

# A Super-Fast Distributed Algorithm for Bipartite Metric Facility Location

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## Abstract

The *facility location* problem consists of a set of *facilities*  $\mathcal{F}$ , a set of *clients*  $\mathcal{C}$ , an *opening cost*  $f_i$  associated with each facility  $x_i$ , and a *connection cost*  $D(x_i, y_j)$  between each facility  $x_i$  and client  $y_j$ . The goal is to find a subset of facilities to *open*, and to connect each client to an open facility, so as to minimize the total facility opening costs plus connection costs. This paper presents the first expected-sub-logarithmic-round distributed  $O(1)$ -approximation algorithm in the *CONGEST* model for the *metric* facility location problem on the complete bipartite network with parts  $\mathcal{F}$  and  $\mathcal{C}$ . Our algorithm has an expected running time of  $O((\log \log n)^3)$  rounds, where  $n = |\mathcal{F}| + |\mathcal{C}|$ . This result can be viewed as a continuation of our recent work (ICALP 2012) in which we presented the first sub-logarithmic-round distributed  $O(1)$ -approximation algorithm for metric facility location on a *clique* network. The bipartite setting presents several new challenges not present in the problem on a clique network. We present two new techniques to overcome these challenges. (i) In order to deal with the problem of not being able to choose appropriate probabilities (due to lack of adequate knowledge), we design an algorithm that performs a random walk over a probability space and analyze the progress our algorithm makes as the random walk proceeds. (ii) In order to deal with a problem of quickly disseminating a collection of messages, possibly containing many duplicates, over the bipartite network, we design a probabilistic hashing scheme that delivers all of the messages in expected- $O(\log \log n)$  rounds.

## 1 Introduction

This paper continues the recently-initiated exploration [2, 3, 10, 12, 20] of the design of sub-logarithmic, or “super-fast” distributed algorithms in low-diameter, bandwidth-constrained settings. To understand the main themes of this exploration, suppose that we want to design a distributed algorithm for a problem on a low-diameter network (we have in mind a clique network or a diameter-2 network). In one sense, this is a trivial task since the entire input could be shipped off to a single node in a single round and that node can simply solve the problem locally. On the other hand, the problem could be quite challenging if we were to impose reasonable constraints on bandwidth that prevent the fast delivery of the entire input to a small number of nodes. A natural example of this phenomenon is provided by the *minimum spanning tree* (MST) problem. Consider a clique network in which each edge  $(u, v)$  has an associated weight  $w(u, v)$  of which only the nodes  $u$  and  $v$  are aware. The problem is for the nodes to compute an MST of the edge-weighted clique such that after the computation, each node knows all MST edges. It is important to note that the problem is defined by  $\Theta(n^2)$  pieces of input and it would take  $\Omega(\frac{n}{B})$  rounds of communication for all of this information to reach a single node (where  $B$  is the number of bits that

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can travel across an edge in each round). Typically,  $B = O(\log n)$ , and this approach is clearly too slow given our goal of completing the computation in a sub-logarithmic number of rounds. Lotker et al. [12] showed that the MST problem on a clique can in fact be solved in  $O(\log \log n)$  rounds in the *CONGEST* model of distributed computation, which is a synchronous, message-passing model in which each node can send a message of size  $O(\log n)$  bits to each neighbor in each round. The algorithm of Lotker et al. employs a clever merging procedure that, roughly speaking, causes the sizes of the MST components to square with each iteration, leading to an  $O(\log \log n)$ -round computation time. The overall challenge in this area is to establish the round complexity of a variety of problems that make sense in low-diameter settings. The area is largely open with few upper bounds and no non-trivial lower bounds known. For example, it has been proved that computing an MST requires  $\Omega\left(\left(\frac{n}{\log n}\right)^{1/4}\right)$  rounds in the *CONGEST* model for diameter-3 graphs [13], but no lower bounds are known for diameter-2 or diameter-1 (clique) networks.

The focus of this paper is the *distributed facility location* problem, which has been considered by a number of researchers [16, 7, 18, 19, 2] in low-diameter settings. We first describe the sequential version of the problem. The input to the facility location problem consists of a set of *facilities*  $\mathcal{F} = \{x_1, x_2, \dots, x_{n_f}\}$ , a set of *clients*  $\mathcal{C} = \{y_1, y_2, \dots, y_{n_c}\}$ , a (nonnegative) *opening cost*  $f_i$  associated with each facility  $x_i$ , and a (nonnegative) *connection cost*  $D(x_i, y_j)$  between each facility  $x_i$  and client  $y_j$ . The goal is to find a subset  $F \subseteq \mathcal{F}$  of facilities to *open* so as to minimize the total facility opening costs plus connection costs, i.e.  $FacLoc(F) := \sum_{x_i \in F} f_i + \sum_{y_j \in \mathcal{C}} D(F, y_j)$ , where  $D(F, y_j) := \min_{x_i \in F} D(x_i, y_j)$ . Facility location is an old and well-studied problem in operations research [1, 4, 5, 9, 21] that arises in contexts such as locating hospitals in a city or locating distribution centers in a region. The *metric facility location* problem is an important special case of facility location in which the connection costs satisfy the following “triangle inequality:” for any  $x_i, x_{i'} \in \mathcal{F}$  and  $y_j, y_{j'} \in \mathcal{C}$ ,  $D(x_i, y_j) + D(y_j, x_{i'}) + D(x_{i'}, y_{j'}) \geq D(x_i, y_{j'})$ . The facility location problem, even in its metric version, is NP-complete and finding approximation algorithms for the problem has been a fertile area of research. There are several constant-factor approximation algorithms for metric facility location (see [11] for a recent example). This approximation factor is known to be near-optimal [8].

More recently, the facility location problem has also been used as an abstraction for the problem of locating resources in wireless networks [6, 17]. Motivated by this application, several researchers have considered the facility location problem in a distributed setting. In [16, 18, 19], as well as in the present work, the underlying communication network is a complete bipartite graph  $G = \mathcal{F} + \mathcal{C}$ , with  $\mathcal{F}$  and  $\mathcal{C}$  forming the bipartition. At the beginning of the algorithm, each node, whether a facility or client, has knowledge of the connection costs (“distances”) between itself and all nodes in the other part. In addition, the facilities know their opening costs. The problem is to design a distributed algorithm that runs on  $G$  in the *CONGEST* model and produces a subset  $F \subseteq \mathcal{F}$  of facilities to *open*. To simplify exposition we assume that every cost in the problem input can be represented in  $O(\log n)$  bits, thus allowing each cost to be transmitted in a single message. Each chosen facility will then open and provide services to any and all clients that wish to connect to it (each client must be served by some facility). The objective is to guarantee that  $FacLoc(F) \leq \alpha \cdot OPT$ , where  $OPT$  is the cost of an optimal solution to the given instance of facility location and  $\alpha$  is a constant. We call this the BIPARTITEFACLOC problem. In this paper we present the first sub-logarithmic-round algorithm for the BIPARTITEFACLOC problem; specifically, our algorithm runs in  $O((\log \log n_f)^2 \cdot \log \log \min\{n_f, n_c\})$  rounds in expectation, where  $n_f = |\mathcal{F}|$  and  $n_c = |\mathcal{C}|$ . All previous distributed approximation algorithms for BIPARTITEFACLOC require a logarithmic number of rounds to achieve near-optimal approximation factors.

## 1.1 Overview of Technical Contributions

In a recent paper (ICALP 2012, [2]; full version available as [3]), we presented an expected- $O(\log \log n)$ -round algorithm in the *CONGEST* model for CLIQUEFACLOC, the “clique version” of BIPARTITEFACLOC. The underlying communication network for this version of the problem is a clique with each edge  $(u, v)$  having an associated (connection) cost  $c(u, v)$  of which only nodes  $u$  and  $v$  are aware (initially). Each node  $u$  also has an opening cost  $f_u$ , and may choose to open as a facility; nodes that do not open must connect to an open facility. The cost of the solution is defined as before – as the sum of the facility

opening costs and the costs of established connections. Under the assumption that the connection costs form a metric, our algorithm for CLIQUEFACLOC yields an  $O(1)$ -approximation. We had hoped that a “super-fast” algorithm for BIPARTITEFACLOC would be obtained in a straightforward manner by extending our CLIQUEFACLOC algorithm. However, it turns out that moving from a clique communication network to a complete bipartite communication network raises several new and significant challenges related to information dissemination and a lack of adequate knowledge. Below we outline these challenges and our solutions to them.

**Overview of solution to CliqueFacLoc.** To solve CLIQUEFACLOC on an edge-weighted clique  $G$  [2, 3] we reduce it to the problem of computing a 2-ruling set in an appropriately-defined spanning subgraph of  $G$ . A  $\beta$ -ruling set of a graph is an independent set  $S$  such that every node in the graph is at most  $\beta$  hops away from some node in  $S$ ; a *maximal independent set* (MIS) is simply a 1-ruling set. The spanning subgraph  $H$  on which we compute a 2-ruling set is induced by clique edges whose costs are no greater than a pre-computed quantity which depends on the two endpoints of the edge in question.

We solve the 2-ruling set problem on the spanning subgraph  $H$  via a combination of deterministic and randomized sparsification. Briefly, each node selects itself with a uniform probability  $p$  chosen such that the subgraph  $H'$  of  $H$  induced by the selected nodes has  $\Theta(n)$  edges in expectation. The probability  $p$  is a function of  $n$  and the number of edges in  $H$ . We next deliver all of  $H'$  to every node. It can be shown that a graph with  $O(n)$  edges can be completely delivered to every node in  $O(1)$  rounds on a clique and since  $H'$  has  $O(n)$  edges in expectation, the delivery of  $H'$  takes expected- $O(1)$  rounds. Once  $H'$  has been disseminated in this manner, each node uses the same (deterministic) rule to locally compute an MIS of  $H'$ . Following the computation of an MIS of  $H'$ , nodes in the MIS and nodes in their 2-neighborhood are all deleted from  $H$  and  $H$  shrinks in size. Since  $H$  is now smaller, a larger probability  $p$  can be used for the next iteration. This increasing sequence of values for  $p$  results in a doubly-exponential rate of progress, which leads to an expected- $O(\log \log n)$ -round algorithm for computing a 2-ruling set of  $H$ . See [2] for more details.

**Challenges for BipartiteFacLoc.** The same algorithmic framework can be applied to BIPARTITEFACLOC; however, challenges arise in trying to implement the ruling-set computation on a bipartite communication network. As in CLIQUEFACLOC [2], we define a particular graph  $H$  on the set of facilities with edges connecting pairs of facilities whose connection cost is bounded above. Note that there is no explicit notion of connection cost between facilities, but we use a natural extension of the facility-client connection costs  $D(\cdot, \cdot)$  and define for each  $x_i, x_j \in \mathcal{F}$ ,  $D(x_i, x_j) := \min_{y \in \mathcal{C}} D(x_i, y) + D(x_j, y)$ . The main algorithmic step now is to compute a 2-ruling set on the graph  $H$ . However, difficulties arise because  $H$  is not a subgraph of the communication network  $G$ , as it was in the CLIQUEFACLOC setting. In fact, initially a facility  $x_i$  does not even know to which other facilities it is adjacent to in  $H$ . This adjacency knowledge is collectively available only to the clients. A client  $y$  witnesses edge  $\{x_i, x_j\}$  in  $H$  if  $D(x_i, y) + D(x_j, y)$  is bounded above by a pre-computed quantity associated with the facility-pair  $x_i, x_j$ . However, (initially) an individual client  $y$  cannot certify the *non-existence* of any potential edge between two facilities in  $H$ ; as, unbeknownst to  $y$ , some other client may be a witness to that edge. Furthermore, the same edge  $\{x_i, x_j\}$  could have many client-witnesses. This “affirmative-only” adjacency knowledge and the duplication of this knowledge turn out to be key obstacles to overcome. For example, in this setting, it seems difficult to even figure out how many edges  $H$  has.

