Complexity of Coloring Graphs without Paths and Cycles

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Abstract. Let P_t and C_ℓ denote a path on t vertices and a cycle on ℓ vertices, respectively. In this paper we study the k-coloring problem for (P_t, C_ℓ)-free graphs. Maffray and Morel, and Bruce, Hoang and Sawada, have proved that 3-colorability of P_5-free graphs has a finite forbidden induced subgraphs characterization, while Hoang, Moore, Recoskie, Sawada, and Vatshelle have shown that k-colorability of P_5-free graphs for $k \geq 4$ does not. These authors have also shown, aided by a computer search, that 4-colorability of (P_5, C_5)-free graphs does have a finite forbidden induced subgraph characterization. We prove that for any k, the k-colorability of (P_6, C_4)-free graphs has a finite forbidden induced subgraph characterization. We provide the full lists of forbidden induced subgraphs for $k = 3$ and $k = 4$. As an application, we obtain certifying polynomial time algorithms for 3-coloring and 4-coloring (P_6, C_4)-free graphs. (Polynomial time algorithms have been previously obtained by Golovach, Paulusma, and Song, but those algorithms are not certifying; in fact they are not efficient in practice, as they depend on multiple use of Ramsey-type results and resulting tree decompositions of very high widths.) To complement these results we show that in most other cases the k-coloring problem for (P_t, C_ℓ)-free graphs is NP-complete. Specifically, for $\ell = 5$ we show that k-coloring is NP-complete for (P_t, C_5)-free graphs when $k \geq 4$ and $t \geq 7$; for $\ell \geq 6$ we show that k-coloring is NP-complete for (P_t, C_ℓ)-free graphs when $k \geq 5$, $t \geq 6$; and additionally, for $\ell = 7$, we show that k-coloring is also NP-complete for (P_t, C_7)-free graphs if $k = 4$ and $t \geq 9$. This is the first systematic study of the complexity of the k-coloring problem for (P_t, C_ℓ)-free graphs. We almost completely classify the complexity for the cases when $k \geq 4, \ell \geq 4$, and identify the last three open cases.

1 Introduction

Since the k-coloring problem is known to be NP-complete for any fixed $k \geq 3$, there has been considerable interest in studying restrictions to various graph classes. For instance the k-coloring problem is polynomially solvable for perfect graphs, since a perfect graph is k-colorable if and only if it has no subgraph isomorphic to K_{k+1}. (In fact the chromatic number of perfect graphs can also be computed in polynomial time [14].) One type of graph class that has been given wide attention in recent years is the class of H-free graphs, for various graphs H [3,4,12,15,24,29]. For example, if H contains a cycle, then k-coloring is NP-complete for H-free graphs. This follows from the fact, proved by Kamiński and Lozin [19] and independently Král, Kratochvíl, Tuza, and Woeginger [20], that, for any fixed $k \geq 3$ and $g \geq 3$, the k-coloring problem is NP-complete for the class of graphs of girth at least g. Similarly, if H is a forest with a vertex of degree at least 3, then k-coloring is NP-complete for H-free graphs; this follows from [17] and [22]. Combining these results we conclude that k-coloring is NP-complete for H-free graphs, as long as H is not a linear forest, i.e., a union of disjoint paths. This focused attention on the case when H is a path. Woeginger and Sgall [29] have proved that 4-coloring is NP-complete for P_{12}-free graphs, and that 5-coloring is NP-complete for P_8-free graphs. Later on, these results were improved by various groups of researchers [3,4,12,21]. The strongest results so far are due to

Huang [18] who has proved that 4-coloring is NP-complete for P_7-free graphs, and that 5-coloring is NP-complete for P_6-free graphs. On the positive side, Hoàng, Kamiński, Lozin, Sawada, and Shu [15] have shown that k-coloring can be solved in polynomial time on P_5-free graphs for any fixed k. These results give a complete classification of the complexity of k-coloring P_t-free graphs for any fixed $k \geq 5$, and leave only 4-coloring P_6-free graphs open for $k = 4$. It should be noted that deciding the complexity of 3-coloring for P_t-free graphs seems difficult. It is not even known that whether or not there exists any t such that 3-coloring is NP-complete on P_t-free graphs. Randerath and Schiermeyer [24] have given a polynomial time algorithm for 3-coloring P_6-free graphs. As far as we know, this result has been extended to 3-coloring P_7-free graphs by Chudnovsky, Maceli, and Zhong [6,7].

One interesting aspect of the k-coloring problem is the number of minimal obstructions, i.e., minimal non-k-colorable graphs. As noted above, there is a unique minimal non-k-colorable perfect graph, namely K_{k+1}. It was shown by Bruce, Hoang and Sawada [5], that the set of minimal non-3-colorable P_5-free graphs is finite, while Hoang, Moore, Recoskie, Sawada, and Vatshelle [16] have shown that the set of minimal non-k-colorable P_5-free graphs is infinite. These authors have also shown, aided by a computer search, that the set of minimal non-4-colorable (P_5, C_5)-free graphs is finite.

In this paper we undertake a systematic examination of k-coloring with inputs restricted to (P_t, C_ℓ)-free graphs. Some results about k-coloring these graphs are known. In addition to the case of 4-coloring (P_5, C_5)-free graphs mentioned just above, it is known that when $\ell = 3$, each k-coloring is polynomial for $t \leq 6$, as (P_6, C_3)-free graphs have bounded cliquewidth. On the other hand, for $t \geq 164$, 4-coloring is NP-complete for (P_t, C_3)-free graphs [12]. When $\ell = 4$, each k-coloring is polynomial for (P_t, C_4)-free graphs [12]. When $\ell \geq 5$, 4-coloring is NP-complete for (P_t, C_ℓ)-free graphs as long as t is large enough with respect to ℓ [12]. (For $\ell = 5$, the bound on t is $t \geq 21$.)

We first focus on the number of minimal obstructions in a case in which polynomial time algorithms are known to exist, namely (P_6, C_4)-free graphs [12]. We prove that, for each k, the set of minimal non-k-colorable (P_6, C_4)-free graphs is finite. We actually describe all the minimal non-k-colorable (P_6, C_4)-free graphs for $k = 3$ and $k = 4$, and then apply these results to derive efficient certifying k-coloring algorithms in these cases. We complement these results by showing that in most cases with $k \geq 4, \ell \geq 4$, the k-coloring problem for (P_t, C_ℓ)-free graphs is NP-complete. Specifically, we prove that k-coloring is NP-complete for (P_t, C_5)-free graphs when $k \geq 4$ and $t \geq 7$, and that k-coloring is NP-complete for (P_t, C_ℓ)-free graphs when $\ell \geq 6$ and $k \geq 5, t \geq 6$. We show that k-coloring is also NP-complete for (P_t, C_7)-free graphs if $k = 4$ and $t \geq 9$. This almost completely classifies the complexity of k-coloring for (P_t, C_ℓ)-free graphs when $\ell \geq 4, k \geq 4$. The few remaining open problems are listed in the last section.

We say that G is *$\mathcal{H}$-free* if it does not contain, as an induced subgraph, any graph $H \in \mathcal{H}$. If $\mathcal{H} = \{H\}$ or $\mathcal{H} = \{H_1, H_2\}$, we say that G is H-free or (H_1, H_2)-free. For two disjoint vertex subsets X and Y we say that X is *complete,* respectively *anti-complete*, to Y if every vertex in X is adjacent, respectively non-adjacent, to every vertex in Y. A graph G is called a *minimal obstruction* for k-coloring if G is not k-colorable but any proper induced subgraph of G is k-colorable. We also call G a *minimal non-k-colorable graph.* A minimal non-$(k-1)$-colorable graph is also called a *k-critical* graph. A graph is *critical* if it is k-critical for some k. We shall use n and m to denote the number of vertices and edges of G, respectively.

2 Imperfect (P_6, C_4)-Free Graphs

In this section, we analyze the structure of imperfect (P_6, C_4)-free graphs. Let G be a connected imperfect (P_6, C_4)-free graph. By the Strong Perfect Graph Theorem [8], G must contain an induced five-cycle, say $C = v_0v_1v_2v_3v_4$. We call a vertex $v \in V \setminus C$ a *p-vertex* with respect to C if v has exactly p neighbors on C, i.e., $|N_C(v)| = p$. We denote by S_p the set of p-vertices for $0 \leq p \leq 5$. In the following all indices are modulo 5. Let $S_1(v_i)$ be the subset of S_1 containing all 1-vertices that have v_i as their neighbor on C. Let $S_3(v_i)$ be the subset of S_3 containing all 3-vertices that have v_{i-1}, v_i and v_{i+1} as their neighbors on C. Let $S_2(v_i, v_{i+1})$ be the subset of S_2 containing all 2-vertices that have v_i and v_{i+1} as their neighbors on C. Note that $S_1 = \bigcup_{i=0}^4 S_1(v_i)$, $S_2 = \bigcup_{i=0}^4 S_2(v_i, v_{i+1})$ and $S_3 = \bigcup_{i=0}^4 S_3(v_i)$.

A subset $S \subseteq V$ is *dominating* if every vertex not in S has a neighbor in S. Brandstädt and Hoàng [2] proved the following fact about induced five-cycles in (P_6, C_4)-free graphs.

Lemma 1. *([2]) Let G be a (P_6, C_4)-free graph without clique cutset. Then every induced C_5 of G is dominating.*

In the rest of this section, we collect some information about imperfect (P_6, C_4)-free graphs. Recall that we assume that G is a connected (P_6, C_4)-free graph, and $v_0v_1v_2v_3v_4$ is an induced five-cycle in G. Then the following properties must hold.

(P0) S_5 and each $S_3(v_i)$ are cliques and $S_4 = \emptyset$.

(P1) $S_1(v_i)$ is complete to $S_1(v_{i+2})$ and anti-complete to $S_1(v_{i+1})$; moreover, if both sets $S_1(v_i)$ and $S_1(v_{i+2})$ are nonempty, then both are cliques.

(P2) $S_2(v_i, v_{i+1})$ is complete to $S_2(v_{i+1}, v_{i+2})$ and anti-complete to $S_2(v_{i+2}, v_{i+3})$; moreover, if both sets $S_2(v_i, v_{i+1})$ and $S_2(v_{i+1}, v_{i+2})$ are nonempty, then both are cliques.

(P3) $S_3(v_i)$ is anti-complete to $S_3(v_{i+2})$.

(P4) $S_1(v_i)$ is anti-complete to $S_2(v_j, v_{j+1})$ if $j \neq i + 2$; moreover, if $y \in S_2(v_{i+2}, v_{i+3})$ is not anti-complete to $S_1(v_i)$, then y is an universal vertex in $S_2(v_{i+2}, v_{i+3})$.

(P5) $S_1(v_i)$ is anti-complete to $S_3(v_{i+2})$.

(P6) $S_2(v_{i+2}, v_{i+3})$ is anti-complete to $S_3(v_i)$.

(P7) One of $S_1(v_i)$ and $S_2(v_{i+3}, v_{i+4})$ is empty, and one of $S_1(v_i)$ and $S_2(v_{i+1}, v_{i+2})$ is empty.

(P8) One of $S_2(v_{i-1}, v_i)$, $S_2(v_i, v_{i+1})$ and $S_2(v_{i+2}, v_{i+3})$ is empty.

(P9) If both $S_1(v_{i-1})$ and $S_1(v_{i+1})$ are nonempty, then $S_2 = \emptyset$; if both $S_1(v_i)$ and $S_1(v_{i+1})$ are nonempty, then $S_2 = S_2(v_i, v_{i+1})$.

(P10) Let $x \in S_3(v_i)$. If both $S_2(v_{i+1}, v_{i+2})$ and $S_2(v_{i+3}, v_{i+4})$ are nonempty, then x is either complete or anti-complete to $S_2(v_{i+1}, v_{i+2}) \cup S_2(v_{i+3}, v_{i+4})$. In the former case, both $S_2(v_{i+1}, v_{i+2})$ and $S_2(v_{i+3}, v_{i+4})$ are cliques. Moreover, if $S_2(v_{i+2}, v_{i+3})$ is also nonempty, then x is anti-complete to $S_2(v_{i+1}, v_{i+2}) \cup S_2(v_{i+3}, v_{i+4})$.

(P11) If $S_1(v_i)$ is not anti-complete to $S_2(v_{i+2}, v_{i+3})$ then $S_1 = S_1(v_i)$.

(P12) If G has no clique cutset, then $S_1(v_i)$ is complete to $S_3(v_i)$.

The proofs of these properties are simple, using the absence of induced copies of P_6 and C_4. The proof of property (P12) also uses Lemma 1.

3 Obstructions to k-coloring

In this section we shall prove our first main result, that for each k, there are only finitely many minimal non-k-colorable (P_6, C_4)-free graphs. In subsequent sections we then describe all minimal non-3-colorable and non-4-colorable (P_6, C_4)-free graphs, and apply these characterizations to obtain polynomial time certifying algorithms for the 3-coloring and the 4-coloring problems on (P_6, C_4)-free graphs.

The following lemma is folklore.

Lemma 2. *A minimal non k-colorable graph G has $\delta(G) \geq k$ and no clique cutset.*

Let P be the graph obtained from the Peterson graph by adding one new vertex that is adjacent to every vertex of P. A graph is called *specific* if it results from replacing each vertex of P by a clique of arbitrary size (including possibly size 0, resulting in deleting the vertex).

Lemma 3. *([2]) Let G be a (P_6, C_4)-free graph without a clique cutset. Then either G is specific, or every induced C_6 of G is dominating. Moreover, there is a linear time algorithm to decide whether or not G is specific.*

We are now ready to prove the main result of this section, the finiteness of the number of minimal obstructions for k-coloring (P_6, C_4)-free graphs. It should be observed that this result is best possible in the sense that there are infinitely many minimal non-k-colorable P_6-free graphs and infinitely many minimal non-k-colorable C_4-free graphs. The former fact follows from [16] where it is shown that there are infinitely many minimal non-k-colorable P_5-free graphs, and the latter fact follows from [10] where it is shown that there are non-k-colorable graphs of arbitrarily high girth.

Theorem 1. *For any k, there are only finitely many minimal non-k-colorable (P_6, C_4)-free graphs.*

Proof. Let G be a (P_6, C_4)-free minimal non-k-colorable graph. By Lemma 2, G has $\delta(G) \geq k$ and no clique cutset. If G contains K_{k+1}, then $G = K_{k+1}$. Thus we assume that that G is K_{k+1}-free. If G contains an induced $C = C_6$, then either G is specific or C is dominating by Lemma 3. In the former case, the size of G is bounded by the definition of specific graph and the fact that G is K_{k+1}-free. In the latter case, we analyze the remaining vertices as to their connection to C, analogously to what we did in the previous section for C being a five-cycle. We define again, for any $X \subseteq C$ the set $S(X)$ to consist of all vertices not in C that have X as their neighborhood on C. Using the fact that G is (P_6, C_4)-free, we derive easily the fact that $S(X) = \emptyset$ if X has size at most two, and that $S(X)$ is a clique and thus of size at most k, if $|X| \geq 3$. Since there are at most 2^6 such set X, we conclude that G has at most $64k$ vertices.

Therefore, we assume from now on that G is K_{k+1}-free, C_6-free, and contains an induced five-cycle $C = v_0v_1v_2v_3v_4$. Since G is K_{k+1}-free, $|S_5| \leq k-2$ and $|S_3(v_i)| \leq k-2$ for each i.

Lemma 4. *If $S_1(v_i)$ is anti-complete to $S_2(v_{i+2}, v_{i+3})$, then both sets are bounded.*

Proof of Lemma 4. It suffices to prove this for $i = 0$. We bound $S_1(v_0)$ as follows. Let A be a component of $S_1(v_0)$ and $x \in S_3(v_4)$. If there exist two vertices $y, z \in A$ such that $xy \in E$ and $xz \notin E$, then we may assume that yz is an edge, by the connectivity of A. Thus, $zyxv_4v_3v_2$ induces a P_6. This is a contradiction and therefore x is either complete or anti-complete to A. Moreover, x is complete to $S_3(v_0)$ if x is complete to A, as G is C_4-free. The same property holds if $x \in S_3(v_1)$. Since G has no clique cutset, A must be complete to a pair of vertices $\{x, y\}$ where $x \in S_3(v_1)$ and $y \in S_3(v_4)$. As G is C_4-free, A must be a clique and so of size at most k. Moreover, the number of components of $S_1(v_0)$ is at most $(k-2)^2$. Otherwise an induced C_4 would arise by the pigeonhole principle and the fact there are at most $(k-2)^2$ pairs of vertices $\{x, y\}$ with $x \in S_3(v_1)$ and $y \in S_3(v_4)$. Hence, $|S_1(v_0)| \leq k(k-2)^2 \leq k^3$.

Let us now consider $S_2(v_2, v_3)$. Let A be a component of $S_2(v_2, v_3)$. Observe first that a vertex $x \in S_3(v_2) \cup S_3(v_3)$, is either complete or anti-complete to A, as G is P_6-free. Let $S'_3(v_3)$ and $S'_3(v_2)$ be the subsets of $S_3(v_3)$ and $S_3(v_2)$ consisting of all vertices that are complete to A, respectively. Moreover, $S'_3(v_3)$ and $S'_3(v_2)$ are complete to each other. Otherwise $v_0v_1t'ztv_4$ would induce a C_6 where $t \in S'_3(v_3)$ and $t' \in S'_3(v_2)$ with $tt' \notin E$, and $z \in A$. So, if A is anti-complete to $S_3(v_1) \cup S_3(v_4)$, then $V' = S_5 \cup \{v_2, v_3\} \cup S'_3(v_2) \cup S'_3(v_3)$ would be a clique cutset of G.

Therefore, the set T of neighbors of $S_3(v_1) \cup S_3(v_4)$ in A is nonempty. Let B be a component of $A \setminus T$. Our goal is to show that $B = \emptyset$ by a similar clique cutset argument. It is not hard to see that every vertex $t \in T$ is either complete or anti-complete to B as G is P_6-free. Let $T' \subseteq T$ be the set of those vertices that are complete to A. By the definition of T', any $t \in T'$ is complete to $\{v_2, v_3\} \cup S'_3(v_2) \cup S'_3(v_3)$. Let $x \in S_5$ and $t \in T'$ be a neighbor of some vertex $y \in S_3(v_1)$. Then $xytv_3 \neq C_4$ implies that $tx \in E$. Hence, T' is complete to S_5.

Next we show that T' is a clique. Let t and t' be any two vertices in T', and $p \in B$. If t is a neighbor of some vertex in $S_3(v_4)$ and t' is a neighbor of some vertex in $S_3(v_1)$, then $v_0v_1t'ptv_4$ would induce a C_6, unless $tt' \in E$. Now we assume that both t and t' are neighbors of some vertex in $S_3(v_4)$. If t and t' have a common neighbor in $S_3(v_4)$, then $tt' \in E$ as G is C_4-free. So we may assume that there exist two distinct vertices $x, x' \in S_3(v_4)$ such that $xt, x't' \in E$ but $xt', x't \notin E$. If $tt' \notin E$, then $C^* = xtpt'x'$ would be an induced C_5. However, this contradicts Lemma 1, since v_1 is anti-complete to C^*. Therefore, T is a clique and so $V' \cup T$ is a clique cutset of G. Thus, $B = \emptyset$ and $A = T$. Since A is an arbitrary component of $S_2(v_2, v_3)$, the above argument shows that $S_2(v_2, v_3)$ is dominated by $S_3(v_1) \cup S_3(v_4)$. Note that for any vertex $x \in S_3(v_1) \cup S_3(v_4)$, the neighbors of x in $S_2(v_2, v_3)$ form a clique and hence have size at most k. This shows that $|S_2(v_2, v_3)| \leq 2k(k-2) \leq 2k^2$. □

Now we consider the following cases.

Case 1. There exists some i such that $S_1(v_i)$ and $S_1(v_{i+2})$ are nonempty.

In this case $S_2 = \emptyset$ by the property (P9). Further, each nonempty $S_1(v_i)$ is a clique by (P1). Hence, the size of G is bounded.

Case 2. There exists some i such that $S_1(v_i)$ and $S_1(v_{i+1})$ are nonempty.

In this case $S_2 = S_2(v_i, v_{i+1})$ by (P9). Further, S_1 and S_2 are anti-complete to each other, hence by Lemma 4, the sizes of $S_1(v_i)$ and $S_2(v_i, v_{i+1})$ are bounded.

Case 3. $S_1 = \emptyset$. Then the size of G is bounded by Lemma 4.

Case 4. There is exactly one $S_1(v_i)$ that is nonempty. We may assume that $S_1(v_0) \neq \emptyset$ and that $S_1(v_0)$ is not anti-complete to $S_2(v_2, v_3)$. If $S_2(v_1, v_2) \neq \emptyset$ or $S_2(v_3, v_4) \neq \emptyset$, then each nonempty $S_2(v_i)$ would be a clique (and hence bounded) as G is C_4-free. So we assume that $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$. By Lemma 4, $S_2(v_0, v_4)$ and $S_2(v_0, v_1)$ are bounded. The remaining sets are $S_1(v_0)$ and $S_2(v_2, v_3)$.

Bounding the size of $S_1(v_0)$. Let $X \subseteq S_1(v_0)$ be the set of vertices that are not anti-complete to $S_2(v_2, v_3)$, let $S'_1(v_0) = S_1(v_0) \setminus X$, and let A be a component of $S'_1(v_0)$. As G is P_6-free, we conclude that any vertex $x \in X \cup S_3(v_1) \cup S_3(v_4)$ is either complete or anti-complete to A. If A has a neighbor in both $S_3(v_1)$ and $S_3(v_4)$, then A must be a clique and thus of size at most k. Further, there are at most k^2 such components.

Hence, we may assume that A is anti-complete to $S_3(v_4)$. Let $X' \subseteq X$ be the set of vertices that are complete to A. We claim that X' is a clique. Let $x_i \in X'$ $(i = 1, 2)$ and $p \in A$. If x_1 and x_2 have a common neighbor $y \in S_2(v_2, v_3)$, then $x_1x_2 \in E$ or x_1px_2y would induce a C_4. So, we assume that there exist $y_i \in S_2(v_2, v_3)$ $(i = 1, 2)$ such that $x_iy_i \in E$ but $x_iy_j \notin E$ for $i \neq j$. Now $x_1y_1y_2x_2p$ is an induced C_5, and it is anti-complete to v_1, which contradicts Lemma 1. Let $S'_3(v_1) \subseteq S_3(v_1)$ be the set of vertices that are complete to A. By C_4-freeness of G it is easy to see that $S'_3(v_1)$ is complete to X'. Let $V' = \{v_0\} \cup S_3(v_0) \cup S'_3(v_1)$. If A is anti-complete to X' or S_5, then G has a clique cutset $V' \cup S_5$ or $V' \cup X'$. So, A has a neighbor $x \in X'$ and $p \in S_5$ with $px \notin E$. As $|X'| \leq k^2$ and $|S_5| \leq k$, there are at most k^3 such pairs of vertices. Hence, there are at most k^3 such components, otherwise by the pigeonhole principle an induced C_4 would arise.

