Trajectory-Based Dynamic Map Labeling

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Abstract. In this paper we introduce *trajectory-based labeling*, a new variant of dynamic map labeling, where a movement trajectory for the map viewport is given. We define a general labeling model and study the active range maximization problem in this model. The problem is \mathcal{NP} -complete and $\mathcal{W}[1]$ -hard. In the restricted, yet practically relevant case that no more than k labels can be active at any time, we give polynomial-time algorithms. For the general case we present a practical ILP formulation with an experimental evaluation as well as approximation algorithms.

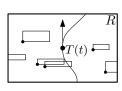
1 Introduction

In contrast to traditional static maps, dynamic digital maps support continuous movement of the map viewport based on panning, rotation, or zooming. Creating smooth visualizations under such map dynamics induces challenging geometric problems, e.g., continuous generalization [11] or dynamic map labeling [2]. In this paper, we focus on map labeling and take a trajectory-based view on it. In many applications, e.g., car navigation, a movement trajectory is known in advance and it becomes interesting to optimize the visualization of the map locally along this trajectory.

Selecting and placing a maximum number of non-overlapping labels for various map features is an important cartographic problem. Labels are usually modeled as rectangles and a typical objective in a static map is to find a maximum (possibly weighted) independent set of labels. This is known to be \mathcal{NP} -complete [6]. There are several approximation algorithms and PTAS's in different labeling models [1, 5], as well as practically useful heuristics [12, 13].

With the increasing popularity of interactive dynamic maps, e.g., as digital globes or on mobile devices, the static labeling problem has been translated into a dynamic setting. Due to the temporal dimension of the animations occurring during map movement, it is necessary to define a notion of *temporal consistency* or *coherence* for map labeling as to avoid distracting effects such as jumping or flickering labels [2]. Previously, consistent labeling has been studied from a global perspective under continuous zooming [3] and continuous rotation [7]. In practice, however, an individual map user with a mobile device, e.g., a tourist or a car driver, is typically interested only in a specific part of a map and it is thus often more important to optimize the labeling locally for a certain trajectory of the map viewport than globally for the whole map.

We introduce a versatile trajectory-based model for dynamic map labeling, and define three label activity models that guarantee concistency. We apply this model to point feature labeling for a viewport that moves and rotates along a differentiable trajectory in a fixed-scale base map in a forward-facing way. Although we present our approach in



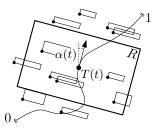


Fig. 1: Illustration of the viewport moving along a trajectory. Left the user's view and right a general view of the map and the viewport.

a very specific problem setting, our model is very general. Our approach can be applied for every dynamic labeling problem that can be expressed as a set of label availability intervals over time and a set of conflict intervals over time for pairs of labels. The exact algorithms hold for the general model, the approximation algorithm itself is also applicable, but the analysis of the approximation ratio requires problem-specific geometric arguments, which must be adjusted to the specific setting.

Contribution. For our specific problem, we show that maximizing the number of visible labels integrated over time in our model is \mathcal{NP} -complete; in fact it is even $\mathcal{W}[1]$ -hard and thus it is unlikely that a fixed-parameter tractable algorithm exists. We present an integer linear programming (ILP) formulation for the general unrestricted case, which is supported by a short experimental evaluation. For the special case of unit-square labels we give an efficient approximation algorithm with different approximation ratios depending on the actual label activity model. Moreover, we present polynomial-time algorithms for the restricted case that no more than k labels are active at any time for some constant k. We note that limiting the number of simultaneously active labels is of practical interest as to avoid overly dense labelings, in particular for dynamic maps on small-screen devices such as in car navigation systems.

2 Trajectory-Based Labeling Model

Let M be a labeled north-facing, fixed-scale map, i.e., a set of points $P=\{p_1,\ldots,p_N\}$ in the plane together with a corresponding set $L=\{\ell_1,\ldots,\ell_N\}$ of labels. Each label ℓ_i is represented by an axis-aligned rectangle of individual width and height. We call the point p_i the *anchor* of the label ℓ_i . Here we assume that each label has an arbitrary but fixed position relative to its anchor, e.g., with its lower left corner coinciding with the anchor. The *viewport* R is an arbitrarily oriented rectangle of fixed size that defines the currently visible part of M on the map screen. The viewport follows a trajectory that is given by a continuous differentiable function $T\colon [0,1]\to \mathbb{R}^2$. For an example see Fig. 1. More precisely, we describe the viewport by a function $V\colon [0,1]\to \mathbb{R}^2\times [0,2\pi]$. The interpretation of $V(t)=(c,\alpha)$ is that at time t the center of the rectangle R is located at c and R is rotated clockwise by the angle α relatively to a north base line of the map. Since R moves along T we define $V(t)=(T(t),\alpha(t))$, where $\alpha(t)$ denotes the direction of T at time t. For simplicity, we sometimes refer to R at time t as V(t). To

ensure good readability, we require that the labels are always aligned with the viewport axes as the viewport changes its orientation, i.e., they rotate around their anchors by the same angle $\alpha(t)$, see Fig. 1. We denote the rotated label rectangle of ℓ at time t by $\ell(t)$.

We say that a label ℓ is present at time t, if $V(t) \cap \ell(t) \neq \emptyset$. As we consider the rectangles $\ell(t)$ and V(t) to be closed, we can describe the points in time for which ℓ is present by closed intervals. We define for each label ℓ the set Ψ_{ℓ} that describes all disjoint subintervals of [0,1] for which ℓ is present, thus $\Psi_{\ell} = \{[a,b] \mid [a,b] \subseteq [0,1]$ is maximal so that ℓ is present at all $t \in [a,b]\}$. Further, we define the disjoint union $\Psi = \{([a,b],\ell) \mid [a,b] \in \Psi_{\ell} \text{ and } \ell \in L\}$ of all Ψ_{ℓ} . We abbreviate $([a,b],\ell) \in \Psi$ by $[a,b]_{\ell}$ and call $[a,b]_{\ell} \in \Psi$ a presence interval of ℓ . In the remainder of this paper we denote the number of presence intervals by n.

Two labels ℓ and ℓ' are in *conflict* with each other at time t if $\ell(t) \cap \ell'(t) \neq \emptyset$. If $\ell(t) \cap \ell'(t) \cap V(t) \neq \emptyset$ we say that the conflict is *present* at time t. As in [7] we can describe the occurrences of conflicts between two labels $\ell, \ell' \in L$ by a set of closed intervals: $C_{\ell,\ell'} = \{[a,b] \subseteq [0,1] \mid [a,b] \text{ is maximal and } \ell \text{ and } \ell' \text{ are in conflict at all } t \in [a,b]\}$. We define the disjoint union $C = \{([a,b],\ell,\ell') \mid [a,b] \in C_{\ell,\ell'} \text{ and } \ell,\ell' \in L\}$ of all $C_{\ell,\ell'}$. We abbreviate $([a,b],\ell,\ell') \in C$ as $[a,b]_{\ell,\ell'}$ and call it a *conflict interval* of ℓ and ℓ' . Two presence intervals $[a,b]_{\ell}$ and $[c,d]_{\ell'}$ are in conflict if there is a conflict $[f,g]_{\ell,\ell'} \in C$ s.t. the intersection of the intervals $[f,g]_{\ell,\ell'} \cap [a,b]_{\ell} \cap [c,d]_{\ell'} \neq \emptyset$.