Thus, an example of a problem we need to solve is this: without knowing the number of edges in  $H$ , how do we correctly pick a probability  $p$  that will induce a random subgraph  $H'$  with  $\Theta(n)$  edges? Duplication of knowledge of  $H$  leads to another problem as well. Suppose we did manage to pick a “correct” value of  $p$  and have induced a subgraph  $H'$  having  $\Theta(n)$  edges. In the solution to CLIQUEFACLOC, we were able to deliver all of  $H'$  to a single node (in fact, to every node). In the bipartite setting, how do we deliver  $H'$  to a single node given that even though it has  $O(n)$  edges, information duplication can cause the sum of the number of adjacencies witnessed by the clients to be as high as  $\Omega(n^2)$ ?

We introduce new techniques to solve each of these problems. These techniques are sketched below.

- **Message dissemination with duplicates.** We model the problem of delivering all of  $H'$  to a single node as the following message-dissemination problem on a complete bipartite graph.

### Message Dissemination with Duplicates (MDD).

Given a bipartite graph  $G = \mathcal{F} + \mathcal{C}$ , with  $n_f := |\mathcal{F}|$  and  $n_c := |\mathcal{C}|$ , suppose that there are  $n_f$  messages that we wish to be known to all client nodes in  $\mathcal{C}$ . Initially, each client possesses some subset of the  $n_f$  messages, with each message being possessed by at least one client. Suppose, though, that no client  $y_j$  has any information about which of its messages are also held by any other client. Disseminate all  $n_f$  messages to each client in the network in expected-sub-logarithmic time.

We solve this problem by presenting an algorithm that utilizes probabilistic hashing to iteratively reduce the number of duplicate copies of each message. Note that if no message exists in duplicate, then the total number of messages held is only  $n_f$ , and each can be sent to a distinct facility which can then broadcast it to every client. The challenge, then, lies in coordinating bandwidth usage so as to avoid “bottlenecks” that could be caused by message duplication. Our algorithm for MDD runs in  $O(\log \log \min\{n_f, n_c\})$  rounds in expectation.

- **Random walk over a probability space.** Given the difficulty of quickly acquiring even basic information about  $H$  (e.g., how many edges does it have?), we have no way of setting the value of  $p$  correctly. So we design an algorithm that performs a random walk over a space of  $O(\log \log n_f)$  probabilities. The algorithm picks a probability  $p$ , uses this to induce a random subgraph  $H'$  of  $H$ , and attempts to disseminate  $H'$  to all clients within  $O(\log \log \min\{n_f, n_c\})$  rounds. If this dissemination succeeds,  $p$  is modified in one way (increased appropriately), otherwise  $p$  is modified differently (decreased appropriately). This technique can be modeled as a random walk on a probability space consisting of  $O(\log \log n_f)$  elements, where the elements are distinct values that  $p$  can take. We show that after a random walk of length at most  $O(\log \log n_f)$ , sufficiently many edges of  $H$  are removed, leading to  $O(\log \log n_f)$  levels of progress. Thus we have a total of  $O((\log \log n_f)^2)$  steps and since in each step an instance of MDD is solved for disseminating adjacencies, we obtain an expected- $O((\log \log n_f)^2 \cdot \log \log \min\{n_f, n_c\})$ -round algorithm for computing a 2-ruling set of  $H$ .

To summarize, our paper makes three main technical contributions. (i) We show (in Section 2) that the framework developed in [2] to solve CLIQUEFACLOC can be used, with appropriate modifications, to solve BIPARTITEFACLOC. Via this algorithmic framework, we reduce BIPARTITEFACLOC to the problem of computing a 2-ruling set of a graph induced by facilities in a certain way. (ii) In order to compute a 2-ruling set of a graph, we need to disseminate graph adjacencies whose knowledge is distributed among the clients with possible duplication. We model this as a message dissemination problem and show (in Section 3), using a probabilistic hashing scheme, how to efficiently solve this problem on a complete bipartite graph. (iii) Finally, we present (in Section 4) an algorithm that performs a random walk over a probability space to efficiently compute a 2-ruling set of a graph, without even basic information about the graph. This algorithm repeatedly utilizes the procedure for solving the message-dissemination problem mentioned above.

## 2 Reduction to the Ruling Set Problem

In this section we reduce BIPARTITEFACLOC to the ruling set problem on a certain graph induced by facilities. The reduction is achieved via the distributed facility location algorithm called LOCATE-FACILITIES and shown as Algorithm 1. This algorithm is complete except that it calls a subroutine, RULINGSET( $H, s$ ) (in Step 4), to compute an  $s$ -ruling set of a certain graph  $H$  induced by facilities. In this section we first describe Algorithm 1 and then present its analysis. It is easily observed that all the steps in Algorithm 1, except the one that calls RULINGSET( $H, s$ ) take a total of  $O(1)$  communication rounds. Thus the running time of RULINGSET( $H, s$ ) essentially determines the running time of Algorithm 1. Furthermore, we show that if  $F^*$  is the subset of facilities opened by Algorithm 1, then  $FacLoc(F^*) = O(s) \cdot OPT$ . In the remaining sections of the paper we show how to implement

RULINGSET( $H, 2$ ) in expected  $O((\log \log n_f)^2 \cdot \log \log \min\{n_f, n_c\})$  rounds. This yields an expected  $O((\log \log n_f)^2 \cdot \log \log \min\{n_f, n_c\})$ -round,  $O(1)$ -approximation algorithm for BIPARTITEFACLOC.

## 2.1 Algorithm

Given  $\mathcal{F}$ ,  $\mathcal{C}$ ,  $D(\cdot, \cdot)$ , and  $\{f_i\}$ , define the *characteristic radius*  $r_i$  of facility  $x_i$  to be the nonnegative real number satisfying  $\sum_{y \in B(x_i, r_i)} (r_i - D(x_i, y)) = f_i$ , where  $B(x, r)$  (the *ball* of radius  $r$ ) denotes the set of clients  $y$  such that  $D(x, y) \leq r$ . This notion of a characteristic radius was first introduced by Mettu and Plaxton [14], who use it to drive their sequential, greedy algorithm. We extend the client-facility distance function  $D(\cdot, \cdot)$  to facility-facility distances; let  $D : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}^+ \cup \{0\}$  be defined by  $D(x_i, x_j) = \min_{y_k \in \mathcal{C}} \{D(x_i, y_k) + D(x_j, y_k)\}$ . With these definitions in place we are ready to describe Algorithm 1. The algorithm consists of three stages, which we now describe.

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### Algorithm 1 LOCATEFACILITIES

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**Input:** A complete bipartite graph  $G$  with partition  $(\mathcal{F}, \mathcal{C})$ ; (bipartite) metric  $D(\cdot, \cdot)$ ; opening costs  $\{f_i\}_{i=1}^{n_f}$ ; a sparsity parameter  $s \in \mathbb{Z}^+$

**Assumption:** Each facility knows its own opening cost and its distances to all clients; each client knows its distances to all facilities

**Output:** A subset of facilities (a *configuration*) to be declared open.

1. Each facility  $x_i$  computes and broadcasts its radius  $r_i$  to all clients;  $r_0 := \min_i r_i$ .
  2. Each client computes a partition of the facilities into classes  $\{V_k\}$  such that  $3^k \cdot r_0 \leq r_i < 3^{k+1} \cdot r_0$  for  $x_i \in V_k$ .
  3. For  $k = 0, 1, \dots$ , define a graph  $H_k$  with vertex set  $V_k$  and edge set:
 
$$\{\{x_i, x_{i'}\} \mid x_i, x_{i'} \in V_k \text{ and } D(x_i, x_{i'}) \leq r_i + r_{i'}\}$$
 (Observe from the definition of facility distance that such edges may be known to as few as one client, or as many as all of them.)
  4. All nodes in the network use procedure RULINGSET( $\cup_k H_k, s$ ) to compute a 2-ruling set  $T$  of  $\cup_k H_k$ .  $T$  is known to every client. We use  $T_k$  to denote  $T \cap V_k$ .
  5. Each client  $y_j$  sends an **open** message to each facility  $x_i$ , if and only if both of the following conditions hold:
    - (i)  $x_i$  is a member of the set  $T_k \subseteq H_k$ , for some  $k$ .
    - (ii)  $y_j$  is not a witness to the existence of a facility  $x_{i'}$  belonging to a class  $H_{k'}$ , with  $k' < k$ , such that  $D(x_i, x_{i'}) \leq 2r_i$ .
  6. Each facility  $x_i$  opens, and broadcasts its status as such, if and only if  $x_i$  received an **open** message from every client.
  7. Each client connects to the nearest open facility.
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**Stage 1. (Steps 1-2)** Each facility knows its own opening cost and the distances to all clients. So in Step 1 facility  $x_i$  computes  $r_i$  and broadcasts that value to all clients. Once this broadcast is complete, each client knows all of the  $r_i$  values. This enables every client to compute the same partition of the facilities into classes as follows (Step 2). Define the special value  $r_0 := \min_{1 \leq i \leq n_f} \{r_i\}$ . Define the class  $V_k$ , for  $k = 0, 1, \dots$ , to be the set of facilities  $x_i$  such that  $3^k \cdot r_0 \leq r_i < 3^{k+1} \cdot r_0$ . Every client computes the class into which each facility in the network falls.

**Stage 2. (Steps 3-4)** Now that the facilities are divided into classes having comparable  $r_i$ 's, and every client knows which facility is in each class, we focus our attention on class  $V_k$ . Suppose  $x_i, x_{i'} \in V_k$ . Then we define  $x_i$  and  $x_{i'}$  to be *adjacent* in class  $V_k$  if  $D(x_i, x_{i'}) \leq r_i + r_{i'}$  (Step 3). These adjacencies define the graph  $H_k$  with vertex set  $V_k$ . Note that two facilities  $x_i, x_{i'}$  in class  $V_k$  are adjacent if and only if there is at least one client *witness* for this adjacency. Next, the network computes an  $s$ -ruling set  $T$  of  $\cup_k H_k$  with procedure RULINGSET() (Step 4). We describe a super-fast implementation of RULINGSET() in Section 4. After a ruling set  $T$  has been constructed, every client knows all the members of  $T$ . Since the  $H_k$ 's are disjoint,  $T_k := T \cap V_k$  is a 2-ruling set of  $H_k$  for each  $k$ .