Hence, it suffices to bound the size of A. Let $R \subseteq S_5$ be the set of vertices that are not anti-complete to A and have a non-neighbor in X'. Let $S'_5 = S_5 \setminus R$. Note that X' and S'_5 are complete to each other. Let $T \subseteq A$ be the set of vertices that are neighbors of R. Since any $r \in R$ has a non-neighbor $x \in X'$, the set $N_A(r)$ is a clique and hence $|N_A(r)| \leq k$. So, $|T| \leq k^2$. Let B be a component of $A \setminus T$. Observe that any $t \in T$ is either complete or anti-complete to B. If not, let $bb' \in E(B)$ with $bt \in E$ but $b't \notin E$. Let $r \in R$ be a neighbor of t, let $x \in X'$ be a non-neighbor of r, and let $y \in S_2(v_2, v_3)$ be a neighbor of x. If $ry \in E$ then $tryx$ would induce a C_4. But now $b'btrv_2y$ induces a P_6.

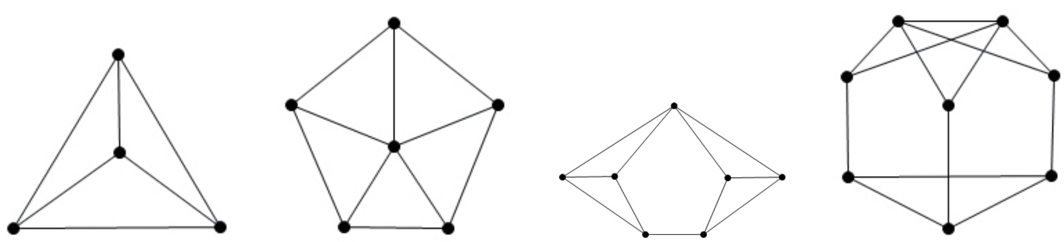
Let $T' \subseteq T$ be the set of vertices that are complete to B. Note that by definition $V^* = S'_5 \cup X' \cup V'$ is a clique. Our goal is to show that $V^* \cup T'$ is a clique. Let $t_i \in T'$. If t_1 and t_2 have a common neighbor in R, then an induced C_4 would arise unless $t_1t_2 \in E$. So, we assume that there exist $r_i \in R$ $(i = 1, 2)$ such that $t_ir_i \in E$ but $t_ir_j \notin E$ for $i \neq j$. Let $b \in B$. Now $t_1r_1r_2t_2b$ induces a C_5. Let $x_i \in X'$ be a non-neighbor of r_i and $y_i \in S_2(v_2, v_3)$ be a neighbor of x_i. Note that for any $r \in S_5$, r is either complete or anti-complete to any edge between $S_1(v_0)$ and $S_2(v_2, v_3)$. If $x_1 = x_2$ or $y_1 = y_2$, $t_1r_1r_2t_2b$ would not be dominating. Hence, $x_1 \neq x_2$, $y_1 \neq y_2$ and $y_ix_j \notin E$ for $i \neq j$. Now $x_1x_2y_2y_1$ induces a C_4. This proves that T' is a clique. By definition, T' is complete to $X' \cup S_3(v_0) \cup \{v_0\}$. Let $q \in S'_3(v_1)$ and $t \in T'$, and $r \in R$ be a neighbor of t. Since $qbtr$ does not induce a C_4, we have $tq \in E$. Now suppose that $q \in S'_5$ and q has a neighbor $b \in B$. As $qbtr$ does not induce a P_4, we have $qt \in E$. Hence, T' is complete to $S'_3(v_1) \cup S'_5$. We have shown that T' is complete to V^* and T' is a clique. So, $V^* \cup T'$ is a clique cutset if $B \neq \emptyset$. Therefore, $A = T$ and has size at most k^2. Thus, $|S_1(v_0)| \leq k^2 + k^2 \times k + k^2 \times k^3 = k^2 + k^3 + k^5$.

Bounding the size of $S_2(v_2, v_3)$. Let $Y \subseteq S_2(v_2, v_3)$ be the set of vertices that are not anti-complete to $S_1(v_0) \cup S_3(v_1) \cup S_3(v_4)$. Let A be a component $S'_2(v_2, v_3) = S_2(v_2, v_3) \setminus Y$. As in previous case, we can show that any $y \in Y$ is either complete or anti-complete to A. Let $Y' \subseteq Y$ be the set of vertices that are complete to A. Since any vertex in $S_2(v_2, v_3)$ that is not anti-complete to $S_1(v_0)$ is a universal vertex in $S_2(v_2, v_3)$, we conclude that Y' is a clique. Let $S'_3(v_3)$ and $S'_3(v_2)$ be the subsets of $S_3(v_3)$ and $S_3(v_2)$ consisting of all vertices that are complete to A, respectively. Let $V' = \{v_3, v_2\} \cup S'_3(v_2) \cup S'_3(v_3)$. If A is anti-complete to S_5 or Y', then $V' \cup S_5$ or $V' \cup Y'$ would be a clique cutset. Hence, A corresponds to a pair of nonadjacent vertices $y \in Y'$ and $r \in S_5$ such that r is not anti-complete to A. By property (P4), each $y \in Y$ is a dominating vertex in $S_2(v_2, v_3)$, and so $|Y'| \leq |Y| \leq k$. Since $|Y'| \leq k$ and $|S_5| \leq k$, there are at most k^2 components of $S'_2(v_2, v_3)$ by the pigeonhole principle and the fact that G is C_4-free.

It suffices to bound the size of A. We define $R \subseteq S_5$, $S'_5 = S_5 \setminus R$ and $T = N_A(R)$ as in the previous case. Then $|T| \leq k^2$. Let B be a component of $A \setminus T$. Note that any $t \in T$ is either complete or anti-complete to B. Let $T' \subseteq T$ be the set of vertices that are complete to A. By definition, $V^* = V' \cup S'_5 \cup Y'$ is a clique. Moreover, T' is complete to $V^* \setminus S'_5$. Let $b \in B$ be a neighbor of $q \in S'_5$, and let $t \in T'$. Then $tb \in E$. Let $r \in R$ be a neighbor of t. Since $btrq$ does not induce a C_4, we have $tq \in E$, as $rb \notin E$ by definition. Hence, T' is complete to vertices in S'_5 that are not anti-complete to B. Finally, we show that T' is a clique. Let $t_i \in T'$ for $i = 1, 2$. Let $r_i \in R$ be a neighbor of t_i. If $r_1 = r_2$, then $t_1t_2 \in E$ or $t_1bt_2r_1$ would induce a C_4. So $r_1 \neq r_2$ and $r_it_j \notin E$ if $i \neq j$. Suppose that $t_1t_2 \notin E$. Then $bt_1r_1r_2t_2$ induces a C_5. Let $y_i \in Y'$ be a non-neighbor of r_i, and let $x_i \in S_1(v_0)$ be a neighbor of y_i ($i = 1, 2$). If $y_1 = y_2$ or $x_1 = x_2$, then $bt_1r_1r_2t_2$ is not dominating, contradicting Lemma 1. Hence, $y_1 \neq y_2$ and $y_ix_j \notin E$. Thus, $x_1x_2 \notin E$. Since $\{y_1, y_2\}$ is complete to A and thus to $\{b, t_1, t_2\}$, the set $\{y_1, y_2, r_1, r_2\}$ induces a disjoint union of two copies of K_2. Moreover, $r_ix_i \notin E$ or $x_ir_it_iy_i$ would induce a C_4. Since $bt_1r_1r_2t_2$ is dominating, we obtain that $r_1x_2 \in E$ and $r_2x_1 \in E$. But then $\{y_1, y_2, x_1, x_2, r_1, r_2\}$ induces a C_6, a contradiction. Hence, $A = T$ and so has size at most k^2. Therefore, $|S_2(v_2, v_3)| \leq k^2 + k^2 \times k^2 = k^4 + k^2$. □

4 Obstructions to 3-Coloring

In this section we explicitly describe all the minimal non-3-colorable (P_6, C_4)-free graphs. We note that [23], in conjunction with [5], describe all minimal non-3-colorable P_5-free graphs, and that [16] describes all minimal non-4-colorable (P_5, C_5)-free graphs.



(a) K_4. (b) W_5. (c) The Hajos graph. (d) F.

Fig. 1. All minimal non-3-colorable (P_6, C_4)-free graphs.

Theorem 2. *There are exactly four minimal non-3-colorable (P_6, C_4)-free graphs, depicted in Figure 1.*

Proof. Let G be a (P_6, C_4)-free minimal non 3-colorable graph. From the first few lines of the proof of Theorem 1 we know that G has $\delta(G) \geq k$, contains no clique cutset, is K_4-free, and contains an induced $C = C_5 = v_0v_1 \dots v_4$. We use the notation S_p, $S_1(v_i)$, $S_2(v_i, v_{i+1})$, and $S_3(v_i)$ from Section 2. From Lemma 1, we have $S_0 = \emptyset$. It is easy to see that $|S_5| \leq 1$. If $|S_5| = 1$, then $G = W_5$. So we may assume that $S_5 = \emptyset$. If there exists an index i such that $S_3(v_i) \neq \emptyset$ and $S_3(v_{i+2}) \neq \emptyset$, then G is the Hajos graph. Hence, at most two $S_3(v_i)$'s are nonempty. Furthermore, each $S_3(v_i)$ is clique and contains at most one vertex, since G is (C_4, K_4)-free. Therefore, $|S_3| \leq 2$. We distinguish three cases.

Case 1. $|S_3| = 2$.

Without loss of generality, assume that $S_3(v_0) = \{x\}$ and $S_3(v_1) = \{y\}$. $xy \notin E$ as G is K_4-free. Also $S_1(v_3) = \emptyset$, otherwise let $t \in S_1(v_3)$ and then $tv_3v_2yv_0x = P_6$. Moreover, x (respectively y) is complete to $S_2(v_3, v_4)$ (respectively $S_2(v_2, v_3)$). Otherwise there exists some vertex $z \in S_2(v_3, v_4)$ with $xz \notin E$. Then $zv_3v_2yv_0x = P_6$. Hence, $S_2(v_3, v_4)$ and $S_2(v_3, v_2)$ are cliques and each of them contains at most one vertex. As $d(v_3) \geq 3$ and $S_1(v_3) = \emptyset$, at least one of them is nonempty. Suppose first that $p \in S_2(v_3, v_4)$ and $q \in S_2(v_2, v_3)$. Then $xp \in E$ and $yq \in E$. It follows from $S_1(v_3) = \emptyset$ and property

(P7) that $S_1 = \emptyset$. Further, $S_2(v_1, v_2) = S_2(v_0, v_4) = \emptyset$ by (P10). Hence we have $S_2 = \{p, q\}$ by (P8), and therefore $N(x) = \{v_4, v_1, v_0, p\}$. Since G is a minimal obstruction, there exists a 3-coloring ϕ of $G - x$. Note that we must have $\phi(v_4) = \phi(q) = \phi(v_1)$ and $\phi(p) = \phi(v_2) = \phi(v_0)$. Consequently, we can extend ϕ to G by setting $\phi(x) = \{1, 2, 3\} \setminus \{\phi(v_0), \phi(v_1)\}$. This contradicts the fact that G is not 3-colorable. Therefore, exactly one of $S_2(v_3, v_4)$ and $S_2(v_3, v_2)$ is empty. Without loss of generality, assume that $S_2(v_3, v_4) = \emptyset$ and let $z \in S_2(v_2, v_3)$. Note that $N(v_3) = \{v_4, v_2, z\}$. Let ϕ be a 3-coloring of $G - v_3$, and note that we must have $\phi(v_4) = \phi(v_1) = \phi(z)$. Thus we can extend ϕ to G. This is a contradiction.

Case 2. $|S_3| = 1$. Without loss of generality, assume that $x \in S_3(v_0)$.

Case 2.1 $S_1(v_0) = \emptyset$.

We claim that in this case $S_2(v_2, v_3) = \emptyset$. Otherwise we let $z \in S_2(v_2, v_3)$. Note that $S_2(v_2, v_3)$ is independent and anti-complete to x since G is (C_4, K_4)-free. By property (P4), the set $S_2(v_2, v_3)$ is anti-complete to S_1. Since $\{v_2, v_3\}$ is not a clique cutset separating $S_2(v_2, v_3)$, one of $S_2(v_3, v_4)$ and $S_2(v_1, v_2)$ is nonempty. We assume by symmetry that $S_2(v_3, v_4) \neq \emptyset$ and let $w \in S_2(v_3, v_4)$. By property (P7), $S_1 = S_1(v_3)$. Moreover, x is anti-complete to $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$. Otherwise consider induced $C_5 = C' = xv_1v_2v_3v_4$. We define S'_3 with respect to C' in the same way as we define S_3. It is easy to check that $|S'_3| \geq 2$ and we are in Case 1. Also, $S_2(v_0, v_4) = \emptyset$. Otherwise let $t \in S_2(v_0, v_4)$. Since xv_0twzv_2 does not induce a P_6, xt must be an edge, and hence $\{x, v_0, v_4, t\}$ would induce a K_4. Therefore, $N(x) = \{v_0, v_1, v_4\}$. If $S_2(v_1, v_2) \neq \emptyset$, then in any 3-coloring ϕ of $G - x$ we would have $\phi(v_1) = \phi(v_4)$ and so ϕ can be extended to G. This contradicts that G is a minimal obstruction. Hence, $S_2(v_1, v_2) = \emptyset$. Note that $S_2 = \{w, z\}$ since G is (C_4, K_4)-free, and hence $N(v_2) = \{v_3, z, v_1\}$. Observe that in any 3-coloring ϕ of $G - v_2$ we have $\phi(z) = \phi(v_4) = \phi(v_1)$. Consequently, we can extend ϕ to G, and this is a contradiction. So the claim follows. By (P7), one of $S_2(v_3, v_4)$ and $S_1(v_2)$ is empty, and one of $S_2(v_1, v_2)$ and $S_1(v_3)$ is empty. On the other hand, $S_2(v_3, v_4) \cup S_1(v_3) \neq \emptyset$ and $S_2(v_1, v_2) \cup S_1(v_2) \neq \emptyset$ as $\delta(G) \geq 3$. This leads to the following two cases.

Case 2.1.a $S_1(v_2) \neq \emptyset$ and $S_1(v_3) \neq \emptyset$ while $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$.

By (P7), the set $S_2(v_0, v_1) = S_2(v_0, v_4) = \emptyset$, and so $S_2 = \emptyset$. Since $\{v_3\}$ is not a clique cutset separating $S_1(v_3)$, we have $S_1(v_1) \neq \emptyset$. Similarly, $S_1(v_4) \neq \emptyset$. Let $u_i \in S_1(v_i)$ for $i \neq 0$. By (P1), each $S_1(v_i)$ is a clique, for $i \neq 0$. Moreover, $|S_1(v_1)| + |S_1(v_3)| = 3$ and $|S_1(v_2)| + |S_1(v_4)| = 3$ as $\delta(G) \geq 3$ and G is K_4-free. If $|S_1(v_1)| = 2$, then $|S_1(v_4)| = 1$ and so $|S_1(v_2)| = 2$. Hence, $\{u_4, v_1, v_2\} \cup S_1(v_1) \cup S_1(v_2)$ induces a Hajos graph. Therefore, $|S_1(v_1)| = |S_1(v_4)| = 1$ and $|S_1(v_2)| = |S_1(v_3)| = 2$. Note that x is anti-complete to $\{u_1, u_4\}$ or G would contain either a C_4 or a W_5 as an induced subgraph. Now G has a 3-coloring: $\{v_1, u_3, u_2, v_4\}$, $\{v_0, v_3, u_1, u'_2\}$, $\{x, u_4, u'_3, v_2\}$ where $u'_2 \in S_1(v_2)$ and $u'_3 \in S_1(v_3)$.

Case 2.1.b $S_2(v_1, v_2) \neq \emptyset$ and $S_2(v_3, v_4) \neq \emptyset$ while $S_1(v_2) = S_1(v_3) = \emptyset$.

Recall that x is anti-complete to $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$. Let $y \in S_2(v_3, v_4)$ and $z \in S_2(v_1, v_2)$. By (P8), $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$. Since $\{v_3, v_4\}$ is not a clique cutset, $S_2(v_3, v_4)$ has a neighbor in $S_1(v_1)$. Similarly, $S_2(v_1, v_2)$ has a neighbor in $S_1(v_4)$. However, this contradicts (P11).

Case 2.2 $S_1(v_0) \neq \emptyset$. Let $y \in S_1(v_0)$.

In this case $xy \in E$ by property (P12). It follows from properties (P7) to (P9) that $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$. If $S_1(v_0)$ is not anti-complete to $S_2(v_2, v_3)$, G would contain F as an induced subgraph and so $G = F$. Hence, we may assume that $S_1(v_0)$ is anti-complete to $S_2(v_2, v_3)$. Therefore, $S_2(v_2, v_3) = \emptyset$ or $\{v_2, v_3\}$ would be a clique cutset of G. Since $\delta(G) \geq 3$, $S_1(v_2) \neq \emptyset$ and $S_1(v_3) \neq \emptyset$. By (P9), $S_2 = \emptyset$. Let $p \in S_1(v_2)$ and $q \in S_1(v_3)$. Note that $pq \notin E$, $py \in E$ and $qy \in E$. Consider induced $C_5 = C' = v_0v_1v_2py$. We define S'_3 and $S'_p(v_0)$ in the same way we define S_3 and $S_p(v_0)$. It is easy to see that $S'_3 = S'_3(v_0) = \{x\}$. By (P1), $S_1(v_0)$ is a clique and hence $S_1(v_0) = \{y\}$. Now we are in Case 2.1 since $S'_1(v_0) = \emptyset$.

Case 3. $|S_3| = 0$, i.e., $V = C \cup S_1 \cup S_2$.

We first claim that now $S_1 \neq \emptyset$. Assume that $S_1 = \emptyset$ and thus $S_2 \neq \emptyset$ or G is 3-colorable. Note that each $S_2(v_i, v_{i+1})$ is an independent set. If there is exactly one nonempty $S_2(v_i, v_{i+1})$, then G is 3-colorable. If there are exactly three nonempty $S_2(v_i, v_{i+1})$'s, then each of them is a clique by property (P2). Since G is K_4-free, each $S_2(v_i, v_{i+1})$ contains only one vertex. Therefore, G has eight vertices and it is easy to check that G is 3-colorable. Let us assume now that there are exactly two nonempty $S_2(v_i, v_{i+1})$. If two $S_2(v_i, v_{i+1})$'s are complete to each other, then we either find a K_4 or conclude that $|S_2| = 2$ so that G is 3-colorable. If two $S_2(v_i, v_{i+1})$'s are anti-complete to each other, G is also 3-colorable. Therefore, we may assume that $S_1(v_0) \neq \emptyset$ and let $x \in S_1(v_0)$. $S_2(v_3, v_4) = S_2(v_1, v_2) = \emptyset$ by (P7). We claim that $S_1(v_3) \neq \emptyset$ and $S_1(v_4) \neq \emptyset$. Otherwise we must have $S_2(v_2, v_3) \neq \emptyset$ and $S_2(v_0, v_4) \neq \emptyset$, and $S_1(v_3) = S_1(v_4) = \emptyset$ since $d(v_3) \geq 3$ and $d(v_4) \geq 3$. By properties (P7) and (P8), the set $S_2(v_0, v_1) = S_1(v_1) = \emptyset$. This contradicts the fact that $\delta(G) \geq 3$. By symmetry, $S_1(v_1) \neq \emptyset$ and $S_1(v_2) \neq \emptyset$. Hence, $S_2 = \emptyset$ and $S_1(v_i)$ is nonempty for each i. Since G is K_4-free, we have $5 \leq |S_1| \leq 7$. It is easy to check that G is 3-colorable if $|S_1| \leq 6$. Thus $|S_1| = 7$ and we may assume that $|S_1(v_0)| = |S_1(v_1)| = 2$. Let $u_i \in S_1(v_i)$ and $u'_0 \in S_1(v_0)$, $u'_1 \in S_1(v_1)$. The subgraph induced by $\{u_3, u_1, v_1, v_0, u_0, u'_0, u'_1\}$ is isomorphic to the Hajos graph. □

5 Obstructions to 4-Coloring

Theorem 3. *There are exactly* 13 *minimal non-4-colorable* (P_6, C_4)*-free graphs, depicted in Figure* 2*.*

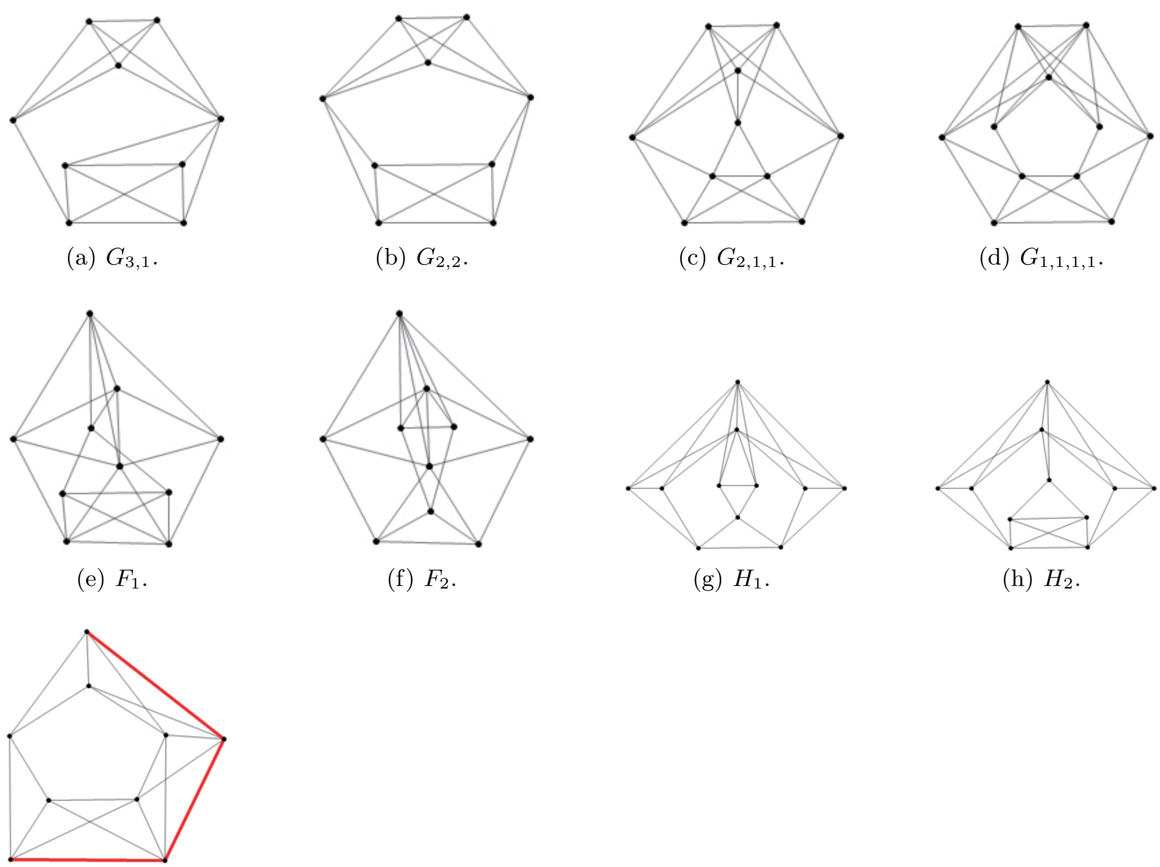
Our proof of Theorem 3 has two parts. The first part deals with the case when G contains an induced W_5. In the second part of the proof, we handle the case when G has no induced W_5. The technique we use is to choose some induced C_5 with a certain minimality condition and derive some additional properties, valid for graphs without induced W_5.