The tuple (P, L, Ψ, C) is called an *instance* of trajectory-based labeling. Note that the essential information of T is implicitly given by Ψ and C and that for each label $\ell \in L$ there can be several presence intervals. In this paper we assume that Ψ and C is given as input. In practice, however, we usually first need to compute Ψ and C given a continuous and differentiable trajectory T. An interesting special case is that T is a continuous, differentiable chain of m circular arcs (possibly of infinite radius), e.g., obtained by approximating a polygonal route in a road network. Niedermann [10] showed that in this case the set Ψ can be computed in $O(m \cdot N)$ time and the set C in $O(m \cdot N^2)$ time. His main observation was that for each arc of T the viewport can in fact be treated as a huge label and that "conflicts" with the viewport correspond to presence intervals. We refer to [10, Chapter 15] for details.

Next we define the *activity* of labels, i.e., when to actually display which of the present labels on screen. We restrict ourselves to closed and disjoint intervals describing the activity of a label ℓ and define the set $\Phi_\ell = \{[a,b] \subseteq [0,1] \mid [a,b] \text{ is maximal such that } \ell \text{ is active at all } t \in [a,b]\}$, as well as the disjoint union $\Phi = \{([a,b],\ell) \mid [a,b] \in \Phi_\ell \text{ and } \ell \in L\}$ of all Φ_ℓ . We abbreviate $([a,b],\ell) \in \Phi$ with $[a,b]_\ell$ and call $[a,b]_\ell \in \Phi$ an *active interval* of ℓ .

It remains to define an *activity model* restricting Φ in order to obtain a reasonable labeling. Here we propose three activity models AM1, AM2, AM3 with increasing flexibility. All three activity models exclude overlaps of displayed labels and guarantee consistency criteria introduced by Been et al. [2], i.e., labels must not flicker or jump. To that end they share the following properties (**A**) a label ℓ can only be active at time t if it is present at time t, (**B**) to avoid flickering and jumping each presence interval of ℓ contains at most one active interval of ℓ , and (**C**) if two labels are in conflict at a time t, then at most one of them may be active at t to avoid overlapping labels.

What distinguishes the three models are the possible points in time when labels can become active or inactive. The first and most restrictive activity model AM1 demands that each activity interval $[a,b]_\ell$ of a label ℓ must coincide with a presence interval of ℓ . The second activity model AM2 allows an active interval of a label ℓ to end earlier than the corresponding presence interval if there is a *witness label* ℓ' for that, i.e., an active interval for ℓ may end at time ℓ if there is a starting conflict interval $[\ell,d]_{\ell,\ell'}$ and the conflicting label ℓ' is active at ℓ . However, AM2 still requires every active interval to begin with the corresponding presence interval. The third activity model AM3 extends AM2 by also relaxing the restriction regarding the start of active intervals. An active interval for a label ℓ may start at time ℓ if a present conflict $[\ell,\ell]_{\ell,\ell'}$ involving ℓ and an active witness label ℓ' ends at time ℓ . In this model active intervals may begin later and end earlier than their corresponding presence intervals if there is a visible reason for the map user to do so, namely the start or end of a conflict with an active witness label.

A common objective in both static and dynamic map labeling is to maximize the number of labeled points. Often, however, certain labels are more important than others. To account for this, each label ℓ can be assigned a weight W_ℓ that corresponds to its significance. Then we define the weight of an interval $[a,b]_\ell \in \Phi$ as $w([a,b]_\ell) = (b-a)\cdot W_\ell$. Given an instance (P,L,Ψ,C) , then with respect to one of the three activity models we want to find an activity Φ that maximizes $\sum_{[a,b]_\ell \in \Phi} w([a,b]_\ell)$; we call this optimization problem General Maxtotal. If we require that at any time t at most k labels are active for some k, we call the problem k-Restricted Maxtotal. In particular the latter problem is interesting for small-screen devices, e.g., car navigation systems, that should not overwhelm the user with additional information.

3 Solving GENERALMAXTOTAL

We first prove that GENERALMAXTOTAL is $\mathcal{N}P$ -complete. The membership of GENERALMAXTOTAL in $\mathcal{N}P$ follows from the fact that the start and the end of an active interval must coincide with the start or end of a presence interval or a conflict interval. Thus, there is a finite number of candidates for the endpoints of the active intervals so that a solution \mathcal{L} can be guessed. Verifying that \mathcal{L} is valid in one of the three models and that its value exceeds a given threshold can obviously be checked in polynomial time.

For the $\mathcal{N}P$ -hardness we apply a straight-forward reduction from the $\mathcal{N}P$ -complete maximum independent set of rectangles problem [6]. We simply interpret the set of rectangles as a set of labels with unit weight, choose a short vertical trajectory T and a viewport R that contains all labels at any point of T. Since the conflicts do no change over time, the reduction can be used for all three activity models. By means of the same reduction and Marx' result [9] that finding an independent set for a given set of axis-parallel unit squares is $\mathcal{W}[1]$ -hard we derive the next theorem.

Theorem 1. GENERALMAXTOTAL is \mathcal{NP} -complete and $\mathcal{W}[1]$ -hard for all activity models AMI-AM3.

As a consequence, GENERALMAXTOTAL is not fixed-parameter tractable unless $W[1] = \mathcal{FPT}$. Note that this also means that for k-RestrictedMaxTotal we cannot

expect to find an algorithm that runs in $O(p(n) \cdot C(k))$ time, where p(n) is a polynomial that depends only on the number n of presence intervals and the computable function C(k) depends only on the parameter k.

3.1 Integer Linear Programming for GENERALMAXTOTAL

Since we are still interested in finding an optimal solution for GENERALMAXTOTAL we have developed integer linear programming (ILP) formulations for all three activity models. We present the formulation for the most involved model AM3 and then argue how to adapt it to the simpler models AM1 and AM2.

We define E to be the totally ordered set of the endpoints of all presence and all conflict intervals and include 0 and 1; see Fig. 2. We call each interval [c, d] between two consecutive elements c and d in E an atomic seg*ment* and denote the i-th atomic segment of Eby E(i). Further, let $X(\ell, i)$ be the set of labels that are in conflict with ℓ during E(i-1), but not during E(i), i.e., the conflicts end with E(i-1). Analogously, let $Y(\ell, i)$ be the

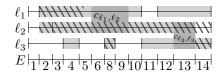


Fig. 2: Depiction of presence intervals (light gray), active intervals (hatched), and conflicts (dark gray).

set of labels that are in conflict with ℓ during E(i+1), but not during E(i), i.e., the conflicts begin with E(i + 1). For each label ℓ we introduce three binary variables $b_i, x_i, e_i \in \{0, 1\}$ and the following constraints.

$$b_i^{\ell} = x_i^{\ell} = e_i^{\ell} = 0 \qquad \forall 1 \le i \le |E| \text{ s.t. } \forall [c, d] \in \Psi_{\ell} : E(i) \cap [c, d] = \emptyset$$
 (1)

$$b_i^\ell = x_i^\ell = e_i^\ell = 0 \qquad \forall 1 \le i \le |E| \text{ s.t. } \forall [c,d] \in \Psi_\ell : E(i) \cap [c,d] = \emptyset \qquad (1)$$

$$\sum_{j \in J} b_j^\ell \le 1 \text{ and } \sum_{j \in J} e_j^\ell \le 1 \qquad \qquad \forall [c,d] \in \Psi_\ell \text{ where } J = \{j \mid E(j) \subseteq [c,d]\} \qquad (2)$$

$$x_i^{\ell} + x_i^{\ell'} \le 1 \qquad \qquad \forall 1 \le i \le |E| \ \forall [c, d]_{\ell, \ell'} \in C : \ E(i) \subseteq [c, d] \qquad (3)$$

$$x_{i-1}^{\ell} + b_i^{\ell} = x_i^{\ell} + e_{i-1}^{\ell} \qquad \forall 1 \le i \le |E| \text{ (set } x_0 = e_0 = 0\text{)}$$
 (4)

$$x_i^{\ell} + x_i^{\ell'} \leq 1 \qquad \qquad \forall 1 \leq i \leq |E| \ \forall [c,d]_{\ell,\ell'} \in C : \ E(i) \subseteq [c,d] \qquad (3)$$

$$x_{i-1}^{\ell} + b_i^{\ell} = x_i^{\ell} + e_{i-1}^{\ell} \qquad \qquad \forall 1 \leq i \leq |E| \ (\text{set } x_0 = e_0 = 0) \qquad (4)$$

$$b_j^{\ell} \leq \sum_{\ell' \in X(\ell,j)} x_{j-1}^{\ell'} \qquad \qquad \forall [c,d]_{\ell} \in \varPsi \ \forall E(j) \subset [c,d]_{\ell} \ \text{with} \ c \not\in E(j) \qquad (5)$$

$$e_j^{\ell} \le \sum_{\ell' \in Y(\ell,j)} x_{j+1}^{\ell'} \qquad \forall [c,d]_{\ell} \in \Psi \, \forall E(j) \subset [c,d]_{\ell} \text{ with } d \notin E(j) \quad (6)$$