**Stage 3. (Steps 5-7)** Finally, a client  $y_j$  sends an **open** message to facility  $x_i$  in class  $V_k$  if (i)  $x_i \in T_k$ , and (ii) there is no facility  $x_{i'}$  of class  $V_{k'}$  such that  $D(x_i, y_j) + D(x_{i'}, y_j) \leq 2r_i$ , and for which  $k' < k$  (Step 5). A facility opens if it receives **open** messages from all clients (Step 6). Lastly, open facilities declare themselves as such in a broadcast, and every client connects to the nearest open facility (Step 7).

Algorithm LOCATEFACILITIES is complete except for the call to the RULINGSET procedure. The remaining sections of the paper describe the implementation and analysis of RULINGSET.

## 2.2 Analysis

The approximation-factor analysis of Algorithm 1 is similar to the analysis of our algorithm for CLIQUEFACLOC [2, 3]. First, we show a lower bound on the cost of *any* solution to BIPARTITEFACLOC. In order to do so, we define  $\bar{r}_j$  (for  $y_j \in \mathcal{C}$ ) as  $\bar{r}_j = \min_{1 \leq i \leq n_f} \{r_i + D(x_i, y_j)\}$ . This concept was introduced and motivated in [2, 3]. Specifically, we show that the cost of any solution to the facility location problem is bounded before by  $\frac{1}{6} \cdot \sum_{j=1}^{n_c} \bar{r}_j$ . Subsequently, we show that the solution computed by Algorithm LOCATEFACILITIES has cost that is  $O(s)$  times  $\sum_{j=1}^{n_c} \bar{r}_j$ . Thus, guaranteeing  $s = O(1)$ , yields an  $O(1)$ -approximation.

### 2.2.1 Approximation Analysis - Lower Bound

We start the lower bound proof by extending the sequential metric facility location algorithm (and analysis) of Mettu and Plaxton [14] on a clique network to the bipartite setting. This part of the analysis closely follows [14] and we include it mainly for completeness. We start by presenting the bipartite version of the Mettu-Plaxton algorithm. The algorithm is greedy in that it considers facilities in non-decreasing order of their  $r$ -values and opens a facility only if there is no already-open facility within 2 times the  $r$ -value of the facility being considered. Below we use the  $D(x, F)$ , where  $x \in \mathcal{F}$  and  $F \subseteq \mathcal{F}$ , to denote  $\min_{x' \in F} D(x, x')$ .

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#### Algorithm 2 Bipartite Mettu-Plaxton Algorithm

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**Input:**  $\mathcal{F}, \mathcal{C}, D(\cdot, \cdot), \{f_i\}$

**Output:** A subset of facilities to open

1. Let  $F_0 = \emptyset$ .
  2. For each facility  $x_i$ , compute the characteristic radius  $r_i$ .
  3. Let  $\varphi$  be a permutation of  $\{1, \dots, n_f\}$  such that for  $1 \leq i < i' \leq n_f$ ,  $r_{\varphi(i)} \leq r_{\varphi(i')}$ .
  4. For  $i = 1$  to  $n_f$ , if  $D(x_{\varphi(i)}, F_{i-1}) > 2r_{\varphi(i)}$ , then set  $F_i = F_{i-1} \cup \{x_{\varphi(i)}\}$ ; else set  $F_i = F_{i-1}$ .
  5. Return  $F_{MP} = F_{n_f}$ .
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The running time of Algorithm 2 is not important to us, but the approximation factor is. Let  $F_{MP}$  denote the subset of facilities opened by algorithm 2. The following series of lemmas (Lemmas 1 to 7) lead to Theorem 1, which shows that  $FacLoc(F_{MP})$  is within 3 times the optimal facility opening cost.

**Lemma 1** *For any facility  $x_i$ , there exists a facility  $x_j \in F_{MP}$  such that  $\varphi^{-1}(j) \leq \varphi^{-1}(i)$  (i.e.  $r_j \leq r_i$ ) and  $D(x_i, x_j) \leq 2r_i$ . (Note that  $x_j$  may be  $x_i$  itself.)*

**Proof.** Suppose not. Then  $D(x_i, F_{\varphi^{-1}(i)-1}) > 2r_i$ , so  $x_i$  should have been added to  $F_{MP}$ , which is a contradiction.  $\square$

**Lemma 2** *Let  $x_i, x_j \in F_{MP}$ . Then  $D(x_i, x_j) > 2 \cdot \max\{r_i, r_j\}$ .*

**Proof.** Without loss of generality, assume that  $\varphi^{-1}(i) < \varphi^{-1}(j)$ . Then  $r_i \leq r_j$ , and since  $x_j$  was added to  $F_{MP}$ , it must have been the case that  $D(x_j, F_{\varphi^{-1}(j)-1})$  was greater than  $2r_j$ . As  $x_i \in F_{\varphi^{-1}(j)-1}$ , we conclude that  $D(x_i, x_j) > 2r_j = 2 \cdot \max\{r_i, r_j\}$ .  $\square$

An important contribution of [14] was a standard way of “charging” the cost of a facility location solution to clients. For a client  $y_j \in \mathcal{C}$  and a facility subset  $F$ , the *charge* of  $y_j$  with respect to  $F$  is defined as

$$\text{charge}(y_j, F) = D(F, y_j) + \sum_{x_i \in F} \max\{0, r_i - D(x_i, y_j)\}$$

Here  $D(F, y)$ , for  $F \subseteq \mathcal{F}$  and  $y \in \mathcal{C}$ , is used as shorthand for  $\min_{x \in F} D(x, y)$ . In the following lemma, we use simple algebraic manipulation to show that for any facility subset  $F$ , the cost of  $F$  is correctly distributed to the “charge” associated with each client, as per the definition  $\text{charge}(y_j, F)$ .

**Lemma 3** *For any subset  $F$ ,  $\sum_{y_j \in \mathcal{C}} \text{charge}(y_j, F) = \text{FacLoc}(F)$ .*

**Proof.** Observe that

$$\begin{aligned} \sum_{y_j \in \mathcal{C}} \text{charge}(y_j, F) &= \sum_{y_j \in \mathcal{C}} D(F, y_j) + \sum_{x_i \in F} \sum_{y_j \in \mathcal{C}} \max\{0, r_i - D(x_i, y_j)\} \\ &= \sum_{y_j \in \mathcal{C}} D(F, y_j) + \sum_{x_i \in F} \sum_{y_j \in B(x_i, r_i)} (r_i - D(x_i, y_j)) \\ &= \sum_{y_j \in \mathcal{C}} D(F, y_j) + \sum_{x_i \in F} f_i \\ &= \text{FacLoc}(F) \end{aligned}$$

□

**Lemma 4** *Let  $y_j$  be a client, let  $F$  be a subset of facilities, and let  $x_i \in F$ . If  $D(x_i, y_j) = D(F, y_j)$ , then  $\text{charge}(y_j, F) \geq \max\{r_i, D(x_i, y_j)\}$ .*

**Proof.** If  $D(x_i, y_j) > r_i$ , then  $\text{charge}(y_j, F) \geq D(F, y_j) = D(x_i, y_j) > r_i$ . If  $D(x_i, y_j) \leq r_i$ , then  $\text{charge}(y_j, F) \geq D(F, y_j) + (r_i - D(x_i, y_j)) = D(x_i, y_j) + (r_i - D(x_i, y_j)) = r_i \geq D(x_i, y_j)$ . □

**Lemma 5** *Let  $y_j$  be a client and let  $x_i \in F_{MP}$ . If  $y_j \in B(x_i, r_i)$ , then  $\text{charge}(y_j, F_{MP}) = r_i$ .*

**Proof.** By Lemma 2, there can be no other facility  $x_{i'} \in F_{MP}$ ,  $i' \neq i$ , such that  $D(x_{i'}, y_j) \leq \max\{r_i, r_{i'}\}$ , for then  $D(x_i, x_{i'})$  would be at most  $2 \cdot \max\{r_i, r_{i'}\}$ . Therefore  $\text{charge}(y_j, F_{MP}) = D(x_i, y_j) + (r_i - D(x_i, y_j)) = r_i$ . □

**Lemma 6** *Let  $y_j$  be a client and let  $x_i \in F_{MP}$ . If  $y_j \notin B(x_i, r_i)$ , then  $\text{charge}(y_j, F_{MP}) \leq D(x_i, y_j)$ .*

**Proof.** If there is no  $x_{i'} \in F_{MP}$  such that  $y_j \in B(x_{i'}, r_{i'})$ , then  $\text{charge}(y_j, F_{MP}) = D(F_{MP}, y_j) \leq D(x_i, y_j)$ . If there is such an  $x_{i'}$ , then by Lemma 2,  $D(x_i, x_{i'}) > 2 \cdot \max\{r_i, r_{i'}\}$ . By Lemma 5, then,

$$\begin{aligned} \text{charge}(y_j, F_{MP}) &= r_{i'} \\ &\leq D(x_i, x_{i'}) - r_{i'} \\ &\leq D(x_i, x_{i'}) - D(x_{i'}, y_j) \\ &\leq (D(x_i, x_{i'}) - D(x_{i'}, y_j) - D(x_i, y_j)) + D(x_i, y_j) \\ &\leq D(x_i, y_j) \end{aligned}$$

□

**Lemma 7** *For any client  $y_j$  and subset  $F$ ,  $\text{charge}(y_j, F_{MP}) \leq 3 \cdot \text{charge}(y_j, F)$ .*

**Proof.** Let  $x_i \in F$  be such that  $D(x_i, y_j) = D(F, y_j)$ . By Lemma 1, there is a facility  $x_{i'} \in F_{MP}$  such that  $\varphi^{-1}(i') \leq \varphi^{-1}(i)$  ( $r_{i'} \leq r_i$ ) and  $D(x_i, x_{i'}) \leq 2r_i$ .

If  $y_j \in B(x_{i'}, r_{i'})$ , then by Lemma 5 we have  $\text{charge}(y_j, F_{MP}) = r_{i'} \leq r_i$ ; thus, by Lemma 4,  $\text{charge}(y_j, F_{MP}) \leq \text{charge}(y_j, F)$ .

If  $y_j \notin B(x_{i'}, r_{i'})$ , then by Lemma 6 we have  $\text{charge}(y_j, F_{MP}) \leq D(x_{i'}, y_j) \leq D(x_{i'}, x_i) + D(x_i, y_j) \leq 2r_i + D(x_i, y_j)$ . Now, by Lemma 4, we see that  $2r_i + D(x_i, y_j) \leq 3 \cdot \max\{r_i, D(x_i, y_j)\} \leq 3 \cdot \text{charge}(y_j, F)$ .  $\square$

The following theorem follows from Lemma 3 and Lemma 7.

**Theorem 1** *For any subset  $F$  of facilities,  $\text{FacLoc}(F_{MP}) \leq 3 \cdot \text{FacLoc}(F)$ .*

Now, as mentioned previously, we define  $\bar{r}_j$  (for  $y_j \in \mathcal{C}$ ) as  $\bar{r}_j = \min_{1 \leq i \leq n_f} \{r_i + D(x_i, y_j)\}$ .

