{S}$ if and only if \mathbf{s} has a factor in \mathcal{C} .

Proof. Suppose that **s** has a factor in \mathcal{C} , so that $\mathbf{s} = pt\mathbf{s}'$ with $t \in \mathcal{C}$. If $\mathbf{s} \in \mathcal{S}$, then $T^{|p|}\mathbf{s} = t\mathbf{s}' \in \mathcal{S}$ by Lemma 5. But $\pi(t\mathbf{s}')$ has prefix $\pi(t)$, in which a forbidden word occurs by the definition of \mathcal{C} , a contradiction.

Conversely, suppose that $\mathbf{s} \notin \mathcal{S}$. Then $\pi(\mathbf{s}) \notin \mathcal{X}$, so there exists a minimal forbidden word v_0 of \mathcal{X} occurring in $\pi(\mathbf{s})$. Let u_0 be the shortest prefix of \mathbf{s} such that $v_0 \subset \pi(u_0)$. Then either u_0 is a minimal cover of v_0 or $v_0 \subset \mathbf{s}$

 $f(\pi({}^{\triangleleft}u_0))$. In the former case we are done, so suppose the latter case holds. Then $v_0 \subset f(\pi(T\mathbf{s}))$ and so $\pi(T\mathbf{s})$ has a factor v_1 such that $v_0 \subset f(v_1)$ and $|v_1| \leq |v_0|$. Since $f(\mathcal{L}) \subset \mathcal{L}$, it follows that v_1 is a forbidden word of \mathcal{X} ; by taking a factor of v_1 if necessary, we may assume v_1 is also minimal. Let u_1 be the shortest prefix of $T\mathbf{s}$ such that $v_1 \subset \pi(u_1)$. Then either u_1 is a minimal cover of v_1 or $v_1 \subset f(\pi({}^{\triangleleft}u_1))$. In the former case $u_1 \in \mathcal{C}$ and so \mathbf{s} has a factor u_1 in \mathcal{C} . In the latter case $v_1 \subset f(\pi(T^2\mathbf{s}))$, and we continue the process. This generates a sequence v_0, v_1, \ldots of minimal forbidden words of \mathcal{X} such that $v_n \subset f(\pi(T^{n+1}\mathbf{s})), v_{n+1} \subset f(v_n),$ and $|v_{n+1}| \leq |v_n|$. Each letter $a \in A$ is growing because $f \in \mathcal{N}$, and therefore the words v_n are pairwise distinct. Thus the length restriction on the v_n 's implies that the sequence v_0, v_1, \ldots is finite with a last element, say, v_k . The fact that there is no element v_{k+1} means that $T^{k+1}\mathbf{s}$ has a prefix u_k that is a minimal cover of v_k . Since $u_k \in \mathcal{C}$ then occurs also in \mathbf{s} , we are done.

Theorem 2. Suppose that $f \in \mathcal{N}$ and that the set \mathcal{C} of minimal covers of minimal forbidden words is a regular language. Then the language of \mathcal{S} is regular. In particular, (\mathcal{S}, T) is a sofic subshift.

Proof. Since \mathcal{C} is regular, so is the complement $S^* \setminus S^*\mathcal{C}S^*$, which we denote by L_0 . Let M_0 be the minimal DFA accepting L_0 . Modify M_0 by removing the states from which there are no arbitrarily long directed walks to accepting states. Remove also the corresponding edges and denote the obtained NFA by M. We claim that the language $\mathcal{L}(\mathcal{S})$ of \mathcal{S} is the language L(M) recognized by M.

If $w \in \mathcal{L}(\mathcal{S})$, then w is in $S^* \setminus S^*\mathcal{C}S^*$ by Lemma 7, so that it is accepted by M_0 . Furthermore, since w has arbitrarily long extensions to the right that are also in $\mathcal{L}(\mathcal{S})$, each accepted by M_0 of course, it follows that w is accepted by M. Conversely, by the construction of M, if $w \in L(M)$, then there exists an infinite walk on the graph of M whose label contains w. The label of this infinite path is in \mathcal{S} .

3 The suffix conjugate of the Fibonacci subshift

Recall the Fibonacci morphism φ for which $0 \mapsto 01$ and $1 \mapsto 0$, the Fibonacci word $\mathbf{f} = \varphi^{\omega}(0)$, and the Fibonacci subshift $(\mathcal{X}_{\varphi}, T)$. The suffix conjugate $(\mathcal{S}_{\varphi}, H_{\varphi})$ of the Fibonacci subshift is guaranteed to exist by Theorem 1. The goal of this section is to give a characterization for \mathcal{S}_{φ} and H_{φ} , and it will be achieved in Theorem 3.

The set of suffixes of φ is $S' = \{0, 1, 01\}$, and we define a bijection c between $S = \{0, 1, 2\}$ and S' by c(0) = 0, c(1) = 1, and c(2) = 01. In this case we have $S_{\varphi} \subset \{0, 1, 2\}^{\mathbb{N}}$.

We will now continue by finding a characterization for the set \mathcal{C}_{φ} of minimal covers of minimal forbidden words of the Fibonacci subshift.