Subject to these constraints we maximize $\sum_{\ell \in L} \sum_{i=1}^{|E|-1} x_i^{\ell} \cdot w(E(i))$. The intended meaning of the variables is that $x_i^{\ell} = 1$ if ℓ is active during E(i) and otherwise $x_i^{\ell} = 0$. Variable $b_i^{\ell} = 1$ if and only if E(i) is the first atomic segment of an active interval of ℓ , and analogously $e_i^{\ell}=1$ if and only if E(i) is the last atomic segment of an active interval of ℓ . Recall the properties of the activity models as defined in Section 2. Constraints (1)–(3) immediately ensure properties (A)–(C), respectively. Constraint (4) means that if ℓ is active during E(i-1) ($x_{i-1}^{\ell}=1$), then it must either stay active during E(i) $(x_i^{\ell}=1)$ or the active interval ends with E(i-1) $(e_{i-1}^{\ell}=1)$, and if ℓ is active during E(i) ($x_i^{\ell} = 1$) then it must be active during E(i-1) ($x_{i-1}^{\ell} = 1$) or the active interval begins with E(i) ($b_i^{\ell}=1$). Constraint (5) enforces that for ℓ to become active with E(j) at least one witness label of $X(\ell, j)$ is active during E(j-1). Analogously, constraint (6) enforces that for ℓ to become inactive with E(j) at least

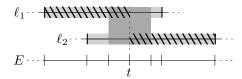


Fig. 3: The light gray intervals show presence intervals, the hatched intervals active intervals and the dark gray intervals conflicts between labels. Further, the bottom line illustrates possible atomic segments, when assuming that there is a third label that induces the segmentation at time t.

one witness label of $Y(\ell,j)$ is active during E(j+1). Note that without the explicit constraints (5) and (6) two conflicting labels could switch activity at any point during the conflict interval rather than only at the endpoints. For an example see Fig. 3. The drawing shows an optimal solution that is valid for the ILP formulation if the constraints (5) and (6) are omitted. In particular ℓ_1 becomes inactive at time t, although t is not the right boundary of the corresponding presence interval and there is no conflict of ℓ_1 that begins at t such that the corresponding opponent is active from t on. Analogous observations can be made for ℓ_2 . Consequently, this solution does not satisfy AM3.

Theorem 2. Given an instance $I = (P, L, \Psi, C)$, the ILP (1)–(6) computes an optimal solution Φ of GENERALMAXTOTAL in AM3. It uses $O(N \cdot (|\Psi| + |C|))$ variables and constraints.

Proof. Every solution of the ILP corresponds to an activity Φ by defining for every label ℓ the set Φ_ℓ as the set of all maximal intervals in $\bigcup_{i:x_i^\ell=1} E(i)$. Conversely, every valid activity Φ in AM3 can be expressed in terms of the variables of the ILP. To show that we first observe that for every valid activity interval $[a,b]_\ell$ in AM3 the endpoints a and b are necessarily endpoints of a conflict interval or a presence interval of ℓ . Thus $[a,b]_\ell$ can be expressed as the union of consecutive atomic segments represented by the variables x_i^ℓ .

It is clear that the objective function computes the weight of a solution Φ correctly. Thus it remains to show that the constraints (1)–(6) indeed model AM3, i.e., every solution of the ILP satisfies AM3 and every activity in AM3 is a solution of the ILP. It follows immediately from the definition of constraints (1)–(3) that they model properties (A)-(C), assuming that the start- and endpoint of every activity interval is indeed marked by setting $b_i^\ell=1$ and $e_i^\ell=1$ for its first and last atomic segments E(i) and E(j). But this is achieved by the constraints (4) as discussed above. Now in AM3 a label can only become active (inactive) at the start (end) of its presence interval or at the end (start) of a conflict interval if the conflicting label is active as a witness. We show that constraint (5) yields that the start of an activity interval is correct according to AM3. The argument for the end of an activity interval follows analogously from constraint (6). Let E(i) be the first atomic segment in an activity interval of the label ℓ . Then by constraint (4) we have $b_i^{\ell} = 1$ and $x_i^{\ell} = 1$. If E(i) is the first segment of a presence interval then this is a valid start according to AM3. Note that constraint (5) is not present in that case and thus does not restrict b_i^{ℓ} . Otherwise let E(i) be not the first segment of a presence interval. Then for this segment the ILP contains constraint (5). If

no conflict interval of ℓ ends with E(i-1) then (5) sets $b_i^\ell=0$ anyways, so this is not possible. If some conflict intervals of ℓ end with E(i-1) but none of them are active in E(i-1) then constraint (5) also yields $b_i^\ell=0$. So the only two possibilities for $b_i^\ell=1$ are that either E(i) is the first segment of a presence interval or E(i) is the first segment after a conflict interval of ℓ for which a witness label is active. Thus every solution of the ILP satisfies AM3.

Conversely, let Φ be valid according to AM3. Since Φ satisfies properties (A)–(C), the corresponding assignment of binary values to the variables x_i^ℓ , b_i^ℓ , and e_i^ℓ satisfy constraints (1)–(4). It remains to show that the constraints (5) and (6) hold. Let $[a,b]_\ell \in \Phi$ be a particular activity interval and let E(i) be the atomic segment starting at a. If a is the start of a presence interval of ℓ then there is no constraint (5) for ℓ and the segment E(i) and thus it is possible to have $b_i^\ell=1$. Otherwise, a is the end of a conflict interval of ℓ with another label ℓ' that is an active witness in the atomic segment E(i-1) ending at a. This means that $x_{i-1}^{\ell'}=1$ and thus constraint (5) is satisfied for $b_i^\ell=1$. Analogous reasoning for the endpoints of all activity intervals and constraint (6) yield that Φ can indeed be represented as a solution to the ILP.

Since the number of atomic segments is $O(|\Psi| + |C|)$ and there are N labels the bound on the size of the ILP follows.

We can adapt the above ILP to AM1 and AM2 as follows. For AM2 we replace the right hand side of constraint (5) by 0, and for AM1 we also replace the right hand side of constraint (6) by 0. This excludes exactly the start- and endpoints of the activity intervals that are forbidden in AM1 or AM2. It is easy to see that these ILP formulations can be modified further to solve k-RESTRICTEDMAXTOTAL by adding the constraint $\sum_{\ell \in L} x_i^\ell \leq k$ for each atomic segment E(i).

Corollary 1. Given an instance $I = (P, L, \Psi, C)$, GeneralmaxTotal and k-RestrictedmaxTotal can be solved in AM1, AM2, and AM3 by an ILP that uses $O(N \cdot (|\Psi| + |C|))$ variables and constraints.