**Lemma 8**  *$\text{FacLoc}(F) \geq (\sum_{j=1}^{n_c} \bar{r}_j)/6$  for any subset  $F \subseteq \mathcal{F}$ .*

**Proof.** Recall that  $F_{MP}$  has the property that no two facilities  $x_i, x_j \in F_{MP}$  can have  $D(x_i, x_j) \leq r_i + r_j$ . Therefore, for a client  $y_j$ , if  $x_{\delta(j)}$  denotes a closest open facility (i.e. an open facility satisfying  $D(x_{\delta(j)}, y_j) = D(F_{MP}, y_j)$ ), then

$$\begin{aligned} \text{FacLoc}(F_{MP}) &= \sum_{j=1}^{n_c} \text{charge}(y_j, F_{MP}) \\ &\geq \sum_{y_j \in \mathcal{C}} [D(x_{\delta(j)}, y_j) + \max\{0, r_{\delta(j)} - D(x_{\delta(j)}, y_j)\}] \\ &= \sum_{y_j \in \mathcal{C}} \max\{r_{\delta(j)}, D(x_{\delta(j)}, y_j)\} \end{aligned}$$

Note that the inequality in the above calculation follows from throwing away some terms of the sum in the definition of  $\text{charge}(y_j, F_{MP})$ .

By the definition of  $\bar{r}_j$ ,  $\bar{r}_j \leq r_{\delta(j)} + D(x_{\delta(j)}, y_j) \leq 2 \cdot \max\{r_{\delta(j)}, D(x_{\delta(j)}, y_j)\}$ . It follows that

$$\text{FacLoc}(F_{MP}) \geq \sum_{y_j \in \mathcal{C}} \frac{\bar{r}_j}{2} = \frac{1}{2} \cdot \sum_{j=1}^{n_c} \bar{r}_j.$$

Therefore  $\text{FacLoc}(F) \geq \text{FacLoc}(F_{MP})/3 \geq (\sum_{j=1}^{n_c} \bar{r}_j)/6$ , for any  $F \subseteq \mathcal{F}$ .  $\square$

### 2.2.2 Approximation Analysis - Upper Bound

Let  $F^*$  be the set of facilities opened by Algorithm 1. We analyze  $\text{FacLoc}(F^*)$  by bounding  $\text{charge}(y_j, F^*)$  for each client  $y_j$ . Recall that  $\text{FacLoc}(F^*) = \sum_{j=1}^{n_c} \text{charge}(y_j, F^*)$ . Since  $\text{charge}(y_j, F^*)$  is the sum of two terms,  $D(F^*, y_j)$  and  $\sum_{x_i \in F^*} \max\{0, r_i - D(x_i, y_j)\}$ , bounding each separately by a  $O(s)$ -multiple of  $\bar{r}_j$ , yields the result.

The  $s$ -ruling set  $T_k \subseteq V_k$  has the property that for any node  $x_i \in V_k$ ,  $D(x_i, T_k) \leq 2 \cdot 3^{k+1} r_0 \cdot s$ , where  $s$  is the sparsity parameter used to procedure  $\text{RULINGSET}()$ . Also, for no two members of  $T_k$  is the distance between them less than  $2 \cdot 3^k r_0$ . Note that here we are using distances from the extension of  $D$  to  $\mathcal{F} \times \mathcal{F}$ .

Now, in our cost analysis, we consider a facility  $x_i \in V_k$ . To bound  $D(x_i, F^*)$ , observe that either  $x_i \in T_k$ , or else there exists a facility  $x_{i'} \in T_k$  such that  $D(x_i, x_{i'}) \leq 2 \cdot 3^{k+1} r_0 \cdot s \leq 6r_i \cdot s$ . Also, if a facility  $x_i \in T_k$  does not open, then there exists another node  $x_{i'}$  in a class  $V_{k'}$ , with  $k' < k$ , such that  $D(x_i, x_{i'}) \leq 2r_j$ .

We are now ready to bound the components of  $\text{charge}(y_j, F^*)$ .

**Lemma 9**  $D(F^*, y_j) \leq (15s + 15) \cdot \bar{r}_j$

**Proof.** First, consider any facility  $x_i$ . Suppose that the class containing  $x_i$  is  $V_k$ . Observe that the result of procedure RULINGSET is that  $x_i$  is within distance  $6r_i \cdot s$  of a facility  $x_{i'} \in T_k$  (which may be  $x_i$  itself). Now, in Algorithm 1,  $x_{i'}$  either opens, or there exists a facility  $x_{i''}$  of a lower class such that  $D(x_{i'}, x_{i''}) \leq 2r_{i'} \leq 6r_i$ . We therefore conclude that within a distance  $(6s + 6) \cdot r_i$  of  $x_i$ , there exists either an open facility or a facility of a class of index less than  $k$ .

Now, let  $x_{j'}$  be a minimizer for  $r_x + D(x, y_j)$  so that  $\bar{r}_j = r_{j'} + D(x_{j'}, y_j)$ , and suppose  $x_{j'} \in V_{k'}$ . By the preceding analysis, there exists within a distance  $(6s + 6) \cdot r_{j'}$  of  $x_{j'}$  either an open facility or a facility  $x_{j''}$  of a class of index  $k'' < k'$ . If it is the latter, then within a distance  $(6s + 6) \cdot r_{j''}$  of  $x_{j''}$  there exists either an open facility or a facility  $x_{j'''}$  of a class of index  $k''' \leq k' - 2$ .

Repeating this argument up to  $k' + 1$  times reveals that there must exist an open facility within a distance  $(6s + 6) \cdot (r_{j'} + r_{j''} + r_{j'''} + \dots)$  of  $x_{j'}$ . (Note that any facility  $x_i$  in class  $V_0$  has an open facility within distance  $6s \cdot r_i$  because every member of  $T_0$  opens.) We can simplify this distance bound by noting that  $r_{j'} > r_{j''}$ ,  $r_{j'} > 3r_{j''}$ ,  $r_{j'} > 9r_{j'''}$ , etc., and so  $D(F^*, x_{j'}) \leq (6s + 6) \cdot (r_{j'} + r_{j''} + \frac{1}{3}r_{j''} + \frac{1}{9}r_{j'''} + \dots) = (6s + 6) \cdot \frac{5}{2}r_{j'} = (15s + 15)r_{j'}$ .

Thus we have

$$\begin{aligned} D(F^*, y_j) &\leq D(F^*, x_{j'}) + D(x_{j'}, y_j) \\ &\leq (15s + 15) \cdot r_{j'} + D(x_{j'}, y_j) \\ &\leq (15s + 15) \cdot (r_{j'} + D(x_{j'}, y_j)) \\ &= (15s + 15) \cdot \bar{r}_j \end{aligned}$$

which completes the proof.  $\square$

**Lemma 10**  $\sum_{x_i \in F^*} \max\{0, r_i - D(x_i, y_j)\} \leq 3 \cdot \bar{r}_j$

**Proof.** We begin by observing that we cannot simultaneously have  $D(x_i, y_j) \leq r_i$  and  $D(x_{i'}, y_j) \leq r_{i'}$  for  $x_i, x_{i'} \in F^*$  and  $i \neq i'$ . Indeed, if this were the case, then  $D(x_i, x_{i'}) \leq r_i + r_{i'}$ . If  $x_i$  and  $x_{i'}$  were in the same class  $V_l$ , then they would be adjacent in  $H$ ; this is impossible, for then they could not both be members of  $T_l$  (for a node in  $V_l$ , membership in  $T_l$  is necessary to join  $F^*$ ). If  $x_i$  and  $x_{i'}$  were in different classes, then assume WLOG that  $r_i < r_{i'}$ . Then  $D(x_i, x_{i'}) \leq r_i + r_{i'} \leq 2r_{i'}$ , and  $x_{i'}$  should not have opened. These contradictions imply that there is at most one open facility  $x_i$  for which  $D(x_i, y_j) \leq r_i$ .

For the rest of this lemma, then, assume that  $x_i \in F^*$  is the unique open node such that  $D(x_i, y_j) \leq r_i$  (if such a  $x_i$  does not exist, there is nothing to prove). Also, let  $x_{j'}$  be a minimizer for  $r_x + D(x, y_j)$  so that  $\bar{r}_j = r_{j'} + D(x_{j'}, y_j)$ .

Now, suppose that  $3 \cdot \bar{r}_j < r_i - D(x_i, y_j)$ . Then  $3 \cdot r_{j'} + 3 \cdot D(x_{j'}, y_j) < r_i - D(x_i, y_j)$ , and so we can conclude that (i)  $3r_{j'} < r_i$  (and  $x_{j'}$  is in a lower class than  $x_i$ ) and (ii)  $D(x_i, x_{j'}) \leq D(x_i, y_j) + D(x_{j'}, y_j) \leq r_i + r_i = 2r_i$ , which implies that  $x_i$  should not have opened. This is a contradiction, and so therefore it must be that  $r_i - D(x_i, y_j) \leq 3\bar{r}_j$ . Since  $x_i$  is unique (if it exists), this completes the proof.  $\square$

We are now ready to present the final result of this section.

**Theorem 2** *Algorithm 1 (LOCATEFACILITIES) computes an  $O(s)$ -factor approximation to BIPARTITE-FACLOC in  $O(\mathcal{T}(n, s))$  rounds, where  $\mathcal{T}(n, s)$  is the running time of procedure RULINGSET( $H, s$ ), called an  $n$ -node graph  $H$ .*

**Proof.** Combining Lemma 9 and Lemma 10 gives

$$\begin{aligned}
\text{FacLoc}(F^*) &= \sum_{j=1}^{n_c} \text{charge}(F^*, y_j) \\
&= \sum_{j=1}^{n_c} \left[ D(F^*, y_j) + \sum_{x_i \in F^*} \max\{0, r_i - D(x_i, y_j)\} \right] \\
&\leq \sum_{j=1}^{n_c} [(15s + 15) \cdot \bar{r}_j + 3\bar{r}_j] \\
&\leq (15s + 18) \cdot \sum_{j=1}^{n_c} \bar{r}_j \\
&\leq 6 \cdot (15s + 18) \cdot \text{OPT}.
\end{aligned}$$

The last inequality follows from the lower bound established in Lemma 8.

Also, noting that all the steps in Algorithm 1, except the one that calls  $\text{RULINGSET}(\cup_k H_k, s)$  take a total of  $O(1)$  communication rounds, we obtain the theorem.  $\square$

**Note on the size of the constant.** The above analysis yields the approximation factor  $90s + 108$ , which amounts to 288, since we describe how to compute a 2-ruling set. This is obviously huge, but we have made no attempt to optimize it. A small improvement in the size of this constant can be obtained by using multiplier  $1 + \frac{1}{\sqrt{2}}$  instead of 3 in the definition of the classes  $V_0, V_1, \dots$  (see [3]). For improved exposition, we use the multiplier 3.