Denote $f_n = \varphi^{n-1}(0)$ for all $n \ge 1$, so that in particular $f_1 = 0$ and $f_2 = 01$. For $n \ge 2$, we let p_n be the word defined by the relation $f_n = p_n ab$, where $ab \in \{01, 10\}$. Then $p_2 = \varepsilon$ and $p_3 = 0$. The words p_n are known as the *bispecial factors* of the Fibonacci word, and they possess the following well-known and easily established properties:

• For all $n \geq 2$, we have

$$f_n f_{n-1} = p_{n+1} ab$$
 and $f_{n-1} f_n = p_{n+1} ba$, (7)

where ab = 10 for even n and ab = 01 for odd n.

• For all $n \geq 2$, we have $\varphi(p_n) 0 = p_{n+1}$.

The minimal forbidden words of the Fibonacci word \mathbf{f} can be expressed in terms of the bispecial factors p_n as follows [13]. For every $n \geq 2$, write

$$d_n = \begin{cases} 1p_n 1 & \text{for } n \text{ even,} \\ 0p_n 0 & \text{for } n \text{ odd.} \end{cases}$$

Then a word is a minimal forbidden word of \mathbf{f} if and only if it equals d_n for some $n \geq 2$. The first few d_n 's are 11, 000, and 10101.

If x is a finite word and y a finite or infinite word, we write $(x <_p y)$ $x \le_p y$ to indicate that x is a (proper) prefix of y. We say two finite words x, y are prefix compatible if one of $x \le_p y$ or $y \le_p x$ holds.

Lemma 8. Let $x, y \in \{0, 1\}^+$ and $k \ge 1$. Then $\varphi^k(x) <_p \varphi^k(y)$ implies $x^{\flat} <_p y^{\flat}$.

Proof. Suppose x^b is not a prefix of y^b . Then x = uat and y = ubs with distinct letters a, b and nonempty words t, s. Then one of $\varphi(at)$ and $\varphi(bs)$ starts with 01 and the other one with 00. Thus $\varphi(x)$ is not a prefix of $\varphi(y)$. The rest follows by induction.

Lemma 9. Let $x, y \in \{0, 1, 2\}^*$ and suppose that $\pi(x) <_p \pi(y)$. Then either $x^{\flat} <_p y^{\flat}$ or x = u01 and y = u2s for some $u \in \{0, 1, 2\}^*$ and nonempty $s \in \{0, 1, 2\}^+$.

Proof. Suppose x^b is not a prefix of y^b . Then x = uat and y = ubs with distinct letters $a, b \in \{0, 1, 2\}$ and nonempty words $t, s \in \{0, 1, 2\}^+$. Lemma 3 applied to finite words gives

$$\pi(x) = \pi(u)\varphi^{|u|} (\pi(at))$$
 and $e(y) = \pi(u)\varphi^{|u|} (\pi(bs))$

Thus $\pi(x) <_p \pi(y)$ implies $\varphi^{|u|}(\pi(at)) <_p \varphi^{|u|}(\pi(bs))$, so that by Lemma 8, we have $\pi(at)^{\flat} <_p \pi(bs)^{\flat}$, or

$$c(a)\varphi(\pi(t))^{\flat} <_p c(b)\varphi(\pi(s))^{\flat}.$$

Since $a \neq b$, it follows that a = 0, b = 2, and $\varphi(\pi(t))^{\flat} = \varepsilon$. The last identity implies t = 1; therefore x = u01 and y = u2s.

Lemma 10. We have $d_3 = \pi(01)0$ and $d_4 = \pi(10)01$. For all $n \ge 0$, we have

$$\pi(021^{2n}2) = d_{2n+5}1$$
 and $\pi(121^{2n+1}2) = d_{2n+6}0$

Proof. Recalling that $f_k f_{k+1} = p_{k+2} ab$ with $ab \in \{01, 10\}$ for all $k \ge 1$, we get

$$\pi(021^{2n}2) = 0\varphi(01)\varphi^{2}(1)\varphi^{3}(1)\cdots\varphi^{2n}(1)\varphi^{2n+1}(1)\varphi^{2n+2}(01)$$

$$= 0\varphi^{2}(0)\varphi^{1}(0)\varphi^{2}(0)\cdots\varphi^{2n-1}(0)\varphi^{2n}(0)\varphi^{2n+3}(0)$$

$$= 0f_{3}f_{2}f_{3}\cdots f_{2n}f_{2n+1}f_{2n+4}$$

$$= 0f_{2n+3}f_{2n+4} = 0p_{2n+5}01 = d_{2n+5}1.$$

Similarly,

$$\pi(121^{2n+1}2) = 1\varphi(01)\varphi^{2}(1)\varphi^{3}(1)\cdots\varphi^{2n}(1)\varphi^{2n+1}(1)\varphi^{2n+2}(1)\varphi^{2n+3}(01)$$

$$= 1\varphi^{2}(0)\varphi^{1}(0)\varphi^{2}(0)\cdots\varphi^{2n-1}(0)\varphi^{2n}(0)\varphi^{2n+1}(0)\varphi^{2n+4}(0)$$

$$= 1f_{3}f_{2}f_{3}\cdots f_{2n+1}f_{2n+2}f_{2n+5}$$

$$= 1f_{2n+4}f_{2n+5} = 1p_{2n+6}10 = d_{2n+6}0.$$

Lemma 11. The forbidden word $d_2 = 11$ does not have covers. The minimal covers of d_3 are the words in O1(0+1+2). The minimal covers of d_4 are the words in (1+2)O(0+1+2). For other forbidden words, we have the following. Let n > 0.

