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Decidability and Complexity via Mosaics of the Temporal Logic of the Lexicographic Products of Unbounded Dense Linear Orders

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Abstract. This article considers the temporal logic of the lexicographic products of unbounded dense linear orders and provides via mosaics a complete decision procedure in nondeterministic polynomial time for the satisfiability problem it gives rise to.

Keywords: Linear temporal logic, lexicographic product, satisfiability problem, decidability, complexity, mosaic method, decision procedure.

1 Introduction

The mosaic method originates in algebraic logic, see [19], where the existence of a model is proved to be equivalent to the existence of a finite set of partial models verifying some conditions. It has also been applied for proving completeness and decidability of temporal logics over linear flows of time. See [5, section 6.4], or [7, 16, 17, 20]. For their use, specialized systems such as temporal logics must be combined with each other. This has led to the development of techniques for the combination of linear flows of time such as the classical operation of Cartesian product [10, 11, 14, 21]. Within the context of modal logic, the operation of lexicographic product of Kripke frames has been introduced as a variant of the operation of Cartesian product. It has also been used for defining the semantical basis of different languages designed for time representation and temporal reasoning from the perspective of non-standard analysis. See [1–3].

In [3], the temporal logic of the lexicographic products of unbounded dense linear orders has been considered and its complete axiomatization has been given. The purpose of this paper is to apply the mosaic method for providing a complete decision procedure in nondeterministic polynomial time for the satisfiability problem this temporal logic gives rise to. Its section-by-section breakdown is as follows. Section 2 formally introduces the lexicographic products of unbounded dense linear orders and presents the syntax and the semantics of the temporal logic we will be working with. Sections 3 defines mosaics and collections of mosaics satisfying saturation properties. In Section 4 and 5, we prove the completeness and soundness, respectively, of the mosaic method. Applying these result we prove in Section 6 that the satisfiability problem for our temporal logic is decidable in nondeterministic polynomial time.

2 Products of Unbounded Dense Linear Orders

Let $\mathcal{F}_1 = (T_1, <_1)$ and $\mathcal{F}_2 = (T_2, <_2)$ be linear orders. Their lexicographic product is the structure $\mathcal{F} = (T, \prec_1, \prec_2)$ where

- $T = T_1 \times T_2$,
- $\prec_1$ and $\prec_2$ are binary relations on T defined by $(s_1, s_2) \prec_1 (t_1, t_2)$ iff $s_1 <_1 t_1$ and $(s_1, s_2) \prec_2 (t_1, t_2)$ iff $s_1 = t_1$ and $s_2 <_2 t_2$.

We define the binary relation $\prec$ on T by $(s_1, s_2) \prec (t_1, t_2)$ iff $(s_1, s_2) \prec_1 (t_1, t_2)$ or $(s_1, s_2) \prec (t_1, t_2)$. The effect of the operation of lexicographic product may be described informally as follows: given two linear orders, their lexicographic product is the structure obtained by replacing each point of the first one by a copy of the second one. The global intuitions underlying such an operation is based upon the fact that, depending on the accuracy required or the available knowledge, one can describe a temporal situation at different levels of abstraction. See [4, section I.2.2], or [8] for details. In Fig. 1 below, we have $s_1 <_1 t_1$ and $s_2 <_2 t_2$. As a result, we have $(s_1, s_2) \prec_2 (s_1, t_2)$, $(s_1, s_2) \prec_1 (t_1, s_2)$, $(s_1, s_2) \prec_1 (t_1, t_2)$, $(s_1, t_2) \prec_1 (t_1, s_2)$, $(s_1, t_2) \prec_1 (t_1, t_2)$ and $(t_1, s_2) \prec_2 (t_1, t_2)$. It is now time to meet the temporal language we will be working with. Let At be a countable set of atomic formulas (with typical members denoted p, q, etc). We define the set $\mathcal{L}_t$ of formulas of our temporal language (with typical members denoted φ, ψ, etc.) as follows:

- $\varphi := p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid G_1\varphi \mid G_2\varphi \mid H_1\varphi \mid H_2\varphi$,

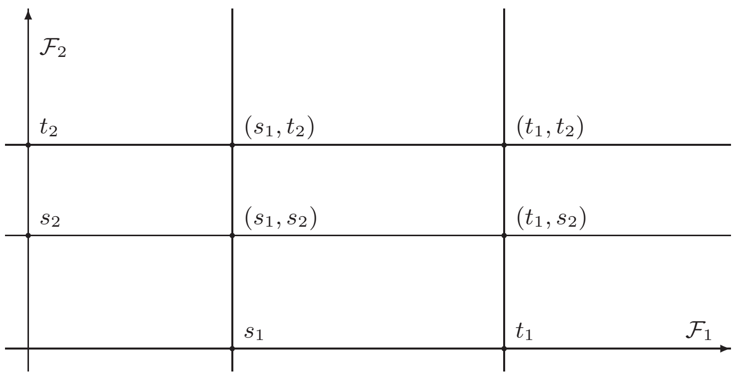

Fig. 1. Illustration of $\prec_1$ and $\prec_2$

the formulas $G_1\varphi$, $G_2\varphi$, $H_1\varphi$ and $H_2\varphi$ being read "φ will be true at each point within the future of but not infinitely close to the present point", "φ will be true at each instant within the future of and infinitely close to the present instant", "φ has been true at each point within the past of but not infinitely close to the present point" and "φ has been true at each point within the past of and infinitely close to the present point". We adopt the standard definitions for the remaining Boolean connectives. As usual, we define for all $i \in \{1, 2\}$,

- $F_i\varphi := \neg G_i \neg\varphi$,
- $P_i\varphi := \neg H_i \neg\varphi$,
- $\Diamond_i\varphi := \varphi \vee F_i\varphi \vee P_i\varphi$.