### 3 Dissemination on a Bipartite Network

In the previous section we reduced  $\text{BIPARTITEFACLOC}$  to the problem of computing an  $s$ -ruling set on a graph  $H = \cup_k H_k$  defined on facilities. Our technique for finding an  $s$ -ruling set involves selecting a set  $M$  of facilities at random, disseminating the induced subgraph  $H[M]$  to every client and then having each client locally compute an MIS of  $H[M]$  (details appear in Section 4). A key subroutine needed to implement this technique is one that can disseminate  $H[M]$  to every client efficiently, provided the number of edges in  $H[M]$  is at most  $n_f$ . In Section 1 we abstracted this problem as the **Message Dissemination with Duplicates** (MDD) problem. In this section, we present a randomized algorithm for MDD that runs in expected  $O(\log \log \min\{n_f, n_c\})$  communication rounds.

Recall that the difficulty in disseminating  $H[M]$  is the fact that the adjacencies in this graph are witnessed only by clients, with each adjacency being witnessed by at least one client. However, an adjacency can be witnessed by many clients and a client is unaware of who else has knowledge of any particular edge. Thus, even if  $H[M]$  has at most  $n_f$  edges, the total number of adjacency observations by the clients could be as large as  $n_f^2$ . Below we use iterative probabilistic hashing to rapidly reduce the number of “duplicate” witnesses to adjacencies in  $H[M]$ . Once the total number of distinct adjacency observations falls to  $48n_f$ , it takes only a constant number of additional communication rounds for the algorithm to finish disseminating  $H[M]$ . The constant “48” falls out easily from our analysis (Lemma 14, in particular) and we have made no attempt to optimize it in any way.

#### 3.1 Algorithm

The algorithm proceeds in iterations and in each iteration a hash function is chosen at random for hashing messages held by clients onto facilities. Denote the universe of possible adjacency messages by  $\mathcal{U}$ . Since messages represent adjacencies among facilities,  $|\mathcal{U}| = \binom{n_f}{2}$ . However, it is convenient for  $|\mathcal{U}|$  to be equal to  $n_f^2$  and so we extend  $\mathcal{U}$  by dummy messages so that this is the case. We now define a family  $\mathcal{H}_{\mathcal{U}}$  of hash functions from  $\mathcal{U}$  to  $\{1, 2, \dots, n_f\}$  and show how to pick a function from this family, uniformly at random. To define  $\mathcal{H}_{\mathcal{U}}$ , fix an ordering  $m_1, m_2, m_3, \dots$  of the messages of  $\mathcal{U}$ . Partition  $\mathcal{U}$  into groups of

size  $n_f$ , with messages  $m_1, m_2, \dots, m_{n_f}$  as the first group, the next  $n_f$  elements as the second group, and so on. The family  $\mathcal{H}_U$  is obtained by independently mapping each group of messages onto  $(1, 2, \dots, n_f)$  via a cyclic permutation. For each group of  $n_f$  messages in  $\mathcal{U}$ , there are precisely  $n_f$  such cyclic maps for it, and so a map in  $\mathcal{H}_U$  can be selected uniformly at random by having each facility choose a random integer in  $\{1, 2, \dots, n_f\}$  and broadcast this choice to all clients (in the first round of an iteration). Each client then interprets the integer received from facility  $x_i$  as the image of message  $m_{(i-1) \cdot n_f + 1}$ .

In round 2, each client chooses a destination facility for each adjacency message in its possession (note that no client possesses more than  $n_f$  messages), based on the hash function chosen in round 1. For a message  $m$  in the possession of client  $y_j$ ,  $y_j$  computes the hash  $h(m)$  and marks  $m$  for delivery to facility  $x_{h(m)}$ . In the event that more than one of  $y_j$ 's messages are intended for the same recipient,  $y_j$  chooses one uniformly at random for *correct* delivery, and marks the other such messages as “leftovers.” During the communication phase of round 2, then, client  $y_j$  delivers as many messages as possible to their correct destinations; leftover messages are delivered uniformly at random over unused communication links to other facilities.

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**Algorithm 3** DISSEMINATE ADJACENCIES

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**Input:** A complete bipartite graph  $G$ , with partition  $(\mathcal{F}, \mathcal{C})$ ; an overlay network  $H$  on  $\mathcal{F}$  with  $|E[H]| \leq n_f$

**Assumption:** For each adjacency  $e'$  in  $H$ , *one or more* clients has knowledge of  $e'$

**Output:** Each client should know the entire contents of  $E[H]$

---

1. **while** *true* **do**
    - Start of Iteration:*
    - 2. Each client  $y_j$  sends the number of distinct messages currently held,  $n_j$ , to facility  $x_1$ .
    - 3. **if**  $\sum_{j=1}^{n_c} n_j \leq 48n_f$  **then**
    - 4. Facility  $x_1$  broadcasts a *break* message to each client.
    - 5. Client  $y_1$ , upon receiving a *break* message, broadcasts a *break* message to each facility.
    - end-if-then**
    - 6. Each facility  $x_i$  broadcasts an integer in  $\{1, \dots, n_f\}$  chosen uniformly at random; this collection of broadcasts determines a map  $h \in \mathcal{H}_U$ .
    - 7. For each adjacency message  $m'$  currently held, client  $y_j$  maps  $m'$  to  $x_{h(m')}$ .
    - 8. For each  $i \in \{1, \dots, n_f\}$ , if  $|\{m' \text{ held by } y_j : h(m') = i\}| > 1$ , client  $y_j$  chooses one message to send to  $x_i$  at random from this set and marks the others as *leftovers*.
    - 9. Each client  $y_j$  sends the messages chosen in Lines 7-8 to their destinations; *leftover* messages are delivered to other facilities (for whom  $y_j$  has no intended message) in an arbitrary manner (such that  $y_j$  sends at most one message to each facility).
    - 10. Each facility  $x_i$  receives a collection of at most  $n_c$  facility adjacency messages; if duplicate messages are received,  $x_i$  discards all but one of them so that the messages held by  $x_i$  are distinct.
    - 11. Each facility  $x_i$  sends its number of distinct messages currently held,  $b_i$ , to client  $y_1$ .
    - 12. Client  $y_1$  responds to each facility  $x_i$  with an index  $c(i) = (\sum_{k=1}^{i-1} b_k \bmod n_c)$ .
    - 13. Each facility  $x_i$  distributes its current messages evenly to the clients in the set  $\{y_{c(i)+1}, y_{c(i)+2}, \dots, y_{c(i)+b_i}\}$  (where indexes are reduced modulo  $n_c$  as necessary).
    - 14. Each client  $y_j$  receives at most  $n_f$  messages; the numbers of messages received by any two clients differ by at most one.
    - 15. Each client discards any duplicate messages held.
    - End of Iteration:*
    - 16. At this point, at most  $48n_f$  total messages remain among the  $n_c$  clients; these messages may be distributed evenly to the facilities in  $O(1)$  communication rounds.
    - 17. The  $n_f$  facilities can now broadcast the (at most)  $2n_f$  messages to all clients in  $O(1)$  rounds.
- 

In round 3, a facility has received a collection of up to  $n_c$  messages, some of which may be duplicates of each other. After throwing away all but one copy of any duplicates received, each facility announces to client  $y_1$  the number of (distinct) messages it has remaining. In round 4, client  $y_1$  has received from each facility its number of distinct messages, and computes for each an index (modulo  $n_c$ ) that allows facilities to coordinate their message transfers in the next round. Client  $y_1$  transmits the indices back to the respective facilities in round 5.

In round 6, facilities transfer their messages back across the bipartition to the clients, beginning at their determined index (received from client  $y_1$ ) and working modulo  $n_c$ . This guarantees that the numbers of messages received by two clients  $y_j, y_{j'}$  in this round can differ by no more than one. (Although it is possible that some of these messages will “collapse” as duplicates.) Clients now possess subsets of the original  $n_f$  messages, and the next iteration can begin.

### 3.2 Analysis

Algorithm 3 is proved correct by observing that (i) the algorithm terminates only when dissemination has been completed; and (ii) for a particular message  $m'$ , in any iteration, there is a nonzero probability that all clients holding a copy of  $m'$  will deliver  $m'$  correctly, after which there will never be more than one copy of  $m'$  (until all messages are broadcast to all clients at the end of the algorithm). The running time analysis of Algorithm 3 starts with two lemmas that follow from our choice of the probabilistic hash function.

**Lemma 11** *Suppose that, at the beginning of an iteration, client  $y_j$  possesses a collection  $S_j$  of messages, with  $|S_j| = n_j$ . Let  $E_{i,j}$  be the event that at least one message in  $S_j$  hashes to facility  $x_i$ . Then the probability of  $E_{i,j}$  (conditioned on all previous iterations) is bounded below by  $1 - e^{-n_j/n_f}$ .*

**Proof.** Let  $R_{j,k}$  be the intersection of  $S_j$  with the  $k$ th group of  $\mathcal{U}$ , so that we have the partition  $S_j = R_{j,1} + R_{j,2} + \dots + R_{j,n_f}$ . Let  $E_{i,j,k}$  be the event that some message in  $R_{j,k}$  hashes to  $x_i$ , so that  $E_{i,j} = \bigcup_{k=1}^{n_f} E_{i,j,k}$ . Due to the nature of  $\mathcal{H}_{\mathcal{U}}$  as maps on  $R_{j,k}$  (a uniformly distributed collection of cyclic injections), the probability of  $E_{i,j,k}$ ,  $\mathbf{P}(E_{i,j,k})$ , is precisely  $|R_{j,k}|/n_f$ . Therefore, the probability of the complement of  $E_{i,j,k}$ ,

$\mathbf{P}(\overline{E_{i,j,k}})$ , is  $1 - |R_{j,k}|/n_f$ . Using the inequality  $1 - nx \leq (1 - x)^n$  (for  $x \in [0, 1]$ ), we can bound  $\mathbf{P}(\overline{E_{i,j,k}})$  above by  $(1 - \frac{1}{n_f})^{|R_{j,k}|}$ .

Next, since the actions of  $h$  (from  $\mathcal{H}_{\mathcal{U}}$ ) on each  $R_{j,k}$  are chosen independently, the events  $\{E_{i,j,k}\}_{k=1}^{n_f}$  are (mutually) independent. Therefore, we have

$$\begin{aligned} \mathbf{P}(\overline{E_{i,j}}) &= \mathbf{P}\left(\overline{\bigcup_{k=1}^{n_f} E_{i,j,k}}\right) \\ &= \mathbf{P}\left(\bigcap_{k=1}^{n_f} \overline{E_{i,j,k}}\right) \\ &= \prod_{k=1}^{n_f} \mathbf{P}(\overline{E_{i,j,k}}) \\ &\leq \prod_{k=1}^{n_f} \left(1 - \frac{1}{n_f}\right)^{|R_{j,k}|} \\ &= \left(1 - \frac{1}{n_f}\right)^{|R_{j,1}| + |R_{j,2}| + \dots + |R_{j,n_f}|} \\ &= \left(1 - \frac{1}{n_f}\right)^{n_j} \end{aligned}$$

Using the inequality  $1 + x \leq e^x$  (for all  $x$ ), we can then bound  $\mathbf{P}(\overline{E_{i,j}})$  above by  $(e^{-\frac{1}{n_f}})^{n_j} = e^{-\frac{n_j}{n_f}}$ . Thus we have  $\mathbf{P}(E_{i,j}) \geq 1 - e^{-\frac{n_j}{n_f}}$ .  $\square$

**Lemma 12** *Suppose that, at the beginning of an iteration, client  $y_j$  possesses a collection  $S_j$  of messages, with  $|S_j| = n_j$ . Let  $M_j \subseteq S_j$  be the subset of messages that are correctly delivered by client  $y_j$  in the*

present iteration. Then the expected value of  $|M_j|$  (conditioned on previous iterations) is bounded below by  $n_j - \frac{n_j^2}{2n_f}$ .