(i) The minimal covers of d_{2n+5} are

$$021^{2n}(2+00+01+02). (8)$$

(ii) The minimal covers of d_{2n+6} are

$$(1+2)21^{2n+1}(2+00+01+02).$$

Proof. We leave verifying the claims on d_2 , d_3 , and d_4 to the reader. The displayed words are minimal covers because they are obtained from the clearly minimal words in Lemma 10 by modifying the first and the last two letters in obvious ways.

To prove that this collection is exhaustive, suppose that u is a minimal cover of d_{2n+5} . Then $d_{2n+5} \subset \pi(u)$, and since d_{2n+5} is not a factor of $\varphi(\pi({}^{\triangleleft}u))$, it follows that u can be written as u=ax with $a\in\{0,1,2\}$ such that $d_{2n+5}\leq_p\pi(bx)$ for some $b\in\{0,1,2\}$. Noticing that d_{2n+5} starts with with 00, we actually must have a=b=0, and so $d_{2n+5}\leq_p\pi(u)$. Lemma 10 says that then $\pi(021^{2n}2)<_p\pi(u)$, so that $021^{2n}<_pu^b$ by Lemma 9. It is readily verified using (7) that $\pi(021^{2n}1)$ is not prefix compatible with d_{2n+5} , and thus either $021^{2n}0\leq_pu$ or $021^{2n}2\leq_pu$. This observation and the minimality of u show that the words in (8) are exactly all the minimal covers of d_{2n+5} .

The case for d_{2n+6} can be handled in the same way, the only difference being that since d_{2n+6} starts with 10, the letters a and b may differ, but then $\{a,b\} = \{1,2\}$.

Theorem 3. The language $\mathcal{L}(S_{\varphi})$ of the suffix conjugate $(S_{\varphi}, H_{\varphi})$ of the Fibonacci subshift $(\mathcal{X}_{\varphi}, T)$ is regular. An infinite word $\mathbf{s} \in S^{\mathbb{N}}$ is in S_{φ} if and only if it is the label of an infinite walk on the graph depicted in Fig. 1b. The mapping $H_{\varphi} \colon S_{\varphi} \to S_{\varphi}$ is given by

$$H_{\varphi}(\mathbf{s}) = \begin{cases} 1\mathbf{z} & \text{if } \mathbf{s} = 2\mathbf{z}; \\ \lambda(x2)\mathbf{z} & \text{if } \mathbf{s} = ax2\mathbf{z} \text{ with } a \in \{0,1\}, \ x \in \{0,1\}^*; \\ \lambda(\mathbf{z}) & \text{if } \mathbf{s} = a\mathbf{z} \text{ with } a \in \{0,1\} \text{ and } \mathbf{z} \in \{0,1\}^{\mathbb{N}}, \end{cases}$$

where λ is the morphism given by $\lambda(1) = 0$, $\lambda(0) = 2$, and $\lambda(2) = 21$.

Proof. Lemma 11 says that the set C_{φ} of all minimal covers of minimal forbidden words of \mathbf{f} is regular. Thus Theorem 2 tells us that $\mathcal{L}(S_{\varphi})$ is regular. Following the proof of that theorem, we first construct the minimal deterministic automaton¹ accepting the language $S^* \setminus S^* C_{\varphi} S^*$ and then remove the states and edges that cannot be on the path of an infinite walk through accepting states. The result is given in Fig. 1a. Notice that the label of each walk starting from state q_0 can be obtained from a walk starting from states q_1 , q_3 , or q_4 . The state q_2 is superfluous for the same reason. The removal of states q_0 and q_2 and the corresponding edges yields in the graph in Fig. 1b.

Using Eq. (6) for constructing the mapping G and then recalling the definition $H = T \circ G$, the given formula for H_{φ} is readily verified. This completes the proof.

Our last goal for this section is to prove Theorem 4. To that end, let us first state a well-known property of the Fibonacci subshift.

Lemma 12. If \mathbf{z} has two T-preimages in \mathcal{X}_{φ} , then $\mathbf{z} = \mathbf{f}$.

Proof. If \mathbf{z} has two T-preimages, then $0\mathbf{z}$, $1\mathbf{z} \in \mathcal{X}_{\varphi}$. This means that all prefixes of \mathbf{z} are so-called left special factors of the Fibonacci word \mathbf{f} . The unique word in \mathcal{X}_{φ} with this property is \mathbf{f} ; see, e.g., [11, Ch. 2].