The notion of a subformula is standard. It is usual to omit parentheses if this does not lead to any ambiguity. The size of a formula φ, in symbols $|\varphi|$, is the number of symbols of φ. A model is a structure $\mathcal{M} = (\mathcal{F}_1, \mathcal{F}_2, V)$ where $\mathcal{F}_1 = (T_1, <_1)$ and $\mathcal{F}_2 = (T_2, <_2)$ are linear orders and $V\colon At \to \wp(T_1 \times T_2)$ is a valuation. Satisfaction is a ternary relation $\models$ between a model $\mathcal{M} = (\mathcal{F}_1, \mathcal{F}_2, V)$, a pair $(s_1, s_2) \in T_1 \times T_2$ and a formula φ. It is defined by induction on φ as usual. In particular, for all $i \in \{1, 2\}$,

- $\mathcal{M}, (s_1, s_2) \models G_i\varphi$ iff $\mathcal{M}, (t_1, t_2) \models \varphi$ for every $(t_1, t_2) \in T_1 \times T_2$ such that $(s_1, s_2) \prec_i (t_1, t_2)$,
- $\mathcal{M}, (s_1, s_2) \models H_i\varphi$ iff $\mathcal{M}, (t_1, t_2) \models \varphi$ for every $(t_1, t_2) \in T_1 \times T_2$ such that $(t_1, t_2) \prec_i (s_1, s_2)$.

As a result, for all $i \in \{1, 2\}$,

- $\mathcal{M}, (s_1, s_2) \models F_i\varphi$ iff $\mathcal{M}, (t_1, t_2) \models \varphi$ for some $(t_1, t_2) \in T_1 \times T_2$ such that $(s_1, s_2) \prec_i (t_1, t_2)$,
- $\mathcal{M}, (s_1, s_2) \models P_i\varphi$ iff $\mathcal{M}, (t_1, t_2) \models \varphi$ for some $(t_1, t_2) \in T_1 \times T_2$ such that $(t_1, t_2) \prec_i (s_1, s_2)$,
- $\mathcal{M}, (s_1, s_2) \models \Diamond_2\varphi$ iff $\mathcal{M}, (s_1, t) \models \varphi$ for some $t \in T_2$.

$\mathcal{M}$ is said to be a model for φ iff there exists $(s_1, s_2) \in T_1 \times T_2$ such that $\mathcal{M}, (s_1, s_2) \models \varphi$. In this case, we shall also say that φ is satisfied in $\mathcal{M}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of linear orders. We shall say that a formula φ is satisfiable with respect to $(\mathcal{C}_1, \mathcal{C}_2)$ iff there exists a linear order $\mathcal{F}_1 = (T_1, <_1)$ in $\mathcal{C}_1$, there exists a linear order $\mathcal{F}_2 = (T_2, <_2)$ in $\mathcal{C}_2$ and there exists a valuation $V\colon At \to \wp(T_1 \times T_2)$ such that $(\mathcal{F}_1, \mathcal{F}_2, V)$ is a model for φ. The temporal logic of $(\mathcal{C}_1, \mathcal{C}_2)$ is the set of all formulas φ such that $\neg\varphi$ is not satisfiable with respect to $(\mathcal{C}_1, \mathcal{C}_2)$. The class of all unbounded dense linear orders will be denoted $\mathcal{C}_{ud}$. [3] considers the temporal logic of $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$ and gives its complete axiomatization. The satisfiability problem of this temporal logic is to

- determine whether a given formula φ is satisfiable with respect to $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$.

In order to provide a complete decision procedure in nondeterministic polynomial time for it, we use mosaics.

3 Mosaics

Until the end of this paper, ξ will denote a fixed formula and Γ will denote the least set of formulas such that $\Diamond_2\xi \in \Gamma$ (recall that $\Diamond_2\varphi$ is defined as $\varphi \vee F_2\varphi \vee P_2\varphi$) and $\top \in \Gamma$, Γ is closed under subformulas and single negations (we identify $\neg\neg\gamma$ with γ) and

- if $G_1\varphi \in \Gamma$ or $H_1\varphi \in \Gamma$, then $G_2\varphi, H_2\varphi \in \Gamma$,
- if $F_1\varphi \in \Gamma$ or $P_1\varphi \in \Gamma$, then $\Diamond_2\varphi \in \Gamma$.

Recall that ξ has at most $|\xi|$ subformulas. Closing this set under single negations gives us at most $2 \times |\xi|$ formulas. Then $\Diamond_2\xi$ yields $2 \times |\xi| + 10$ formulas (subformulas and their negations). The first requirement above introduces at most 2 new formulas plus their negations, and the last requirement introduces at most 10 new formulas (with negations), for every φ. Thus we get that $|\Gamma| \leq 14 \times (2 \times |\xi| + 10) + 2$ (the 2 is for $\top$ and $\bot$).

Let λ be a function such that $\mathrm{dom}(\lambda) \subseteq \Gamma$ is closed under single negations and $\mathrm{ran}(\lambda) \subseteq \{0, 1\}$. We say that λ is *adequate* if

- $\top \in \mathrm{dom}(\lambda)$ and $\lambda(\top) = 1$,
- for every $\gamma \in \mathrm{dom}(\lambda)$, we have $\lambda(\neg\gamma) = 1 - \lambda(\gamma)$,
- for every $\gamma \vee \rho \in \mathrm{dom}(\lambda)$, we have $\lambda(\gamma \vee \rho) \geq \lambda(\gamma)$ provided that $\gamma \in \mathrm{dom}(\lambda)$, $\lambda(\gamma \vee \rho) \geq \lambda(\rho)$ provided that $\rho \in \mathrm{dom}(\lambda)$, and $\lambda(\gamma \vee \rho) = \max\{\lambda(\gamma), \lambda(\rho)\}$ if both $\gamma, \rho \in \mathrm{dom}(\lambda)$.

The reason for the complicated form of the last condition is that generally we do not require that $\mathrm{dom}(\lambda)$ is closed under subformulas. However, when we state a requirement that $\lambda(\gamma) \in \{0, 1\}$, then we implicitly require that $\gamma \in \mathrm{dom}(\lambda)$.

Definition 1. *Let $i \in \{1, 2\}$ and (σ, τ) be a pair of adequate functions. We define the following* coherence *properties.*

G_i-coherence $\sigma(G_i\varphi) = 1$ *implies* $\tau(G_i\varphi) = \tau(\varphi) = 1$.
H_i-coherence $\tau(H_i\varphi) = 1$ *implies* $\sigma(H_i\varphi) = \sigma(\varphi) = 1$.