**Proof.** Let  $M_{j,i} \subseteq M_j$  be the subset of messages correctly delivered by  $y_j$  to  $x_i$  (in the present iteration), so that we have the partition  $M_j = M_{j,1} + M_{j,2} + \dots + M_{j,n_f}$ . Observe that  $|M_{j,i}|$  is 1 if at least one message in  $S_j$  hashes to  $x_i$ , and 0 otherwise. Therefore  $|M_{j,i}|$  is equal to  $1_{E_{i,j}}$ , the indicator random variable for event  $E_{i,j}$ , and so the expected value of  $|M_{j,i}|$ ,  $\mathbf{P}(|M_{j,i}|)$ , is equal to  $\mathbf{P}(E_{i,j})$ . By linearity of expectation, we have

$$\mathbf{E}(|M_j|) = \sum_{i=1}^{n_f} \mathbf{E}(|M_{j,i}|) = \sum_{i=1}^{n_f} \mathbf{P}(E_{i,j}) \geq n_f \cdot \left(1 - e^{-\frac{n_j}{n_f}}\right)$$

Now, the function  $1 - e^{-x}$  has the (alternating) Taylor series

$$1 - e^{-x} = 1 - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right) = x - \frac{x^2}{2} + \frac{x^3}{6} - \dots$$

Since  $0 \leq n_j \leq n_f$ , with  $x = n_j/n_f$  we have an alternating series with terms of decreasing magnitude; thus we have the lower bound

$$\mathbf{E}(|M_j|) \geq n_f \cdot \left(1 - e^{-\frac{n_j}{n_f}}\right) \geq n_f \cdot \left(\frac{n_j}{n_f} - \frac{n_j^2}{2n_f^2}\right) = n_j - \frac{n_j^2}{2n_f}.$$

□

By Lemma 12, the number of incorrectly delivered messages in  $S_j$  is bounded above (in expectation) by  $\frac{n_j^2}{2n_f}$ . Informally speaking, this implies that the sequence  $n_f, \frac{n_f}{2}, \frac{n_f}{2^3}, \frac{n_f}{2^7}, \dots$  bounds from above the number of incorrectly delivered messages (in expectation) in each iteration. This doubly-exponential rate of decrease in the number of undelivered messages leads to the expected-doubly-logarithmic running time of the algorithm.

We now step out of the context of a single client and consider the progress of the algorithm on the whole. Using Lemma 12, we derive the following recurrence for the expected total number of messages held by all clients at the beginning of each iteration.

**Lemma 13** *Suppose that the algorithm is at the beginning of iteration  $I$ ,  $I \geq 2$ , and let  $T_I$  be the total number of messages held by all clients (i.e.  $T_I = \sum_{j=1}^{n_c} n_j(I)$ , where  $n_j(I)$  is the number of messages held by client  $y_j$  at the beginning of iteration  $I$ ). Then the conditional expectation of  $T_{I+1}$  given  $T_I$ ,  $\mathbf{E}(T_{I+1} | T_I)$ , satisfies*

$$\mathbf{E}(T_{I+1} | T_I) \leq \begin{cases} n_f + \frac{(T_I + n_c)^2}{2n_f \cdot n_c} & \text{if } T_I > n_c \\ n_f + \frac{T_I}{2n_f} & \text{if } T_I \leq n_c \end{cases}$$

**Proof.** Since  $I \geq 2$ , at the beginning of iteration  $I$ , the  $T_I$  messages are evenly spread among all clients; the numbers  $n_j, n_{j'}$  of messages held by two distinct clients  $y_j, y_{j'}$  differ by no more than 1. Therefore,  $n_j \leq T_I/n_c + 1$  for all  $j$ . As well, if  $T_I \leq n_c$ , then  $n_j = 1$  for  $T_I$  values of  $j$ , and 0 otherwise.

The number of messages remaining after iteration  $I$  is bounded above by the number of messages not correctly delivered during iteration  $I$ , plus  $n_f$  (for each collection of identical messages that ‘collapse’ at a given facility  $x_i$ , one such message is kept and passed back to some client). Therefore,  $T_{I+1} \leq$

$n_f + \sum_{j=1}^{n_c} |S_j \setminus M_j|$  (where  $S_j$  is as defined in Lemma 11, and  $M_j$  as in Lemma 12). We then have

$$\begin{aligned}
\mathbf{E}(T_{I+1} \mid T_I) &\leq n_f + \sum_{j=1}^{n_c} \mathbf{E}(|S_j \setminus M_j| \mid T_I) \\
&\leq n_f + \sum_{j=1}^{n_c} \frac{n_j^2}{2n_f} \\
&\leq n_f + \sum_{j=1}^{n_c} \frac{(\frac{T_I}{n_c} + 1)^2}{2n_f} \\
&= n_f + n_c \cdot \frac{(\frac{T_I}{n_c} + 1)^2}{2n_f} \\
&= n_f + \frac{(T_I + n_c)^2}{2n_f \cdot n_c}
\end{aligned}$$

If  $T_I \leq n_c$ , we get also that

$$\begin{aligned}
\mathbf{E}(T_{I+1} \mid T_I) &\leq n_f + \sum_{j=1}^{n_c} \frac{n_j^2}{2n_f} \\
&= n_f + \frac{T_I}{2n_f}
\end{aligned}$$

□

We now define a sequence of variables  $t_i$  (via the recurrence below) that bounds from above the expected behavior of the sequence of  $T_I$ 's established in the previous lemma. Let  $t_1 = n_f \cdot \min\{n_f, n_c\}$ ,  $t_i = \frac{1}{2}t_{i-1}$  for  $2 \leq i \leq 5$ , and for  $i > 5$ , define  $t_i$  by

$$t_i = \begin{cases} 2n_f + \frac{(t_{i-1} + n_c)^2}{n_f \cdot n_c} & \text{if } t_{i-1} > n_c \\ 2n_f + \frac{t_{i-1}}{n_f} & \text{if } t_{i-1} \leq n_c \end{cases}$$

The following lemma establishes that the  $t_i$ 's fall rapidly.

**Lemma 14** *The smallest index  $i$  for which  $t_i \leq 48n_f$  is at most  $\log \log \min\{n_f, n_c\} + 2$ .*

**Proof.** Equivalently, we concern ourselves with  $t'_i = \frac{4t_i}{n_f}$  and determine the number of rounds required before this quantity becomes bounded above by 192.

If  $n_c \leq 48$  or  $n_f \leq 48$ , then  $t'_1 = 4 \min\{n_f, n_c\} \leq 192$  and we are done. So we assume that both  $n_c$  and  $n_f$  are greater than 48. Now rewrite the recursion for  $t_i$  ( $i > 5$ ) as

$$t_i \leq \begin{cases} 2n_f + \frac{(2t_{i-1})^2}{n_f \cdot n_c} & \text{if } t_{i-1} > n_c \\ 2n_f + \frac{t_{i-1}}{n_f} & \text{if } t_{i-1} \leq n_c \end{cases}$$

Correspondingly, write the recurrence for  $t'_i$  as follows.

$$t'_i \leq \begin{cases} 8 + \frac{(t'_{i-1})^2}{n_c} & \text{if } t'_{i-1} > \frac{4n_c}{n_f} \\ 8 + \frac{t'_{i-1}}{n_f} & \text{if } t'_{i-1} \leq \frac{4n_c}{n_f} \end{cases}$$

We now prove the following claim by induction: for each  $i = 5, 6, \dots$ ,  $t'_i \leq \min\{4n_f, n_c/4\}$ . The base case concerns  $t'_5$ . Since  $t'_5 = \frac{1}{4} \min\{n_f, n_c\}$ , the claim is clearly true for  $i = 5$ . Assuming that the claim is true for  $t'_i$ , we now consider  $t'_{i+1}$ . First note that since  $t'_i \leq \min\{4n_f, n_c/4\}$ , it follows that  $(t'_i)^2 \leq n_c \cdot n_f$  and  $(t'_i)^2 \leq n_c^2/16$ . Now we consider the two possible cases of the recursion to get a bound on  $t'_{i+1}$ .

1. If  $t'_i > 4n_c/n_f$ , then  $t'_{i+1} \leq 8 + \frac{n_c n_f}{n_c} = 8 + n_f$ . Since  $n_f \geq 48$ , we have that  $8 + n_f \leq 4n_f$  and therefore  $t'_{i+1} \leq 4n_f$ . Similarly, if  $t'_i > 4n_c/n_f$ , we also have that  $t'_{i+1} \leq 8 + \frac{n_c}{16}$ . Again, since  $n_c \geq 48$ , it follows that  $8 + \frac{n_c}{16} \leq \frac{n_c}{4}$ . Hence,  $t'_{i+1} \leq n_c/4$ .
2. If  $t'_i \leq 4n_c/n_f$ , then  $t'_{i+1} \leq 8 + 4n_f/n_f = 12$ . Since both  $n_f$  and  $n_c$  are greater than 48,  $12 < \min\{4n_f, n_c/4\}$ . Hence,  $t'_{i+1} < \min\{4n_f, n_c/4\}$ .

To finish the proof, we now consider two cases.

**Case 1:**  $n_c \geq n_f^2$ . In this case,  $4n_c/n_f \geq 4n_f$ . According to the inductive claim proved above,  $t'_i \leq 4n_f$  for all  $i = 5, 6, \dots$ . Therefore,  $t'_5 \leq 4n_c/n_f$  and Case 2 of the recurrence applies and yields  $t'_6 \leq 8 + 4n_f/n_f = 12$ , completing the proof.

**Case 2:**  $n_c < n_f^2$ . For notational convenience, let us use  $R$  to denote  $c \cdot \log \log \min\{n_f, n_c\}$ . For  $i = 5, 6, \dots, R-1$ , we assume that  $t'_i > 4n_c/n_f$ . Otherwise, Case 2 of the recurrence applies and  $t'_{i+1} \leq 8 + 4n_f/n_f = 12$  and we are done. Also, for  $i = 5, 6, \dots, R-1$ , we assume that  $(t'_i)^2/n_c > 4$ . Otherwise,  $t'_{i+1} \leq 8 + 4 = 12$  (using Case 1 of the recurrence) and we are done.

Now define  $t''_i = 3t'_i$ ; we bound  $t''_i$  above by a sequence that falls at a double-exponential rate. Given that  $t'_i > 4n_c/n_f$  for  $i = 5, 6, \dots, R-1$ , we see that

$$3t'_{i+1} \leq 24 + \frac{3(t'_i)^2}{n_c}.$$

Furthermore, given that  $(t'_i)^2/n_c > 4$  for  $i = 5, 6, \dots, R-1$ , we see that

$$3t'_{i+1} \leq \frac{3(t'_i)^2}{n_c} + \frac{3(t'_i)^2}{n_c} = \frac{9(t'_i)^2}{n_c} = \frac{(t''_i)^2}{n_c}.$$

Thus  $t''_{i+1} \leq (t''_i)^2/n_c$ . Now,  $t''_5 \leq \frac{3}{4}n_c$ , and so by induction,  $t''_{5+j} \leq (\frac{3}{4})^{2^j} \cdot n_c$ . Thus the smallest  $j$  for which  $t''_{5+j} \leq 192$  is at most  $2 + \log \log n_c$ , which in this case is also  $2 + \log \log \min\{n_f, n_c\}$ .