Lemma 5. *Let G be a (P_6, C_4)-free minimal non-*4*-colorable graph with an induced W_5. Then G either is one of four minimal non-*3*-colorable graphs with an additional dominating vertex or G is F_1 or F_2 from Figure* 2*.*

Proof. If G is perfect, then $G = K_5$. Hence, we assume that G is imperfect and K_5-free. Let $C = v_0 \ldots v_4$ be an induced C_5. If $|S_5| \geq 2$, then G is W_5 with an additional dominating vertex. Hence we may assume that every induced C_5 has at most one 5-vertex. In particular, $|S_5| = 1$. Let $S_5 = \{w\}$. Note that S_5 is complete to S_3. Hence, if there exists i such that $S_3(v_i) \neq \emptyset$ and $S_3(v_{i+2}) \neq \emptyset$, then G is the Hajos graph with an additional dominating vertex. So there are at most two $S_3(v_i)$ are nonempty. Further, $|S_3(v_i)| \leq 1$ as G contains no K_5. So $|S_3| \leq 2$.

Case 1. $|S_3| = 2$. Let $x \in S_3(v_0)$ and $y \in S_3(v_1)$. Then $xy \notin E$ as G contains no K_5. If $t \in S_1(v_3)$, then $tv_3v_4xv_1y$ would induce a P_6. So, $S_1(v_3) = \emptyset$. Also, x is complete to $S_2(v_3, v_4)$. Otherwise let $z \in S_2(v_3, v_4)$ with $xz \notin E$. Then $zv_3v_2yv_0x$ would induce a P_6. By symmetry, y is complete to $S_2(v_2, v_3)$. Note that $N(v_3) = \{v_2, v_4, w\} \cup S_2(v_3, v_4) \cup S_2(v_3, v_2)$. Now let ϕ be a 4-coloring of $G - v_3$. Note that $\phi(v_4) = \phi(v_1)$, $\phi(x) = \phi(y)$ and $\phi(v_0) = \phi(v_2)$. As v_4xv_0 induces a triangle, we may assume that $\phi(v_4) = 1$, $\phi(v_0) = 2$, $\phi(x) = 3$. Hence, $\phi(w) = 4$. Since x is complete to $S_2(v_3, v_4)$ and y is complete to $S_2(v_2, v_3)$, any vertex t in $S_2(v_3, v_4) \cup S_2(v_3, v_2)$ has $\phi(t) \neq 3$. Hence, only colors 1, 2, 4 appear on $N(v_3)$ and so we can extend ϕ to G by setting $\phi(v_3) = 3$.

Case 2. $|S_3| = 0$. We claim that $S_1 \neq \emptyset$. If not, $S_2 \neq \emptyset$. If there is exactly one nonempty $S_2(v_i, v_{i+1})$ then $S_2(v_i, v_{i+1}) \cup \{w\}$ must be bipartite otherwise a K_5 or W_5 with an additional dominating vertex would arise. It is easy to see G is 4-colorable. Now suppose that there are exactly three nonempty $S_2(v_i, v_{i+1})$. We may assume that $S_2(v_4, v_3)$, $S_2(v_3, v_2)$ and $S_2(v_2, v_1)$ are nonempty. Observe that each $S_2(v_i, v_{i+1})$ is a clique now and thus contains at most two vertices. Further, $|S_2(v_i, v_{i+1})| + |S_2(v_{i+1}, v_{i+2})| \leq 3$. Let $p \in S_2(v_4, v_3)$, $r \in S_2(v_3, v_2)$ and $q \in S_2(v_2, v_1)$. Suppose that $wr \in E$. Then the fact that $wrqv_1$ does not induce a C_4 implies that $wq \in E$. By symmetry, $wp \in E$. Let $r' \in S_2(v_3, v_2)$. $wqr'v_3 \neq C_4$ implies that $wr' \in E$. Therefore, w is either complete or anti-complete to S_2. In the former case, w is a dominating vertex and hence $G - w$ is a minimal non-3-colorable graph. In the latter case, it is easy to check that G is 4-colorable.



(a) $G_{3,1}$. (b) $G_{2,2}$. (c) $G_{2,1,1}$. (d) $G_{1,1,1,1}$. (e) F_1. (f) F_2. (g) H_1. (h) H_2. (i) G_{P_4}.

Fig. 2. 9 nontrivial minimal non-4-colorable (P_6, C_4)-free graphs.

Suppose now that there are exactly two nonempty $S_2(v_i, v_{i+1})$. If the two sets are complete to each other, then it is same as the above case. So let us assume that the two sets are anti-complete to each other. Without loss of generality, assume that $S_2(v_0, v_1) \neq \emptyset$ and $S_2(v_2, v_3) \neq \emptyset$. Since $G - v_4$ is 4-colorable, both $\{w\} \cup S_2(v_2, v_3)$ and $\{w\} \cup S_2(v_0, v_1)$ are bipartite. In fact, $T = \{w\} \cup S_2(v_0, v_1) \cup S_2(v_2, v_3)$ is also bipartite. If not, let Q be an induced odd cycle in T. As $Q \nsubseteq \{w\} \cup S_2(v_0, v_1)$ and $Q \nsubseteq \{w\} \cup S_2(v_2, v_3)$, Q contains a vertex in both $S_2(v_0, v_1)$ and $S_2(v_3, v_4)$. As $S_2(v_0, v_1)$ is anti-complete to $S_2(v_2, v_3)$, Q must contain w and Q is not a triangle. However, $Q - w$ is connected and hence is fully contained in $S_2(v_0, v_1)$ or $S_2(v_2, v_3)$. This is a contradiction. We therefore can 4-color G as following: $\phi(v_0) = \phi(v_2) = 1$, $\phi(v_1) = \phi(v_3) = 2$, $\phi(v_4) = 3$, $\phi(w) = 4$, and color one partite of T with color 3 and the other with color 4.

Therefore, we may assume that $S_1(v_0) \neq \emptyset$. Going through the same argument for Case 3 in Theorem 2, we conclude that $S_1(v_i) \neq \emptyset$ for each i and $S_2 = \emptyset$. Moreover, w is either complete or anti-complete to S_1 as G is C_4-free. In the former case, w is a dominating vertex and hence $G - w$ is a minimal minimal non-3-colorable graph. In the latter case, we let $u_i \in S_1(v_i)$ for each $0 \leq i \leq 4$, and $wv_2u_2u_4u_1u_3$ induce a P_6.

Case 3. $|S_3| = 1$. Let $x \in S_3(v_0)$. We distinguish two cases.

Case 3.1 $S_1(v_0) = \emptyset$. We claim that $S_2(v_2, v_3) = \emptyset$. Otherwise let $z \in S_2(v_2, v_3)$. By (P7), we have $S_1(v_1) = S_1(v_4) = \emptyset$. Note that $S_2(v_2, v_3)$ is bipartite and is anti-complete to x. Since $\{v_2, v_3\}$ does not separate $S_2(v_2, v_3)$, one of $S_2(v_3, v_4)$ and $S_2(v_1, v_2)$ is nonempty. By symmetry, we assume that $p \in S_2(v_3, v_4)$. By properties (P7) to (P9), we have $S_1 = S_1(v_3)$ and $S_2(v_0, v_1) = \emptyset$. In fact, $S_1(v_3) = \emptyset$ otherwise $\{v_3, w\}$ would separate $S_1(v_3)$. Moreover, x is anti-complete to $S_2(v_3, v_4)$. If not, we may assume $xp \in E$ and consider induced $C_5 = C' = C \setminus \{v_0\} \cup \{x\}$. Observe that $w \in S'_5$ and $\{v_0, p\} \subseteq S'_3$, so we are in Case 1. Going through the same argument in Case 2 we conclude that w is either complete or anti-complete to S_2. In the former case w is a dominating vertex of G and we are done. Therefore, we assume w is anti-complete to S_2. Note also that $2 \leq |S_2| \leq 5$. In the following we either find a minimal obstruction or show G is 4-colorable. Consider first that $S_2 = S_2(v_4, v_3) \cup S_2(v_3, v_2)$. If $S_2 = \{p, z\}$, G has a 4-coloring ϕ: $\{v_4, v_1, z\}$, $\{v_2, v_0, p\}$, $\{x, v_3\}$, $\{w\}$. If there exists $p' \in S_2(v_4, v_3)$ or $z' \in S_2(v_4, v_3)$, then we can extend ϕ by adding p' or z' to $\{w\}$. So, we assume that $S_2(v_4, v_0) \neq \emptyset$ and let $r \in S_2(v_0, v_4)$. The fact that $v_2zprv_0x \neq P_6$ implies that $xr \in E$, hence $S_2(v_4, v_0) = \{r\}$ as G is K_5-free. If $S_2(v_4, v_3) = \{p, p'\}$, then $\{w, x, v_0, v_3, v_4, p, p', r\}$ induces a graph that is not 4-colorable. Note that v_4 is a dominating vertex in this subgraph, and hence G is the Hajos graph with an additional dominating vertex. Thus, $S_2(v_4, v_3) = \{p\}$. Note that $S_2(v_3, v_2)$ might contain a vertex $z' \neq z$ or not. In either case, G has a 4-coloring: $\{v_4, v_1, z\}$, $\{v_2, v_0, p\}$, $\{x, v_3\}$, $\{w, r, z'\}$. Finally, we assume that $S_2(v_0, v_4) = \emptyset$ and let $r \in S_2(v_2, v_1)$. If $S_2(v_2, v_3) = \{z, z'\}$, then $S_2(v_4, v_3) = \{p\}$ and $S_2(v_1, v_2) = \{r\}$ since G is K_5-free. G has a 4-coloring: $\{v_4, v_1, z\}$, $\{v_2, v_0, p\}$, $\{x, v_3, r\}$, $\{w, z'\}$. Hence, $S_2(v_2, v_3) = \{z, \}$. By $\delta(G) \geq 4$ we have $S_2(v_4, v_3) = \{p, p'\}$ and $S_2(v_1, v_2) = \{r, r'\}$. In this case G has a 4-coloring: $\{v_4, v_1, z\}$, $\{v_2, v_0, p\}$, $\{x, v_3, r\}$, $\{w, p', r'\}$.

Therefore, $S_2(v_2, v_3) = \emptyset$. Consider first that $S_2(v_1, v_2) \neq \emptyset$ and $S_2(v_3, v_4) \neq \emptyset$ but $S_1(v_2) = S_1(v_3) = \emptyset$. Then $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$ by (P7) and the fact that $S_2(v_2, v_3) = \emptyset$. Since $\{v_3, v_4, w\}$ is not a clique cutset separating $S_2(v_3, v_4)$, $S_2(v_3, v_4)$ has a neighbor in $S_1(v_1)$. Similarly, $S_2(v_1, v_2)$ has a neighbor in $S_1(v_4)$. However, this contradicts property (P11). Hence, we must have $S_1(v_2) \neq \emptyset$ and $S_1(v_3) \neq \emptyset$ but $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$. By (P7), we have $S_2 = \emptyset$. $S_1(v_1) \neq \emptyset$, since $\{v_3, w\}$ is not a clique cutset separating $S_1(v_3)$. Similarly, $S_1(v_4) \neq \emptyset$. Let $u_i \in S_1(v_i)$ for $i \neq 0$. Note that w is either complete or anti-complete to S_1. In the former case w is a dominating vertex and we are done. In the latter case we find an induced $P_6 = wv_2u_2u_4u_1u_3$.

Case 3.2 $S_1(v_0) \neq \emptyset$. Let $y \in S_1(v_0)$. $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$ by (P7). Consider first that $S_2(v_2, v_3) = \emptyset$. Since $d(v_2) \geq 4$ and $d(v_3) \geq 4$, we have $S_1(v_2) \neq \emptyset$ and $S_1(v_3) \neq \emptyset$. By properties (P7) to (P9), the set $S_2 = \emptyset$. Let $p \in S_1(v_3)$ and $q \in S_1(v_2)$. Consider induced $C_5 = C' = v_0v_1v_2qy$. If w is complete to S_1, w is a dominating vertex in G and we are done. Hence, w is anti-complete to S_1. Note that $S_1(v_0)$ is a clique and thus contains at most two vertices. Suppose first that $S_1(v_0) = \{y, y'\}$. If $S_1(v_3) = \{p, p'\}$, then $\{v_4, v_0, v_3, w, x, p, p', y, y'\}$ induces a G_{P_4} with respect to $C_5 = v_0y'p'v_3w$. Thus, $G = G_{P_4}$ but this contradicts that G contains an induced W_5. Hence, $S_1(v_3) = \{p\}$ and $S_1(v_2) = \{q\}$. As $d(p) \geq 4$ and $d(q) \geq 4$, both $S_1(v_1)$ and $S_1(v_4)$ are nonempty. Let $u_i \in S_1(v_i)$ for each i, and so G contains an induced $P_6 = wv_2u_2u_4u_1u_3$. Hence, $S_1(v_0) = \{y\}$. If $|S_1(v_3)| = |S_1(v_2)| = 3$, then $G = G_{3,1}$ which is W_5-free. Thus we assume that $|S_1(v_3)| \leq 2$. Note that $S_1(v_1) \neq \emptyset$ as $d(p) \geq 4$. Let $t \in S_1(v_1)$, and so $wv_1tpyq = P_6$.

Therefore, $S_2(v_2, v_3) \neq \emptyset$. Let $z \in S_2(v_2, v_3)$. As $\{v_2, v_3, w\}$ is not a clique cutset separating $S_2(v_2, v_3)$, we may assume that $yz \in E$. If $wy \in E$, then the fact that $wyzv_3 \neq C_4$ implies that $wz \in E$. Hence G is the graph F with an additional dominating vertex. If $wz \in E$, G is the graph F with an additional dominating vertex. Therefore, w is anti-complete to $\{y, z\}$. By (P11), we have $S_1 = S_1(v_0)$. Further, $S_2(v_0, v_1) = \emptyset$ otherwise $\{v_0, v_1, x, w\}$ would be a clique cutset. Similarly, $S_2(v_0, v_4) = \emptyset$. Hence, $S_2 = S_2(v_2, v_3)$. Note that $S_1(v_0) \cup S_2(v_2, v_3)$ contains no induced C_5, since v_1 is anti-complete to $S_1(v_0) \cup S_2(v_2, v_3)$. If $S_1(v_0) \cup S_2(v_2, v_3)$ is not bipartite, it must contain a triangle, and hence $G = F_1$ or $G = F_2$. Therefore, we assume that $S_1(v_0) \cup S_2(v_2, v_3)$ is triangle-free and the edges between $S_1(v_0)$ and $S_2(v_2, v_3)$ form a matching. As $d(y) \geq 4$ and $d(z) \geq 4$, y and z have a neighbor $y' \in S_1(v_0)$ and $z' \in S_2(v_2, v_3)$, respectively. Note $y'z' \notin E$ or $z'y'yz = C_4$. If w is complete to $\{y', z'\}$, $\{w, y, y', z, z'x, v_0, v_2, v_3\}$ would induce a $G_{3,1}$ with respect to $C_5 = wy'yzz'$. If $wy' \in E$, then $wz' \notin E$ and hence $v_1wy'yzz' = P_6$. Thus, $wy' \notin E$. Similarly, $wz' \notin E$. By (P7), the vertex z is universal in $S_2(v_2, v_3)$, and so z' cannot have a neighbor different from z, as otherwise A K_5 would arise. As $d(z') \geq 4$, z' must have a neighbor y'' in $S_1(v_0)$. Note that $y'' \notin \{y, y'\}$. Applying the argument for $\{z, y\}$ to $\{z', y''\}$, we conclude that w is anti-complete to $\{z', y''\}$. y'' is not complete to $\{y, y'\}$ or K_5 would arise. If $y''y \in E$, then $y''yzz' = C_4$. If $y''y' \in E$, then $y''y \notin E$ and thus $y''y'yzv_3w = P_6$. As $d(y'') \geq 4$, y'' has a neighbor $y''' \in S_1(v_0)$. $y''' \notin \{y, y', y''\}$. Moreover, y''' is not complete to $\{y, y'\}$. If $y'''y \in E$, then $y'''y' \notin E$ and thus $y'yy'''y''z'v_2 = P_6$. By symmetry, $y'''y' \notin E$. Now $y'''y''z'zyy' = P_6$. □

The following holds under the assumption that G has no induced W_5.

Observation 1 *Let G be a (P_6, C_4)-free graph without an induced W_5. Let $C = v_0v_1v_2v_3v_4$ be an induced C_5 of G. Then the following properties hold.*

(1) If both $S_1(v_{i-1})$ and $S_1(v_{i+1})$ are nonempty then $S_3(v_i)$ is anti-complete to $S_1(v_{i-1})$ and $S_1(v_{i+1})$.

(2) If both $S_2(v_{i-1}, v_i)$ and $S_2(v_i, v_{i+1})$ are nonempty, then $S_3(v_i)$ is complete to $S_2(v_{i-1}, v_i)$ and $S_2(v_i, v_{i+1})$.

(3) Let $x \in S_3(v_{i-1}) \cup S_3(v_{i+1})$. Suppose that $pq \in E$ where $p \in S_1(v_i)$ and $q \in S_2(v_{i+2}, v_{i+3})$. Then x is anti-complete to $\{p, q\}$.

Lemma 6. *Suppose that G is a (P_6, C_4)-free minimal non-4-colorable graph without an induced W_5. Then $G \in \{G_{3,1}, G_{2,2}, G_{2,1,1}, G_{1,1,1,1}, H_1, H_2, G_{P_4}\}$ (see Figure 2).*

We postpone the lengthy proof of this lemma to the Appendix.

6 The Complexity of k-Coloring

We now apply our results to the questions of complexity of k-coloring (P_6, C_4)-free graphs. Reference [12] gives a linear time algorithm for k-coloring (P_t, C_4)-free graphs for any k, t. However, that algorithm depends on Ramsey-type results, and end up using tree-decompositions with very high widths. We offer more practical algorithms for 3-coloring and 4-coloring (P_6, C_4)-free graphs. Our algorithms are linear time, once a clique cutset decomposition is given. Moreover, our algorithms are certifying algorithms. Indeed, they are based on our characterizations of minimal non-k-colorable (P_6, C_4)-free graphs, and when no coloring is found, they exhibit a forbidden induced subgraph from Theorems 2 and 3.

The proof of Theorem 2 can be easily turned into a linear time algorithm for 3-coloring (P_6, C_4)-free graphs without clique cutset. We first test if G is chordal. If so, we can test whether or not G is 3-colorable. Otherwise we have an induced $C = C_\ell$ for some $\ell \geq 4$. Up to this point every step can be done in linear time [13]. If $\ell = 4$ or $\ell \geq 7$ then G is not (P_6, C_4)-free. If $\ell = 5$ we follow the above proof, and it can be readily checked that every step can be performed in linear time. The remaining case is $\ell = 6$, and we can now assume G is also C_5-free. By Lemma 3, either G is specific or C is dominating. In the former case, a k-coloring of G or a K_4 can be found in linear time. Therefore, we assume that C is dominating. We define p-vertices and S_p with respect to C. We either find that G is not (P_6, C_4)-free or the vertices of G consist of $C \cup S_6 \cup S_3$. Finally, in linear time we either find a K_4 or conclude that G has at most 13 vertices, in which case a 3-coloring of G can be obtained by brute force. A similar algorithm applies to the problem of 4-coloring (P_6, C_4)-free graphs. Thus we have the following result.

Theorem 4. *There exist linear time certifying algorithms for* 3*-coloring and* 4*-coloring* (P_6, C_4)*-free graphs, given a clique cutset decomposition of the input graph.*

We note that a clique cutset decomposition can be obtained in time $O(mn)$ [27].

We now complement our results by proving most of the remaining problems of k-coloring (P_t, C_ℓ)-free graphs NP-complete (at least as long as $k \geq 3$ and $\ell > 3$).

Recently, Huang [18] proved that the 5-coloring problem for P_6-free graphs is NP-complete, and that the 4-coloring problem for P_7-free graphs is also NP-complete. The proof used the following framework. We call a k-critical graph *nice* if G contains three independent vertices $\{c_1, c_2, c_3\}$ such that the clique number $\omega(G - \{c_1, c_2, c_3\}) = \omega(G) = k - 1$. For example, any odd cycle of length at least 7 is a nice 3-critical graph.

We give a reduction from 3-SAT, as in [18]. Let I be any 3-SAT instance with variables $X = \{x_1, x_2, \ldots, x_n\}$ and clauses $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$, and let H be a nice k-critical graph with three specified independent vertices $\{c_1, c_2, c_3\}$. We construct a new graph $G_{H,I}$ as follows.

- Introduce for each variable x_i a *variable component* T_i which is isomorphic to K_2, labeled by $x_i\bar{x}_i$. Call these vertices X*-type.*
- Introduce for each variable x_i a vertex d_i. Call these vertices D*-type.*
- Introduce for each clause $C_j = y_{i_1} \vee y_{i_2} \vee y_{i_3}$ a *clause component* H_j which is isomorphic to H, where y_{i_t} is either x_{i_t} or $\bar{x_{i_t}}$. Denote three specified independent vertices in H_j by $c_{i_t j}$ for $t = 1, 2, 3$. Call $c_{i_t j}$ C*-type* and all remaining vertices U*-type.*

For any C-type vertex c_{ij} we call x_i or $\bar{x}_i$ its *corresponding literal vertex*, depending on whether $x_i \in C_j$ or $\bar{x}_i \in C_j$.

- Make each U-type vertex adjacent to each D-type and X-type vertices.
- Make each C-type vertex c_{ij} adjacent to d_i and its corresponding literal vertex.