3.2 Experiments.

We have evaluated the ILP in all three models using Open Street Map data of the city center of Karlsruhe (Germany) which contains more than 2,000 labels. To this end we generated 1,000 shortest paths on the road network of Karlsruhe by selecting source and target vertices uniformly at random and transformed those shortest paths into trajectories consisting of circular arcs. We fixed the viewport's size to that of a typical mobile device (640×480 pixels) and considered the map scales 1:2000, 1:3000, and 1:4000, which corresponds to areas with dimensions $339m \times 254m$, $508m \times 381m$ and $678m \times 508m$, respectively. The experiments were performed on a single core of an AMD Opteron 6172 processor running Linux 3.4.11. The machine is clocked at 2.1 Ghz, and has 256 GiB RAM. Our implementation is written in C++, uses Gurobi 5.1. as ILP solver, and was compiled with GCC 4.7.1 using optimization -03.

For plots and a table depicting the results of the experimental evaluation see Fig. 4. We observe that for a scale factor of 1:2000 the running times for the vast majority of instances remained below one second, while no instances required more than ten

seconds to be solved. Since for the scale factors 1:3000 and 1:4000 the density of labels increases, the running times increase, too. Still 75% of the instances were solved in less than three seconds.

It is remarkable that for the scale factor 1:2000 over 99% of the instances and for the scale factor 1:3000 over 75% of the instances can be solved in less than a second, while for the scale factor 1:4000 over 75% of the instances can still be solved in less than three seconds. However, for 1:3000 there are runs that needed almost 50 seconds and for 1:4000 there are runs that needed almost 475 seconds. Note that due to these outliers, for a scale factor of 1:4000 the average running time lies above the third quartile, but still does not exceed six seconds. Two instances for 1:4000 exceeded a timeout of 600 seconds, were aborted and not included in the analysis. For the scale factors 1:2000 and 1:3000 the average running time is less than one second. Considering the same scale factor the three models do not differ much from each other, except for some outliers. As the number of labels and conflicts to be considered depends on the applied scale factor and the concrete trajectory the table summarizes the number of considered labels and conflicts in maximum and average over all trajectories. As to be expected for a scale factor of 1:4000 the number of considered conflicts is significantly greater than for a scale factor of 1:2000. This also explains the different running times.

In conclusion, our brief evaluation indicates that the ILP formulations are indeed applicable in practice.

3.3 Approximation of GENERALMAXTOTAL

In this section we describe a simple greedy algorithm for GENERALMAXTOTAL in all three activity models assuming that all labels are unit squares anchored at their lower-left corner. Further, we assume that the weight of each presence interval $[a,b]_\ell$ is its length $w([a,b]_\ell)=b-a$.

Starting with an empty solution Φ , our algorithm GREEDYMAXTOTAL removes the longest interval I from Ψ and adds it to Φ , i.e., I is set active. Then, depending on the activity model, it updates all presence intervals that have a conflict with I in Ψ and continues until the set Ψ is empty.

For AM1 the update process simply removes all presence intervals from Ψ that are in conflict with the newly selected interval I. For AM2 and AM3 let $I_j \in \Psi$ and let I_j^1, \ldots, I_j^k be the longest disjoint sub-intervals of I_j that are not in conflict with the selected interval I. We assume that I_j^1, \ldots, I_j^k are sorted by their left endpoint. The update operation for AM2 replaces every interval $I_j \in \Psi$ that is in conflict with I with I_j^1 . In AM3 we replace I_j by I_j^1 , if I_j^1 is not fully contained in I. Otherwise, I_j is replaced by I_j^k . Note that this discards some candidate intervals, but the chosen replacement of I_j is enough to prove the approximation factor. Note that after each update all intervals in Ψ are valid choices according to the specific model. Hence, we can conclude that the result Φ of GREEDYMAXTOTAL is also valid in that model.

In the following we analyze the approximation quality of GREEDYMAXTOTAL. To that end we first introduce a purely geometric packing lemma. Similar packing lemmas have been introduced before, but to the best of our knowledge for none of them it is sufficient that only one prescribed corner of the packed objects lies within the container.

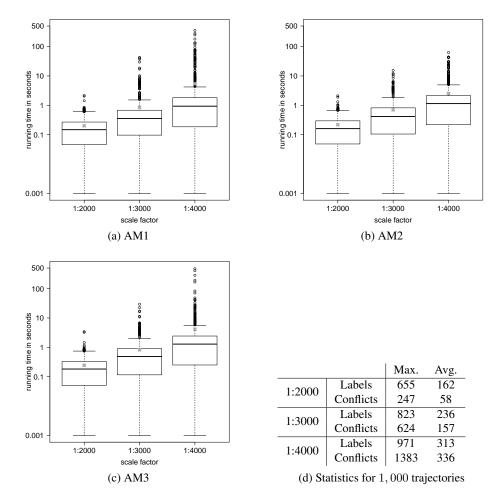


Fig. 4: Results of the experimental evaluation. In order to limit the vertical axis in the plots we rounded up all running times below 1ms to 1ms.

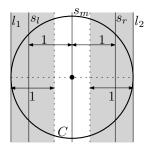


Fig. 5: Illustration for the proof of Lemma 1.

Lemma 1. Let C be a circle of radius $\sqrt{2}$ in the plane and let Q be a set of non-intersecting closed and axis-parallel unit squares with their bottom-left corner in C. Then Q cannot contain more than eight squares.

Proof. First, we show that Q cannot contain more than nine squares and extend the result to the claim of the lemma. We begin by proving the following claim.

(S) At most three squares of Q can be stabbed by a vertical line. In order to prove (S) let $Q' \subseteq Q$ be a set of squares that is stabbed by an arbitrary vertical line l and let q_t be the topmost square stabbed by l and let q_b be the bottommost square stabbed by l. Since both the bottom-left corner of q_t and q_b are in C, their vertical distance is at most $2\sqrt{2}$. Consequently, there can be at most one other square in Q' that lies in between q_t and q_b , which shows the claim (S).

Now let l_1 be the left vertical tangent of C and let l_2 be its right vertical tangent; see Fig. 5. We define $Q_l \subseteq Q$ to be the set of squares whose bottom-left corner has distance of at most 1 to l_1 . Hence, there must be a vertical line that stabs all squares in Q_l . By (S) it follows that $|Q_l| \leq 3$. We can analogously define the set $Q_r \subseteq Q$ whose bottom-left corner has distance of at most one to the vertical line l_2 . By the same argument it follows that $|Q_r| \leq 3$. Further, the bottom-left corners of the squares $Q_m = Q \setminus \{Q_l, Q_r\}$ must be contained in a vertical strip of width $2\sqrt{2}-2 < 1$. Hence, there is a vertical line that stabs all squares of Q_m and $|Q_m| \leq 3$ follows. We conclude that the set Q contains at most nine squares; in fact, $|Q| \leq 8$ as we show next.

For the sake of contradiction we assume that |Q|=9, i.e., $|Q_l|=|Q_m|=|Q_r|=3$. We denote the topmost square in Q_l by t_l and the bottommost square by b_l , and define t_r and b_r for Q_r analogously. Further, let s_m be the vertical line through the center of C, let s_l be the vertical line that lies one unit to the left of s_m and let s_r be the vertical line that lies one unit to the right of s_m . Note that the length of the segment of s_l and s_r that is contained in C has length 2. Since the bottom-left corners of t_l and b_l must have vertical distance strictly greater than 2, both squares must lie to the right of s_l . Hence, t_l and t_l intersect t_l and t_l intersect t_l and t_l must lie to the left of t_l and, hence intersect t_l and squares of t_l and t_l must lie to the left of t_l and, hence intersect t_l and square of t_l and that intersects t_l and the left of t_l and t_l must lie to the left of t_l and

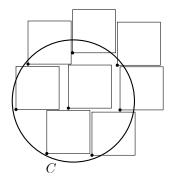


Fig. 6: Example configuration of eight axis-aligned, non-intersecting, unit-squares with their bottom-left corner inside a circle C with radius $\sqrt{2}$.