□

**Lemma 15** *For  $i > 5$ , if  $T_I \leq t_i$ , then the conditional probability (given iterations 1 through  $I-1$ ) of the event that  $T_{I+1} \leq t_{i+1}$  is bounded below by  $\frac{1}{2}$ .*

**Proof.** If  $i > 5$  and  $T_I \leq t_i$ , then by Lemma 13,  $\mathbf{E}(T_{I+1}) \leq \frac{1}{2}t_{i+1}$ . Therefore, by Markov's inequality,  $\mathbf{P}(T_{I+1} > t_{i+1}) \leq \frac{1}{2}$  and  $\mathbf{P}(T_{I+1} \leq t_{i+1}) \geq \frac{1}{2}$ . □

**Theorem 3** *Algorithm 3 solves the dissemination problem in  $O(\log \log \min\{n_f, n_c\})$  rounds in expectation.*

**Proof.** Let  $\tau_i = \min_{\{I: T_I \leq t_{i+1}\}}\{I\} - \min_{\{I: T_I \leq t_i\}}\{I\}$ . Conceptually,  $\tau_i$  is the number of rounds necessary for the total number of messages remaining to decrease from  $t_i$  to  $t_{i+1}$ . Thus, the running time of Algorithm 3 is  $O(1) + \sum_{i=1}^{O(\log \log \min\{n_f, n_c\})} \tau_i$ . By linearity of expectation, the expected running time is then  $O(1) + \sum_{i=1}^{O(\log \log \min\{n_f, n_c\})} \mathbf{E}(\tau_i)$ . By Lemma 15, if  $T_I \leq t_i$ , then regardless of past history, there is at least a probability- $\frac{1}{2}$  chance that  $T_{I+1}$  will be less than  $t_{i+1}$ . It follows that  $\tau_i$  is dominated by an  $\text{Exp}(\frac{1}{2})$  (exponential) random variable, and so  $\mathbf{E}(\tau_i) \leq 2$ . Therefore the expected running time of Algorithm 3 is  $O(\log \log \min\{n_f, n_c\})$ . □

## 4 Computing a 2-Ruling Set of Facilities

In this section, we show how to efficiently compute a 2-ruling set on the graph  $H$  (with vertex set  $\mathcal{F}$ ) constructed in Algorithm 1 (LOCATEFACILITIES). Our algorithm (called FACILITY2RULINGSET and described as Algorithm 4) computes a 2-ruling set in  $H$  by performing *iterations* of a procedure that

combines randomized and deterministic sparsification steps. In each iteration, each facility chooses (independently) to join the *candidate set*  $M$  with probability  $p$ . Two neighbors in  $H$  may both have chosen to join  $M$ , so  $M$  may not be independent in  $H$ . We would therefore like to select an MIS of the graph induced by  $M$ ,  $H[M]$ . In order to do this, the algorithm attempts to communicate all known adjacencies in  $H[M]$  to every client in the network, so that each client may (deterministically) compute the same MIS. The algorithm relies on Algorithm DISSEMINATEADJACENCIES (Algorithm 3) developed in Section 3 to perform this communication.

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**Algorithm 4** FACILITY2RULINGSET

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**Input:** Complete bipartite graph  $G$  with partition  $(\mathcal{F}, \mathcal{C})$  and  $H$ , an overlay network on  $\mathcal{F}$ .

**Output:** A 2-ruling set  $T$  of  $H$

1.  $i := 1; p := p_1 = \frac{1}{8 \cdot n_f^{1/2}}; T := \emptyset$
  2. **while**  $|E(H)| > 0$  **do**  
     *Start of Iteration:*
  3.  $M := \emptyset$
  4. Each facility  $x$  joins  $M$  with a probability  $p$ .
  5. Run Algorithm DISSEMINATEADJACENCIES for  $7 \log \log \min\{n_f, n_c\}$  iterations to communicate the edges in  $H[M]$  to all clients in the network.
  6. **if** Algorithm DISSEMINATEADJACENCIES completes within the allotted number of iterations **then**
  7.     Each client computes the same MIS  $L$  on  $M$  using a deterministic algorithm.
  8.      $T := T \cup L$
  9.     Remove  $M \cup N(M)$  from  $H$ .
  10.      $i := i + 1; p := p_i = \frac{1}{8 \cdot n_f^{2^{-i}}}$
  11. **else**
  12.      $i := i - 1; p := p_i = \frac{1}{8 \cdot n_f^{2^{-i}}}$
  13. **if**  $|E(H)| = 0$  **then break;**  
     *End of Iteration:*
  14. Output  $T$ .
- 

For Algorithm DISSEMINATEADJACENCIES to terminate quickly, we require that the number of edges in  $H[M]$  be  $O(n_f)$ . This requires the probability  $p$  to be chosen carefully as a function of  $n_f$  and the number of edges in  $H$ . Due to the lack of aggregated information, nodes of the network do not generally know the number of edges in  $H$  and thus the choice of  $p$  may be “incorrect” in certain iterations. To deal with the possibility that  $p$  may be too large (and hence  $H[M]$  may have too many edges), the dissemination procedure is not allowed to run indefinitely – rather, it is cut off after  $7 \log \log \min\{n_f, n_c\}$  iterations of disseminating hashing. If dissemination was successful, i.e. the subroutine completed prior to the cutoff, then each client receives complete information about the adjacencies in  $H[M]$ , and thus each is able to compute the same MIS in  $H[M]$ . Also, if dissemination was successful, then  $M$  and its neighborhood,  $N(M)$ , are removed from  $H$  and the next iteration is run with a larger probability  $p$ . On the other hand, if dissemination was unsuccessful, the current iteration of FACILITY2RULINGSET is terminated and the next iteration is run with a smaller probability  $p$  (to make success more likely the next time).

To analyze the progress of the algorithm, we define two notions – *states* and *levels*. For the remainder of this section, we use the term *state* (of the algorithm) to refer to the current probability value  $p$ .

The probability  $p$  can take on values  $\left(\frac{1}{8 \cdot n_f^{2^{-i}}}\right)$  for  $i = 0, 1, \dots, \Theta(\log \log n_f)$ . We use the term *level* to refer to the progress made up until the current iteration. Specifically, the  $j$ th level  $L_j$ , for  $j = 0, 1, \dots, \Theta(\log \log n_f)$ , is defined as having been reached when the number of facility adjacencies remaining in  $H$  becomes less than or equal to  $l_j = 8 \cdot n_f^{1+2^{-j}}$ . In addition, we define one special level  $L_*$  as the level in which no facility adjacencies remain. These values for the states and levels are chosen so that, once level  $L_i$  has been reached, one iteration run in state  $i + 1$  has at least a probability- $\frac{1}{2}$  chance of advancing progress to level  $L_{i+1}$ .

## 4.1 Analysis

It is easy to verify that the set  $T$  computed by Algorithm 4 (FACILITY2RULINGSET) is a 2-ruling set and we now turn our attention to the expected running time of this algorithm. The algorithm halts exactly when level  $L_*$  is reached (this termination condition is detected in Line 15), and so it suffices to bound the expected number of rounds necessary for progress (removal of edges from  $H$ ) to reach level  $L_*$ . The following lemmas show that quick progress is made when the probability  $p$  matches the level of progress made thus far.

**Lemma 16** *Suppose  $|E(H)| \leq l_i$  (progress has reached level  $L_i$ ) and in this situation one iteration is run in state  $i + 1$  (with  $p = p_{i+1}$ ). Then in this iteration, the probability that Algorithm DISSEMINATEADJACENCIES succeeds is at least  $\frac{3}{4}$ .*

**Proof.** Let  $a$  refer to the number of adjacencies (edges) in  $H[M]$ . With  $p = p_{i+1}$ ,  $\mathbf{E}(a) = |E(H)| \cdot p_{i+1}^2 \leq l_i \cdot p_{i+1}^2$ . Plugging the values of  $l_i$  and  $p_{i+1}$  into this bound, we see that  $\mathbf{E}(a) \leq \frac{n_f}{8}$ . By Markov's inequality,  $\mathbf{P}(a > n_f) \leq \frac{\mathbf{E}(a)}{n_f} = \frac{1}{8}$ .

Let  $T_d$  be the number of iterations that dissemination *would* run for if it were allowed to run to completion in this iteration. (Recall that, regardless of  $T_d$ , we always terminate dissemination after  $7 \log \log \min\{n_f, n_c\}$  iterations.) By Theorem 3,  $\mathbf{E}(T_d \mid a \leq n_f) \leq \log \log \min\{n_f, n_c\}$ . Therefore,  $\mathbf{P}(T_d > 7 \log \log \min\{n_f, n_c\} \mid a \leq n_f)$  is bounded above by  $\frac{1}{7}$  (again, using Markov's inequality). If  $E_c$  is the event that  $a > n_f$ , and  $E_T$  is the event that  $T_d > 7 \log \log \min\{n_f, n_c\}$ , then we can bound  $\mathbf{P}(E_T)$  above by

$$\begin{aligned} \mathbf{P}(E_T) &\leq \mathbf{P}(E_c \cup E_T) \\ &= \mathbf{P}(E_c) + \mathbf{P}(E_T \cap \overline{E_c}) \\ &= \mathbf{P}(E_c) + \mathbf{P}(E_T \mid \overline{E_c}) \cdot \mathbf{P}(\overline{E_c}) \\ &\leq \frac{1}{8} + \mathbf{P}(E_T \mid \overline{E_c}) \cdot \frac{7}{8} \\ &\leq \frac{1}{8} + \frac{1}{7} \cdot \frac{7}{8} \\ &= \frac{1}{4} \end{aligned}$$

So with probability at least  $\frac{3}{4}$ , dissemination succeeds (completes in the time allotted).  $\square$

**Lemma 17** *Suppose  $|E(H)| \leq l_i$  (progress has reached level  $L_i$ ). Then, after one iteration run in state  $i + 1$  (with  $p = p_{i+1}$ ), the probability that level  $L_{i+1}$  will be reached (where  $|E(H)| \leq l_{i+1}$ ) is at least  $\frac{1}{2}$ .*