1. *A* 1-mosaic *is a pair (σ, τ) such that σ and τ are adequate functions, (σ, τ) satisfies G_1- and H_1-coherence and the following* transfer *conditions.*
   G-transfer $\sigma(G_1\varphi) = 1$ *implies* $\tau(G_2\varphi) = \tau(H_2\varphi) = 1$.
   H-transfer $\tau(H_1\varphi) = 1$ *implies* $\sigma(G_2\varphi) = \sigma(H_2\varphi) = 1$.
2. *A* 2-mosaic *is a pair (σ, τ) such that σ and τ are adequate functions with full domain* $\mathrm{dom}(\sigma) = \mathrm{dom}(\tau) = \Gamma$*, (σ, τ) satisfies G_2- and H_2-coherence and the following* uniformity *conditions.*
   G_1-uniformity $\sigma(G_1\varphi) = \tau(G_1\varphi)$.
   H_1-uniformity $\sigma(H_1\varphi) = \tau(H_1\varphi)$.

As an example of mosaics take a model $\mathcal{M} = (\mathcal{F}_1, \mathcal{F}_2, V)$ with unbounded, dense linear orders $\mathcal{F}_1 = (T_1, <_1)$ and $\mathcal{F}_2 = (T_2, <_2)$ and a valuation $V\colon At \to$

$\wp(T_1 \times T_2)$. Denote $T := T_1 \times T_2$. Define for every $(s,t) \in T$, $\lambda_{(s,t)} \colon \Gamma \to \{0,1\}$ by

$$\lambda_{(s,t)}(\gamma) := \begin{cases} 1 & \text{if } \mathcal{M},(s,t) \models \gamma \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for every $\gamma \in \Gamma$. It is straightforward to check that

- for every $(s,t),(u,v) \in T$ with $s <_1 u$, the pair $(\lambda_{(s,t)}, \lambda_{(u,v)})$ is a 1-mosaic (with full domain)
- for every $(s,t),(s,u) \in T$ with $t <_2 u$, the pair $(\lambda_{(s,t)}, \lambda_{(s,u)})$ is a 2-mosaic.

For every $s \in T_1$, define κ_s as follows:

$$\kappa_s(\gamma) := \begin{cases} 1 & \text{if } \lambda_{(s,u)}(\gamma) = 1 \text{ for every } u \in T_2 \\ 0 & \text{if } \lambda_{(s,u)}(\gamma) = 0 \text{ for every } u \in T_2 \\ \text{undefined} & \text{otherwise} \end{cases} \tag{2}$$

for every $\gamma \in \Gamma$. Then κ_s is an adequate function and (κ_s, κ_t) is a 1-mosaic whenever $s <_1 t$. An i-SSM will correspond to a flow of time in dimension i.

Definition 2. *Let $i \in \{1,2\}$. An* i-saturated set of mosaics*, an i-SSM, is a collection M of i-mosaics such that M satisfies the Density, No-endpoints and the corresponding* i-saturation *conditions below.*

Density *If $(\sigma,\tau) \in M$, then there is μ such that $(\sigma,\mu),(\mu,\tau) \in M$.*
No-endpoints *If $(\sigma,\tau) \in M$, then there are μ,ν such that $(\mu,\sigma) \in M$ and $(\tau,\nu) \in M$.*
F_1-saturation—insertion *If $(\sigma,\tau) \in M$, $\sigma(F_1\varphi) = 1$ and $\tau(F_1\varphi) = \tau(\Diamond_2\varphi) = 0$, then there is μ such that $\mu(\Diamond_2\varphi) = 1$ and $(\sigma,\mu),(\mu,\tau) \in M$.*
F_2-saturation—insertion *If $(\sigma,\tau) \in M$, $\sigma(F_2\varphi) = 1$ and $\tau(F_2\varphi) = \tau(\varphi) = 0$, then there is μ such that $\mu(\varphi) = 1$ and $(\sigma,\mu),(\mu,\tau) \in M$.*
P_1-saturation—insertion *If $(\sigma,\tau) \in M$, $\tau(P_1\varphi) = 1$ and $\sigma(P_1\varphi) = \sigma(\Diamond_2\varphi) = 0$, then there is μ such that $\mu(\Diamond_2\varphi) = 1$ and $(\sigma,\mu),(\mu,\tau) \in M$.*
P_2-saturation—insertion *If $(\sigma,\tau) \in M$, $\tau(P_2\varphi) = 1$ and $\sigma(P_2\varphi) = \sigma(\varphi) = 0$, then there is μ such that $\mu(\varphi) = 1$ and $(\sigma,\mu),(\mu,\tau) \in M$.*
F_1-saturation—expansion *If $(\sigma,\tau) \in M$ and $\tau(F_1\varphi) = 1$, then there is μ such that $\mu(\Diamond_2\varphi) = 1$ and $(\tau,\mu) \in M$.*
F_2-saturation—expansion *If $(\sigma,\tau) \in M$ and $\tau(F_2\varphi) = 1$, then there is μ such that $\mu(\varphi) = 1$ and $(\tau,\mu) \in M$.*
P_1-saturation—expansion *If $(\sigma,\tau) \in M$ and $\sigma(P_1\varphi) = 1$, then there is μ such that $\mu(\Diamond_2\varphi) = 1$ and $(\mu,\sigma) \in M$.*
P_2-saturation—expansion *If $(\sigma,\tau) \in M$ and $\sigma(P_2\varphi) = 1$, then there is μ such that $\mu(\varphi) = 1$ and $(\mu,\sigma) \in M$.*

That is, besides the Density and No-endpoint conditions, a 1-SSM should satisfy the F_1-saturation and P_1-saturation conditions and a 2-SSM should satisfy the F_2-saturation and P_2-saturation conditions. We say that M is an i-SSM for φ if there is $(\mu, \nu) \in M$ such that $\mu(\varphi) = 1$ or $\nu(\varphi) = 1$. Let us continue with our example. The collections

$$\{(\lambda_{(s,t)}, \lambda_{(u,v)}) : (s,t) \prec_1 (u,v)\} \text{ and } \{(\kappa_s, \kappa_u) : s <_1 u\}$$

of 1-mosaics are 1-SSMs. Fix $s \in T_1$ and consider the set M_s of 2-mosaics defined by

$$M_s := \{(\lambda_{(s,t)}, \lambda_{(s,u)}) : t, u \in T_2, t <_2 u\}.$$

It is easy to check that M_s is a 2-SSM for every $s \in T_1$. A 1-supermosaic will be a 1-mosaic of two 2-SSMs.