We say that an infinite word \mathbf{x} is in the *strictly positive orbit* of \mathbf{z} if $T^k\mathbf{z} = \mathbf{x}$ for some k > 0 and that \mathbf{x} is the *strictly negative orbit* of \mathbf{z} if $T^k\mathbf{x} = \mathbf{z}$ for some k > 0. Also, an infinite word \mathbf{x} is said to have a *tail* \mathbf{z} if $\mathbf{x} = u\mathbf{z}$ for some finite word u.

Since $\mathbf{f} = 01\varphi(1)\varphi^2(1)\cdots$, we have $\mathbf{f} = \pi(21^{\omega})$. Thus Theorem 3 gives

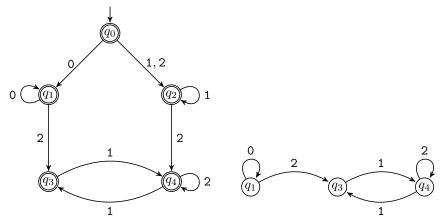
$$T\mathbf{f} = T \circ \pi(2\mathbf{1}^{\omega}) = \pi \circ H_{\omega}(2\mathbf{1}^{\omega}) = \pi(\mathbf{1}^{\omega}).$$

Similarly,

$$T^2 \mathbf{f} = \pi(0^{\omega})$$
 and $T^3 \mathbf{f} = \pi(2^{\omega})$

The word $T^2\mathbf{f}$ divides the shift orbit of the Fibonacci word in the following sense.

¹This automaton as well as the one in the proof Theorem 5 was computed using Petri Salmela's FAFLA Python package for finite automata and formal languages. http://coyote.dy.fi/~pesasa/fafla/



- (a) An NFA accepting the language of $\mathcal{L}(\mathcal{S}_{\varphi})$.
- (b) A graph for the sequences in S_{φ} .

Figure 1: The suffix conjugate of the Fibonacci subshift.

Theorem 4. Let $\mathbf{s} \in \mathcal{S}_{\varphi}$. Then

- (i) $\pi(\mathbf{s})$ is in the strictly positive orbit of $T^2\mathbf{f}$ if and only if \mathbf{s} has a tail 2^ω .
- (ii) $\pi(\mathbf{s})$ is in the strictly negative orbit of $T^2\mathbf{f}$ if and only if \mathbf{s} has a tail $\mathbf{1}^{\omega}$.

Proof. We begin by proving (i). If $\pi(s)$ is in the strictly positive orbit of $T^2\mathbf{f}$, then there exists k > 0 such that

$$\pi(\mathbf{s}) = T^{k+2}\mathbf{f} = T^k \circ \pi(\mathbf{0}^\omega) = \pi \circ H_\varphi^k(\mathbf{0}^\omega) = \pi \circ H_\varphi^{k-1}(\mathbf{2}^\omega).$$

Since π is injective, we have $\mathbf{s}=H_{\varphi}^{k-1}(2^{\omega})$. Thus we see from the characterization of H_{φ} given in Theorem 3 that \mathbf{s} has a tail 2^{ω} .

Conversely, suppose that s has a tail 2^{ω} . Then there exists an integer $m \geq 0$ such that both $\pi(\mathbf{s})$ and $T^3\mathbf{f}$ have a tail $f^m(\pi(2^\omega))$. This implies that there exist finite words u, v and $\mathbf{z} \in \{0, 1\}^{\mathbb{N}}$ such that $\pi(\mathbf{s}) = u\mathbf{z}$ and $T^3\mathbf{f} = v\mathbf{z}$ and the only common suffix of u and v is the empty word ε .

Case 1: $u \neq \varepsilon$ and $v \neq \varepsilon$. Then $0\mathbf{z}, 1\mathbf{z} \in \mathcal{X}_{\varphi}$, and so by Lemma 12, we have $\mathbf{z} = \mathbf{f}$. But \mathbf{f} cannot be a tail of $T^3\mathbf{f}$ because \mathbf{f} is aperiodic, a contradiction.

Case 2: $u \neq \varepsilon$ and $v = \varepsilon$. Then both $T^{|u|-1} \circ \pi(\mathbf{s})$ and $T^2\mathbf{f}$ are T-preimages

of $T^3\mathbf{f}$, so that $T^{|u|-1} \circ \pi(\mathbf{s}) = T^2\mathbf{f}$ again by Lemma 12. Thus

$$\pi(\mathbf{0}^\omega) = T^2\mathbf{f} = T^{|u|-1} \circ \pi(\mathbf{s}) = \pi \circ H_\varphi^{|u|-1}(\mathbf{s}),$$

from which we get $H_{\varphi}^{|u|-1}(\mathbf{s}) = 0^{\omega}$ by the injectivity of π . But this is not possible because s has a tail 2^{ω} and Theorem 3 says that H_{φ} preserves such

Case 3: $u = \varepsilon$. Then $\pi(\mathbf{s})$ is in the strictly positive orbit of $T^2\mathbf{f}$, and this is what we wanted to prove.