Fig. 6 shows that the bound is tight. Based on Lemma 1 we now show that for any label with anchor p there is no point of time $t \in [0,1]$ for which there can be more than eight active labels whose anchors are within distance $\sqrt{2}$ of p. We call a set $X \subseteq \Psi$ conflict-free if it contains no pair of presence intervals that are in conflict. Further, we say that X is in conflict with $I \in \Psi$ if every element of X is in conflict with I, and we say that X contains $t \in [0,1]$ if every element of X contains t.

Lemma 2. For every $t \in [0, 1]$ and every $I \in \Psi$ any maximum cardinality conflict-free set $X_I(t) \subseteq \Psi$ that is in conflict with I and contains t satisfies $|X_I(t)| \le 8$.

Proof. Assume that there is a time t and an interval I such that there is a set $X_I(t)$ that contains more than eight intervals. Let ℓ be the label that corresponds to I. For an interval $I' \in X_I(t)$ to be in conflict with I the anchors of the two corresponding labels must have a distance of at most $\sqrt{2}$. Hence, there are $|X_I(t)|$ labels corresponding to the intervals in $X_I(t)$ with anchors of distance at most $\sqrt{2}$ to the anchor of ℓ . By Lemma 1 we know that two of these labels must overlap. This implies that there is a conflict between the corresponding intervals contained in $X_I(t)$, which is a contradiction. \square

With this lemma we can finally obtain the approximation guarantees for GREEDYMAX-TOTAL for all activity models.

Theorem 3. Assuming that all labels are unit squares and w([a,b]) = b-a, GREEDY-MAXTOTAL is a 1/24-, 1/16-, 1/8-approximation for AM1-AM3, respectively, and needs $O(n \log n)$ time for AM1 and $O(n^2)$ time for AM2 and AM3.

Proof. To show the approximation ratios, we consider an arbitrary step of GREEDY-MAXTOTAL in which the presence interval $I=[a,b]_\ell$ is selected from Ψ . Let C^I_ℓ be the set of presence intervals in Ψ that are in conflict with I.

Consider the model AM1. Since I is the longest interval in Ψ when it is chosen, the intervals in C_ℓ^I must be completely contained in J=[a-w(I),b+w(I)]. As C_ℓ^I contains all presence intervals that are in conflict with I it is sufficient to consider J to bound the effect of selecting I. Obviously, the interval J is three times as long as I. By

Lemma 2 we know that for any $X_I(t)$ it holds that $|X_I(t)| \le 8$ for all $t \in J$. Hence, in an optimal solution there can be at most eight active labels at each point $t \in J$ that are discarded when $[a, b]_{\ell}$ is selected. Thus, the cost of selecting $[a, b]_{\ell}$ is at most $3 \cdot 8 \cdot w(I)$.

For AM2 we apply the same arguments, but restrict the interval J to J=[a,b+w(I)], which is only twice as long as I. To see that consider for an interval $[c,d]_{\ell'}\in C^I_\ell$ the prefix [c,a] if it exists. If [c,a] does not exist (because a< c), removing $[c,d]_{\ell'}$ from Ψ changes Ψ only in the range of J. If [c,a] exists, then again Ψ is only changed in the range of I, because by definition $[c,d]_{\ell'}$ is shortened to an interval that at least contains [c,a] and is still contained in Ψ . Thus, the cost of selecting I is at most $2\cdot 8w(I)$.

Analogously, for AM3 we can argue that it is sufficient to consider the interval J=[a,b]. By definition of the update operation of GREEDYMAXTOTAL at least the prefix or suffix subinterval of each $[c,d]_{\ell'} \in C^I_\ell$ remains in Ψ that extends beyond I (if such an interval exists). Thus, selecting I influences only the interval J and its cost is at most 8w(I). The approximation bounds of 1/24, 1/16, and 1/8 follow immediately.

We use a heap to achieve the time complexity $O(n \log n)$ of GREEDYMAXTOTAL for AM1 since each interval is inserted and removed exactly once. For AM2 and AM3 we use a linear sweep to identify the longest interval contained in Ψ . In each step we need O(n) time to update all intervals in Ψ , and we need a total of O(n) steps. Thus, GREEDYMAXTOTAL needs $O(n^2)$ time in total for AM2 and AM3.

4 Solving k-RESTRICTEDMAXTOTAL

Corollary 1 showed that k-RESTRICTEDMAXTOTAL can be solved by integer linear programming in all activity models. In this section we prove that unlike GENERAL-MAXTOTAL the problem k-RESTRICTEDMAXTOTAL can actually be solved in polynomial time. We give a detailed description of our algorithm for AM1, and then show how it can be extended to AM2. Note that solving k-RESTRICTEDMAXTOTAL is related to finding a maximum cardinality k-colorable subset of n intervals in interval graphs. This can be done in polynomial time in both n and k [4]. However, we have to consider additional constraints due to conflicts between labels, which makes our problem more difficult. First, we discuss how to solve the case for k=1, then give an algorithm that solves k-RESTRICTEDMAXTOTAL for k=2, and extend this result recursively to any constant k>2. Since the running times of the presented algorithms are, even for small k, prohibitively expensive in practice, we finally propose an approximation algorithm for k-RESTRICTEDMAXTOTAL.

4.1 An Algorithm for 2-RESTRICTEDMAXTOTAL in AM1

We start with some definitions before giving the actual algorithm. We assume that the intervals of $\Psi=\{I_1,\ldots,I_n\}$ are sorted in non-decreasing order by their left endpoints; ties are broken arbitrarily. First note that for the case that at most one label can be active at any given point in time (k=1), conflicts between labels do not matter. Thus, it is sufficient to find an independent subset of Ψ of maximum weight. This is equivalent to finding a maximum weight independent set on interval graphs, which can be done in O(n) time using dynamic programming given n sorted intervals [8]. We denote this

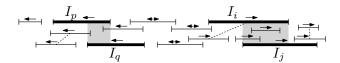


Fig. 7: Illustration of presence intervals. Intervals that are in conflict are connected by a dotted line. Both (I_i, I_j) and (I_p, I_q) are separating pairs. The intervals of $L_2[i, j]$ $(R_2[p, q])$ are marked by a left (right) arrow.

algorithm by \mathcal{A}_1 . Let $L_1[I_j]$ be the set of intervals that lie completely to the left of the left endpoint of I_j . Algorithm \mathcal{A}_1 basically computes a table \mathcal{T}_1 indexed by the intervals in Ψ , where an entry $\mathcal{T}_1[I_j]$ stores the value of a maximum weight independent set Q of $L_1[I_j]$ and a pointer to the rightmost interval in Q.

We call a pair of presence intervals (I_i, I_j) , i < j, a separating pair if I_i and I_j overlap and are not in conflict with each other. Further, a separating pair $\boldsymbol{v} = (I_p, I_q)$ is smaller than another separating pair $\boldsymbol{w} = (I_i, I_j)$ if and only if p < i or p = i and q < j. This induces a total order and we denote the ordered set of all separating pairs by $S_2 = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_z\}$. The weight of a separating pair \boldsymbol{v} is defined as $w(\boldsymbol{v}) = \sum_{I \in \boldsymbol{v}} w(I)$.