Definition 3. *Let M be a 2-SSM. We define λ_M by*

$$\lambda_M(\gamma) := \begin{cases} 1 & \text{if } \mu(\gamma) = \nu(\gamma) = 1 \text{ for every } (\mu, \nu) \in M \\ 0 & \text{if } \mu(\gamma) = \nu(\gamma) = 0 \text{ for every } (\mu, \nu) \in M \\ \text{undefined} & \text{otherwise} \end{cases} \tag{3}$$

for every $\gamma \in \Gamma$. Observe that λ_M is an adequate function.

A 1-supermosaic *is a pair (M, N) of 2-SSMs such that (λ_M, λ_N) is a 1-mosaic.*

In our example, for every $s, t \in T_1$ such that $s <_1 t$, the pair (M_s, M_t) is a 1-supermosaic. A saturated set of 1-supermosaics will correspond to a flow (in dimension 1) of flows (in dimension 2).

Definition 4. *A* saturated set of 1-supermosaics*, a 1-SSS, is a collection Σ of 1-supermosaics such that $\{(\lambda_M, \lambda_N) : (M, N) \in \Sigma\}$ is a 1-SSM.*

We say that Σ is a 1-SSS for φ if there is $(M, N) \in \Sigma$ such that $\lambda_M(\Diamond_2\varphi) = 1$ or $\lambda_N(\Diamond_2\varphi) = 1$. Observe that then there is a 2-mosaic (σ, τ) in one of the 2-SSMs in one of the 1-supermosaics of Σ such that $\sigma(\varphi) = 1$ or $\tau(\varphi) = 1$. In our example the set

$$\Sigma_{\mathcal{M}} := \{(M_s, M_t) : s, t \in T_1, s <_1 t\}$$

is a 1-SSS and in fact a 1-SSS for ξ if there is $(s,t) \in T_1 \times T_2$ such that $\mathcal{M}, (s,t) \models \xi$. Our running example should convince the reader that satisfiability of ξ implies the existence of a 1-SSS for ξ. But we want to be more economical in creating the 1-SSS; we will describe the procedure for creating a smaller 1-SSS in Section 5. First we show how to create a model from a 1-SSS, though.

4 Completeness

In this section we show the completeness of the mosaic approach. We will need the following.

Definition 5. *Let $i \in \{1, 2\}$ and $\mathcal{W} = (W, <, \lambda)$ be a structure such that $(W, <)$ is a linear order and λ_q is an adequate function for every $q \in W$. We say that $\mathcal{W}$ is i-*consistent *if it satisfies for every $q \in W$, the corresponding i-*completness *and i-*soundness *conditions below.*

G_i-soundness *If $\lambda_q(G_i\gamma) = 1$, then $\lambda_p(G_i\gamma) = \lambda_p(\gamma) = 1$ for every $p \in W$ such that $q < p$.*

H_i-soundness *If $\lambda_q(H_i\rho) = 1$, then $\lambda_p(H_i\rho) = \lambda_p(\rho) = 1$ for every $p \in W$ such that $p < q$.*

F_1-completeness *If $\lambda_q(F_1\gamma) = 1$, then there is $p \in W$ such that $q < p$ and $\lambda_p(\Diamond_2\gamma) = 1$.*

F_2-completeness *If $\lambda_q(F_2\gamma) = 1$, then there is $p \in W$ such that $q < p$ and $\lambda_p(\gamma) = 1$.*

P_1-completeness *If $\lambda_q(P_1\rho) = 1$, then there is $p \in W$ such that $p < q$ and $\lambda_p(\Diamond_2\gamma) = 1$.*

P_2-completeness *If $\lambda_q(P_2\rho) = 1$, then there is $p \in W$ such that $p < q$ and $\lambda_p(\gamma) = 1$.*

That is, a 1-consistent structure must satisfy the G_1-soundness, H_1-soundness, F_1-completeness and P_1-completeness conditions, and a 2-consistent structure satisfies the G_2-soundness, H_2-soundness, F_2-completeness and P_2-completeness conditions.

Let $\mathcal{W} = (W, <, \lambda)$ be a structure such that $(W, <)$ is a linear order and λ_q is an adequate function for every $q \in W$. A *future defect* of $\mathcal{W}$ is a pair (q, γ) such that $\lambda_q(F_i\gamma) = 1$ but there is no $p > q$ such that λ_p satisfies the requirements in the F_i-completeness condition above. Past defects are defined similarly. Below we will construct a i-complete (i.e., without defects) and i-sound structure from an i-SSM, see Case 1 and 2 in the proof of Lemma 1 below, where we construct the required future and past witnesses for the defects. In addition, we need that the constructed structure is dense and without endpoints. That is why we will need Case 3 and 4, where we construct new successors and predecessors for each point in the linear order, which in the limit of the construction yields a dense linear order without endpoints.

Lemma 1. *Let $i \in \{1, 2\}$. Assume that M is an i-SSM for φ. Then there is an i-consistent structure $\mathcal{Q}_M = (Q_M, <, \lambda)$ such that $(Q_M, <)$ is isomorphic to the rationals $\mathbb{Q}$ and $\lambda_q(\varphi) = 1$ for some $q \in Q_M$.*

Proof. We will define the order $(Q_M, <)$ and the adequate functions λ_q by induction. To this end let us have a countable enumeration D of potential defects $\{(q, \gamma, k) : q \in \mathbb{Q}, \gamma \in \Gamma, k \in \{1, 2, 3, 4\}\}$ such that every item appears infinitely often. The value of k will indicate the type of the potential defect: future, past, successor, predecessor.

By assumption there is an i-mosaic $(\mu, \nu) \in M$ such that $\mu(\varphi) = 1$ or $\nu(\varphi) = 1$. In the base step of the construction we define the finite order $Q_1 = \{0, 1\}$ with $0 < 1$ and functions $\lambda_0 = \mu$ and $\lambda_1 = \nu$. Obviously the soundness conditions restricted to Q_1 hold.