We refer to [18] for the proofs of the following two lemmas.

Lemma 7. *Let H be a nice k-critical graph. Suppose $G_{H,I}$ is the graph constructed from H and a 3-SAT instance I. Then I is satisfiable if and only if $G_{H,I}$ is $(k+1)$-colorable.*

Lemma 8. *Let H be a nice k-critical graph. Suppose $G_{H,I}$ is the graph constructed from H and a 3-SAT instance I. If H is P_t-free where $t \geq 6$, then $G_{H,I}$ is P_t-free as well.*

To obtain NP-completeness results for (P_t, C_ℓ)-free graphs, we need an additional lemma.

Lemma 9. *Let $\ell \geq 6$. If H is C_ℓ-free, then $G_{H,I}$ is C_ℓ-free.*

Proof. Let $Q = v_1 \dots v_\ell$ be an induced C_ℓ in $G_{H,I}$. Let C_i (respectively $\bar{C_i}$) be the set of C-type vertices that connect to x_i (respectively $\bar{x_i}$). Let $G_i = G[\{T_i \cup \{d_i\} \cup C_i \cup \bar{C_i}\}]$. Note that $G - U$ is disjoint union of G_i, $i = 1, 2, \dots, n$. If $Q \cap U = \emptyset$, then $Q \subseteq G_i$ for some i. It is easy to see that G_i is C_ℓ-free as $\ell \geq 6$. Thus, $Q \cap U \neq \emptyset$. Without loss of generality, we assume that v_1 is a U-type vertex where v_1 is in the jth clause component H_j. If v_2 and v_ℓ are both in H_j, then $Q \subseteq H_j$, which contradicts our assumption that $H_j = H$ is C_ℓ-free. If v_2 and v_ℓ are both in $X \cup D$, then as U-type vertices are complete to X-type and D-type vertices, all other vertices on Q are of C-type. This is impossible since C is independent. The last case is v_ℓ is in H_j and v_2 is in $X \cup D$. Similar to the second case, we have $v_4, v_5, \dots v_{l-1}$ are C-type vertices. This contradicts that $v_4 v_5$ is an edge. $\square$

The following theorem follows now directly from the above lemmas.

Theorem 5. *Let $\ell \geq 6$. Then k-coloring is NP-complete for (P_t, C_ℓ)-free graphs whenever there exists a (P_t, C_ℓ)-free nice $(k-1)$-critical graph.*

We apply Theorem 5 to derive a series of hardness results on (P_t, C_ℓ)-free graphs for various values of k and t.

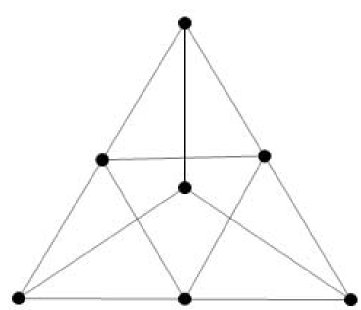


Fig. 3. G_1.

Theorem 6. *Let $k \geq 5$, $t \geq 6$ and $\ell \geq 6$ be fixed integers. Then k-coloring is NP-complete for (P_t, C_ℓ)-free graphs.*

Proof. It is easy to check that the graph G_1 shown in Figure 3 is a nice 4-critical (P_6, C_ℓ)-free graph for any fixed $\ell \geq 6$. Applying Theorem 5 with G_1 will complete our proof. $\square$

Theorem 7. *4-coloring is NP-complete for (P_t, C_ℓ)-free graphs when $t \geq 7$ and $\ell \geq 6$ with $\ell \neq 7$; and 4-coloring is NP-complete for (P_t, C_ℓ)-free graphs when $t \geq 9$ and $\ell \geq 6$ with $\ell \neq 9$.*

Proof. It is easy to check that C_7 is a nice 3-critical (P_t, C_ℓ)-free graph for any $t \geq 7$ and $\ell \geq 6$ except $\ell = 7$, and that C_9 is a nice 3-critical (P_t, C_ℓ)-free graph for any $t \geq 9$ and $\ell \geq 6$ except $\ell = 9$. Applying Theorem 5 with C_7 and C_9 will complete the proof. $\square$

We shall use a different reduction to prove the next result.

Theorem 8. *4-coloring is NP-complete for (P_7, C_5)-free graphs.*

Proof. We reduce NOT-ALL-EQUAL 3-SATISFIABILITY with positive literals only (NAE 3-SAT PL for short) to our problem. The NAE 3-SAT PL is NP-complete [26] and is defined as follows. Given a set $X = \{x_1, x_2, \ldots, x_n\}$ of logical variables, and a set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of three-literal clauses over X in which all literals are positive, does there exist a truth assignment for X such that each clause contains at least one true literal and at least one false literal? Given an instance I of NAE 3-SAT PL we construct a graph G_I as follows.

- For each variable x_i we introduce a single vertex named as x_i. Call these vertices X-type.
- For each variable x_i we introduce a "truth assignment" component F_i where F_i is isomorphic to P_4 whose vertices are labeled by $d_i e'_i e_i d'_i$.
- For each clause $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$ we introduce two copies of C_7 denoted by H_j and H'_j. Choose three independent vertices of H_j and name them as $c_{i_1 j}$, $c_{i_2 j}$ and $c_{i_3 j}$. Choose three independent vertices of H'_j and name them as $c'_{i_1 j}$, $c'_{i_2 j}$ and $c'_{i_3 j}$. Call these vertices C-type and C'-type, respectively. The remaining vertices in clause components are said to be of U-type.
- Make each U-type vertex adjacent to each X-type vertex and each vertex in F_i for $1 \leq i \leq n$.
- Make each C-type vertex c_{ij} adjacent to x_i and d_i and make each C'-type vertex c'_{ij} adjacent to x_i and d'_i.

This completes the construction of G_I. It is easy to see that d_i and d'_i have no common neighbor in $G - U$ and same for e_i and e'_i.

Claim 1. *The instance I is satisfiable if and only if G_I is 4-colorable.*

Proof. Suppose first that G_I is 4-colorable and ϕ is a 4-coloring of G_I. Without loss of generality, we may assume that the two adjacent U-type vertices in H_1 receive color 1 and 2, respectively. Now as U is complete to $X \cup F$, it follows that each x_i and each vertex in F_i receives color 3 or 4. Further, $\phi(d_i) \neq \phi(d'_i)$ for each i. We define a truth assignment as follows.

- We set x_i to be TRUE if $\phi(x_i) = \phi(d_i)$ and to be FALSE if $\phi(x_i) \neq \phi(d_i)$.

We show that every clause C_j contains at least one true literal and one false literal. Suppose x_{i_1}, x_{i_2}, and x_{i_3} are all TRUE. Then it implies that $\phi(d'_{i_j}) \neq \phi(x_{i_j})$ for all $j = 1, 2, 3$. As a result, c'_{i_j} must be colored with color 1 or 2 under ϕ. Moreover, all U-type vertices in H'_j are colored with 1 or 2 under ϕ. This contradictions the fact that $H'_j = C_7$ is not 2-colorable. If x_{i_1}, x_{i_2}, and x_{i_3} are all FALSE we would reach a similar contradiction. Conversely, suppose that every clause C_j contains at least one true literal and one false literal. We define a 4-coloring ϕ as follows.

- Set $\phi(x_i) = 3$ if x_i is TRUE and $\phi(x_i) = 4$ if x_i is FALSE.
- We color vertices in F_i alternately with color 3 and 4 starting from setting $\phi(d_i) = 3$.
- Let $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$ be a clause. Without loss of generality, we may assume that x_{i_1} is TRUE and x_{i_2} is FALSE. It follows from the definition of ϕ that $\phi(x_{i_1}) = \phi(d_{i_1}) = 3$. Hence, we can color $c_{i_1 j}$ with color 4, so that $H_j - c_{i_1 j}$ can be colored with colors 1 and 2. Similarly, we can 4-color H'_j. $\square$

Claim 2. *G_I is C_5-free.*

Proof. Let $Q = v_1 \ldots v_5$ be an induced C_5 in G_I. Let C_i (respectively C'_i) be the set of C-type (respectively C'-type) vertices that are adjacent to x_i. Let G_i be the subgraph of G_I induced by

$\{x_i\} \cup C_i \cup C_i' \cup F_i$. Note that $G - U$ is disjoint union of G_i. Suppose first that $Q \cap U = \emptyset$. Note that both e_i and e_i' have degree 2 in G_i. If Q contains e_i or e_i', then Q contains F_i as an induced subgraph and thus the fifth vertex of Q would be a common neighbor of d_i and d_i', a contradiction. So $Q \cap \{e_i, e_i'\} = \emptyset$. If $Q \cap \{d_i, d_i'\} = \emptyset$, then Q is a star which is impossible. Without loss of generality, we assume that $d_i \in Q$. If d_i' is also in Q, then there would be a common neighbor of d_i and d_i'. So $d_i' \notin Q$. Then the two neighbors of d_i on Q must be of C or C'-type, and so the other two vertices have to be of X-type, which is not possible. Hence, $Q \cap U \neq \emptyset$. Suppose v_1 is of U-type and from H_j. If both v_2 and v_5 are of X-type or F-type, then v_3 and v_4 have to be C or C'-type. But this is a contradiction as $C \cup C'$ is independent. If both v_2 and v_5 are in H_j, then $Q \subseteq H_j$, which is impossible as $H_j = C_7$ is C_5-free. So we assume that v_2 is of X-type or F-type and v_5 is in H_j. Then v_5 must be C or C'-type. Moreover, v_4 must be of C or C'-type as it is not adjacent to v_1 or v_2. This is impossible since v_4v_5 is an edge. □

Claim 3. *G_I is P_7-free.*

Proof. Let P be an induced P_7 in G_I. We first consider the case $P \cap U \neq \emptyset$. Let $u \in P$ be an U-type vertex and u is in some clause component H_j. For any vertex x on P we denote by x^- and x^+ the left and right neighbor of x on P, respectively. Suppose first that u is an endvertex of P. Then u^+ is in $X \cup F$ or H_j. If u^+ is in H_j, then $P \subseteq H_j$, which is a contradiction since $H_j = C_7$ is P_7-free. So u^+ is in $X \cup F$. If u^{++} is in $C \cup C'$, then $|P| = 3$, a contradiction. So u^{++} is in U. But now u^{+++} must be in $C \cup C'$, and thus $|P| = 4$, a contradiction. Hence, u must have degree 2 on P. If u^- and u^+ are both in H_j, then $P \subseteq H_j$, a contradiction. If u^- and u^+ are both in $X \cup F$, then u^{--} and u^{++} are both of C- or C'-type. Hence, $|P| \leq 5$ since $C \cup C'$ is independent. So we may assume that u^+ is in H_j and u^- is in $X \cup F$. Now u^+ must be of C- or C'-type and hence an endvertex of P. Therefore, $|P| \leq 2 + 4 - 1 = 5$.

We have shown that $P \cap U = \emptyset$. So $P \subseteq G_i$ for some i. Now we show that $|P \cap C_i| = 1$. Otherwise assume that $|P \cap C_i| = 2$. Let c_1 and c_2 be the vertices in $P \cap C_i$. If c_1 and c_2 are not at distance 2 on P, then x_i and d_i are not on P otherwise P would not be induced. However, x_i and d_i are the only neighbors of C-type vertices in G_i, a contradiction. So, c_1 and c_2 must be at distance 2 on P. If they are connected by d_i, then $x_i \notin P$ and vice versa. But now $|P| = 3$, since $C_i \cup C_i'$ is independent. Therefore, $|P \cap C| \leq 1$ and similarly $|P \cap C'| \leq 1$. So, we must have $F_i \cup \{x_i\} \subseteq P$, and thus $P = C_7$, a contradiction. □

The following result is a direct corollary of Theorem 8.

Theorem 9. *Let $k \geq 4$ and $t \geq 7$. Then k-coloring is NP-complete for (P_t, C_5)-free graphs.*

7 Conclusions

We have undertaken a first systematic study of the k-coloring problem for graphs without an induced cycle C_ℓ and an induced path P_t. We have shown that while for many values of k, t and ℓ these problems are NP-complete, the case of (P_6, C_4)-free graphs offers much structure to be exploited. In particular, we have shown that there are for each k only finitely many non-k-colorable (P_6, C_4)-free graphs.

For $k = 3$ and $k = 4$, we were able to describe these minimal obstructions explicitely, and so obtained certifying polynomial time (linear time if a clique cutset decomposition is given) algorithms for coloring (P_6, C_4)-free graphs. However, for larger k, we do not know certifying k-coloring algorithms for (P_6, C_4)-free graphs.

Our hardness results come close to classifying the complexity all cases of k-coloring for (P_t, C_ℓ)-free graphs. There seem to be two stubborn cases about which not much can be said with the current tools, when $k = 3$ or $\ell = 3$. (But note [6,7].) Beyond these cases, our results leave only the following remaining open problems.

Problem 1. What is the complexity of k-coloring (P_6, C_5)-free graphs for $k \geq 4$?

Problem 2. What is the complexity of 4-coloring (P_6, C_6)-free graphs?

Problem 3. What is the complexity of 4-coloring (P_t, C_7)-free graphs for $t = 7$ and $t = 8$?

In [18] Huang conjectured that 4-coloring is polynomial time solvable for P_6-free graphs. If the problems in Problem 1 for $k = 4$ or Problem 2 are polynomial, this would add evidence to the conjecture.

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Appendix

Proof of Lemma 6. By Lemma 5, we may assume that no induced C_5 has a 5-vertex. Let $C = v_0 \dots v_4$ be an induced C_5 such that $|S_3|$ is as small as possible. As the graph $G_{3,1}$ is a minimal obstruction, we obtain that $|S_3| \le 7$. Suppose first $|S_3| = 0$. It is easy to check that either G contains a K_5 or is 4-colorable if $S_1 = \emptyset$. Hence, we may assume that $S_1(v_0) \neq \emptyset$. Going through the same argument as in Case 2 of Lemma 5, we conclude that each $S_1(v_i) \neq \emptyset$ for each i. If two $S_1(v_i)$ have size at least 3, then G either contains K_5 or $G_{3,1}$. Now suppose that $|S_1(v_0)| = 3$. Thus $|S_1(v_2)| = |S_1(v_3)| = 1$ or K_5 arises. If $|S_1(v_1)| = |S_1(v_4)| = 2$, then $G = G_{2,2}$. Otherwise one of $S_1(v_1)$ and $S_1(v_4)$ has size 1 in which case it is easy to check G is 4-colorable. Now we assume that each $|S_1(v_i)| \le 2$. If all but one $S_1(v_i)$ have size 2, then $G = G_{P_4}$. Otherwise, there are at least two $|S_1(v_i)| = 1$. It is easy to check G is 4-colorable. Therefore, $1 \le |S_3| \le 7$.

Case 1. $|S_3| = 1$. Let $x \in S_3(v_0)$. Suppose first that $S_1 = \emptyset$. If $S_2(v_2, v_3) = \emptyset$, then both $S_2 = S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ have at least two vertices as $d(v_2) \ge 4$ and $d(v_3) \ge 4$. By (P8), $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$. As $d(x) \ge 3$, we have that x is not anti-complete to S_2 and hence complete to S_2 by (P10). Now G contains $G_{3,1}$ as an induced subgraph and so $G = G_{3,1}$. Thus $S_2(v_2, v_3) \neq \emptyset$. By (P8), one of $S_2(v_0, v_1)$ and $S_2(v_0, v_4)$ is empty, say $S_2(v_0, v_1)$. As $d(v_1) \ge 4$ we have $S_2(v_1, v_2) \neq \emptyset$ and thus $S_2(v_0, v_4) = \emptyset$. As $d(v_4) \ge 4$ we have $S_2(v_3, v_4) \neq \emptyset$. By (P10), x must be anti-complete to S_2. But now $d(x) = 3$ contradicting $\delta(G) \ge 4$.

Therefore, $S_1 \neq \emptyset$. Suppose first that $S_1(v_0) \neq \emptyset$. Going through the same argument as in Case 2 of Lemma 5 we conclude that $S_1(v_i) \neq \emptyset$ for each i and $S_2 = \emptyset$. It is easy to check that either $G \in \{G_{3,1}, G_{2,2}, G_{P_4}\}$ or G is 4-colorable. So, $S_1(v_0) = \emptyset$. We first show that $S_1(v_2)$ is anti-complete to $S_2(v_0, v_4)$. If not, let $x \in S_1(v_2)$ be adjacent to $y \in S_2(v_0, v_4)$. By (P11), $S_1 = S_1(v_2)$. Further, $S_2(v_0, v_1) = S_2(v_4, v_3) = \emptyset$, and one of $S_2(v_1, v_2)$ and $S_2(v_3, v_2)$ is empty by properties (P7) to (P9). As $\delta(G) \ge 4$ there are at least two 3-vertices adjacent to v_1 or v_3. This is impossible as $|S_3| = 1$. Now if $S_1(v_2) \neq \emptyset$, then $S_1(v_4) \neq \emptyset$ as v_2 does not separate $S_1(v_2)$ and by (P9) we have $S_2 = \emptyset$. As $d(v_3) \ge 4$ $S_1(v_3) \neq \emptyset$ and hence $S_1(v_1) \neq \emptyset$. Now by Observation 1, we have x is anti-complete to S_1, contradicting $\delta(G) \ge 4$. Therefore, $S_1(v_2) = \emptyset$. Similarly, $S_1(v_3) = \emptyset$. Now we may assume that $S_1(v_1) \neq \emptyset$. Then $S_2(v_2, v_3) = \emptyset$. As $\delta(G) \ge 4$, both $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ have at least two vertices and so $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$. By (P10), x must be anti-complete to S_2 or $G = G_{3,1}$. By Observation 1 and $d(x) \ge 4$ we have $S_1(v_4) = \emptyset$. But now $\{v_1, x\}$ is a clique cutset separating $S_1(v_1)$.

Case 2. $|S_3| = 2$. We distinguish two subcases.

Case 2.1 There exists some i such that $S_1(v_i) \neq \emptyset$ and $S_1(v_{i+1}) \neq \emptyset$. Without loss of generality, assume that $x \in S_1(v_0)$ and $y \in S_1(v_1)$. By (P9), $S_2 = S_2(v_0, v_1)$. Suppose first that $S_2(v_0, v_1) \neq \emptyset$. Then $S_1(v_2) = S_1(v_4) = \emptyset$. As $d(v_2) \ge 4$ and $d(v_4) \ge 4$ we have $|S_3(v_3)| = 2$. Note that $S_1(v_3)$ is a clique since $S_1(v_0) \neq \emptyset$ and is complete to $S_3(v_3)$. So, $|S_1(v_3)| \le 1$ or K_5 would arise. Note that $S_2(v_0, v_1)$ is bipartite. If $S_1(v_3) = \emptyset$, then it is easy to check that G is 4-colorable. So we assume that $S_1(v_3) = \{w\}$. If w has two neighbors in $S_2(v_0, v_1)$, then $G = G_{2,1,1}$. Thus w has at most one neighbor in $S_2(v_0, v_1)$. If $|S_1(v_0)| = 3$ or $|S_1(v_1)| = 3$, then $G = G_{3,1}$. Thus $|S_1(v_i)| \le 2$ for $i = 0, 1$. Now it is easy to check that G is 4-colorable. Therefore, $S_2(v_0, v_1) = \emptyset$ and thus $S_2 = \emptyset$. We consider two subcases.

Case 2.1.a There exists some i such that $|S_3(v_i)| = 2$. Suppose that $i = 0$ (or $i = 1$). As $d(v_2) \ge 4$ and $d(v_3) \ge 4$, we have that $|S_1(v_i)| \ge 2$ for $i = 2, 3$. By (P12), x is complete to $S_3(v_0)$ and hence $S_1(v_0) = \{x\}$. If $|S_1(v_1)| \ge 2$, then $G = G_{2,2}$. So assume that $S_1(v_1) = \{y\}$. If $|S_1(v_i)| = 3$ for some $i \in \{2, 3\}$, then $G = G_{3,1}$. If $|S_1(v_4)| \ge 2$ then $G = G_{2,2}$. Hence, $|S_1(v_3)| = |S_1(v_2)| = 2$ and $|S_1(v_4)| \le 1$. Let $u_i \in S_1(v_i)$ for nonempty $S_1(v_i)$ and let $u_i' \in S_1(v_i)$ with $u_i' \neq u_i$ for $i = 2, 3$. Let $S_3(v_0) = \{z, z'\}$. If u_4 exists, then $S_3(v_0)$ is anti-complete to $S_1 \setminus \{x\}$ by Observation 1. Thus G has a 4-coloring: $\{v_1, v_4, x\}$, $\{v_0, v_2, u_4, u_3\}$, $\{v_3, y, u_2, z\}$, $\{u_2', u_3', z'\}$. If u_4 does not exist, then y may or may not be adjacent to S_3. In either case, G has a 4-coloring: $\{v_1, v_4, x\}$, $\{v_0, v_3, u_2, y\}$, $\{v_2, u_3, z\}$, $\{z', u_2', u_3'\}$.

Now suppose that $i = 4$ (or $i = 2$). As $d(v_i) \geq 4$ we have $|S_1(v_i)| \geq 2$ for $i = 1, 2$. We may assume that $S_1(v_4) = \emptyset$ or we are in the case $i = 0$. Since $\{v_1\}$ does not separate $S_1(v_1)$, $S_1(v_3) \neq \emptyset$. By Observation 1, S_3 is anti-complete to S_1. Note that $|S_1(v_3)| + |S_1(v_0)| \leq 3$ otherwise $G = G_{3,1}$ or $G = G_{2,2}$. Let $u_i, u_i' \in S_1(v_i)$ for $i = 2, 3$ and $S_3(v_4) = \{z, z'\}$. If each $S_1(v_i)$ has size less than 3, then G has a 4-coloring: $\{v_0, v_3, y\}$, $\{v_2, v_4, u_1', x\}$, $\{v_1, u_2, u_3, z\}$, $\{u_2', u_3', z'\}$. So assume without loss of generality that $|S_1(v_2)| = 3$ and hence $S_1(v_0) = \{x\}$. It is easy to check G is also 4-colorable.

Finally, suppose that $i = 3$. As $d(v_i) \geq 4$ we have $|S_1(v_i)| \geq 2$ for $i = 0, 1$. If $S_1(v_3) = \emptyset$, then as G has no clique cutset, $S_1(v_j) \neq \emptyset$ for $j = 1, 4$, and we are in the case $i = 4$. So $S_3(v_1) = \{z\}$. Note that $|S_1(v_0)| = |S_1(v_1)| = 2$ or $G = G_{3,1}$. Moreover, each of $S_1(v_1)$ and $S_1(v_4)$ has size at most 1 or $G = G_{2,2}$. Now it is easy to check that G is 4-colorable.