We observe that a separating pair $v=(I_i,I_j)$ contained in a solution of 2-RE-STRICTEDMAXTOTAL splits the set of presence intervals into two independent subsets. Specifically, a left (right) subset $L_2[v]$ ($R_2[v]$) that contains only intervals which lie completely to the left (right) of the intersection of I_i and I_j and are neither in conflict with I_i nor I_j ; see Fig. 7.

We are now ready to describe our dynamic programming algorithm \mathcal{A}_2 . For ease of notation we add two dummy separating pairs to S_2 . One pair v_0 with presence intervals strictly to the left of 0 and one pair v_{z+1} with presence intervals strictly to the right of 1. Since all original presence intervals are completely contained in [0,1] every optimal solution contains both dummy separating pairs. Our algorithm computes a one-dimensional table \mathcal{T}_2 , where for each separating pair v there is an entry $\mathcal{T}_2[v]$ that stores the value of the optimal solution for $L_2[v]$. We compute \mathcal{T}_2 from left to right starting with the dummy separating pair v_0 and initialize $\mathcal{T}_2[v_0] = 0$. Then, we recursively define $\mathcal{T}_2[v_j]$ for every $v_j \in S_2$ as $\mathcal{T}_2[v_j] = \max_{i < j} \{\mathcal{T}_2[v_i] + w(v_i) + \mathcal{A}_1(v_i, v_j) \mid v_i \in S_2, v_i \subseteq L_2[v_j], v_j \subseteq R_2[v_i]\}$. Additionally, we store a backtracking pointer to the predecessor pair that yields the maximum value. In other words, for computing $\mathcal{T}_2[v_j]$ we consider all possible direct predecessors $v_i \in S_2$ with $i < j, v_i \cap v_j = \emptyset$, and no conflict with v_j . Each such v_i induces a candidate solution whose value is composed of $\mathcal{T}_2[v_i], w(v_i)$, and the value of an optimal solution of algorithm \mathcal{A}_1 for the intervals between v_i and v_j with v_i and v_j active.

Since by construction $L_2[v_{z+1}] = \Psi \cup v_0$, the optimal solution to 2-RESTRIC-TEDMAXTOTAL is stored in $\mathcal{T}_2[v_{z+1}]$ once v_0 is removed. To compute a single entry $\mathcal{T}_2[v_j]$ our algorithm needs to consider all possible separating pairs preceding v_j , and for each of them obtain the optimal solution from algorithm \mathcal{A}_1 under some additional constraints. For the call $\mathcal{A}_1(v_i, v_j)$ in the recursive equation above, we distinguish two cases. If the rightmost endpoint of v_i is to the left of the leftmost endpoint of v_j then we run algorithm \mathcal{A}_1 on the set of intervals $L_2[v_j] \cap R_2[v_i]$ and obtain the value $\mathcal{A}_1(v_i, v_j)$.

Otherwise, there is an overlap between an interval I_a of v_i and an interval I_b of v_j . Since for k=2 no other interval can cross this overlap, we actually make two calls to \mathcal{A}_1 , once on the set $R_2[v_i] \cap L_2[(I_a,I_b)]$ and once on the set $R_2[(I_a,I_b)] \cap L_2[v_j]$. We add both values to obtain $\mathcal{A}_1(v_i,v_j)$. Since we run algorithm \mathcal{A}_1 for each of O(z) separating pairs, the time complexity to compute a single entry of \mathcal{T}_2 is O(nz). To compute the whole table the algorithm repeats this step O(z) times, which yields a total time complexity of $O(nz^2)$. Note that the number of separating pairs z is in $O(n^2)$.

We prove the correctness of the algorithm by contradiction. Assume that there exists an instance for which our algorithm does not compute an optimal solution and let OPT be an optimal solution. This means, that there is a smallest separating pair v_j for which the entry in $\mathcal{T}_2[v_j]$ is less than the value of OPT for $L_2[v_j]$. Note that v_j cannot be the dummy separating pair v_0 since $\mathcal{T}_2[v_0]$ is trivially correct. Let v_i be the rightmost separating pair in OPT that precedes v_j and is disjoint from it (possibly $v_i = v_0$). Since there is no other disjoint separating pair between v_i and v_j in OPT, all intervals in OPT between v_i and v_j form a subset of $R_2[v_i] \cap L_2[v_j]$ that is a valid configuration for k=1. We can obtain an optimal solution for k=1 of the intervals in $R_2[v_i] \cap L_2[v_j]$ by computing $\mathcal{A}_1(v_i,v_j)$ as described above. Since, by assumption, $\mathcal{T}_2[v_i]$ is optimal, \mathcal{A}_1 is correct [8], and our algorithm explicitly considers all possible preceding separating pairs including v_i , the entry $\mathcal{T}_2[v_j]$ must be at least as good as OPT for $L_2[v_j]$. This is a contradiction and the correctness of \mathcal{A}_2 follows.

Theorem 4. Algorithm A_2 solves 2-RESTRICTEDMAXTOTAL in AM1 in $O(nz^2)$ time and O(z) space, where z is the number of separating pairs in the input instance.

4.2 An Algorithm for k-RESTRICTEDMAXTOTAL in AM1

In the following we extend the dynamic programming algorithm A_2 to a general algorithm A_k for the case k > 2. To this end, we extend the definition of separating pairs to separating k-tuples. A separating k-tuple v is a set of k presence intervals that are not in conflict with each other and that have a non-empty intersection $Y_v = \bigcap_{I \in v} I$. We say a separating k-tuple v is smaller than a separating k-tuple w if Y_v begins to the left of Y_w . Ties are broken arbitrarily. This lets us define the ordered set $S_k = \{v_1, \dots, v_z\}$ of all separating k-tuples of a given set of presence intervals. We say a set C of presence intervals is k-compatible if no more than k intervals in C intersect at any point and there are no conflicts in C. Two separating k-tuples v and w are k-compatible if they are disjoint and $v \cup w$ is k-compatible. The definitions of the sets $R_2[v]$ and $L_2[v]$ extend naturally to the sets $R_k[v]$ and $L_k[v]$ of all intervals completely to the right (left) of Y_v and not in conflict with any interval in v. Now, we recursively define the algorithm A_k that solves k-RESTRICTEDMAXTOTAL given a pair of active k-compatible boundary k-tuples. Note that in the recursive definition these boundary tuples may remain k-dimensional even in $A_{k'}$ for k' < k. For A_k we define as boundary tuples two k-compatible dummy separating k-tuples v_0 and v_{z+1} with all presence intervals strictly to the left of 0 and to the right of 1, respectively. The algorithm fills a one-dimensional table \mathcal{T}_k . Similarly to the case k=2, each entry $\mathcal{T}_k[v]$ stores the value of the optimal solution for $L_k[v]$, i.e., the final solution can again be obtained from $\mathcal{T}_k[v_{z+1}]$. We initialize $\mathcal{T}_k[v_0] = 0$. Then, the remaining entries of \mathcal{T}_k can be obtained

by computing $\mathcal{T}_k[v_j] = \max_{i < j} \{\mathcal{T}_k[v_i] + w(v_i) + \mathcal{A}_{k-1}(\tilde{v}_i, \tilde{v}_j) \mid v_i \in S_k, v_i \subseteq L_k[v_j] \cup v_0, v_j \subseteq R_k[v_i] \cup v_{z+1}, v_0 \cup v_{z+1} \cup v_i \cup v_j \text{ is } k\text{-compatible}\}$, which uses the algorithm \mathcal{A}_{k-1} recursively on a suitable subset of presence intervals between the boundary tuples \tilde{v}_i and \tilde{v}_j . Here \tilde{v}_i is defined as the union of the tuple v_i and all intervals in $v_0 \cup v_{z+1}$ that intersect the right endpoint of Y_{v_i} ; analogously \tilde{v}_j is defined as the union of the tuple v_j and all intervals in $v_0 \cup v_{z+1}$ that intersect the left endpoint of Y_{v_i} . This makes sure that in each subinstance all active intervals that are relevant for that particular subinstance are known. Note that by the k-compatibility condition \tilde{v}_i and \tilde{v}_j contain at most k elements each. In fact, $\mathcal{A}_{k-1}(\tilde{v}_i, \tilde{v}_j)$ uses \tilde{v}_i and \tilde{v}_j as boundary k-tuples (and thus does not create dummy boundary tuples) and the set $R_k[v_i] \cap L_k[v_j]$ as the set of presence intervals from which separating (k-1)-tuples can be formed.