For the inductive step assume that we constructed a sound structure $\mathcal{Q}_n$ consisting of a finite order $(Q_n, <) = (q_0 < q_1 < \ldots < q_n)$ and adequate functions λ_q for $q \in Q_n$ such that $(\lambda_{q_j}, \lambda_{q_{j+1}}) \in M$ for every $j < n$. Let $D(n) = (q, \gamma, k)$. If $q \notin Q_n$, then we define $\mathcal{Q}_{n+1} := \mathcal{Q}_n$. Otherwise we consider the following four cases. If none of the four cases below holds, then we let $\mathcal{Q}_{n+1} := \mathcal{Q}_n$.

Case 1 $k = 1$, $\lambda_q(F_i\gamma) = 1$ and for every $r \in Q_n$ with $q < r$, we have $\lambda_r(\Diamond_2\gamma) = 0$ in case $i = 1$, or $\lambda_r(\gamma) = 0$ in case $i = 2$.

We will construct the required witness in the future of q. First assume that for every r with $q < r$, we have $\lambda_r(F_i\gamma) = 1$ (note that this includes the case $q = q_n$). The i-mosaic $(\lambda_{q_{n-1}}, \lambda_{q_n}) \in M$. By F_i-saturation—expansion there is an i-mosaic $(\lambda_{q_n}, \mu) \in M$ such that

- $\mu(\Diamond_2\gamma) = 1$ if $i = 1$
- $\mu(\gamma) = 1$ if $i = 2$.

In this case we define $(Q_{n+1}, <) := (q_0 < q_1 < \ldots < q_n < p)$ for some $p \in \mathbb{Q}$ such that $q_n < p$ and let $\lambda_p := \mu$. Next assume that there is r with $q < r$ such that $\lambda_r(F_i\gamma) = 0$. Let m be such that q_m is minimal in Q_n with respect to this property. Consider the i-mosaic $(\lambda_{q_{m-1}}, \lambda_{q_m}) \in M$. By F_i-saturation—insertion there is an adequate function μ such that $(\lambda_{q_{n-1}}, \mu), (\mu, \lambda_{q_n}) \in M$ and

- $\mu(\Diamond_2\gamma) = 1$ if $i = 1$
- $\mu(\gamma) = 1$ if $i = 2$.

Then we let $(Q_{n+1}, <) := (q_0 < \ldots < q_{m-1} < p < q_m < \ldots < q_n)$ for some $p \in \mathbb{Q}$ such that $q_{m-1} < p < q_m$ and define $\lambda_p := \mu$.

Case 2 $k = 2$, $\lambda_q(P_i\gamma) = 1$ and for every $r \in Q_n$ with $r < q$ we have $\lambda_r(\Diamond_2\gamma) = 0$ if $i = 1$, or $\lambda_r(\gamma) = 0$ if $i = 2$.

A completely analogous construction to Case 1 provides the required witness in the past of q.

Case 3 $k = 3$.

We will construct a new successor for q. In the case $q = q_n$, consider $(\lambda_{q_{n-1}}, \lambda_{q_n}) \in M$. By the No-endpoints condition, we have an i-mosaic $(\lambda_{q_n}, \mu) \in M$. We define $(Q_{n+1}, <) := (q_0 < q_1 < \ldots < q_n < p)$ for some $p \in \mathbb{Q}$ such that $q_n < p$ and let $\lambda_p := \mu$. Now assume that $q = q_m < q_n$. Consider the i-mosaic $(\lambda_{q_m}, \lambda_{q_{m+1}}) \in M$. Then there is an adequate function μ such that $(\lambda_{q_m}, \mu), (\mu, \lambda_{q_{m+1}}) \in M$, by the Density condition. Then we let $(Q_{n+1}, <) := (q_0 < \ldots < q_m < p < q_{m+1} < \ldots < q_n)$ for some $p \in \mathbb{Q}$ such that $q_{m-1} < p < q_m$ and define $\lambda_p := \mu$.

Case 4 $k = 4$.

A completely analogous construction to Case 3 provides a new predecessor of q.

It is easy to check that $\mathcal{Q}_{n+1}$ is sound in every case, since it consists of elements of M. Let $\mathcal{Q}_M = (Q_M, <) := \bigcup_{n\in\omega} \mathcal{Q}_n$ (recall that we defined λ_q in the step we created q for every $q \in Q_M$). It is easy to see that $(Q_M, <)$ is a countable linear order which is dense and does not have endpoints (by Case 3 and 4), hence isomorphic to $\mathbb{Q}$. Since we considered every potential defects infinitely often, it follows that $\mathcal{Q}_M$ does not contain any future or past defect in dimension i. That is, if $i = 1$ and $\lambda_q(F_1\gamma) = 1$, then there is $p \in Q_M$ such that $q < p$ and $\lambda_p(\Diamond_2\gamma) = 1$, and if $i = 2$ and $\lambda_q(F_2\gamma) = 1$, then there is $p \in Q_M$ such that $q < p$ and $\lambda_p(\gamma) = 1$ (and similarly for past formulas). Hence $\mathcal{Q}_M$ is i-consistent. $\square$

Next we apply Lemma 1 in both dimensions to construct a model from a 1-SSS.

Lemma 2. *If there is a 1-SSS for ξ, then there is a model $\mathcal{M}$ for ξ. Furthermore, $\mathcal{M}$ can be chosen to be the lexicographic product of the rationals with some valuation V: $\mathcal{M} = (\mathbb{Q}, \mathbb{Q}, V)$.*

Proof. Let Σ be a 1-SSS for ξ. Thus Σ is a collection of 1-supermosaics (M, N) such that $\{(\lambda_M, \lambda_N) : (M, N) \in \Sigma\}$ is a 1-SSM. Furthermore, there is $(M, N) \in \Sigma$ such that $\lambda_M(\Diamond_2\xi) = 1$ or $\lambda_N(\Diamond_2\xi) = 1$.