Let us then prove (ii). If $\pi(\mathbf{s})$ is in the negative orbit of $T^2\mathbf{f}$, then $T^k\pi(\mathbf{s})=T^2\mathbf{f}$ for some k>0. Then

$$\pi(\mathbf{1}^{\omega}) = T\mathbf{f} = T^{k-1} \circ \pi(\mathbf{s}) = \pi \circ H_{\omega}^{k-1}(\mathbf{s}),$$

so that $H_{\varphi}^{k-1}(\mathbf{s}) = \mathbf{1}^{\omega}$ by the injectivity of π . The characterization of H_{φ} given in Theorem 3 shows, then, that \mathbf{s} must have a tail $\mathbf{1}^{\omega}$.

Conversely, suppose that **s** has a tail 1^{ω} . Since $T\mathbf{f} = \pi(1^{\omega})$, there exists an integer $m \geq 0$ such that both $\pi(\mathbf{s})$ and $T\mathbf{f}$ have a common tail $f^m(\pi(1^{\omega}))$. Thus $\pi(\mathbf{s}) = u\mathbf{z}$ and $T\mathbf{f} = v\mathbf{z}$ for some finite words u, v and $\mathbf{z} \in \{0, 1\}^{\mathbb{N}}$ such that the only common suffix of u and v is the empty word ε .

Case 1: $v \neq \varepsilon$ and $u \neq \varepsilon$. Then $0\mathbf{z}, 1\mathbf{z} \in \mathcal{X}_{\varphi}$, so that $\mathbf{z} = \mathbf{f}$, contradicting the fact that \mathbf{f} cannot be a tail of $T\mathbf{f}$.

Case 2: $v \neq \varepsilon$ and $u = \varepsilon$. Then

$$\pi(\mathbf{s}) = T^{|v|} \circ T\mathbf{f} = T^{|v|} \circ \pi(\mathbf{1}^{\omega}) = \pi \circ H_{\varphi}^{|v|}(\mathbf{1}^{\omega}) = \pi \circ H_{\varphi}^{|v|-1}(\mathbf{0}^{\omega}),$$

so that $\mathbf{s} = H_{\varphi}^{|v|-1}(0^{\omega})$. Once again, the characterization of H_{φ} shows that 1^{ω} cannot be a tail of \mathbf{s} , a contradiction.

Case 3: $v = \varepsilon$. Then $\pi(\mathbf{s})$ is in the strictly negative orbit of $T^2\mathbf{f}$, which is want we wanted to show.

4 The suffix conjugate of the Thue-Morse subshift

Let μ be the Thue-Morse morphism $0 \mapsto 01$, $1 \mapsto 10$ and $\mathbf{t} = \mu^{\omega}(0)$ the Thue-Morse word. Let \mathcal{X}_{μ} denote the shift orbit closure of \mathbf{t} , so that the Thue-Morse subshift is (\mathcal{X}_{μ}, T) . In this section we will characterize its suffix conjugate $(\mathcal{S}_{\mu}, H_{\mu})$ defined in Theorem 1.

Here the set of suffixes is $S' = \{0, 1, 01, 10\}$ and $S = \{0, 1, 2, 3\}$, and we let c be the bijection between S and S' given by

$$c(0) = 0,$$
 $c(1) = 1,$ $c(2) = 01,$ $c(3) = 10.$

The minimal forbidden words of the Thue-Morse word are 000, 111,

$$0\mu^{2n}(010)0$$
, $0\mu^{2n}(101)0$, $1\mu^{2n}(010)1$, $1\mu^{2n}(101)1$

and

$$1\mu^{2n+1}(010)0$$
, $1\mu^{2n+1}(101)0$, $0\mu^{2n+1}(010)1$, $0\mu^{2n+1}(101)1$

for all $n \ge 0$; see [13, 16].

Let us introduce a shorthand. For $x, y, z \in \{0, 1\}$ and $k \geq 0$, we write

$$\gamma(k, x, y, z) = x\mu^k(y\overline{y}y)z.$$

Here the overline notation $\bar{\cdot}$ swaps 0's and 1's. The minimal forbidden words of t can then be written as xxx and

$$\gamma(2n, x, x, x), \quad \gamma(2n, x, \overline{x}, x), \quad \gamma(2n+1, x, x, \overline{x}), \quad \gamma(2n+1, x, \overline{x}, \overline{x}), \quad (9)$$

for all $n \geq 0$ and $x \in \{0, 1\}$. Furthermore,

- $\mu(\gamma(k, x, y, z)) = x\gamma(k+1, \overline{x}, y, z)\overline{z}$, and
- $\gamma(k, x, y, z)$ is a forbidden word if and only if $\gamma(k-1, \overline{x}, y, z)$ is a forbidden word, where $k \ge 1$.

The mapping λ defined in (3) and (5) in the current case is

$$\lambda(0) = 2, \qquad \lambda(1) = 3, \qquad \lambda(2) = 21, \qquad \lambda(3) = 30.$$

The next lemma is analogous to Lemma 9 for the Fibonacci subshift.