Case 2.1.b Each $S_3(v_i)$ has at most one vertex. Let N be the set of v_i such that $S_3(v_i) \neq \emptyset$. Then there are six possible cases.

Suppose first that $N = \{v_0, v_1\}$. Let $t \in S_3(v_0)$ and $r \in S_3(v_1)$. Since $xtv_4v_3v_2r \neq P_6$, we have $rt \in E$ or $rx \in E$. Similarly, the fact that $yrv_2v_3v_4t \neq P_6$ implies that $rt \in E$ or $yt \in E$. If $rt \notin E$, then xr and yt are edges and so $txry = C_4$. Hence, $rt \in E$. As $d(v_3) \geq 4$, $|S_1(v_3)| \geq 2$. Similarly, both $S_1(v_2)$ and $S_1(v_4)$ are nonempty. By Observation 1, t (respectively r) is anti-complete to $S_1 \setminus S_1(v_0)$ (respectively $S_1 \setminus S_1(v_1)$). Note that $|S_1(v_3)| = 2$ or $G = G_{3,1}$. If $S_1(v_0)$ has two vertices, then $\{v_0, v_1, t, r, y\} \cup S_1(v_0) \cup S_1(v_3)$ induces a G_{P_4} with respect to $tv_1u_1v_3v_0$ and $u_3'u_0'v_0r$ where $u_i, u_i' \in S_1(v_i)$ for each i. Hence, $S_1(v_0) = \{x\}$. Similarly, $S_1(v_1) = \{y\}$. If $|S_1(v_2)| = 3$, then $G = G_{2,2}$. So $|S_1(v_i)| \leq 2$ for $i = 2, 4$. Now it is easy to check G is 4-colorable.

Now suppose that $N = \{v_1, v_2\}$. Let $t \in S_3(v_2)$ and $r \in S_3(v_1)$. As $\delta(G) \geq 4$ we have $|S_1(v_4)| \geq 2$ and $|S_1(v_3)| \geq 1$. By Observation 1, we have $ty \notin E$. Since $tv_3v_4wyr \neq P_6$, we have $rt \in E$, where $w \in S_1(v_4)$. We may assume that $S_1(v_2) = \emptyset$ or we are in the case $N = \{v_0, v_1\}$. Note that r is anti-complete to $S_1(v_0)$ as $S_1(v_3) \neq \emptyset$ and G is C_4-free. Since $d(x) \geq 4$ we have $|S_1(v_0)| + |S_1(v_3)| = 4$. If $|S_1(v_4)| = 3$, then $|S_1(v_1)| = 1$. Also, $|S_1(v_3)| \leq 2$ or $G = G_{3,1}$. Now G is 4-colorable. So, $|S_1(v_4)| = 2$. As $\delta(G) \geq 4$ we have $|S_1(v_1)| = 2$. If $|S_1(v_3)| \geq 2$ then $\{v_1, v_2, v_3, u_3', u_1', u_3, u_1, r, t\}$ induces a G_{P_4} with respect to $v_1v_2v_3u_3'u_1'$ and u_3u_1rt where $u_i, u_i' \in S_1(v_i)$. So, $|S_1(v_3)| = 1$ and then $|S_1(v_0)| = 3$. Now $S_1(v_0) \cup \{v_0, u_3, r, v_1, u_1, u_1'\}$ induces a $G_{2,2}$ with respect to induced $K_5 - e = S_1(v_0) \cup \{v_0, u_3\}$ and $K_4 = rv_1u_1u_1'$. This completes the proof of $N = \{v_1, v_2\}$.

Let $N = \{v_2, v_3\}$. As $\delta(G) \geq 4$, both $S_1(v_1)$ and $S_1(v_4)$ are nonempty, and $S_1(v_0)$ has at least two vertices. If one of $S_1(v_2)$ and $S_1(v_3)$ is nonempty, we are in one of previous cases. But now $\{v_0\}$ is a clique cutset separating $S_1(v_0)$.

Let $N = \{v_0, v_3\}$ and let $r \in S_3(v_0)$, $t \in S_3(v_3)$. As $\delta(G) \geq 4$ we have $S_1(v_i) \neq \emptyset$ for $i \neq 4$. Let $u_i \in S_1(v_i)$. By G is C_4-free, we have r (respectively t) is anti-complete to $S_1(v_1)$ (respectively $S_1(v_2)$). Then as $d(u_1) \geq 4$ and $d(u_2) \geq 4$, we have $|S_1(v_1)| + |S_1(v_3)| = 4$ and $|S_1(v_2)| + |S_1(v_0)| = 4$. If $|S_1(v_0)| = 3$, then $|S_1(v_2)| = |S_1(v_2)| = 1$, and so $|S_1(v_1)| = 3$. Now $G = G_{3,1}$. So each $S_1(v_i)$ has size 2. But now $\{r, v_4, t, v_3, v_0\} \cup S_1(v_0) \cup S_1(v_1)$ induces a G_{P_4}.

Let $N = \{v_0, v_2\}$. As in the case where $N = \{v_0, v_3\}$, we obtain that each $S_1(v_i) \neq \emptyset$. Moreover, each $S_1(v_i)$ has size 2 except $S_1(v_1)$. Hence, $G = G_{P_4}$.

The case $N = \{v_2, v_4\}$ is the same as $N = \{v_0, v_3\}$. This completes the proof of Case 2.1.

Case 2.2 One of $S_1(v_i)$ and $S_1(v_{i+1})$ is empty for each i. Hence, there are at most two nonempty $S_1(v_i)$. We consider following three cases.

Suppose first that there are exactly two $S_1(v_i)$ that are nonempty. Without loss of generality, we assume that $S_1(v_0)$ and $S_1(v_2)$ are nonempty. By (P9), we have $S_2 = \emptyset$. As $d(v_1) \geq 4$ and $d(v_4) \geq 4$, we have $|S_3(v_0)| = 2$ but this contradicts $d(v_3) \geq 4$.

Now we suppose that $S_1(v_0) \neq \emptyset$ while $S_1(v_i) = \emptyset$ for $i \neq 0$. Let $x \in S_1(v_0)$. Note that $S_2(v_2, v_1) = S_2(v_3, v_4) = \emptyset$.

We first claim that S_3 is not anti-complete to S_1. If not, x has a neighbor $y \in S_2(v_2, v_3)$ or $\{v_0\}$ would be a clique cutset. Further, one of $S_2(v_0, v_4)$ and $S_2(v_0, v_1)$ is empty, say $S_2(v_0, v_4)$. Since $d(v_4) \geq 4$, we have $S_3(v_1) = S_3(v_2) = \emptyset$. Also, $S_3(v_0) = \emptyset$ by our assumption. If $S_3(v_4) \neq \emptyset$, then $|S_2(v_0, v_1)| \geq 1$ since $d(v_1) \geq 4$. Since $\{v_0, v_1\}$ does not separate $S_2(v_0, v_1)$, $S_2(v_0, v_1)$ has a neighbor t in $S_3(v_4)$ and hence $ty \in E$ by the property (P10). Now $tyxv_0 = C_4$. So it must be the case that $|S_3(v_3)| = 2$. Then $|S_2(v_0, v_1)| \geq 1$ since $d(v_1) \geq 4$ and so $\{v_0, v_1\}$ is a clique cutset separating $S_2(v_0, v_1)$.

Hence, S_3 is not anti-complete to S_1, and thus $|S_3(v_3) \cup S_3(v_2)| \leq 1$. By $d(v_2) \geq 4$ and $d(v_3) \geq 4$ we have that $S_2(v_2, v_3) \neq \emptyset$. We first consider the case that $S_1(v_0)$ is anti-complete to S_2. If $S_3(v_3) \neq \emptyset$, then G would have a clique cutset separating $S_1(v_0)$. So, $S_3(v_3) = S_3(v_2) = \emptyset$. Further, there is no $S_3(v_i)$ having size 2 or G would have a clique cutset. Let $S_3 = \{r, t\}$. If $r \in S_3(v_4)$ and $t \in S_3(v_0)$, then $rt \notin E$ or clique cutset would arise. Thus, $S_2(v_0, v_1) \neq \emptyset$ as $d(v_1) \geq 4$. As $\{v_2, v_3\}$ is not a clique cutset, r is not anti-complete to $S_2(v_2, v_3)$ and hence complete to $S_2(v_2, v_3)$ and $S_2(v_0, v_1)$. As $d(v_2) \geq 2$, $|S_2(v_2, v_3)| \geq 2$ and thus $S_2(v_0, v_1) = \{q\}$. As $d(q) \geq 4$ we have $qt \in E$ and thus $rqtv_4 = C_4$. By symmetry, it is impossible for $r \in S_3(v_0)$ and $t \in S_3(v_1)$. Finally, it is impossible for $S_3(v_4)$ and $S_3(v_1)$ to be nonempty by properties (P7) to (P9) and $\delta(G) \geq 4$. Therefore, we may assume that x has a neighbor y in $S_2(v_2, v_3)$. Without loss of generality, we assume that $S_3(v_2) = \emptyset$. Next we distinguish two cases by properties (P7) to (P9).

(I) $S_2(v_0, v_1) = \emptyset$. Then $S_3 = S_3(v_0) \cup S_3(v_1)$ by $d(v_1) \geq 4$. If $|S_3(v_0)| = 2$, then $|S_2(v_2, v_3)| \geq 2$ by $d(v_3) \geq 4$. If x has a different neighbor y' in $S_2(v_2, v_3)$, then $G = G_{2,1,1}$. If there is an edge other than xy between $S_1(v_0)$ and $S_2(v_2, v_3)$, then $G = G_{1,1,1,1}$. Hence, $S_2(v_2, v_3) \setminus \{y\}$ is anti-complete to S_1 and thus $\{v_2, v_3\}$ is a clique cutset by (P4) to (P6). If $|S_3(v_1)| = 2$, then we are in the case S_3 is anti-complete to S_1. Now let $t \in S_3(v_1)$ and $r \in S_3(v_0)$. By $\delta(G) \geq 4$ we have $|S_2(v_2, v_3)| \geq 2$ and $|S_2(v_0, v_4)| \geq 1$. As $\{v_0, v_4, r\}$ does not separate $S_2(v_0, v_4)$, t is not anti-complete to $S_2(v_0, v_4)$ and hence complete to $S_2(v_0, v_4)$ and $S_2(v_2, v_3)$. Thus $S_2(v_0, v_4) = \{q\}$. As $d(q) \geq 4$ we have $qr \in E$ and thus $rt \in E$ or $qrv_1t = C_4$. But now it is easy to see G contains $G_{3,1}$ as an induced subgraph.

(II) $S_2(v_0, v_4) = \emptyset$. So, $S_3(v_i) = \emptyset$ for $i = 1, 2$. Note that it is impossible that $|S_3(v_3)| = 2$ by our assumption. If $|S_3(v_4)| = 2$, then we are in the case where S_3 is anti-complete to S_1. If $|S_3(v_0)| = 2$, then the only edge between $S_1(v_0)$ and $S_2(v_2, v_3)$ is xy or $G \in \{G_{1,1,1,1}, G_{2,1,1}\}$. As G has no clique cutset, $S_1(v_0) = \{x\}$ and $S_2(v_0, v_1) = \emptyset$. Note that $S_2(v_2, v_3)$ is bipartite and thus G is 4-colorable. If $|S_3(v_3)| = |S_3(v_0)| = 1$, then $S_2(v_0, v_1) \neq \emptyset$ by $d(v_1) \geq 4$ and thus $\{v_0, v_1\} \cup S_3(v_0)$ would be a clique cutset. If $|S_3(v_4)| = |S_3(v_0)| = 1$, then it is same as the third case in **(I)**. Finally, $|S_3(v_3)| = |S_3(v_4)| = 1$. Let $r \in S_3(v_4)$ and $t \in S_3(v_3)$. Then $|S_2(v_0, v_1)| \geq 2$ as $d(v_1) \geq 4$. As G has no clique cutset, r is not anti-complete to $S_2(v_0, v_1)$ and thus complete to S_2. Thus, $S_2(v_0, v_1) = \{p, p'\}$ and $S_2(v_2, v_3) = \{y\}$. Since $tv_3yxv_0p \neq P_6$, we have $ty \in E$ and so $rt \in E$ or $yrv_4t = C_4$. Now $G = G_{3,1}$.

Finally, we assume that $S_1 = \emptyset$. Consider first that $|S_3(v_0)| = 2$. If $S_2(v_2, v_3) = \emptyset$, then both $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ contain at least two vertices since $d(v_2) \geq 4$ and $d(v_3) \geq 4$. As G has no clique cutset, there exists $t \in S_3(v_0)$ that is complete to S_2 by (P9). But now $G = G_{3,1}$. So, let $x \in S_2(v_2, v_3)$. Then one of $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ is nonempty, say $y \in S_2(v_3, v_4)$. If $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$, then $|S_2(v_3, v_2)| = 2$ and $|S_2(v_3, v_4)| = 1$. Now G is 4-colorable. Hence, either $S_2(v_1, v_2) \neq \emptyset$ or

$S_2(v_0, v_4) \neq \emptyset$. In the former case, S_3 is anti-complete to S_2 or C_4 occurs and thus G is 4-colorable. In the latter case, we have $|S_2(v_3, v_2)| = 2$, $|S_2(v_3, v_4)| = 1$ and $|S_2(v_0, v_4)| \leq 2$. Note that any $t \in S_2(v_0, v_4)$ is not complete to $S_1(v_0)$ or K_5 would occur, and hence G is 4-colorable. Hence, no $S_3(v_i)$ has size 2. Suppose that $|S_3(v_0)| = |S_3(v_2)| = 1$. If $S_2(v_3, v_4) = \emptyset$, then $S_2(v_0, v_4) \neq \emptyset$ and so $S_3(v_0) \cup \{v_0, v_4\}$ is a clique cutset. So, $S_2(v_3, v_4) \neq \emptyset$. Now as $d(v_0) \geq 2$ and $d(v_2) \geq 2$ we have three $S_2(v_i, v_{i+1})$ are nonempty, contradicting the property (P7).

So, there must be the case that $|S_3(v_0)| = |S_3(v_1)| = 1$. Let $r \in S_3(v_0)$ and $t \in S_3(v_1)$. If $S_2(v_2, v_3) = \emptyset$, then $|S_2(v_1, v_2)| \geq 1$ and $|S_2(v_3, v_4)| \geq 2$. As G has no clique cutset, r is not anti-complete to $S_2(v_3, v_4)$ and thus complete to S_2. So, $|S_2(v_1, v_2)| = 1$ and $|S_2(v_3, v_4)| = 2$. Let $q \in S_2(v_1, v_2)$. Note that $qt \in E$ as $d(q) \geq 4$. Hence, $rt \in E$ or $qrv_1t = C_4$. Now $G = G_{3,1}$. Therefore, $S_2(v_2, v_3) \neq \emptyset$. By symmetry, $S_2(v_4, v_3) \neq \emptyset$. Let $p \in S_2(v_3, v_2)$ and $q \in S_2(v_4, v_3)$. If $S_2 = S_2(v_3, v_4) \cup S_2(v_3, v_2)$, then G has a 4-coloring ϕ: $\{v_0, v_2, q\}$, $\{v_1, v_4, p\}\}$, $\{r, p'\}$, $\{v_3, t\}$ if $S_2(v_3, v_2) = \{p, p'\}$. If $S_2(v_4, v_3) = \{q, q'\}$, then G has a 4-coloring by replacing $\{r, p'\}$, $\{v_3, t\}$ in ϕ with $\{t, q'\}$, $\{v_3, r\}$. Hence we assume by symmetry that $S_2(v_1, v_2) \neq \emptyset$. Let $s \in S_2(v_1, v_2)$. By (P9) and C_4-freeness of G, we have r is anti-complete to S_2 and thus $rt \in E$ since $d(r) \geq 4$. Suppose $S_2(v_2, v_3) = \{p\}$. If $S_2(v_1, v_2) = \{s, s'\}$, then t is not complete to $\{s, s'\}$, say $ts' \notin E$, since G is K_5-free. Hence G has a 4-coloring: $\{r, q, v_2\}$, $\{v_0, v_3, s\}$, $\{v_4, v_1, p\}$, $\{t, s', q'\}$ where $q' \in S_2(v_3, v_4)$. Finally, suppose that $S_2(v_2, v_3) = \{p, p'\}$. Then $S_2(v_3, v_4) = \{q\}$ and $S_2(v_1, v_2) = \{s\}$. If t is complete to $\{p, p', s\}$, then K_5 would arise. Otherwise it is easy to check that G is 4-colorable. This completes the proof of Case 2.

In the remaining of the proof, we shall frequently consider some induced $C_5 = C'$ with $C' \neq C$ or $C_5 = C_t$ by modifying C with respect to some vertex $t \notin C$. We can then define p-vertices with respect to C' and C_t as well. We adapt those definitions by using the notation S'_p and S^t_p. For example, S'_1 is the set of 1-vertices with respect to C', and S^t_1 is the set of 1-vertices with respect to C_t, and so on. Let $s = (s_1, \ldots, s_5)$ be an integer vector. We say that C is of *type* s if $S_3(v_i)$ has size s_i for each $0 \leq i \leq 4$.

Case 3. $|S_3| = 3$. There are four possible configurations.

C **is of type (2,1,0,0,0)**. Let $S_3(v_0) = \{x, x'\}$ and $S_3(v_1) = \{y\}$. We may assume that $xy \notin E$. If $t \in S_1(v_3)$ then $tv_3v_4xv_1y = P_6$. So, $S_1(v_3) = \emptyset$. Let $C_x = C \setminus \{v_0\} \cup \{x\}$ and $C_y = C \setminus \{v_0\} \cup \{y\}$. As $xy \notin E$, we have $S^x_3 \cap S_2 \neq \emptyset$ and $S^y_3 \cap S_2 \neq \emptyset$. Let $p \in S^x_3 \cap S_2$ and $q \in S^y_3 \cap S_2$. Note that $xp \in E$ and $qy \in E$ by definition of p and q. Suppose first that $p \in S_2(v_1, v_2)$. Then $py \notin E$ or $pyv_0x = C_4$. If $q \in S_2(v_2, v_3)$, then $qpv_1y = C_4$. So, $q \in S_2(v_0, v_4)$. By (P8), $S_2(v_3, v_4) = S_2(v_3, v_2) = \emptyset$. Now $d(v_3) = 2$ as $S_1(v_3) = \emptyset$. Therefore, $p \in S_2(v_3, v_4)$. Then $py \in E$ or $pyv_1x = C_4$. If $q \in S_2(v_0, v_4)$, then $qx \notin E$ or $qxv_1y = C_4$. Also, $qp \in E$ and so $pqv_0x = C_4$. Thus $q \in S_2(v_2, v_3)$. Now by (P7) to (P9) and the fact that $xp, qy \in E$, we have $S_2 = S_2(v_2, v_3) \cup S_2(v_3, v_4)$ and $S_1 = \emptyset$. Thus, $2 \leq |S_2| \leq 3$. Suppose first that $S_2(v_2, v_3) = \{q, q'\}$. Then G has a 4-coloring: $\{v_1, v_4, q\}$, $\{v_0, v_2, p\}$, $\{y, x, v_3\}$, $\{x', q'\}$. Now suppose that $S_2(v_3, v_4) = \{p, p'\}$. If $x'y \notin E$ then $N(y) = \{v_0, v_1, v_2, q\}$. Since G **is a minimal obstruction**, $G - y$ has a 4-coloring ϕ. Note that $\phi(v_1) = \phi(v_4) = \phi(q)$. Hence, ϕ can be extended to G, a contradiction. So, $x'y \in E$ and so $x'p \notin E$ or $x'pqy = C_4$. Now G has a 4-coloring: $\{v_1, v_4, q\}$, $\{v_0, v_2, p'\}$, $\{y, x, v_3\}$, $\{x', p\}$.

C **is of type (2,0,1,0,0)**. Let $S_3(v_0) = \{x, x'\}$ and $S_3(v_2) = \{y\}$. We first claim that $S_1(v_3) = \emptyset$. Otherwise let $t \in S_1(v_3)$. Suppose that $S_1(v_1) \neq \emptyset$. Let $p \in S_1(v_1)$. Then $tp \in E$. By (P9), $S_2 = \emptyset$. Let $C' = v_3tpv_1v_2$. Note that $x, x' \notin S'_3$ as $\{x, x'\}$ is anti-complete to $\{t, v_3, v_2\}$. So, $|S'_3 \cap S_1| \geq 2$ by the minimality of C. It is straightforward to check that $|S'_3 \cap (S_1(v_1) \cup S_1(v_3))| \geq 2$ and thus $|S_1(v_1)| + |S_1(v_3)| \geq 4$. So, $|S_1(v_1)| + |S_1(v_3)| = 4$ by G is K_5-free. As $d(v_2) \geq 4$ we have $S_1(v_2) \neq \emptyset$. Let $q \in S_1(v_2)$. Since $\{v_2, y\}$ is not a clique cutset separating $S_1(v_2)$, $S_1(v_0) \cup S_1(v_4) \neq \emptyset$. Suppose that $t' \in S_1(v_4)$. Then $t'q \in E$. Let $C'' = v_4t'qv_2v_3$. Similar as above we have that $|S_1(v_2)| + |S_1(v_4)| = 4$. If $S_1(v_0) \neq \emptyset$ then $G = G_{P_4}$. If $|S_1(v_1)| = 3$ or $|S_1(v_4)| = 3$ then $G = G_{3,1}$. So, each $S_1(v_i)$ has size

2. But now $\{x, x', v_0, v_1, v_4\} \cup S_1(v_1) \cup S_1(v_4)$ induce a $G_{2,2}$. Thusm $S_1(v_4) = \emptyset$ and so $S_1(v_0) \neq \emptyset$. Since $S_1(v_0)$ is complete to $S_3(v_0)$, we have $\{x, x', v_0, v_1\} \cup S_1(v_0) \cup S_1(v_1) \cup S_1(v_3)$ induces a $G_{2,2}$ or $G_{3,1}$. So, $S_1(v_1) = \emptyset$. Let $p \in S_1(v_0)$. $pt \in E$. Note that $S_1(v_0) = \{p\}$ or K_5 would arise. Also, $S_2 = \emptyset$. Let $C' = v_0ptv_3v_4$. As y is anti-complete to $\{v_0, v_4, p\}$, $y \notin S_3'$ and so $S_3' \cap S_1 \neq \emptyset$. Let $t' \in S_3' \cap S_1$ and it is easy to see that $t' \in S_1(v_3)$. Thus, $S_1(v_3) = \{t, t'\}$ or $\{x, x'\} \cup C' \cup S_1(v_3)$ induces a $G_{3,1}$. Now by (P11), we have $S_1(v_3)$ is anti-complete to $S_2(v_0, v_1)$. Hence, $\{t, t'\}$ is complete to y as $d(t) \geq 4$ and $d(t') \geq 4$. Now G contains $G_{2,1,1}$ as an induced subgraph. So far, we have showed that $S_1(v_1) = S_1(v_0) = \emptyset$ if $S_1(v_3) \neq \emptyset$. As $\{v_3, y\}$ is not a clique cutset separating $S_1(v_3)$, we may assume that t has a neighbor $p \in S_2(v_0, v_1)$. By Observation 1 (3), y is anti-complete to $\{p, t\}$. Then the fact that $yv_3tpv_0x(x') \neq P_6$ implies that p is complete to $\{x, x'\}$ and so $\{p, v_0, v_1, x, x'\}$ induces a K_5.