We apply Lemma 1 to get the 1-consistent structure $\mathcal{Q}_\Sigma = (Q_\Sigma, <_1, \lambda)$ such that $(Q_\Sigma, <_1)$ is isomorphic to $\mathbb{Q}$ and $\lambda_q(\Diamond_2\xi) = 1$ for some $q \in Q_\Sigma$. By the construction of $\mathcal{Q}_\Sigma$, for every $q \in Q_\Sigma$, there is a 2-SSM M such that $\lambda_q = \lambda_M$. Thus we can assume that there is a function $f\colon q \mapsto M$ with domain Q_Σ. For every $q \in Q_\Sigma$, we apply Lemma 1 to $f(q) = M$. Hence, we get a 2-consistent structure $\mathcal{Q}_{f(q)} = (Q_{f(q)}, <_2, \lambda)$ such that $(Q_{f(q)}, <_2)$ is isomorphic to $\mathbb{Q}$, and for every $r_q \in Q_{f(q)}$, the adequate function λ_{r_q} has full domain Γ (since $f(q) = M$ is a 2-SSM).

Let us replace every $q \in Q_\Sigma$ with the copy $(Q_{f(q)}, <_2)$ of $\mathbb{Q}$ (say, mapping q to 0). Thus we get a grid $\mathbb{Q} \times \mathbb{Q}$ such that the elements (q, r) have the property that $r = r_q \in Q_{f(q)}$. Hence to every $(q, r) \in \mathbb{Q} \times \mathbb{Q}$ we can associate a full adequate function $\lambda_{(q,r)} := \lambda_{r_q}$.

Let $\mathcal{M} = (\mathbb{Q}, \mathbb{Q}, V)$ be the model defined by the valuation V:

$$V(p) := \{(q, r) \in \mathbb{Q} \times \mathbb{Q} : \lambda_{(q,r)}(p) = 1\}$$

for every atomic proposition p. An easy formula-induction, using 1-consistency of $(Q_\Sigma, <_1, \lambda)$ and 2-consistency of $(Q_{f(q)}, <_2)$, establishes that

$$\mathcal{M}, (q, r) \models \varphi \text{ iff } \lambda_{(q,r)}(\varphi) = 1$$

for every $\varphi \in \Gamma$. Since we have $\lambda_q(\Diamond_2\xi) = 1$ for some $q \in Q_\Sigma$, we also get that $\lambda(q, r)(\xi) = 1$ for some $r \in Q_{f(q)}$ by F_2/P_2-completeness. Hence $\mathcal{M}, (q, r) \models \xi$, that is, $\mathcal{M}$ is a model satisfying ξ. $\square$

5 Soundness

For the reverse direction we also compute an upper bound on the size of the required 1-SSS.

Definition 6. *Let $\mathcal{W}_i = (W_i, <_i, \lambda)$ be an i-consistent structure. For $i = 1$ we define the following* transfer *conditions: for every $p, q \in W_1$ such that $p <_1 q$,*

G-transfer $\lambda_p(G_1\varphi) = 1$ *implies* $\lambda_q(G_2\varphi) = \lambda_q(H_2\varphi) = 1$,
H-transfer $\lambda_q(H_1\varphi) = 1$ *implies* $\lambda_p(G_2\varphi) = \lambda_p(H_2\varphi) = 1$.

For $i = 2$ we define the following uniformity *conditions: for every $p, q \in W_2$,*

G_1-uniformity $\lambda_p(G_1\varphi) = \lambda_q(G_1\varphi)$,
H_1-uniformity $\lambda_p(H_1\varphi) = \lambda_q(H_1\varphi)$.

We will need the following technical lemma.

Lemma 3. *Fix $i \in \{1,2\}$. Let $\mathcal{W}_i = (W_i, <_i, \lambda)$ be an i-consistent structure such that $(W_i, <_i)$ is a dense, linear order without endpoints. Assume that $\mathcal{W}_i$ satisfies the G- and H-transfer conditions if $i = 1$, and the G_1- and H_1-uniformity conditions if $i = 2$.*

Let $u \in W_i$ and $\gamma \in \Gamma$ such that $\lambda_u(\gamma) = 1$. Then there is an i-SSM M_i for γ of size at most $(4 \times |\Gamma|)^2$. In fact, M_i can be chosen such that for some $U_i \subseteq W_i$ with $|U_i| \leq 4 \times |\Gamma|$

$$M_i = \{(\lambda_u, \lambda_v) : u, v \in U_i, (\exists u' \in W_i)(\exists v' \in W_i) u \equiv u' \ \&\ v \equiv v' \ \&\ u' <_i v'\}$$

where $w \equiv w'$ iff $\lambda_w = \lambda_{w'}$.

Proof. For every $w, w' \in W_i$, we let $w \equiv w'$ iff $\lambda_w = \lambda_{w'}$. Note that there are finitely many equivalence classes, since Γ is finite. For every formula $\varphi \in \Gamma$, let $W_i(\varphi) := \{w \in W_i : \lambda_w(\varphi) = 1\}$. Let $\overline{w}_\varphi$ be a *maximal* element of $W_i(\varphi)$ (provided that $W_i(\varphi)$ is not empty) in the following sense:

$$(\forall w' \in W_i(\varphi))(\exists w'' \in W_i(\varphi)) w' \leq_i w'' \ \&\ w'' \equiv \overline{w}_\varphi.$$

The existence of a maximal element can be easily shown. For every $\varphi \in \Gamma$ choose a maximal element $\overline{w}_\varphi$ from W_φ. Similarly, for every $\varphi \in \Gamma$, choose a minimal element $\underline{w}_\varphi$ from $W_i(\varphi)$. Let

$$W_i^- = \{\overline{w}_\varphi, \underline{w}_\varphi : \varphi \in \Gamma\}.$$

Note that $|W_i^-| \leq 2 \times |\Gamma|$.

The problem with W_i^- is that it may not contain "enough" points to cure density defects. Indeed, consider the *unique points* in W_i^-, i.e., those $w \in W_i^-$ such that for every $w' \neq w$, $\lambda_w \neq \lambda_{w'}$. Note that the set X of unique points can be linearly ordered: $x_1 <_i x_2 <_i \ldots <_i x_m$. Now consider two unique points $x_j, x_{j+1} \in W_i^-$ such that there is no $z \in W_i$ with $x_j <_i z <_i x_{j+1}$ in W_i and $z \equiv z' \in W_i^-$ for some z'. Then we would not be able to insert a point into the mosaic $(\lambda_{x_j}, \lambda_{x_{j+1}})$.