Lemma 13. Let $x, y \in \{0, 1, 2, 3\}^*$ and suppose that $\pi(x) <_p \pi(y)$. Then one of the following holds:

- (i) $x^{\flat} <_{p} y^{\flat}$
- (ii) $x = uz\overline{z}$ and $y = u\lambda(z)s$, where $z \in \{0, 1\}$ and s has prefix z or $\lambda(z)$.
- (iii) $x = u\lambda(z)z$ and y = uzs, where $z \in \{0,1\}$ and s has prefix \overline{zz} or $\overline{z}\lambda(\overline{z})$.

Proof. Suppose that x^{\flat} is not a prefix of y^{\flat} . Then x = uat and y = ubs with distinct letters $a, b \in \{0, 1, 2, 3\}$ and nonempty words $t, s \in \{0, 1, 2, 3\}^+$. Lemma 3 gives

$$\pi(x) = \pi(u)\mu^{|u|}\big(\pi(at)\big) \qquad \text{and} \qquad \pi(y) = \pi(u)\mu^{|u|}\big(\pi(bs)\big),$$

so that $\pi(x) <_p \pi(y)$ implies $\mu^{|u|}(\pi(at)) <_p \mu^{|u|}(\pi(bs))$. Since μ is an injective and $|\mu(0)| = |\mu(1)|$, it follows that $\pi(at) <_p \pi(bs)$, or

$$c(a)\mu\big(\pi(t)\big) <_p c(b)\mu\big(\pi(s)\big).$$

Since $a \neq b$, this implies that $\{a,b\} = \{z,\lambda(z)\}$ for some $z \in \{0,1\}$. If a=z and $b=\lambda(z)$, a simple analysis shows that we have $t=\overline{z}$ and s starts with z or with $\lambda(z)$; this corresponds to option (ii). Similarly, if $a=\lambda(z)$ and b=z, then t=z and s starts with \overline{zz} or $\overline{z}\lambda(\overline{z})$; this corresponds to option (iii).

Our next goal is to characterize the minimal covers of the minimal forbidden words of \mathcal{X}_{μ} . We begin with the next lemma, whose easy verification is left to the reader.

Lemma 14. Let $x \in \{0,1\}$. The forbidden words xxx and $\gamma(0,x,x,x)$ do not have covers. For other forbidden words, we have the following.

(i) The minimal covers of $\gamma(0, x, \overline{x}, x)$ are in

$$(x + \lambda(\overline{x}))\overline{x}(\overline{x} + \lambda(\overline{x}))$$
 and $\lambda(x)x(x + \lambda(x))$.

(ii) The minimal covers of $\gamma(1, x, x, \overline{x})$ are in

$$(x + \lambda(\overline{x}))x\overline{x}(\overline{x} + \lambda(\overline{x}))$$
 and $(x + \lambda(\overline{x}))\lambda(x)(x + \lambda(x))$.

(iii) The minimal covers of $\gamma(1, x, \overline{x}, \overline{x})$ are in

$$(x + \lambda(\overline{x}))\overline{x}x(\overline{x} + \lambda(\overline{x}))$$
 and $(x + \lambda(\overline{x}))\overline{x}\lambda(x)$.

We will characterize the minimal covers of the remaining forbidden words in Lemma 16 with the help of the following result.

Lemma 15. For $x, y, z \in \{0, 1\}$ and $k \ge 2$, we have

$$\gamma(k, x, y, z) = \pi \left(x \lambda(y) \overline{y}^{k-2} \lambda(\overline{y}) \right) z. \tag{10}$$

Furthermore, $x\lambda(y)\overline{y}^{k-1}w$ is a minimal cover of $\gamma(k, x, y, \overline{y})$ with

$$\gamma(k, x, y, \overline{y}) <_{p} \pi(x\lambda(y)\overline{y}^{k-1}w) \tag{11}$$

if and only if $w \in \{y, \lambda(y)\}.$

Proof. Observe that

$$\mu^{k}(y) = y\overline{y}\mu(\overline{y})\mu^{2}(\overline{y})\cdots\mu^{k-1}(\overline{y})$$

$$= y\overline{y}y\mu^{2}(\overline{y})\cdots\mu^{k-1}(\overline{y})$$

$$= \mu(c(\lambda(y)))\mu^{2}(\overline{y})\cdots\mu^{k-1}(\overline{y})$$

so that

$$x\mu^{k}(y) = x\mu(c(\lambda(y)))\mu^{2}(\overline{y})\cdots\mu^{k-1}(\overline{y}) = \pi(x\lambda(y)\overline{y}^{k-2}).$$
 (12)

Consequently, we have

$$x\mu^k(y\overline{y}y)z=x\mu^k(y)\mu^k\big(c(\lambda(\overline{y}))\big)z=\pi\big(x\lambda(y)\overline{y}^{k-2}\lambda(\overline{y})\big)z,$$

verifying (10). We also have

$$x\mu^k(y\overline{y}y)=x\mu^k(y)\mu^k(\overline{y})\mu^k(y)=\pi\big(x\lambda(y)\overline{y}^{k-1}\big)\mu^k(y),$$

where the latter identity is due to (12) and Lemma 3. From this it follows that

$$\gamma(k, x, y, \overline{y}) = x\mu^k(y\overline{y}y)\overline{y} <_p \pi(x\lambda(y)\overline{y}^{k-1}w)$$

if and only if $w \in \{y, \lambda(y)\}$, in which case $x\lambda(y)\overline{y}^{k-1}w$ is a minimal cover of $\gamma(k, x, y, \overline{y})$.