Therefore, $S_1(v_3) = \emptyset$. Next we claim that $S_2(v_3, v_2) \neq \emptyset$. If not, we have $S_2(v_3, v_4) \neq \emptyset$ and $S_2(v_1, v_2) \neq \emptyset$ as $d(v_2) \geq 4$ and $d(v_3) \geq 4$. So, $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$ and $S_1 = S_1(v_1) \cup S_1(v_4)$. Suppose that $q \in S_2(v_1, v_2)$ is adjacent a vertex $t \in S_1(v_4)$. By Observation 1 (3), $\{x, x'\}$ is anti-complete to $\{q, t\}$. Hence, $\{x, x'\}$ is anti-complete to S_2. Also, $S_1(v_1) = \emptyset$. Let $C' = v_4v_3v_2qt$. Note that $x, x' \notin S_3'$ and so $|S_3' \cap (S_1 \cup S_2)| \geq 2$. Note that $S_3' \cap S_2(v_3, v_4) = \emptyset$. If $|S_3' \cap S_1(v_4)| \geq 2$ or $|S_3' \cap S_2(v_4)| \geq 2$, then $\{x, x', v_0, v_1, v_4\} \cup S_1(v_4) \cup S_2(v_4)$ induces a $G_{3,1}$ or $G_{2,2}$. Thus, there exists a vertex $q \in S_3' \cap S_2(v_1, v_2)$ with $q' \neq q$. Now $qy \in E$ as $xv_4tqv_2y \neq P_6$. Since $q'q \in E$, $q'y \notin E$ or K_5 would arise. But then $yv_2q'tv_4x = P_6$. Therefore, $S_1(v_4)$ is anti-complete to $S_2(v_1, v_2)$. Since $\{v_1, v_2, y\}$ is not a clique cutset separating $S_2(v_1, v_2)$, $S_2(v_1, v_2)$ is not anti-complete to $\{x, x'\}$. Thus, we may assume that x' is complete to S_2 by (P9). Now note that $S_1(v_1)$ is anti-complete to $S_2(v_3, v_4)$ by Observation 1 (3). If $t \in S_1(v_4)$ and $t' \in S_1(v_1)$, then $\{x, x'\}$ is anti-complete to $\{t, t'\}$ by Observation 1 (1). Also, $ty \notin E$. As $d(t) \geq 4$, we have $|S_1(v_1)| + |S_1(v_4)| = 4$. Then $\{x, x', v_0, v_1, v_4\} \cup S_1(v_1) \cup S_1(v_4)$ induces a $G_{2,2}$ or $G_{3,1}$. So, if $S_1(v_4) \neq \emptyset$, then $S_1(v_1) = \emptyset$ and thus $\{v_4, x, x'\}$ would be a clique cutset. Hence, $S_1(v_4) = \emptyset$. Since x' is complete to S_2, we have that $2 \leq |S_2| \leq 3$. Moreover, $py \notin E$ or $pyv_1x' = C_4$. Thus y is anti-complete to $S_2(v_3, v_4)$. Next we show that $S_1(v_1)$ is a clique. Let $t \in S_1(v_1)$ and A be the component of $S_1(v_0)$ containing t. Since $\{v_1, x, x'\}$ is not a clique cutset separating A, A is not anti-complete to y and hence complete to y. Further, since $v_0x'pv_3yt \neq P_6$, we have $x't \in E$ and thus A is complete to x'. Hence, A is a clique. By G is C_4-free, $S_1(v_1) = A$ and $|A| \leq 2$ by G is K_5-free. If $S_2(v_3, v_4) = \{p\}$ then $xp \in E$ as $d(p) \geq 4$. Thus, $S_2(v_1, v_2) = \{q\}$ or K_5 would arise. Note that $qy \notin E$ or $G = G_{P_4}$ with respect to xqv_2v_3p. Now $S_1 = \emptyset$ or if $t \in S_1(v_1)$ then $tx'qv_2y$ and v_1 induce a W_5. Now G has a 4-coloring: $\{v_0, p, q, y\}$, $\{v_4, v_1\}$, $\{x, v_3\}$, $\{x', v_2\}$. So, we assume that $S_2(v_4, v_3) = \{p, p'\}$ and thus $S_2(v_1, v_2) = \{q\}$. Now x is anti-complete to S_2 or K_5 would arise. If $|S_1(v_1)| = 2$, then $G = G_{3,1}$. So, $S_1(v_1)$ contains at most one vertex. If $S_1(v_1) = \{t\}$, then $qy \notin E$ or $qytx' = C_4$. Now G has a 4-coloring: $\{v_1, v_4\}$, $\{x', v_3\}$, $\{t, v_2, p, v_0\}$, $\{x, y, p', q\}$. So, $S_1(v_1) = \emptyset$. Also, $qy \in E$ as $d(q) \geq 4$. Now $\{x', v_4, p, p', v_3\} = K_5 - e$ and $\{v_1, q, v_2, y\} = K_4$ induce a $G_{2,2}$.

Therefore, let $p \in S_2(v_2, v_3)$. As $\{v_2, v_3, y\}$ is not a clique cutset separating $S_2(v_2, v_3)$ the following three cases are possible. First we suppose that $S_2(v_1, v_2) \neq \emptyset$. Let $q \in S_2(v_1, v_2)$. By Observation 1 (2), y is complete to $S_2(v_1, v_2) \cup S_2(v_2, v_3)$. Thus, $S_2(v_2, v_3) = \{p\}$ and $S_2(v_2, v_3) = \{q\}$. Further, $S_1 = S_1(v_2)$ and so $S_1(v_2) = \emptyset$ or $\{v_2, y\}$ would be a clique cutset. Suppose that $S_2(v_3, v_4) \neq \emptyset$. Note that $\{x, x'\}$ is anti-complete to S_2. If $S_2(v_3, v_4) = \{r\}$, then G has a 4-coloring: $\{v_1, v_4, p\}$, $\{v_0, v_3, q\}$, $\{x, r, v_2\}$, $\{x', y\}$. So, $S_2(v_3, v_4) = \{r, r'\}$. Then y is not complete to $\{r, r'\}$, say $yr' \notin E$ or $\{r, r', p, v_3, y\}$ would induce a K_5. Then G has a 4-coloring: $\{v_1, v_4, p\}$, $\{v_0, v_3, q\}$, $\{x, r, v_2\}$, $\{x', y, r'\}$. So, we may assume that $S_2(v_3, v_4) = \emptyset$. If $S_2(v_0, v_1) \neq \emptyset$, then G has a 4-coloring as above. Suppose that $r \in S_2(v_0, v_1)$. The fact that $v_3pqrv_0x \neq P_6$ implies that $xq \in E$ or $xr \in E$. Similarly, $x'q \in E$ or $x'r \in E$. Also, the fact that xq (respectively $x'q$) is an edge implies that xr (respectively $x'r$) is an edge, since G is C_4-free. Hence, q is not complete to $\{x, x'\}$, say $qx' \notin E$ or $\{x, x', v_0, v_1, r\}$ would induce a K_5. As $qx' \notin E$, $x'r \in E$. Hence, $xr \notin E$ and so $xq \in E$. Now $xx'rq = C_4$.

Therefore, we may assume that $S_2(v_1, v_2) = \emptyset$. Suppose that $S_2(v_3, v_4) \neq \emptyset$. Let $q \in S_2(v_3, v_4)$. Note that $S_1 = \emptyset$ since $S_1(v_3) = \emptyset$. Suppose that $r \in S_2(v_0, v_4)$. Note that r is not complete to $\{x, x'\}$, say $xr \notin E$. As $v_2pqrv_0x \neq P_6$, we have $xq \in E$. But now $xqrv_0 = C_4$. So, $S_2(v_0, v_4) = \emptyset$. Thus $2 \leq |S_2| \leq 3$. Now as $yv_2pqv_4v_0 \neq P_6$, we have $yp \in E$ or $yq \in E$. Also, if $yq \in E$ then $yp \in E$ or $yv_2pq = C_4$. So, $yp \in E$ and thus $S_2(v_2, v_3) = \{p\}$. Suppose first that $S_2(v_3, v_4) = \{q\}$. Note that q is not complete to $\{x, y\}$ or $\{x', y\}$. Thus G has a 4-coloring: $\{v_1, v_4, p\}$, $\{q, y, v_0\}$, $\{x, v_2\}$, $\{x', v_3\}$ if $qy \notin E$, and otherwise we move q from $\{q, y, v_0\}$ to $\{x, v_2\}$. Now suppose that $S_2(v_3, v_4) = \{q, q'\}$. Then we may assume that $qy \notin E$ or K_5 would arise. Also, $\{q, q'\}$ is not anti-complete to $\{x, x'\}$. If $q'y \notin E$, then q and q' are in the same place thus we may assume that $qx \notin E$. Now G has a 4-coloring: $\{v_1, v_4, p\}$, $\{q', y, v_0\}$, $\{q, x, v_2\}$, $\{x', v_3\}$. Otherwise $q'y \in E$ and so q' is anti-complete to $\{x, x'\}$. Then G has a 4-coloring: $\{v_1, v_4, p\}$, $\{q, y, v_0\}$, $\{q, x, v_2\}$, $\{x', v_3\}$.

Now we may assume that $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$ and p has a neighbor $t \in S_1(v_0)$. Then t is complete to $\{x, x'\}$. Also $S_1 = S_1(v_0)$ by (P11). Let $C' = v_0v_4v_3p$. Clearly, $y \notin S_3'$ as y is anti-complete to $\{v_4, v_0, t\}$. Thus $S_3' \cap (S_1 \cup S_2) \neq \emptyset$. It is easy to check that $S_3' \cap S_1 \subseteq S_1(v_0) \cap S_3'(t)$ and $S_3' \cap S_2(v_2, v_3) \subseteq S_1(v_0) \cap S_3'(p)$. Let $r \in S_3'$. If $r \in S_1(v_0)$, then $rt \in E$ and so $\{x, x', v_0, r, t\}$ induces a K_5. Hence, $r \in S_2(v_2, v_3)$ and now $\{x, x'v_0, v_1, v_4, t, v_3, v_2, p, r\}$ induces a $G_{2,1,1}$.

C **is of type (1,0,1,1,0)**. Let $x \in S_3(v_0)$, $y \in S_3(v_2)$ and $z \in S_3(v_3)$. We first show that $S_1(v_0) = \emptyset$. If $yz \in E$, then if $t \in S_1(v_0)$ we have $G = G_{2,2}$. If $yz \notin E$, then $yv_2zv_4v_0t = P_6$. Next we claim that $yz \in E$. Otherwise let $C_y = C \setminus \{v_2\} \cup \{y\}$ and $C_z = C \setminus \{v_3\} \cup \{z\}$. As $yz \notin E$, we have that $S_3^y \cap (S_1 \cup S_2) \neq \emptyset$ and $S_3^z \cap (S_1 \cup S_2) \neq \emptyset$. Let $p \in S_3^y \cap (S_1 \cup S_2)$ and $q \in S_3^z \cap (S_1 \cup S_2)$. Then $py, qz \in E$ by definition of p and q. Since G is C_4-free, $pz, qy \notin E$. We consider the case $p \in S_2(v_0, v_1)$ first. If $q \in S_2(v_0, v_4)$, then x is complete to $S_2(v_0, v_1) \cup S_2(v_0, v_4)$ by Observation 1 (2). Note that $S_1 = \emptyset$. Further, $S_2(v_1, v_2) = S_2(v_3, v_4) = S_2(v_2, v_3) = \emptyset$ by (P7) to (P9). As G is K_5-free, $S_2 = \{p, q\}$. Hence, G has a 4-coloring: $\{x, y, z\}$, $\{p, v_4, v_2\}$, $\{q, v_1, v_3\}$, $\{v_0\}$. Thus $q \in S_2(v_1, v_2)$. Now $pqv_2y = C_4$ as $qy \notin E$. Hence, $p \in S_2(v_3, v_4)$. If $q \in S_2(v_1, v_2)$, then $S_2(v_0, v_1) = S_2(v_0, v_4) = \emptyset$ and so $N(v_0) = \{v_1, v_4, x\}$, a contradiction. Thus, $q \in S_2(v_0, v_4)$. But now $pqzv_3 = C_4$. Therefore $yz \in E$. Recall that $S_1(v_0) = \emptyset$. As $d(v_0) \geq 4$, we may assume that there exists a vertex $p \in S_2(v_0, v_1)$. Since G has no clique cutset, the following four cases are possible.

Case a. $S_2(v_0, v_4) \neq \emptyset$. Let $q \in S_2(v_0, v_4)$. By Observation 1 (2), x is complete to $S_2(v_0, v_1) \cup S_2(v_0, v_4)$ and so $S_2(v_0, v_1) = \{p\}$ and $S_2(v_0, v_4) = \{q\}$. Note that $S_1 = \emptyset$. If $S_2 = \{p, q\}$, then G has a 4-coloring: $\{q, v_1, v_3\}$, $\{p, v_2, v_4\}$, $\{y, x\}$, $\{z, v_0\}$. Now by symmetry, we may assume that $r \in S_2(v_1, v_2)$. Then z is anti-complete to S_2 and so G has a 4-coloring by adding r to $\{z, v_0\}$ if $S_2(v_1, v_2) = \{r\}$. So, let $S_2(v_1, v_2) = \{r, r'\}$. As G is K_5-free, y is not complete to $\{r, r'\}$, say $yr' \notin E$. Then $yp \in E$ since $yv_2r'pqv_4 \neq P_6$. But now $pr'v_2y = C_4$.

Case b. $S_2(v_1, v_2) \neq \emptyset$. Let $q \in S_2(v_1, v_2)$. We may assume that $S_2(v_0, v_4) = \emptyset$. Suppose that $r \in S_2(v_2, v_3)$. Then $yr, yq \in E$ by Observation 1 (2). Hence, $zr \notin E$ or $\{v_2, v_3, y, z, r\}$ would induce a K_5. So, $zq \notin E$ or $v_3rqz = C_4$. But then $zv_3rpqv_0 = P_6$. So, $S_2(v_2, v_3) = \emptyset$. Hence $2 \leq |S_2| \leq 3$. Also, $S_1 = S_1(v_1)$ and $S_1(v_1)$ is a clique and thus $|S_1(v_1)| \leq 2$. Suppose that $t \in S_1(v_1)$. Then $tyzv_4v_0p \neq P_6$ implies that $yp \in E$ and so $yq \in E$ or $ypqv_2 = C_4$. Since $v_3v_2qpv_0x \neq P_6$, we have either $xp \in E$ or $xq \in E$. In any case, we have an induced C_4 as t is complete to $\{x, y\}$. So, $S_1(v_1) = \emptyset$. If $S_2 = \{p, q\}$, then G has a 4-coloring: $\{v_0, v_3, q\}$, $\{v_2, v_4, p\}$, $\{x, y\}$, $\{z, v_1\}$. Suppose that $S_2(v_0, v_1) = \{p, p'\}$. Now since $v_3v_2qpv_0x \neq P_6$, we have either $xp \in E$ or $xq \in E$. If $xq \in E$, then x is complete to $\{p, p'\}$ by G is C_4-free and hence $\{x, v_0, v_1, p, p'\}$ induces a K_5. So, $xq \notin E$ and thus $xp \in E$. Replacing the argument for $\{p', q\}$, we have $xp' \in E$ and so K_5 would arise. If $S_2(v_1, v_2)$ contains two vertices, we would derive a similar contradiction.

Case c. $S_2(v_0, v_1)$ is not anti-complete to $S_1(v_3)$. We now may assume that $S_2(v_0, v_4) = S_2(v_1, v_2) = \emptyset$. Without loss of generality, we assume that p has a neighbor $q \in S_1(v_3)$. So, $S_1 = S_1(v_3)$ by (P11).

Moreover, $S_2(v_2, v_3) = \emptyset$ or $\{v_2, v_3, y, z\}$ would be a clique cutset. Now $px \in E$ since $v_2v_3qpv_0x \neq P_6$ and so p is the only neighbor of q in $S_2(v_0, v_1)$ or K_5 would arise. Let $C' = v_1v_2v_3qp$. Note that $v_0, v_4, x \notin S'_3$ and $y, z \in S'_3$. Thus, $S'_3 \cap (S_1 \cup S_2) \neq \emptyset$ by the minimality of C. Let $t \in S'_3 \cap (S_1 \cup S_2)$. It is easy to check that $t \in S_1(v_3)$ or $t \in S_2(v_0, v_1)$. If $t \in S_2(v_0, v_1)$, then t must be complete to $\{v_1, p, q\}$, contradicting the fact that p is the only neighbor of q. Hence, $t \in S_1(v_3)$ and t must be complete to $\{v_3, q, p\}$. By (P12), z is complete to $\{q, t\}$. Now $G = H_2$.

Case d. Now we may assume that $py \in E$. If $S_2(v_2, v_3) \neq \emptyset$, then $S_2(v_3, v_4) = \emptyset$ and so $\{v_2, v_3, y, z\}$ would be a clique cutset since $S_1(v_0) = \emptyset$. So, $S_2(v_2, v_3) = \emptyset$. Suppose that $S_1(v_3) \neq \emptyset$. Then $S_1(v_1) \neq \emptyset$ or $\{v_3, y, z\}$ would be a clique cutset. But then $S_2 = \emptyset$ by (P7) to (P9), a contradiction. So, $S_1(v_3) = \emptyset$. Thus, $S_1 = S_1(v_1)$ and $S_2 = S_2(v_0, v_1) \cup S_2(v_3, v_4)$. Next we claim that $xp \notin E$. Otherwise $xp \in E$. Let $C' = xpyzv_4$. It is easy to check that $v_1, v_3 \in S'_3$ but $v_0, v_2 \notin S'_3$. So, $S'_3 \cap (S_1 \cup S_2) \neq \emptyset$. Let $t \in S'_3 \cap (S_1 \cup S_2)$. As $S_1(v_1)$ is anti-complete to $\{v_4, z, p\}$, $t \notin S_1(v_1)$ and so $t \in S_2$. If $t \in S_2(v_0, v_1)$, then t is complete to $\{x, p, y\}$. If $t \in S_2(v_3, v_4)$, then as $py \in E$ we have $yt \in E$ and so $xt \notin E$ or $xtyv_1 = C_4$. Hence, t is complete to $\{v_4, z, y\}$. If $S'_3 \cap S_2 = \{t\}$, then $|S'_3| = 3$ and we are in one of previous two cases. Thus, there exists another vertex $t' \neq t$ with $t' \in S'_3 \cap S_2$. If $t, t' \in S_2(v_0, v_1)$, then $\{v_0, v_1, t, t', p\}$ would induce a K_5. If $t, t' \in S_2(v_4, v_3)$, then $\{y, z, t, t', v_3\}$ would induce a K_5. Hence, $t \in S_2(v_0, v_1)$ and $t' \in S_2(v_4, v_3)$. But now $G = G_{3,1}$. Therefore, $xp \notin E$. Now let $C'' = v_0pyv_3v_4$. As $xp \notin E$, $x \notin S''_3$. Also, $v_2 \notin S''_3$ but $z, v_1 \in S''_3$. Hence, $S''_3 \cap (S_1 \cup S_2) \neq \emptyset$. By the same argument as above, we either find an induced K_5 or $G_{2,2}$ or we are in one of previous two cases.

C **is of type (1,1,0,0,1)**. Let $x \in S_3(v_0)$, $y \in S_3(v_1)$ and $z \in S_3(v_4)$. We first suppose that $S_2(v_2, v_3) = \emptyset$. As $\delta(G) \geq 4$, we have the following two cases. Suppose first that $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ are nonempty but $S_1(v_2) = S_1(v_3) = \emptyset$. By (P9), we may assume that $S_1(v_4) = \emptyset$. Now x is complete to S_2 or $\{v_1, v_2, y\}$ would be a clique cutset. Also, x is complete to $\{y, z\}$, otherwise considering $C_y = C \setminus \{v_1\} \cup \{y\}$ or $C_z = C \setminus \{v_4\} \cup \{z\}$ will obtain by the minimality of C that $S_2(v_0, v_1) \cup S_2(v_0, v_4) \cup S_2(v_2, v_3) \neq \emptyset$ which contradicts our assumption and (P8). But now G contains $G_{3,1}$ as an induced subgraph. So, $S_1(v_2)$ and $S_1(v_3)$ are nonempty and $S_2(v_2, v_1) \cup S_2(v_3, v_4) = \emptyset$. Thus, $S_2 = \emptyset$ and hence x is complete to $\{y, z\}$ by the minimality of C. Let $p \in S_1(v_3)$ and $q \in S_1(v_2)$. Suppose that $t \in S_1(v_0)$. Let $C' = xtpv_3v_4$. Then $v_1, v_2, y \notin S'_3$. Hence, $S'_3 \cap S_1 \neq \emptyset$. Let $r \in S'_3 \cap S_1$. It is easy to check that $r \in S_1(v_0) \cup S_1(v_3)$. We claim that $S'_3 \cap S_1(v_0) \neq \emptyset$. Otherwise $r \in S_1(v_3)$. Then $r \in S'_3(p)$ as r is anti-complete to $\{v_4, x\}$. If $S'_3 = \{v_0, z, r\}$, then we are in the case C **is of type (1,0,1,1,0)**. So, $|S'_3 \cap S_1(v_3)| \geq 2$ and thus $G = G_{2,2}$. Therefore, we may assume that $r \in S_1(v_0)$. If $S_1(v_3) = \{p, p'\}$, then $C_5 = xtpv_3z$ and $P_4 = v_4v_0rt$ induce a G_{P_4}. So, $S_1(v_3) = \{p\}$ and $S_1(v_2) = \{q\}$. Now let $C'' = tqv_2v_3p$. Clearly, $x, v_0, v_1, v_4 \notin S''_3$ and so $S''_3 \cap S_1 \neq \emptyset$. Let $s \in S''_3 \cap S_1$ Clearly, $s \in S_1(v_0) \cup S_1(v_2) \cup S_1(v_3)$. As $s \notin \{p, q\}$, we have $s \in S_1(v_0)$. Hence, $s = r$. By the minimality of C, y and z must be in S''_3. This implies that y is complete to $\{t, q, v_2\}$. So, $ry \in E$ or $rqyx = C_4$. But then $\{x, v_0, y, r, t\}$ induces a K_5. We have shown that $S_1(v_0) = \emptyset$. As G has no clique cutset, $S_1(v_1) \neq \emptyset$ and $S_1(v_4) \neq \emptyset$. Let $u_i \in S_1(v_i)$ for $i = 1, 4$. Note that x is anti-complete to S_1 by Observation 1 (1). If $|S_1(v_1)| \geq 2$, say $u_1, u'_1 \in S_1(v_1)$, then $G = G_{P_4}$ with respect to xyu_1u_4z whose 3-vertices are $v_4v_0v_1u'_1$. Hence, $S_1(v_i) = \{u_i\}$. Note that $pz \notin E$ or $zpu_1u_4 = C_4$. Thus, z is anti-complete to $S_1(v_3)$ and y is anti-complete to $S_1(v_2)$. By $\delta(G) \geq 4$, we must have $|S_1(v_2)| = |S_1(v_3)| = 3$. It is easy to check G is 4-colorable.