**Proof.** In the present iteration, run with  $p = p_{i+1}$ , we first ignore the success or failure of dissemination (within  $7 \log \log n_f$  iterations of hashing), and assume instead that dissemination runs as long as necessary to succeed. Consider, in this modified scenario, the expected number of edges that will remain in  $H$ . The number of edges can be calculated as twice the sum of degrees, and we can bound the expected degree in  $H$  of a facility  $x$  above by the current degree of  $x$  multiplied by the probability that  $x$  remains active. The probability that  $x$  remains active is at most  $(1 - p_{i+1})^{\deg_H(x)}$  (the probability that no neighbor of  $x$  becomes a candidate). In turn, this quantity is less than or equal to  $e^{-p_{i+1} \cdot \deg_H(x)}$ . Thus, if  $m$  refers to the number of edges remaining in  $H$  after the present iteration (again, with dissemination running to

completion), we have

$$\begin{aligned}
\mathbf{E}(m) &= \frac{1}{2} \sum_{x \in \mathcal{F}} \deg(x) \cdot e^{-p_{i+1} \cdot \deg(x)} \\
&= \frac{1}{2p_{i+1}} \sum_{x \in \mathcal{F}} p_{i+1} \cdot \deg(x) \cdot e^{-p_{i+1} \cdot \deg(x)} \\
&\leq \frac{1}{2p_{i+1}} \sum_{x \in \mathcal{F}} \frac{1}{e} \\
&= \frac{1}{2e \cdot p_{i+1}} \cdot n_f \\
&= \frac{1}{2e} \cdot 8n^{\frac{1}{2^{i+1}}} \cdot n_f \\
&= \frac{1}{2e} \cdot l_{i+1}
\end{aligned}$$

The inequality in the above calculation (Line 3) follows from the fact that  $x \cdot e^{-x} \leq e^{-1}$  for all real  $x$ . Since the unconditional expected value satisfies  $\mathbf{E}(m) \leq \frac{l_{i+1}}{2e}$ , the conditional expectation  $\mathbf{E}(m \mid \overline{E_T})$  is bounded above by  $\frac{4}{3} \cdot \mathbf{E}(m) = \frac{2}{3e} \cdot l_{i+1}$ . Recall the definition of the event  $E_T$  from the proof of Lemma 16. Therefore, using Markov's inequality, the probability that  $m > l_{i+1}$  given  $\overline{E_T}$  is no greater than  $\left(\frac{2l_{i+1}}{3e}\right) / l_{i+1} = \frac{2}{3e} < \frac{1}{3}$ .

Thus we have  $\mathbf{P}(m \leq l_{i+1} \mid \overline{E_T}) > \frac{2}{3}$ , and using  $\mathbf{P}(\overline{E_T}) \geq \frac{3}{4}$  (from Lemma 16),

$$\mathbf{P}(m \leq l_{i+1}) \geq \mathbf{P}(\{m \leq l_{i+1}\} \cap \overline{E_T}) = \mathbf{P}(m \leq l_{i+1} \mid \overline{E_T}) \cdot \mathbf{P}(\overline{E_T}) > \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$$

□

Thus, once level  $L_i$  has been reached, we can expect that only a constant number of iterations run in state  $i + 1$  would be required to reach level  $L_{i+1}$ . Therefore, the question is, ‘‘How many iterations of the algorithm are required to execute state  $i + 1$  enough times?’’ To answer this question, we abstract the algorithm as a stochastic process that can be modeled as a (non-Markov) simple random walk on the integers  $0, 1, 2, \dots, \Theta(\log \log n_f)$  with the extra property that, whenever the random walk arrives at state  $i + 1$ , a (fair) coin is flipped. We place a bound on the expected number of steps before this coin toss comes up heads.

First, consider the return time to state  $i + 1$ . In order to prove that the expected number of iterations (steps) necessary before either  $|E(H)| \leq l_{i+1}$  or  $p = p_{i+1}$  is  $O(\log \log n_f)$ , we consider two regimes –  $p > p_{i+1}$  and  $p < p_{i+1}$ . When  $p$  is large (in the regime consisting of probability states intended for fewer edges than currently remain in  $H$ ), it is likely that a single iteration of Algorithm 4 will generate a large number of adjacencies between candidate facilities. Thus, dissemination will likely not complete before ‘‘timing out,’’ and it is likely that  $p$  will be decreased prior to the next iteration. Conversely, when  $p$  is small (in the regime consisting of probability states intended for more edges than currently remain in  $H$ ), a single iteration of Algorithm 4 will likely generate fewer than  $n_f$  adjacencies between candidate facilities, and thus it is likely that dissemination will complete before ‘‘timing out.’’ In this case,  $p$  will advance prior to the next iteration. This analysis is accomplished in the following lemmas and leads to the subsequent theorem.

**Lemma 18** *Consider a simple random walk on the integers  $[0, i]$  with transition probabilities  $\{p_{j,k}\}$  satisfying  $p_{j,j+1} = \frac{3}{4}$  ( $j = 0, \dots, i - 1$ ),  $p_{j,j-1} = \frac{1}{4}$ , ( $j = 1, \dots, i$ ),  $p_{i,i} = \frac{3}{4}$ , and  $p_{0,0} = \frac{1}{4}$ . For such a random walk beginning at 0, the expected hitting time of  $i$  is  $O(i)$ .*

**Proof.** This is an exercise in probability; see [15].

□

**Lemma 19** *When  $j \leq i$ , the expected number of iterations required before returning to state  $i + 1$  is  $O(\log \log n_f)$ .*

**Proof.** By Lemma 18, it suffices to show that when  $j < i$ , the probability of successful dissemination in state  $j$  is at least  $\frac{3}{4}$ . By the proof of Lemma 16, this would be true were the current iteration run with  $p = p_i$ . Since  $p_j < p_i$ , the probability of successful dissemination is greater in state  $j$  than in state  $i$ , and the lemma follows.  $\square$

**Lemma 20** *When  $j > i$ , the expected number of iterations required before returning to state  $i + 1$  or advancing to at least level  $L_{i+1}$  is  $O(\log \log n_f)$ .*

**Proof.** By Lemma 18, it suffices to show that when  $j > i$ , the probability of either unsuccessful dissemination in state  $j$  or progression to level  $L_j$  is at least  $\frac{3}{4}$ . Therefore, consider the modified scenario where dissemination is always run to completion; we will show that the probability of progression to level  $L_j$  in this scenario is at least  $\frac{3}{4}$ .

Recall from the proof of Lemma 17 that, if  $m$  refers to the number of edges remaining in  $H$  after the present iteration (with dissemination run to completion), we have

$$\begin{aligned}
\mathbf{E}(m) &= \frac{1}{2} \sum_{x \in \mathcal{F}} \deg(x) \cdot e^{-p_j \cdot \deg(x)} \\
&= \frac{1}{2p_j} \sum_{x \in \mathcal{F}} p_j \cdot \deg(x) \cdot e^{-p_j \cdot \deg(x)} \\
&= \frac{1}{2p_j} \sum_{x \in \mathcal{F}} \frac{1}{e} \\
&= \frac{1}{2e \cdot p_j} \cdot n_f \\
&= \frac{1}{2e} \cdot 8n^{\frac{1}{2^j}} \cdot n_f \\
&= \frac{1}{2e} \cdot l_j
\end{aligned}$$

Thus, by Markov's inequality,  $\mathbf{P}(m > l_j) \leq \frac{1}{2e} < \frac{1}{4}$ , and so the probability of progression to level  $L_j$  (when dissemination is allowed to run to completion) is at least  $\frac{3}{4}$ , which completes the proof of the lemma.  $\square$

**Lemma 21** *Suppose that Algorithm 4 has reached level  $L_i$ , and let  $T_{i+1}$  be a random variable representing the number of iterations necessary before reaching level  $L_{i+1}$ . Then  $\mathbf{E}(T_{i+1}) = O(\log \log n_f)$ .*

**Proof.** Fix  $i$  and let  $N$  be the number of returns to state  $i + 1$  prior to progressing to level  $L_{i+1}$ . Let  $S_k$  be the number of iterations run between return  $k - 1$  and  $k$  to state  $i + 1$ , so that  $T_{i+1} = \sum_{k=1}^N S_k$ . In the random walk abstraction of the algorithm,  $N$  depends only on a series of coin flips which are themselves independent of all other history. Each  $S_k$  is also independent of any coin flip and so also of  $N$  (again we emphasize that this is only true when abstracting the algorithm to a random walk).

Now, by conditioning on  $N = t$ , we see that

$$\begin{aligned}
\mathbf{E}(T_{i+1}) &= \sum_{t=0}^{\infty} \mathbf{E}(T_{i+1} \mid N = t) \\
&= \sum_{t=0}^{\infty} \mathbf{E}\left(\sum_{k=1}^N S_k \mid N = t\right) \cdot \mathbf{P}(N = t) \\
&= \sum_{t=0}^{\infty} \mathbf{E}\left(\sum_{k=1}^t S_k \mid N = t\right) \cdot \mathbf{P}(N = t) \\
&= \sum_{t=0}^{\infty} \mathbf{E}\left(\sum_{k=1}^t S_k\right) \cdot \mathbf{P}(N = t) \\
&= \sum_{t=0}^{\infty} \sum_{k=1}^t \mathbf{E}(S_k) \cdot \mathbf{P}(N = t) \\
&\leq \sum_{t=0}^{\infty} \sum_{k=1}^t O(\log \log n_f) \cdot \mathbf{P}(N = t) \\
&= \sum_{t=0}^{\infty} t \cdot O(\log \log n_f) \cdot \mathbf{P}(N = t) \\
&\leq \sum_{t=0}^{\infty} t \cdot O(\log \log n_f) \cdot \left(\frac{1}{2}\right)^t \\
&= O(\log \log n_f) \cdot \sum_{t=0}^{\infty} t \cdot \left(\frac{1}{2}\right)^t \\
&= O(\log \log n_f) \cdot O(1) \\
&= O(\log \log n_f)
\end{aligned}$$

□

**Theorem 4** *Algorithm 4 has an expected running time of  $O((\log \log n_f)^2 \cdot \log \log \min\{n_f, n_c\})$  rounds in the `CONGEST` model.*

**Proof.** Once a certain level  $L_i$  has been reached, the expected time for Algorithm `FACILITY2RULINGSET` to reach level  $L_{i+1}$  is  $O((\log \log n_f) \cdot \log \log \min\{n_f, n_c\})$  (where the factor of  $O(\log \log \min\{n_f, n_c\})$  is the upper bound on the running time of the dissemination subroutine). Since there are  $O(\log \log n_f)$  levels to progress through before reaching  $L_*$  and terminating, the algorithm has an expected running time of  $O((\log \log n_f)^2 \cdot \log \log \min\{n_f, n_c\})$  rounds. □

## 5 Concluding Remarks

Our expectation is that the Message Dissemination with Duplicates (MDD) problem and its solution via probabilistic hashing will have applications in other distributed algorithms in low-diameter settings. This problem may also serve as a candidate for lower bounds research. In particular, the results in this paper raise the question of whether  $\Omega(\log \log \min\{n_c, n_f\})$  is a lower bound on the number of rounds it takes to solve MDD. Alternately, it will be interesting (and surprising) to us if MDD was solved in  $O(1)$  rounds.

Our two papers (the current paper and [3]) on super-fast algorithms yielding  $O(1)$ -approximation for metric facility location, lead naturally to similar questions for the non-metric version of the problem. In particular, we are interested in super-fast algorithms, hopefully running in  $O(\text{poly}(\log \log n))$  rounds, that

yield a logarithmic-approximation to the non-metric facility location problem on cliques and complete bipartite networks.

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