So let us expand W_i^- with the required witnesses for density defects. Take the enumeration $x_0 <_i x_1 <_i \ldots <_i x_m$ of unique points. Since $<_i$ is a dense order, there are infinitely many points in each open interval $]x_j, x_{j+1}[= \{x : x_j <_i x <_i x_{j+1}\}$. Thus, we can choose a point $s \in]x_j, x_{j+1}[$ such that there are infinitely many points t in $]x_j, x_{j+1}[$ with $\lambda_s = \lambda_t$. Let us denote such a chosen s by s_j for every $0 \leq j < m$. Define $U_i := W_i^- \cup \{s_j : 0 \leq j < m\}$. Note that $|U_i| \leq 4 \times |\Gamma|$. We claim that

$$M_i := \{(\lambda_u, \lambda_v) : u, v \in U_i, (\exists u' \in W_i)(\exists v' \in W_i) u \equiv u' \ \&\ v \equiv v' \ \&\ u' <_i v'\}$$

is the required i-SSM. Clearly, $|M_i| \leq (4 \times |\Gamma|)^2$. The elements of M_i are G_i- and H_i-coherent because $\mathcal{W}_i$ is sound. The transfer (for $i = 1$) and uniformity (for $i = 2$) conditions also hold, since $\mathcal{W}_i$ has the corresponding properties. Thus every element of M_i is an i-mosaic. It remains to show the saturation conditions.

For the F_i-saturation—expansion requirement assume that $(\lambda_u, \lambda_v) \in M$ and $\lambda_v(F_i\varphi) = 1$. Let $v' \in W$ such that $v \equiv v'$. Since $\lambda_{v'}(F_i\varphi) = 1$ and $\mathcal{W}_i$ is F_i-complete, there is $z \in W_i$ such that $v' <_i z$ and

– $\lambda_z(\Diamond_2\varphi) = 1$ in case $i = 1$,
– $\lambda_z(\varphi) = 1$ in case $i = 2$.

Let $\overline{w}$ be the maximal element in $W_1(\Diamond_2\varphi)$ or $W_2(\varphi)$ (depending on the value of i) that we put in U_i. By the maximality of $\overline{w}$, there is $z' \in W_i$ such that $z \leq_i z'$ and $z' \equiv \overline{w}$. By this observation, we get that $(\lambda_v, \lambda_{\overline{w}}) \in M_i$ as required.

For the F_i-saturation—insertion requirement we work out only the case $i = 2$, since the case $i = 1$ is completely analogous. So assume that $(\lambda_u, \lambda_v) \in M_2$, $\lambda_u(F_2\varphi) = 1$ and $\lambda_v(F_2\varphi) = \lambda_v(\varphi) = 0$. Let $u', v' \in W_2$ such that $u \equiv u'$, $v \equiv v'$ and $u' <_2 v'$. Since $\mathcal{W}_2$ is F_2-complete, there is $z \in W_2$ such that $u' <_2 z$ and $\lambda_z(\varphi) = 1$. Let $\overline{w}$ be the maximal element of $W_2(\varphi)$ that we put in U_2. Then there is $z' \in W_2$ such that $z \leq_2 z'$ and $z' \equiv \overline{w}$. Hence $(\lambda_u, \lambda_{\overline{w}}) \in M_2$. Furthermore, $z' <_2 v'$, since $\lambda_{v'}(F_2\varphi) = \lambda_{v'}(\varphi) = 0$ and $\mathcal{W}_2$ is G_2-sound. That is, $(\lambda_{\overline{w}}, \lambda_v) \in M_2$ as well. Thus we can insert the appropriate mosaics into (λ_u, λ_v).

Checking the saturation conditions for past formulas is completely analogous. The No-endpoints requirement follows from the fact that $(W_i, <_i)$ is an unbounded linear order. Indeed, let $(\lambda_u, \lambda_v) \in M_i$ and $v' \in W_i$ such that $v \equiv v'$. Then there is $w \in W_i$ such that $v' <_i w$ and $\lambda_w(\top) = 1$. Let $\overline{w}$ be the be the maximal element of $W_i(\top)$. Then $(\lambda_v, \lambda_{\overline{w}}) \in M_i$ as required.

It remains to show the Density condition. Let $(\lambda_u, \lambda_v) \in M_i$ be an arbitrary mosaic and $u', v' \in W_i$ such that $u' <_i v'$, $u \equiv u'$ and $v \equiv v'$. If either u' or v' is not unique, then we can insert either (λ_u, λ_u) or (λ_v, λ_v) into (λ_u, λ_v). So assume that both u' and v' are unique, say $u' = x_j$ and $v' = x_{j+k}$. But we defined $s_j \in \,]x_j, x_{j+1}[$ in this case, and we have $(\lambda_u, \lambda_{s_j})$ and $(\lambda_{s_j}, \lambda_v)$ in M_i. Hence we can insert the required mosaics into (λ_u, λ_v) in this case as well.

Finally note that we chose a representative from the equivalence class $W_i(\gamma)$, hence M_i is indeed an i-SSM for γ. □

We are ready to state the reverse of Lemma 2. In the proof, we will apply Lemma 3 in both the "vertical" and "horizontal" dimensions.

Lemma 4. *If ξ is satisfiable, then there is a 1-SSS Σ_1 for ξ of size polynomial in terms of the size of ξ. In fact, the number of elements in Σ_1 is bounded by $(4 \times |\Gamma|)^4$.*

Proof. Assume that ξ is satisfied in a model, say, $\mathcal{M} = ((T_1, <_1), (T_2, <_2), V)$ and $\mathcal{M}$,$(s, t) \models \xi$. We recall the definition of $\lambda_{(p,q)}$ from (1): for every $(p, q) \in T_1 \times T_2$,

$$\lambda_{(p,q)}(\gamma) = \begin{cases} 1 & \text{if } \mathcal{M}, (p, q) \models \gamma \\ 0 & \text{otherwise} \end{cases}$$

for every $\gamma \in \Gamma$. In particular, $\lambda_{(s,t)}(\xi) = 1$.