Therefore, we may assume that $p \in S_2(v_2, v_3)$. By (P7) to (P9), there are at most two nonempty $S_1(v_i)$. If there exists i such that $S_1(v_i) \neq \emptyset$ and $S_1(v_{i+1}) \neq \emptyset$, then $i = 2$ as $S_2(v_2, v_3) \neq \emptyset$. Thus $\{v_2, y\}$ is a clique cutset separating $S_1(v_2)$.

Case a. $S_1(v_i) \neq \emptyset$ for some i. As $S_2(v_2, v_3) \neq \emptyset$, $S_1(v_1) = S_1(v_4) = \emptyset$. So, $i \in \{0, 2, 3\}$. Suppose first that $i = 2$ (or $i = 3$) and let $t \in S_1(v_2)$. As $\{v_2, y\}$ is not a clique cutset, $S_1(v_2)$ is not anti-complete to $S_2(v_0, v_4)$. We may assume that t has a neighbor $q \in S_2(v_0, v_4)$. By Observation 1 (3), y

is anti-complete to $\{q,t\}$ and thus anti-complete to $S_2(v_0,v_4)\cup S_2(v_2,v_3)$. Let $C'=qtv_2v_3v_4$. Clearly, $x,v_0,v_1\notin S_3'$. If $z\in S_3'$, then $z\in S_3'(v_4)$ and if $y\in S_3'$, then $z\in S_3'(t)$. As $x\notin S_3'$, $|S_3'\cap(S_1\cup S_2)|\geq 1$. Let $r\in S_3'\cap(S_1\cup S_2)$. If $r\in S_1(v_2)=S_1$, then $r\in S_3'(t)$. Also, $qx\in E$ $pv_2tqv_0x\neq P_6$. Hence, q is the only neighbor of t in S_2 and so $r\notin S_2(v_0,v_4)$. Clearly, $r\notin S_2(v_2,v_3)$. If $r\in S_2(v_3,v_4)$, then r must be complete to $\{q,v_3,v_4\}$ and hence $r\in S_3'(v_4)$. So, $S_3'=S_3'(v_4)\cup S_3'(t)$. Now as $|S_3'|\geq 3$ either we are in one of previous cases or G contains $G_{3,1}$ as an induced subgraph.

Therefore, we may assume that $i=0$. Let $C_x=C\setminus\{v_0\}\cup\{x\}$. If x is not complete to $\{y,z\}$, then by the minimality of C, we have $S_3^x\cap(S_2(v_1)\cup S_2(v_4))\neq\emptyset$, which contradicts (P7). Hence, $xy,xz\in E$. Suppose that p has a neighbor $q\in S_1(v_0)$. Let $C'=v_0v_1v_2pq$. Note that z is not complete to $\{p,q\}$ by Observation 1 (3) and hence $z\notin S_3'$. Also, $v_3,v_4\notin S_3'$. Thus, $S_3'\cap(S_1\cup S_2)\neq\emptyset$. Let $t\in S_3'\cap(S_1\cup S_2)$. If $t\in S_1(v_0)$, t is complete to $\{v_0,p,q\}$ and then $G=H_1$. Clearly, $t\notin S_2(v_0,v_1)\cup S_2(v_0,v_4)$. If $t\in S_2(v_2,v_3)$, t is complete to $\{q,p,v_2\}$ and then G is not 4-colorable and $G=H_2$. We have shown that $S_1(v_0)$ is anti-complete to $S_2(v_2,v_3)$. As $\{v_2,v_3\}$ does not separate $S_2(v_2,v_3)$, $S_2(v_2,v_3)$ is not anti-complete to $\{y,z\}$. Without loss of generality, assume that $py\in E$. As before we can show that $S_1(v_0)$ is complete to $\{y,z\}$ and thus a clique. So, $S_1(v_0)=\{q\}$ or K_5 would arise. Also, $pz\in E$ or $pzxy=C_4$. As $d(p)\geq 4$, there exists $p'\in S_2(v_2,v_3)$ with $pp'\in E$. Note that in any 4-coloring ϕ of G, $\phi(p')=\phi(y)=\phi(z)$. So if p' is not anti-complete to $\{y,z\}$ G is not 4-colorable. Specifically, if $p'y\in E$ then $\{y,p,p',v_3,v_2\}=K_5-e$ and $\{z,x,v_0,q\}=K_4$ induces a $G_{3,1}$. If $p'z\in E$, then $\{q,v_0,x,z,y\}=K_5-e$ and $\{v_2,v_3,p,p'\}=K_4$ induce a $G_{2,2}$. Thus, we assume that p' is anti-complete to $\{y,z\}$. As $d(p')\geq 4$, there exists $p''\in S_2(v_2,v_3)$ with $p'p''\in E$. Moreover, $pp''\notin E$ or K_5 would arise, and $p''y\notin E$ or $p''ypp'=C_4$. Then the fact that $p''p'pyqz\neq P_6$ implies that $zp''\in E$, and thus $v_4zp''p'py=P_6$.

Case b. $S_1=\emptyset$. Recall that $p\in S_2(v_2,v_3)$. We first show that x is complete to $\{y,z\}$. Otherwise suppose $xy\notin E$. Since $yv_1xv_4v_3p\neq P_6$, we have $yp\in E$ and so $zp\notin E$. Since p is an arbitrary vertex in $S_2(v_2,v_3)$, we have that y is complete to $S_2(v_2,v_3)$, and z is anti-complete to $S_2(v_2,v_3)$. Hence, $xz\in E$ by symmetry. Let $C_x=C\setminus\{v_0\}\cup\{x\}$. As $xy\notin E$, $S_3^x\cap S_2\neq\emptyset$. Let $q\in S_3^x\cap S_2$. $xq\in E$. Suppose that $r\in S_2(v_0,v_4)$. Then $S_2(v_1,v_2)=\emptyset$. Hence, $q\in S_2(v_3,v_4)$. By (P7) to (P9) and the fact that $yp\in E$, we have $yr\in E$ and thus $yrqp=C_4$. So, $S_2(v_0,v_4)=\emptyset$. If $q\in S_2(v_1,v_2)$ then $pq\in E$. Note that $qy\notin E$ or $qyv_0x=C_4$. Then $pqv_1y=C_4$. Thus, $q\in S_2(v_3,v_4)$. As $xq\in E$, $S_2(v_1,v_2)=\emptyset$ by (P9). Thus, $S_2=S_2(v_4,v_3)\cup S_2(v_2,v_3)$ and $2\leq|S_2|\leq 3$. If $S_2=\{p,q\}$, then G has a 4-coloring: $\{x,v_3\}$, $\{v_0,v_2,q\}$, $\{y,z\}$, $\{v_1,v_4,p\}$. Suppose now that $S_2(v_4,v_3)=\{q,q'\}$. As $v_2yv_0xqq'\neq P_6$, we have $xq'\in E$. As $\{x,v_4,z,q,q'\}$ does not induce a K_5, z is not complete to $\{q,q'\}$, say $zq'\notin E$. Then G has a 4-coloring by adding q' to $\{y,z\}$. Finally, $S_2(v_2,v_3)=\{p,p'\}$. Then G has a 4-coloring $\{x,y,v_3\}$, $\{v_0,v_2,q\}$, $\{p',z\}$, $\{v_1,v_4,p\}$ as z is anti-complete to $S_2(v_2,v_3)$.

Therefore, $xy,xz\in E$. Next we show that $S_2(v_1,v_2)=S_2(v_3,v_4)=\emptyset$. By symmetry, we may assume that $S_2(v_1,v_2)\neq\emptyset$. Let $q\in S_2(v_1,v_2)$. Then $S_2(v_0,v_4)=\emptyset$. As $v_4v_3pqv_1y\neq P_6$, we have $py\in E$ or $qy\in E$. Suppose first that $py\in E$. Then $qy\in E$ or $pqv_1y=C_4$. Let $C'=ypv_3v_4v_0$. Clearly, $v_1\notin S_3'$ as v_1 is anti-complete to $\{v_4,v_3,p\}$, and $x\in S_3'(v_0)$, $z\in S_3'(v_4)$ and $v_2\in S_3'(p)$. If $S_3'=\{x,z,v_2\}$ then we are in the case C **is of type (1,0,1,1,0)**. So, let $r\in S_3'\setminus\{x,z,v_2\}$. Clearly, $r\notin S_2(v_0,v_1)\cup S_2(v_1,v_2)$. If $r\in S_2(v_2,v_3)$ then $r\in S_3'(p)$ and $G=G_{2,2}$. So, $r\in S_2(v_3,v_4)$. As $py\in E$, $pz\notin E$ and then $rz\in E$ since $v_1v_2prv_4z\neq P_6$. Hence, y and z are complete to $S_2(v_1,v_2)$ and $S_2(v_3,v_4)$, respectively. So, $S_2(v_1,v_2)=\{q\}$ and $S_2(v_3,v_4)=\{r\}$. If $S_2(v_2,v_3)=\{p,p'\}$ and $p'y\in E$, then $p'\in S_3'(p)$ and thus $G=G_{2,2}$. Note x is anti-complete to S_2. Now G has a 4-coloring: $\{x,q,v_3\}$, $\{v_0,r,v_2\}$, $\{v_1,z,p\}$, $\{v_4,y,p'\}$.

Now we have shown that $py\notin E$ and thus $qy\in E$. Since p is an arbitrary vertex in $S_2(v_2,v_3)$, we may assume that y is anti-complete to $S_2(v_2,v_3)$. Also, replacing any $q'\in S_2(v_1,v_2)$ we obtain $yq'\in E$ and so $S_2(v_1,v_2)=\{q\}$ or K_5 would arise. If $S_2(v_3,v_4)\neq\emptyset$, then z is anti-complete to $S_2(v_1,v_2)$

and complete to $S_2(v_3, v_4)$ by symmetry. Thus, $S_2(v_3, v_4) = \{r\}$ and G has a 4-coloring: $\{x, q, v_3\}$, $\{v_0, r, v_2\}$, $\{y, z, p\}$, $\{v_4, v_1, p'\}$, where p' might be another vertex in $S_2(v_2, v_3)$. If $S_2(v_0, v_1) \neq \emptyset$, then y is complete to $S_2(v_0, v_1) \cup S_2(v_1, v_2)$ and thus $S_2(v_0, v_1) = \{r\}$. Also, $xr \notin E$ or $\{x, y, v_0, v_1, r\}$ would induce a K_5. Note that z is anti-complete to S_2 and thus G has a 4-coloring: $\{x, r, v_2\}$, $\{v_0, q, v_3\}$, $\{y, z, p\}$, $\{v_4, v_1, p'\}$, where p' might be another vertex in $S_2(v_2, v_3)$. Finally, we have $S_2 = \{q\} \cup S_2(v_2, v_3)$. If $S_2(v_2, v_3) = \{p, p'\}$ and z is complete to $\{p, p'\}$, then $\{p, p, v_3, v_2, z\} = K_5 - e$ and $\{x, y, v_0, v_1\}$ induce a $G_{2,2}$. Otherwise in case of $S_2(v_2, v_3) = \{p, p'\}$, we may assume $p'z \notin E$ and thus G has a 4-coloring: $\{x, v_2\}$, $\{v_0, q, v_3\}$, $\{y, z, p'\}$, $\{v_4, v_1, p'\}$

Therefore, $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$. As $\{v_2, v_3\}$ is not a clique cutset, $S_2(v_2, v_3)$ is not anti-complete to $\{y, z\}$. By symmetry, we may assume that $py \in E$. Let $C' = ypv_3v_4v_0$. $v_1 \notin S_3'$. Clearly, $x \in S_3'(v_0)$, $z \in S_3'(v_4)$ and $v_2 \in S_3'(p)$. If $S_3' = \{x, z, v_2\}$ then we are in the case C **is of type (1,0,1,1,0)**. So, let $t \in S_3' \setminus \{x, z, v_2\}$. Note that $t \notin S_2(v_0, v_1)$ as $S_2(v_0, v_1)$ is anti-complete to $\{v_3, v_4, p\}$. If $t \in S_2(v_2, v_3)$, $t \in S_3'(p)$ and thus $G = G_{2,2}$. So, $t \in S_2(v_0, v_4)$ and $t \in S_3'(v_0)$, namely t is complete to $\{v_0, v_4, y\}$. Thus, $xt \in E$. By (P9), y is complete to S_2 and hence $2 \leq |S_2| \leq 3$. Note that $zt \notin E$ or $\{v_0, v_4, x, t, z\}$ would induce a K_5. Now if $S_2(v_2, v_3) = \{p, p'\}$ then $p' \in S_3'(p)$ and $G = G_{2,2}$. So, $S_2(v_2, v_3) = \{p\}$. If $S_2 = \{p, t\}$, then G has a 4-coloring ϕ: $\{x, v_3\}$, $\{v_0, v_2\}$, $\{p, t, z, v_1\}$, $\{v_4, y\}$. If $S_2(v_0, v_4) = \{t, t'\}$ then $t'x \notin E$ or $\{v_0, v_4, x, t, t'\}$ would induce a K_5. Then G has a 4-coloring by adding t' to $\{x, v_3\}$ in ϕ. This completes the proof of Case 3.

Note that if $S_3(v_i)$ has two vertices then $S_3(v_i)$ is not complete to $S_3(v_{i+1})$ as G is K_5-free. Moreover if $S_3(v_{i+1})$ also has two vertices, then there is at most one edge between $S_3(v_i)$ and $S_3(v_{i+1})$ as G is (K_5, C_4)-free.

Case 4. $|S_3| = 4$. There are five possible configurations for S_3.

C **is of type (2,2,0,0,0)**. Let $S_3(v_0) = \{x, x'\}$ and $S_3(v_1) = \{y, y'\}$. As G is (K_5, C_4)-free, we may assume that x is anti-complete to $\{y, y'\}$ and y is anti-complete to $\{x, x'\}$. Let $C' = C \setminus \{v_0\} \cup \{x\}$. Note that $y, y' \notin S_3'$. It is easy to check that $S_3' \cap (S_1 \cup S_2) \subseteq S_3' \cap (S_2(v_1, v_2) \cup S_2(v_3, v_4))$. Hence, $|S_3' \cap (S_2(v_1, v_2) \cup S_2(v_3, v_4))| \geq 2$ by the minimality of C. Suppose that $p \in S_3' \cap S_2(v_3, v_4)$. Note that $px \in E$. Then as $x'xpv_3v_2y \neq P_6$, we have $x'p \in E$. Further, $S_3' \cap S_2(v_3, v_4)$ is a clique and thus $|S_3' \cap S_2(v_3, v_4)| \leq 1$ or K_5 would arise. Next we show that $|S_3' \cap S_2(v_1, v_2)| \leq 1$. If not, let p, p' be two vertices in $S_3' \cap S_2(v_1, v_2)$. Then $\{p, p', y, y', x, x', v_0, v_1\}$ contains a W_5. Therefore, we may assume $q \in S_2(v_1, v_2)$ and $p \in S_2(v_3, v_4)$. Moreover, x is complete to $\{p, q\}$ by definition. As shown above, we obtain that $x'p \in E$. So, $\{x, x'\}$ is complete to $S_2(v_1, v_2) \cup S_2(v_3, v_4)$ and $S_2(v_3, v_2) = \emptyset$ by (P9). Thus, $S_2(v_1, v_2) = \{q\}$ and $S_2(v_1, v_2) = \{p\}$. If $t \in S_1(v_3)$, then $tv_3v_4xv_1y = P_6$. So, $S_1(v_3) = \emptyset$ and now $N(v_3) = \{v_2, v_4, p\}$ which contradicts that $\delta(G) \geq 4$.

C **is of type (1,1,0,2,0)**. Let $S_3(v_3) = \{x, x'\}$, $S_3(v_0) = \{z\}$ and $S_3(v_1) = \{y\}$. Note that $yz \notin E$ or $G = G_{2,2}$. Let $C' = C \setminus \{v_0\} \cup \{z\}$. As $y \notin S_3'$ we have $S_3' \cap (S_2(v_0, v_1) \cup S_2(v_3, v_4)) \neq \emptyset$. Let p be such a vertex. If $p \in S_2(v_3, v_4)$, then p is not complete to $\{x, x'\}$, say $xp \notin E$. Now $yv_1zpv_3x = P_6$. Therefore, $p \in S_2(v_1, v_2)$. Let $C'' = C \setminus \{v_1\} \cup \{y\}$. By symmetry, we obtain that there exists $q \in S_3'' \cap S_2(v_0, v_4)$. Note that $pz \in E$ and $qy \in E$ by definition of p and q. Further, $qz \notin E$ or $qzv_1y = C_4$. If $xp \notin E$, then $xq \notin E$ by (P9) and thus $qv_0zpv_2x = P_6$. Thus $xp \in E$ and now $zv_4xp = C_4$.

C **is of type (2,1,0,0,1)**. Let $S_3(v_0) = \{x, x'\}$, $S_3(v_1) = \{z\}$ and $S_3(v_4) = \{y\}$. As G is K_5-free, each of $\{y, z\}$ is not complete to $\{x, x'\}$. We may assume that $zx \notin E$. If $t \in S_1(v_2)$ then $tv_2v_1x(x')v_4y = P_6$. Thus, $S_1(v_2) = \emptyset$. Similarly $S_1(v_3) = \emptyset$. Let $C' = C \setminus \{v_1\} \cup \{z\}$. By the minimality of C, we have $S_3' \cap (S_2(v_0, v_4) \cup S_2(v_2, v_3)) \neq \emptyset$. We first show that $S_3' \cap S_2(v_2, v_3) = \emptyset$. Otherwise let $p \in S_3' \cap S_2(v_2, v_3)$. Note that $pz \in E$, and $py \notin E$ or $pyv_0z = C_4$. As $xv_1zpv_3y \neq P_6$, we have $yx \in E$ and so $x'y \notin E$. Moreover, $x'z \in E$ since $x'v_1zpv_3y \neq P_6$, and so $yxx'z = P_4$. Let $C'' = C \setminus \{v_4\} \cup \{y\}$. Then there exists $q \in S_3'' \cap (S_2(v_0, v_1) \cup S_2(v_2, v_3))$. It is clear that $qy \in E$ by definition of q. As $py \notin E$

and $qy \in E$, we have $q \in S_2(v_2, v_3)$ by (P9). Note that $p \neq q$. Also $qz \notin E$ or $qzv_0y = C_4$. Then $pq \in E$ since $qyxx'zp \neq P_6$. Let $C_x = C \setminus \{v_0\} \cup \{x\}$. Then there exists $r \in S_3^x \cap (S_2(v_1, v_2) \cup S_2(v_3, v_4))$ by the minimality of C. If $r \in S_2(v_1, v_2)$, then $rxyq = C_4$. Thus, $r \in S_2(v_3, v_4)$. Symmetrically considering $C_{x'} = C \setminus \{v_0\} \cup \{x'\}$ we obtain that there exists $r' \in S_2(v_1, v_2)$. However, this contradicts (P9), since $xr \in E$. Therefore, $S_3' \cap S_2(v_0, v_4) \neq \emptyset$. Symmetrically considering $C'' = C \setminus \{v_4\} \cup \{y\}$ we can conclude that $S_3'' \cap S_2(v_0, v_1) \neq \emptyset$. Hence, $S_2(v_2, v_3) = \emptyset$. Since $d(v_2) \geq 4$ and $d(v_3) \geq 4$, we have $S_2(v_1, v_2) \neq \emptyset$ and $S_2(v_3, v_4) \neq \emptyset$. This contradicts (P8).

C **is of type (2,0,1,0,1)**. Let $S_3(v_0) = \{x, x'\}$, $S_3(v_2) = \{z\}$ and $S_3(v_4) = \{y\}$. We may assume that $xy \notin E$. If $t \in S_1(v_2)$ then $tv_2v_1xv_4y = P_6$. So, $S_1(v_2) = \emptyset$. Let $C_x = C \setminus \{v_0\} \cup \{x\}$ and $C_y = C \setminus \{v_4\} \cup \{y\}$. Then there exists $p \in S_3^y \cap (S_2(v_0, v_1) \cup S_2(v_2, v_3))$. and $q \in S_3^x \cap (S_2(v_2, v_1) \cup S_2(v_4, v_3))$. Note that $py \in E$ and $qx \in E$ by definition. We first claim that $S_3^y \cap S_2(v_3, v_2) = \emptyset$. If not, suppose that $p \in S_2(v_2, v_3)$. Note that $pz \in E$ or $zv_2pyv_0x = P_6$. If $q \in S_2(v_3, v_4)$, then $qy \in E$ or $yv_4qp = C_4$. But then $qyv_0x = C_4$. So, $q \in S_2(v_1, v_2)$. Now $S_2(v_0, v_1) = S_2(v_3, v_4) = \emptyset$ by the fact that $py, qx \in E$ and (P9). Moreover, $S_2(v_0, v_4) = \emptyset$. By Observation 1 (2), z is complete to S_2 and hence $S_2(v_3, v_2) = \{p\}$ and $S_2(v_1, v_2) = \{q\}$. Note that $S_1 = \emptyset$ as $S_1(v_2) = \emptyset$. Thus, G has a 4-coloring: $\{v_1, v_4, p\}$, $\{v_0, v_3, q\}$, $\{v_2, x, y\}$, $\{x', z\}$. Therefore, $p \in S_2(v_0, v_1)$. Suppose first that $q \in S_2(v_3, v_4)$. Then $S_2(v_1, v_2) = S_2(v_2, v_3) = \emptyset$ by (P8). Thus, $d(v_2) = 3$, a contradiction. Hence, $q \in S_2(v_1, v_2)$. Note that $px, qy \notin E$. If $x'y \in E$ then $px' \in E$ or $yx'v_1p = C_4$, and so $\{v_1, p, y, v_4, x\} \cup \{v_0\}$ induces a W_5. So, $x'y \notin E$. Hence, $|S_3^y \cap S_2(v_0, v_1)| \geq 2$ by the minimality of C and the above argument. Let p and p' be two vertices in $S_3^y \cap S_2(v_0, v_1)$, and then $\{p, p', x, x', y, v_0, v_1, v_4\}$ contains a W_5.