Theorem 5. Algorithm A_k solves k-RESTRICTEDMAXTOTAL in AM1 in $O(n^{k^2+k-1})$ time and $O(n^k)$ space.

Proof. We show the correctness of A_k by induction on k. Theorem 4 shows that the statement is true for k=2. Let k>2. Since A_k only considers solutions where adjacent separating k-tuples are k-compatible with each other and the boundary k-tuples, we cannot produce an invalid solution, i.e., a solution with conflicts or more than k active intervals at any point. We prove the correctness by contradiction. So assume that there is an instance Ψ for which \mathcal{A}_k does not compute an optimal solution and let OPT be an optimal solution. There must be a smallest separating k-tuple v_i , j > 0, for which $\mathcal{T}_k[v_j]$ is less than the value of OPT for $L_k[v_j]$. Let v_i , i < j be the rightmost disjoint separating k-tuple in OPT that precedes v_i such that the set $v_0 \cup v_i \cup v_{z+1}$ is k-compatible. By our assumption $\mathcal{T}_k[v_i]$ has the same value as OPT on $L_k[v_i]$. For the set of intervals $L_k[v_i] \cap R_k[v_i]$ there are at most k-1 active intervals at any point (otherwise v_i is not rightmost). This means that when we run algorithm A_{k-1} on that instance with the boundary tuples \tilde{v}_i and \tilde{v}_i , i.e., v_i and v_i enriched by all relevant intervals in $v_0 \cup v_{z+1}$, we obtain by induction a solution that is at least as good as the restriction of OPT to that instance. Since v_i is a valid predecessor k-tuple for v_i the algorithm A_k considers it. So $\mathcal{T}_k[v_j] \geq \mathcal{T}_k[v_i] + w(v_i) + A_{k-1}(\tilde{v}_i, \tilde{v}_j)$, which is at least as good as OPT restricted to $L_k[v_i]$. This is a contradiction and proves the correctness.

For proving the time and space complexity let z_i be the number of separating i-tuples in an instance for $1 < i \le k$. Each z_i is in $O(n^i)$. We again use induction on k. For \mathcal{A}_2 Theorem 4 yields $O(n^5)$ time and $O(n^2)$ space, which match the bounds to be shown. So let k > 2. The table \mathcal{T}_k has $O(z_k) \subseteq O(n^k)$ entries and each of the recursive computations of \mathcal{A}_{k-1} need $O(n^{k-1})$ space by the induction hypothesis. Thus the overall space is dominated by \mathcal{T}_k and the bound follows. Checking whether a separating k-tuple $v_i \in S_k$ is a feasible predecessor for a particular v_j can easily be done in $O(k^2)$ time, which is dominated by the time to compute $\mathcal{A}_{k-1}(\tilde{v}_i, \tilde{v}_j)$. So for the running time we observe that each entry in \mathcal{T}_k makes $O(z_k)$ calls to \mathcal{A}_{k-1} and hence the overall running time is indeed $O(n^{2k} \cdot n^{(k-1)^2 + (k-1)-1}) = O(n^{k^2 + k-1})$.

4.3 Extending the algorithm for k-RESTRICTEDMAXTOTAL to AM2

With some modifications and at the expense of another polynomial factor in the running time we can extend algorithm A_k of the previous section to the activity model

AM2, which shows that k-RESTRICTEDMAXTOTAL in AM2 can still be solved in polynomial time. In the following we give a sketch of the modifications. The important difference between AM1 and AM2 is that presence intervals can be truncated at their right side if there is an active conflicting witness label causing the truncation. We need two modifications to model this behavior. First, we create for each original presence interval $I_i = [a_i, b_i]$ in Ψ at most n prefix intervals $I_i^j = [a_i, c_{ij}]$, where c_{ij} is the start of the first conflict between I_i and $I_j \in \Psi$. Each interval I_i^j inherits the conflicts of I_i that intersect I_i^j . We obtain a modified set of presence intervals $\Psi' = \Psi \cup \{I_i^j \mid I_i, I_j \in \Psi \text{ and } I_i, I_j \text{ in conflict}\}$ of size $O(n^2)$. We create mutual conflicts among all intervals that are prefixes of the same original interval. This will enforce that at most one of them is active. We still have to take care that a truncated interval I_i^j can only be active if I_j (or a prefix of I_j) is active at c_{ij} as a witness.

In order to achieve this we instantiate the algorithm $\mathcal{A}_{k'}$ for every $k' \leq k$ not only with its two boundary k-tuples \tilde{v}_0 and \tilde{v}_{z+1} but also with a set W of at most k witness intervals that are k-compatible and must be made active at some stage of the algorithm. In a valid solution we have $W \subseteq L_{k'}[v] \cup v$ for the leftmost separating k'-tuple v, since otherwise more than k' intervals are active in Y_v . However, the truncated intervals in v themselves define a family of $O(n^{k'})$ possible witness sets W(v) to be respected to the right of v. So when we compute the table entry for a separating k'-tuple v_j and consider a particular predecessor k'-tuple v_i we must in fact iterate over all possible witness sets $W(v_i)$ as well. We need to make sure that v_j is $W(v_i)$ -compatible, i.e., $v_j \cup W(v_i)$ is k-compatible and $W(v_i) \subseteq L_{k'}[v_j] \cup v_j$. For the recursive call to $\mathcal{A}_{k'-1}(\tilde{v}_i, \tilde{v}_j)$ the initial witness set W' consists of $W(v_i) \setminus v_j$, i.e., those witness intervals of $W(v_i)$ that are not part of v_j .

The increase in running time is caused by dealing with $O(n^2)$ intervals in Ψ' and by the fact that instead of one call to $\mathcal{A}_{k-1}(\tilde{\boldsymbol{v}}_i,\tilde{\boldsymbol{v}}_j)$ in the computation of table \mathcal{T}_k we make $O(n^k)$ calls, one for each possible witness set of \boldsymbol{v}_i . By an inductive argument one can show that the running time is in $O(n^{3k^2+2k})$.

Theorem 6. k-RESTRICTEDMAXTOTAL in AM2 can be solved in polynomial time.

It remains open whether k-RESTRICTEDMAXTOTAL can be solved in polynomial time in AM3. Another extension of the dynamic programming algorithm is unlikely, since in AM3 the left and right subinstances created by a separating k-tuple v may have dependencies and thus cannot be solved independently any more. This is because a single original presence interval I can have subintervals both in $L_k[v]$ and $R_k[v]$, which cannot simultaneously be active.

4.4 Approximation of k-RESTRICTEDMAXTOTAL

Since the running times of our algorithms for k-RESTRICTEDMAXTOTAL are, even for small k, prohibitively expensive in practice, we propose an approximation algorithm for k-RESTRICTEDMAXTOTAL based on GREEDYMAXTOTAL.

Our algorithm GREEDYRESTRICTEDMAXTOTAL is a simple extension of GREEDY-MAXTOTAL. Recall that GREEDYMAXTOTAL greedily removes the longest interval I from Ψ and adds it to the set Φ that contains the active intervals of the solution. Then, it

updates all intervals contained in Ψ that are in conflict with I. This process is repeated until Ψ is empty. For approximating k-RESTRICTEDMAXTOTAL we need to ensure that there is no point in time t that is contained in more than k intervals in Φ . We call intervals which we cannot add to Φ without violating this property *invalid*.