For every $p \in T_1$, consider $\mathcal{W}_2^p = (\{p\} \times T_2, \prec_2, \lambda)$ (where $(p, q) \prec_2 (p, r)$ iff $q <_2 r$). Observe that $(p, q) \models G_1\varphi$ iff $(p, r) \models G_1\varphi$ (and $(p, q) \models H_1\varphi$ iff $(p, r) \models H_1\varphi$) for every $q, r \in T_2$. Hence $\mathcal{W}_2^p$ is a 2-consistent structure that satisfies G_1- and H_1-uniformity. By applying Lemma 3 we get a 2-SSM M_2^p for ξ (and in fact for $\Diamond_2\xi$) such that

$$M_2^p = \{(\lambda_{(p,q)}, \lambda_{(p,r)}) : q, r \in U_2^p, (\exists q' \in T_2)(\exists r' \in T_2) q \equiv q' \ \& \ r \equiv r' \ \& \ q' <_2 r'\}$$

where $u \equiv v$ iff $\lambda_{(p,u)} = \lambda_{(p,v)}$, and $U_2^p \subseteq T_2$ such that $|U_2^p| \leq 4 \times |\Gamma|$.

Next we recall the definition λ_M for 2-mosaics M from (3):

$$\lambda_M(\gamma) := \begin{cases} 1 & \text{if } \mu(\gamma) = \nu(\gamma) = 1 \text{ for every } (\mu, \nu) \in M \\ 0 & \text{if } \mu(\gamma) = \nu(\gamma) = 0 \text{ for every } (\mu, \nu) \in M \\ \text{undefined} & \text{otherwise} \end{cases}$$

for every $\gamma \in \Gamma$. Note that $\lambda_N(\Diamond_2\xi) = 1$ for $N = M_2^s$. We define $\lambda_p := \lambda_N$ with $N = M_2^p$ for every $p \in T_1$. It is straightforward to verify that $(T_1, <_1, \lambda)$ is a 1-consistent structure that satisfies G- and H-transfer. Hence we can apply Lemma 3. Thus there is a 1-SSM Σ_1 for $\Diamond_2\xi$ such that

$$\Sigma_1 = \{(\lambda_p, \lambda_q) : p, q \in U_1, (\exists p' \in T_1)(\exists q' \in T_1) p \equiv p' \ \& \ q \equiv q' \ \& \ p' <_1 q'\}$$

where $u \equiv v$ iff $\lambda_u = \lambda_v$ and $U_1 \subseteq T_1$ such that $|U_1| \leq 4 \times |\Gamma|$. Since every M_2^p is a 2-SSM, we get that (M_2^p, M_2^q) is indeed a 1-supermosaic for every $(\lambda_p, \lambda_q) \in \Sigma_1$, whence Σ_1 is a 1-SSS for ξ.

Finally, let us compute an upper bound on the size of Σ_1. Recall that $|U_1| \leq 4 \times |\Gamma|$, whence there are at most $(4 \times |\Gamma|)^2$ many 1-supermosaics in Σ_1. The size of the 1-supermosaics is also bounded by $(4 \times |\Gamma|)^2$, since $|U_2^p| \leq 4 \times |\Gamma|$. Thus the size of Σ_1 is bounded by $(4 \times |\Gamma|)^4$. □

6 Complexity

We are ready to provide a complete decision procedure in nondeterministic polynomial time for the satisfiability problem of the temporal logic of the lexicographic products of unbounded dense linear orders.

Theorem 1. *The satisfiability problem with respect to $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$ is decidable in nondeterministic polynomial time.*

Proof. Given a formula ξ, let us proceed as follows.

1. Compute the least full domain Γ of formulas containing ξ; recall that $|\Gamma| \leq 14 \times (2 \times |\xi| + 10) + 2$.
2. nondeterministically choose a collection Σ_1 of 1-mosaics consisting of 2-mosaics of cardinality bounded by $(4 \times |\Gamma|)^4 \leq (4 \times 14 \times (2 \times |\xi| + 10) + 2)^4$.
3. Check whether Σ is indeed a 1-SSS for ξ.

By Lemma 2 and 4 the above decision procedure is complete. □

Recall that in Lemma 2 we constructed a model $(\mathbb{Q}, \mathbb{Q}, V)$ for ξ based on the rationals from a 1-SSS for ξ, the existence of which is equivalent to satisfiability of ξ by Lemma 4. Thus we have the following.

Theorem 2. *The logic of $(\mathcal{C}_{ud}, \mathcal{C}_{ud})$ coincides with the logic of the lexicographic product $(\mathbb{Q}, \mathbb{Q})$ of the rationals with the standard ordering.*

7 Conclusion

Temporal logics in which one can assign a proper meaning to the association of statements about different grained temporal domains have been considered. See [8, 12, 18] for details. Nevertheless, it seems that the results concerning the issues of axiomatization/completeness and decidability/complexity presented in [3] and in this paper constitute the first steps towards a temporal logic based on different levels of abstraction. Much remains to be done.

For example, one may consider the lexicographic products of special linear flows of time like $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. Concerning the issues of axiomatization and completeness, could transfer results for completeness similar to the ones obtained by Kracht and Wolter [13] within the context of independently axiomatizable bimodal logics be obtained in our lexicographic setting? Concerning the issues of decidability and complexity, all normal extensions of $S4.3$, as proved in [6, 9], possess the finite model property and all finitely axiomatizable normal extensions of $K4.3$, as proved in [23], are decidable. Moreover, it follows from [15] that actually all finitely axiomatizable temporal logics of linear time flows are CoNP-complete. Is it possible to obtain similar results in our lexicographic setting? Or could undecidability results similar to the ones obtained by Reynolds and Zakharyaschev [21] within the context of the products of the modal logics determined by arbitrarily long linear orders be obtained in our lexicographic setting?

There is also the question of associating with $<_1$ and $<_2$ the until-like connectives U_1 and U_2 and the since-like connectives S_1 and S_2, the formulas $\varphi U_1 \psi$, $\varphi U_2 \psi$, $\varphi S_1 \psi$ and $\varphi S_2 \psi$ being read as one reads the formulas $\varphi U \psi$ and $\varphi S \psi$ in classical temporal logic, this time with $<_1$ and $<_2$. As yet, nothing has been done concerning the issues of axiomatization/completeness and decidability/complexity these new temporal connectives give rise to.

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