Lemma 16. Let $x, y \in \{0, 1\}$ and $k \geq 2$. A word is a minimal cover of $\gamma(k, x, y, y)$ if and only if it is in

$$(x + \lambda(\overline{x}))\lambda(y)\overline{y}^{k-2}\lambda(\overline{y})(y + \lambda(y)). \tag{13}$$

A word is a minimal cover of $\gamma(k, x, y, \overline{y})$ if and only if it is in

$$(x + \lambda(\overline{x}))\lambda(y)\overline{y}^{k-2} \left[\lambda(\overline{y})(\overline{y} + \lambda(\overline{y})) + \overline{y}(y + \lambda(y))\right]. \tag{14}$$

Proof. We see from (10) that the words in (13) really are minimal covers of $\gamma(k, x, y, y)$. (Notice here that c(x) is a suffix of $c(\lambda(\overline{x}))$ and c(y) is a prefix of $c(\lambda(y))$.) Further, we see from (10) and (11) that the words in (14) really are minimal covers of $\gamma(k, x, y, \overline{y})$.

Let us show the converse. Let $u \in \{0,1,2,3\}^*$ be a minimal cover of $\gamma(k,x,y,z)$ with $z \in \{y,\overline{y}\}$. Then there exists $a,b \in \{x,\lambda(\overline{x})\}$ and $t \in \{0,1,2,3\}^*$ such that u=at and $\gamma(k,x,y,z) \leq_p \pi(bt)$. Since both possibilities for a are accounted for in (13) and (14), we may assume that a=b, and so u=bt. Then (10) gives

$$\pi(x\lambda(y)\overline{y}^{k-2}\lambda(\overline{y})) = \gamma(k, x, y, z)z^{-1} <_{p} \pi(u). \tag{15}$$

Now Lemma 13 applies, and since its options (ii) and (iii) are clearly not possible here, we get

$$x\lambda(y)\overline{y}^{k-2} <_p u^{\flat},$$

and so we have $u = x\lambda(y)\overline{y}^{k-2}w$ for some $w \in \{0,1,2,3\}^+$. The minimality of u implies |w| = 2. Equation (15) and Lemma 3 then imply

$$\mu^k (\pi(\lambda(\overline{y}))) z <_p \mu^k (\pi(w)),$$
 or equivalently, $\pi(\lambda(\overline{y})) z <_p \pi(w).$

If z = y, then $w \in \lambda(\overline{y})(y + \lambda(y))$. If $z = \overline{y}$, then either

$$w \in \overline{y}(y + \lambda(y))$$
 or $w \in \lambda(\overline{y})(\overline{y} + \lambda(\overline{y})),$

and this completes the proof.

Theorem 5. The language $\mathcal{L}(\mathcal{S}_{\mu})$ of the suffix conjugate $(\mathcal{S}_{\mu}, H_{\mu})$ of the Thue-Morse subshift (\mathcal{X}_{μ}, T) is regular. An infinite word $\mathbf{s} \in S^{\mathbb{N}}$ is in \mathcal{S}_{μ} if and only if it is the label of an infinite walk on the graph depicted in Fig. 2. The mapping $H_{\mu} \colon \mathcal{S}_{\mu} \to \mathcal{S}_{\mu}$ is given by

$$H_{\mu}(\mathbf{s}) = \begin{cases} 1\mathbf{z} & \text{if } \mathbf{s} = 2\mathbf{z}; \\ 0\mathbf{z} & \text{if } \mathbf{s} = 3\mathbf{z}; \\ \lambda(\mathbf{z}) & \text{if } \mathbf{s} = a\mathbf{z} \text{ with } a \in \{0, 1\} \text{ and } \mathbf{z} \in \{0, 1\}^{\mathbb{N}}; \\ \lambda(x2)\mathbf{z} & \text{if } \mathbf{s} = ax2\mathbf{z} \text{ with } a \in \{0, 1\}, \ x \in \{0, 1\}^{*}; \\ \lambda(x3)\mathbf{z} & \text{if } \mathbf{s} = ax3\mathbf{z} \text{ with } a \in \{0, 1\}, \ x \in \{0, 1\}^{*}, \end{cases}$$

where λ is the morphism given by $\lambda(0) = 2$, $\lambda(1) = 3$, $\lambda(2) = 21$, and $\lambda(3) = 30$.

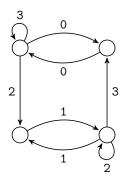


Figure 2: The suffix conjugate of the Thue-Morse subshift.

Proof. The set C_{μ} is obtained from (9) using Lemmas 14 and 16. From these it is clear that C_{μ} is a regular language. Thus $\mathcal{L}(S_{\mu})$ is regular by Theorem 2. Taking the steps outlined in the proof of that theorem and removing the superfluous states and edges, as in the proof of Theorem 3, we get the graph depicted in Fig. 2. Finally, the values of H_{μ} are obtained directly from the definition $H = T \circ G$ and (6).