Our modification of GREEDYMAXTOTAL is as follows. After adding an interval I to Φ and handling conflicts as before, we remove intervals from Ψ that became invalid. We say that we *ensure that* I *is valid*. Note that we cannot shorten those intervals because then we could not ensure that adding an interval from Ψ to Φ is valid according to our model.

In order to prove approximation ratios we first introduce the following lemma that describes the structure of a solution of k-RESTRICTEDMAXTOTAL.

Lemma 3. Let S be a set of intervals such that there is no number that is contained in more than k intervals from S. Then, there is a partition of S into k sets M_1, \ldots, M_k , such that no two intersecting intervals are in the same set M_i .

Proof. Let I_1, \ldots, I_m be all intervals of S sorted by their left endpoints in non-decreasing order. In the following we describe how to construct the partition.

We start with empty sets M_1,\ldots,M_k . First, add I_1 to M_1 . Assume that the first i-1 intervals have been added to the sets M_1,\ldots,M_k . We describe how to add I_i . If there is an empty set M_j then, we simply add I_i to M_j . Otherwise, Let I_{i_1},\ldots,I_{i_k} be the rightmost intervals in the sets M_1,\ldots,M_k , respectively. We denote the set containing those intervals by R. Let $\mathcal{I} = \bigcap_{I \in R} I$. If \mathcal{I} is not empty then, due to the order of the intervals, the interval I_i cannot begin to the left of \mathcal{I} . It also cannot begin in \mathcal{I} because otherwise there would be a number that is contained in k+1 intervals in S. Let $M_x, 1 \leq x \leq k$ be the set that contains the interval $I \in R$ with leftmost right endpoint among the intervals in R. Since I_i lies completely to the right of \mathcal{I} it must also lie completely to the right of I. Thus we can assign I_i to the set M_x without introducing intersections. If \mathcal{I} is empty, then there must be an interval $I \in R$ with right endpoint to the left of another $I' \in R$. Let $M_x, 1 \leq x \leq k$ be the set that contains the interval I. Due to the order of the intervals the interval I_i lies completely to the right of I and hence we assign I_i to M_x without introducing intersections. This concludes the proof.

With this lemma we now can prove the following theorem that makes an statement about the approximation ratio of GREEDYRESTRICTEDMAXTOTAL

Theorem 7. Assuming that all labels are unit squares and w([a,b]) = b-a, GREEDY-RESTRICTEDMAXTOTAL is a $1/\min\{3+3k,27\}$, $1/\min\{3+2k,19\}$, $1/\min\{3+k,11\}$ -approximation for AM1-AM3, respectively, and needs $O(n^2)$.

Proof. We begin by proving its correctness and then we show its time complexity.

Consider the step in which we add an interval I = [a, b] to Φ , and let J = [a - w(I), b + w(I)]. Let \mathcal{L} be a fixed, but arbitrary optimal solution. If $I \in \mathcal{L}$, there is no lost weight compared to the optimal solution when choosing I.

Thus, assume that $I \not\in \mathcal{L}$. Let $C(I) \subseteq \mathcal{L}$ be the set of intervals that are in conflict with I. Identically to the proof of Theorem 3 we can argue that $w(C(I)) = \sum_{I \in C} w(I) \le (4-X) \cdot 8 \cdot w(I)$ considering activity model AMX with $X \in \{1,2,3\}$.

We now show that at most 3w(I) weight of the optimal solution is lost when ensuring that I is valid.

By Lemma 3 we can partition \mathcal{L} into k sets M_1, \ldots, M_k such that no two intersecting intervals are in the same set M_i . If I is in \mathcal{L} , then we do not lose any weight compared to the optimal solution. Hence, assume that I is not in \mathcal{L} . Take any M_i , $1 \le i \le k$, remove all intervals of $\mathcal{L} \setminus \Phi$ that intersect I, and add I to M_i . We denote the set of removed intervals by R. In the following we bound the cost of removing the intervals in R. If there are intervals in R that are longer than I, then we have already accounted for them in previous steps. This relies on the fact that we consider the intervals sorted by their length in non-ascending order, and, hence if those longer intervals are not in Φ , we must have removed them in an earlier step. Thus, we only need to bound the length of intervals in R which have length at most w(I). Those intervals must lie in I, and due to the definition of M_i they must be disjoint. Hence, the cost is bounded by 3w(I).

All together choosing I causes that at most $3w(I) + (4 - X) \cdot 8 \cdot w(i)$ weight is lost compared to the optimal solution considering activity model AMX with $X \in \{1, 2, 3\}$. Finally, this yields an approximation factor of 1/27, 1/19, 1/11 for AM1-3, respectively.

For k < 8 we can improve $w(C(I)) \le (4-X) \cdot 8 \cdot w(i)$ to $w(C(I)) \le (4-X) \cdot k \cdot w(i)$ considering activity model AMX with $X \in \{1,2,3\}$ because we know that I cannot be in conflict with more than k intervals of the optimal solution. Thus, we can bound the loss of choosing I by $3w(I) + (4-X) \cdot k \cdot w(i)$. In total this yields the claimed approximation ratios for the three activity models.

Finally, we argue the correctness of the claimed running time of $O(n^2)$. Since the worst-case running time of GREEDYMAXTOTAL is $O(n^2)$ we only need to argue that we can delete those intervals from Φ , which are not valid anymore, in O(n) time per step. To do this we simply sort the intervals in Ψ in non-decreasing order by their left-endpoint. We also maintain Φ in the same way. Then, we can check for non-valid intervals with a simple linear sweep over Ψ and Φ . Hence, each iteration of the algorithm requires O(n) time, which yields a total running time of $O(n^2)$.

5 Conclusions

We have introduced a trajectory-based model for dynamic map labeling that satisfies the consistency criteria demanded by Been et al. [2], even in a stronger sense, where each activity change of a label must be explainable to the user by some witness label. Our model transforms the geometric information specified by trajectory, viewport, and labels into the two combinatorial problems Generalmaxtotal and k-restricted maxtotal that are expressed in terms of presence and conflict intervals. Thus our algorithms apply to any dynamic labeling problem that can be transformed into such an interval-based problem; the analysis of the approximation ratios, however, requires problem-specific geometric arguments, which must be adjusted accordingly.

We showed that GENERALMAXTOTAL is \mathcal{NP} -complete and $\mathcal{W}[1]$ -hard and presented an ILP model, which we also implemented and evaluated, and constant-factor approximation algorithms for our three different activity models. The problem k-RE-STRICTEDMAXTOTAL, where at most k labels can be visible at any time, can be solved in polynomial time $O(n^{f(k)})$ in activity models AM1 and AM2 for any fixed k, where f

is a polynomial function. Due to the $\mathcal{W}[1]$ -hardness of GENERALMAXTOTAL we cannot expect to find better results for the running times, apart from improving upon the function f. We therefore also presented an $O(n^2)$ -time approximation algorithm for k-RESTRICTEDMAXTOTAL in all three activity models.

It remains open whether k-RESTRICTEDMAXTOTAL is polynomially solvable in activity model AM3. Further, the analysis of the approximation algorithms for both k-RESTRICTEDMAXTOTAL and GENERALMAXTOTAL significantly relies on the assumption that labels are unit squares. Thus, the question arises, whether constant-factor approximations exist when this assumption is dropped or softened, e.g., to labels of unit-height. To answer this question we think that deeper insights into the structure of conflicts are necessary, e.g., does the geometric information based on trajectory, viewport and labels imply a useful structure on the induced label conflict graph?

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