Using the characterization of H_{μ} from Theorem 5, it is readily verified that

$$\mathbf{t} = \mu^{\omega}(0) = \pi(21^{\omega}), \qquad T\mathbf{t} = \pi(1^{\omega}), \qquad T^{2}\mathbf{t} = \pi(3^{\omega}).$$

Our last goal for this section is to establish Theorem 6, for which we need the lemma.

Lemma 17. If \mathbf{x} has two T-preimages in \mathcal{X}_{μ} , then either $\mathbf{x} = \mathbf{t}$ or $\mathbf{x} = \overline{\mathbf{t}}$.

Proof. Since \mathbf{x} is aperiodic, it must have infinitely many prefixes p such that both p0 and p1 occur in \mathbf{x} . Since 0p and 1p occur in \mathbf{x} as well, the words p are so-called bispecial factors of \mathbf{t} , whose form is known [2, Prop. 4.10.5]. They are

$$\varepsilon$$
, 0, 1, $\mu^m(01)$, $\mu^m(10)$, $\mu^m(010)$, $\mu^m(101)$

for
$$m \ge 0$$
. Therefore either $\mathbf{x} = \mu^{\omega}(0) = \mathbf{t}$ or $\mathbf{x} = \mu^{\omega}(1) = \overline{\mathbf{t}}$.

We say that an infinite word \mathbf{x} is in the *positive orbit* of \mathbf{z} if $T^k\mathbf{z} = \mathbf{x}$ for some $k \geq 0$ and that \mathbf{x} is the *negative orbit* of \mathbf{z} if $T^k\mathbf{x} = \mathbf{z}$ for some $k \leq 0$. Notice that in the previous section we used the notions strictly positive and strictly negative orbits. The next result is analogous to Theorem 4.

Theorem 6. Let $\mathbf{s} \in \mathcal{S}_{\mu}$. Then

- (i) $\pi(\mathbf{s})$ is in the positive orbit of $T^2\mathbf{t}$ if and only if 3^{ω} is a tail of \mathbf{s} .
- (ii) $\pi(s)$ is in the negative orbit of Tt if and only if 1^{ω} is a tail of s.

Proof. This proof is nearly identical to the proof of Theorem 4; therefore we will only prove (i) and leave the proof of (ii) to the reader. If $\pi(\mathbf{s})$ is in the positive orbit of $T^2\mathbf{t} = \pi(3^{\omega})$, then there exists $k \geq 0$ such that

$$\pi(\mathbf{s}) = T^{k+2}\mathbf{t} = T^k \circ \pi(3^\omega) = \pi \circ H_\mu^k(3^\omega),$$

so that $\mathbf{s} = H_{\mu}^{k}(3^{\omega})$ by the injectivity of π . Now the characterization of H_{μ} given in Theorem 5 shows that \mathbf{s} must have tail 3^{ω} .

Conversely, suppose that 3^{ω} is a tail of **s**. Since $T^2\mathbf{t} = \pi(3^{\omega})$, this implies that $\pi(\mathbf{s})$ and $T^2\mathbf{t}$ have a common tail of the form $\mu^m(\pi(3^{\omega}))$ for some $m \geq 0$. Therefore there exist finite words u, v and $\mathbf{z} \in \{0, 1\}^{\mathbb{N}}$ such that $\pi(\mathbf{s}) = u\mathbf{z}$ and $T^2\mathbf{t} = v\mathbf{z}$ and the longest common suffix of u and v is the empty word ε .

Case 1: $u \neq \varepsilon$ and $v \neq \varepsilon$. Then $0\mathbf{z}, 1\mathbf{z} \in \mathcal{X}_{\mu}$, so that $\mathbf{z} = \mathbf{t}$ or $\mathbf{z} = \overline{\mathbf{t}}$ by Lemma 17. But this is a contradiction because neither \mathbf{t} nor $\overline{\mathbf{t}}$ can be a tail of $T^2\mathbf{t}$.

Case 2: $u \neq \varepsilon$ and $v = \varepsilon$. Now both $T^{|u|-1} \circ \pi(\mathbf{s})$ and $T\mathbf{t}$ are T-preimages of $T^2\mathbf{t}$, so that $T^{|u|-1} \circ \pi(\mathbf{s}) = T\mathbf{t}$ by Lemma 17. Furthermore,

$$\pi(\mathbf{1}^{\omega}) = T\mathbf{t} = T^{|u|-1} \circ \pi(\mathbf{s}) = \pi \circ H_{\mu}^{|u|-1}(\mathbf{s}),$$

so the the injectivity of π implies $\mathbf{1}^{\omega} = H_{\mu}^{|u|-1}(\mathbf{s})$. But this is impossible because \mathbf{s} has a tail $\mathbf{3}^{\omega}$, which is preserved by H_{μ} according to Theorem 5.

Case 3: $u = \varepsilon$. Then $\pi(\mathbf{s})$ is a tail of $T^2\mathbf{t}$, and this is what we wanted to prove.

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