C **is of type (2,0,0,1,1)**. Let $S_3(v_0) = \{x, x'\}$, $S_3(v_3) = \{z\}$ and $S_3(v_4) = \{y\}$. We may assume that $xy \notin E$. If $t \in S_1(v_2)$ then $tv_2v_1xv_4y = P_6$. So $S_1(v_2) = \emptyset$. Let $C_x = C \setminus \{v_0\} \cup \{x\}$ and $C_y = C \setminus \{v_4\} \cup \{y\}$. Then there exists $q \in S_3^y \cap (S_2(v_0, v_1) \cup S_2(v_2, v_3))$. and $p \in S_3^x \cap (S_2(v_2, v_1) \cup S_2(v_4, v_3))$ by minimality of C. $px, qy \in E$ by definition of p and q. Suppose first that $p \in S_2(v_3, v_4)$. $py \notin E$ or $pyv_0x = C_4$. If $q \in S_2(v_2, v_3)$ then $pqyv_4 = C_4$. So $q \in S_2(v_0, v_1)$. As $S_2(v_3, v_4) \neq \emptyset$ and $S_2(v_0, v_1) \neq \emptyset$, we have $S_2(v_1, v_2) = S_2(v_2, v_3) = \emptyset$. Now $d(v_2) = 3$ since $S_1(v_2) = \emptyset$, a contradiction. Thus $p \in S_2(v_1, v_2)$. p is anti-complete to $\{y, z\}$ since G is C_4-free. zv_2pxv_0y implies that $yz \in E$. If $S_3^x \cap S_2 = \{p\}$, then we are in the case C is of type (2,0,1,0,1). So we let $p' \in S_3^x \cap S_2(v_1, v_2)$ with $p' \neq p$. $p'x \in E$. So x' is not complete to $\{p, p'\}$, say $x'p \notin E$. $x'xpv_2v_3y$ implies that $x'y \in E$. Now we consider q. If $q \in S_2(v_2, v_3)$ then $S_2(v_0, v_1) = S_2(v_3, v_4) = \emptyset$ by the fact that $xp, qy \in E$ and (P9). Also, $S_2(v_0, v_4) = \emptyset$. Now $S_2 = \{p, p', q\}$ and $S_1 = \emptyset$. G has a 4-coloring: $\{v_1, v_4, q\}$, $\{x, y, v_2\}$, $\{v_0, v_3, p'\}$, $\{z, p, x'\}$. Thus $q \in S_2(v_0, v_1)$. Then $qx' \in E$ or $x'yqv_1 = C_4$. But now $\{v_1, v_4, x, y, q\} \cup \{v_0\}$ induces a W_5.

C **is of type (1,1,1,1,0)**. Let $S_3(v_0) = \{x\}$, $S_3(v_1) = \{y\}$, $S_3(v_2) = \{z\}$ and $S_3(v_3) = \{w\}$. Note that $\{x, y, z, w\}$ does not induce a P_4 or $G = G_{P_4}$. So, there are at most two edges in $\{x, y, z, w\}$. We shall consider two subcases.

Case a. There is at most one edge in $\{x, y, z, w\}$. Suppose that $yz \notin E$. Without loss of generality, assume $xy \notin E$. Let $C_y = C \setminus \{v_1\} \cup \{y\}$. As $x, z \notin S_3^y$ we have $|S_3^y \cap S_2| \geq 2$. If $|S_3^y \cap S_2(v_0, v_4)| \geq 2$ or $|S_3^y \cap S_2(v_3, v_2)| \geq 2$, then $|S_3^y \cap S_2| \geq 3$ or we are in one of previous four cases. Thus, $S_3^y \cap S_2(v_3, v_2) \neq \emptyset$ and $S_3^y \cap S_2(v_0, v_4) \neq \emptyset$ or K_5 would arise. Also, y is complete to $S_2(v_0, v_4)$ and $S_2(v_3, v_2)$ and hence $S_2 = S_2(v_3, v_2) \cup S_2(v_0, v_4)$ by (P7) to (P9). But now $C_z = C \setminus \{v_2\} \cup \{z\}$ has $|S_3^z| < 4$, which contradicts the minimality of C. So, it must be the case that $yz \in E$ and $xy, zw \notin E$. Consider C_y and C_z as above. Let $p \in S_3^y \cap (S_2(v_0, v_4) \cup S_2(v_2, v_3))$ by the minimality of C. Suppose that $S_3^y \cap S_2(v_0, v_4) = \emptyset$. Then $p \in S_2(v_2, v_3)$. Note that $py \in E$ by definition of p, and $pz \notin E$ or $pyzv_3 = C_4$. So $S_3^y \cap S_2(v_2, v_3) = \{p\}$ or K_5 would arise. Now $|S_3^y| = 4$ and we are in one of previous four cases. So, we may assume that $p \in S_2(v_0, v_4)$. By symmetry, there exists a vertex $q \in S_3^z \cap S_2(v_4, v_3)$. by definition of p and q, $py, qz \in E$. Now $pyzq = C_4$.

Case b. There are two edges in $\{x, y, z, w\}$. Suppose first that $xy, wz \in E$ but $yz \notin E$. Define C_y and C_z as above. As $y \notin S_3^z$ and $z \notin S_3^y$, we have $S_3^y \cap S_2 \neq \emptyset$ and $S_3^z \cap S_2 \neq \emptyset$. We claim that $S_3^y \cap S_2(v_2, v_3) \neq \emptyset$. Otherwise, let $p \in S_3^y \cap S_2(v_0, v_4)$. Note that $py \in E$, and $px \in E$ or $v_4pyx = C_4$. Also, $S_3^y \cap S_2(v_0, v_4)$ is a clique and hence $S_3^y \cap S_2(v_0, v_4) = \{p\}$ or K_5 would arise. Now $S_3^y = \{x, p, v_1, w\}$ with $x, p \in S_3^y(v_0)$ and so we are in one of four previous cases. Hence, the claim holds. Similarly, $S_3^z \cap S_2(v_0, v_1) \neq \emptyset$. Let $p \in S_3^y \cap S_2(v_2, v_3)$ and $q \in S_3^z \cap S_2(v_0, v_1)$. Note that $py, qz \in E$. Also, $qy \notin E$ or $qyv_2z = C_4$. As $yxv_4wzq \neq P_6$, we have $qx \in E$. Also, $qy \notin E$ or $\{v_0, v_1, q, x, y\}$ would induce a K_5. Then $\{v_2, z, q, x, y\} \cup \{v_1\}$ induces a W_5.

Now we consider the case $xy, yz \in E$ but $zw \notin E$. Let $C_z = C \setminus \{v_2\} \cup \{z\}$ and $C_w = C \setminus \{v_3\} \cup \{w\}$. As $zw \notin E$, we have that $S_3^z \cap S_2 \neq \emptyset$ and $S_3^w \cap S_2 \neq \emptyset$. We claim that $S_3^z \cap S_2(v_3, v_4) \neq \emptyset$. If not, there exists $p \in S_3^z \cap S_2(v_0, v_1)$. Note that $pz \in E$, and so $py \in E$ or $v_0yzp = C_4$. So, $S_3^z \cap S_2(v_0, v_1) = \{p\}$ or K_5 would arise. Hence, $S_3^z = \{x, y, v_2, p\}$ with $y, p \in S_3^z(v_1)$ and we are in one of previous four cases. So, the claim holds and let $p \in S_3^z \cap S_2(v_3, v_4)$. Note that $pz \in E$ and $py \notin E$. Also, $pw \notin E$ or $pwv_2z = C_4$, and $px \notin E$ or $pxv_1z = C_4$. Let $q \in S_3^w \cap S_2$. $qw \in E$. If $q \in S_2(v_0, v_4)$, then $pq \in E$ and thus $wv_3pq = C_4$. So, $q \in S_2(v_1, v_2)$. Also, $qz \notin E$ or $qzv_3w = C_4$, and $qx \notin E$ or $qxv_4w = C_4$. Hence, x is anti-complete to $S_2(v_1, v_2) \cup S_2(v_3, v_4)$. As $qwv_4xyz \neq P_6$, we have $qy \in E$. Note that $S_2(v_3, v_2) = \emptyset$ by the fact $wp \notin E$ and Observation 1 (2). Thus, $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$. So, $S_1 = S_1(v_1) \cup S_1(v_4)$. Now consider $C^* = xyzpv_4$. Note that $v_0 \in S_3^*(x)$, $v_1 \in S_3^*(y)$, and $v_3 \in S_3^*(p)$. but $v_2, q, w \notin S_3^*$. By the minimality of C, we have $S_3^* \cap (S_1 \cup S_2) \neq \emptyset$. Let $r \in S_3^* \cap (S_1 \cup S_2)$. If $r \in S_2(v_3, v_4)$, then r must be in $S_3^*(p)$ as r is anti-complete $\{x, y\}$. Thus $G = G_{2,2}$. Moreover, any vertex $t \in S_2(v_1, v_2)$ is anti-complete to $\{x, v_4, p\}$, and any vertex $t \in S_1(v_4)$ is anti-complete to $\{p, y, z\}$. Therefore, $r \in S_1(v_1)$. If r is complete to $\{x, y, z\}$, then there exists $r' \in S_3^* \cap S_1(v_1)$ with $r' \neq r$ otherwise $|S_3^*| = 4$ and we are in one of four pervious cases. Note that r' must be complete to $\{p, y, z\}$. Hence, in any case there exists a vertex $r \in S_1(v_1)$ that is complete to $\{p, y, z\}$ but $px \notin E$. Now $wv_3prv_1x = P_6$.

Case 5. $|S_3| = 5$. There are five possible configurations for S_3.

C **is of type (2,2,0,0,1)**. $S_3(v_0) = \{x, x'\}$, $S_3(v_1) = \{y, y'\}$, $S_3(v_4) = \{w\}$. We may assume that y is anti-complete to $\{x, x'\}$ and x is anti-complete to $\{y, y'\}$. If $t \in S_1(v_3)$ then $tv_3v_4xv_1y = P_6$. So, $S_1(v_3) = \emptyset$. Let $C' = C \setminus \{v_1, v_4\} \cup \{y, w\}$ be an induced C_5. Let $p \in S_3' \cap S_2$ by $x \notin S_3'$ and the minimality of C. Suppose first that $p \in S_2(v_0, v_1)$. Note that $S_2(v_3, v_4) \cup S_2(v_2, v_3) \neq \emptyset$ by $d(v_3) \geq 4$. Let $q \in S_2(v_3, v_4) \cup S_2(v_2, v_3)$. Without loss of generality, we assume that $q \in S_2(v_2, v_3)$. Since $qv_3v_4xv_1y(y') \neq P_6$, we have q is complete to $\{y, y'\}$. As q is an arbitrary vertex in $S_2(v_3, v_4)$, we have $S_(v_2, v_3)$ is complete to $\{y, y'\}$ and so $S_2(v_2, v_3) = \{q\}$. Note that $S_2(v_3, v_2) = \emptyset$ by (P8) and so $N(v_3) = \{v_2, v_4, q, w\}$. Now as G **is a minimal obstruction**, $G - v_3$ has a 4-coloring ϕ. Note that $\phi(q) = \phi(v_1) = \phi(v_4)$ and therefore we can extend ϕ to G, a contradiction. As $x, x' \notin S_3'$, there exists two different vertices p and q in $S_3' \cap S_2$. If $p, q \in S_2(v_0, v_4)$, then $\{p, q, v_0, v_4, w\}$ induces a K_5. Note that $S_2(v_2, v_3)$ is complete to $\{y, y'\}$ and $S_2(v_2, v_3)$ contains at most one vertex. Hence, we may assume that $p \in S_2(v_0, v_4)$ and $S_2(v_2, v_3) = \{q\}$. By the fact that $yq \in E$ and (P10), we have $S_2(v_3, v_4) = \emptyset$. Hence, we derive a similar contradiction as above.

C **is of type (0,1,0,2,2)**. $S_3(v_3) = \{x, x'\}$, $S_3(v_4) = \{y, y'\}$, $S_3(v_1) = \{w\}$. We may assume that y is anti-complete to $\{x, x'\}$ and x is anti-complete to $\{y, y'\}$. Let $C' = C \setminus \{v_1, v_4\} \cup \{y, w\}$ be an induced C_5. Then $x, x' \notin S_3'$ and hence $S_3' \cap (S_1 \cup S_2)$ contains at least two vertices. Let p and q be such two vertices. Let $t \in S_1(v_0)$. If t is not anti-complete to $\{y, y'\}$, say $ty \in E$, then $tyv_4xv_2v_1 = P_6$. Thus $p, q \notin S_1(v_0)$. Now suppose that $q \in S_2(v_0, v_4)$. Then q is complete to $\{w, y\}$. Note that $qy' \notin E$. Then the fact that $y'yqwv_2x \neq P_6$ implies that $qx \in E$ and thus $qxv_2w = C_4$. Thus, $p, q \notin S_2(v_0, v_4)$. If $p, q \in S_2(v_0, v_1)$, then $\{p, q, v_0, v_1, w\}$ would induce a K_5. Now let $p, q \in S_2(v_2, v_3)$. If $\{p, q\}$ is complete to y or w, then $G = G_{3,1}$ otherwise G would contain an induced W_5. Hence, we may assume that $py \in E$ and $qw \in E$. Thus $pw, qy \notin E$. By (P10), $S_2(v_0, v_1) = S_2(v_0, v_4) = \emptyset$. Also, $S_1(v_1) = \emptyset$ since $S_2(v_3, v_2) \neq \emptyset$. By $d(v_1) \geq 4$ we have $S_2(v_1, v_2) \neq \emptyset$ and thus $S_2(v_2, v_3) = \{p, q\}$. Now let $C_y = C \setminus \{v_4\} \cup \{y\}$. Then $|S_3^y \cap S_2(v_2, v_3)| \geq 2$. But this is impossible since $qy \notin E$. Therefore, $p \in S_2(v_0, v_1)$ and $q \in S_2(v_2, v_3)$. By definition of q, $py \in E$ and hence $S_2(v_1, v_2) = \emptyset$ by (P10). Moreover, $S_2(v_4, v_0) = \emptyset$ by (P7). Now consider $C_x = C \setminus \{v_3\} \cup \{x\}$ and thus $|S_3^x| < 5$ contradicting the minimality of C.

C **is of type (2,1,1,0,1)**. $S_3(v_0) = \{x, x'\}$, $S_3(v_1) = \{y\}$, $S_3(v_2) = \{z\}$, $S_3(v_4) = \{w\}$. If $yz \in E$ and one of $\{x, x'\}$ is complete to $\{y, w\}$, then $G = G_{P_4}$. Hence, either $yz \notin E$ or no vertex in $\{x, x'\}$ is complete to $\{y, w\}$. Let $C' = C \setminus \{v_1, v_4\} \cup \{y, w\}$ be an induced C_5. Thus $|S_3' \cap (S_1 \cup S_2)| \geq 2$. Let $p, q \in S_3'$. Note that $p, q \in S_2(v_0, v_1) \cup S_2(v_0, v_4) \cup S_2(v_2, v_3)$. If $\{p, q\} \subseteq S_2(v_0, v_1)$ or $\{p, q\} \subseteq S_2(v_0, v_4)$, then K_5 would arise. Next we show that $\{p, q\} \nsubseteq S_2(v_2, v_3)$. If not, then both p and q are adjacent to exactly one of $\{y, w\}$. If $pw \in E$, then the fact that $zv_2pwv_0x(x') \neq P_6$ implies that $pz \in E$. We may assume that $xy \notin E$. If $py \in E$, Since $wv_3pyv_1x \neq P_6$, we have $wx \in E$. Thus, $x'w \notin E$. As $wv_3pyv_1x' \neq P_6$, we have $x'y \in E$. Now $zy \in E$ since $wxx'yv_2z \neq P_6$. Hence, $pz \in E$ or $ypv_3z = C_4$. We have showed if $p \in S_2(v_3, v_2)$ then $pz \in E$. Therefore, $pq \notin E$ or $\{p, q, v_2, v_3, z\}$ would induce a K_5. Further, y or w cannot be complete to $\{p, q\}$. Thus, we may assume that $py \in E$ and $qz \in E$. By previous argument we have that $\{y, x, x', w\}$ induces a P_4 and hence $qwxx'yp = P_6$.

Therefore, three cases remain. If $p \in S_2(v_0, v_1)$ and $q \in S_2(v_0, v_4)$, then $pq \in E$ by (P1) to (P3). By Observation 1 (2), we have $\{x, x'\}$ is complete to $\{p, q\}$ and thus $\{x, x', v_0, p, q\}$ induces a K_5. If $p \in S_2(v_0, v_4)$ and $q \in S_2(v_2, v_3)$, then p is complete to $\{y, w\}$ by definition. By (P9), we have $yq \in E$ and $wq \notin E$. We may assume that $xy \notin E$. Thus $wxx'y = P_4$ as shown above. Also $px \notin E$ or $pxv_1y = C_4$ and hence $px' \notin E$ or $px'xw = C_4$. Now we have $wxx'yp$ is an induced C_5 with v_0 being a 5-vertex. Finally, let $p \in S_2(v_0, v_1)$ and $q \in S_2(v_2, v_3)$. By definition, p is complete to $\{y, w\}$. By (P9), $qw \in E$ and $qy \notin E$. Moreover, $qz \in E$, and $pz \in E$ or $zqwp = C_4$. If x is complete to $\{y, w\}$, then $px \in E$ or $xwpy = C_4$ and thus $\{x, y, v_0, v_1, p\}$ would induce a K_5. Hence, none of $\{x, x'\}$ is complete to $\{y, w\}$. Therefore, $yz \in E$ or $|S_3' \cap (S_1 \cup S_2)| \geq 3$, which is impossible by previous argument. Now $\{p, v_1, v_2, q, w, v_0, y, z, v_3\}$ induces a G_{P_4} with respect to $C^* = wv_0yzv_3$ and $S_3^* = \{q, v_2, v_1, p\}$ for which qv_2v_1p induces a P_4.

C **is of type (1,1,2,0,1)**. $S_3(v_0) = \{x\}$, $S_3(v_1) = \{y\}$, $S_3(v_2) = \{z, z'\}$, $S_3(v_4) = \{w\}$. Note that $xw \notin E$ or $G = G_{2,2}$. We may assume that $yz \notin E$. Let $C' = C \setminus \{v_1, v_4\} \cup \{y, w\}$. Hence, $|S_3' \cap (S_1 \cup S_2)| \geq 2$. Let $p, q \in S_3' \cap (S_1 \cup S_2)$. If $p \in S_1(v_0)$, then p is complete to $\{x, y, w\}$ and hence $pxv_4w = C_4$. If $|S_3' \cap (S_1 \cup S_2)| \geq 2$ or $|S_3' \cap (S_1 \cup S_2)| \geq 2$ then K_5 would arise. Next we show that $\{p, q\} \nsubseteq S_2(v_2, v_3)$. If not, let $q, p \in S_2(v_2, v_3)$. Note that p is not complete to $\{z, z'\}$, say $zp \notin E$. If $pw \in E$ then $zv_2pwv_0x = P_6$. Hence, y is complete to $\{p, q\}$. But now $\{v_1, v_2, v_3, y, z, z', p, q\}$ contains an induced W_5. Therefore, three cases remains. If $p \in S_2(v_0, v_1)$ and $q \in S_2(v_0, v_4)$, then x is complete to $\{p, q\}$ by Observation 1 (2). By definition of p, we have $pw \in E$ and thus $pxv_4w = C_4$. If $p \in S_2(v_0, v_1)$ and $q \in S_2(v_2, v_3)$, then $wp \in E$. By (P9), we have $wq \in E$. Now $zv_2qwv_0x = P_6$ or $z'v_2qwv_0x = P_6$. Finally, $p \in S_2(v_0, v_4)$ and $q \in S_2(v_2, v_3)$. By definition of p, we have $py \in E$ and hence $qy \in E$ by (P9). We may assume that $qz \notin E$. Then $zy \notin E$ or $v_3qyz = C_4$. Thus the fact that $zv_3qyv_0x \neq P_6$ implies that $xy \in E$ and so $xp \in E$ or $v_4xyp = C_4$. Now $qv_3wpxv_1 = P_6$.

C **is of type (1,1,1,1,1)**. Let $S_3(v_i) = \{u_i\}$ for each i. Note that there are at most 3 edges within S_3 or $G = G_{P_4}$. We consider the following three cases.

Case a. S_3 has at most two edges and does not induce a P_3. Without loss of generality, we may assume that $u_0u_1, u_1u_2, u_3u_4 \notin E$. Let $C' = C \setminus \{v_1, v_4\} \cup \{u_1, u_4\}$. Note that $u_0, u_2, u_3 \notin S_3'$ and hence $|S_3' \cap (S_1 \cup S_2)| \geq 3$ by the minimality of C. Let $p \in S_3' \cap (S_1 \cup S_2)$. If $p \in S_1(v_1)$, then $pu_0 \in E$ by properties (P11) and (P12). and thus $pu_0v_1u_1 = C_4$. Hence, $S_3' \cap S_1 = \emptyset$. If $|S_3' \cap S_2(v_0, v_1)| \geq 2$ or $|S_3' \cap S_2(v_0, v_1)| \geq 2$, then K_5 would occur. Now suppose that $p, q, r \in S_3' \cap S_2(v_2, v_3)$. If u_1 or u_4 is complete to $\{p,q,r\}$, then K_5 would occur. So, we may assume that $pu_1, qu_1 \in E$ and $ru_4 \in E$. Since $rv_3pu_1v_0u_0 \neq P_6$, we have $rp \in E$. Replacing q with p we have $rq \in E$ and so $\{v_2, v_3, p, q, r\}$ induces a K_5. By (P8), we have that $|S_3' \cap S_2(v_2, v_3)| = 2$ and $S_3' \cap (S_2(v_0, v_1) \cup S_2(v_0, v_4)) \neq \emptyset$. Suppose that $p \in S_2(v_0, v_1)$. We repeat the argument for $C'' = C \setminus \{v_1, v_3\} \cup \{u_1, u_3\}$ and obtain that $S_2(v_0, v_4) \neq \emptyset$. This contradicts (P8). Hence, let $p \in S_2(v_0, v_4)$ and $q, r \in S_2(v_2, v_3)$. Note that $pu_1 \in E$ by definition of p and hence u_1 is complete to $\{p, q, r\}$. So, $S_2(v_2, v_3) = \{q, r\}$ and $S_2(v_0, v_4) = \{r\}$. But this contradicts the fact that $|S_3'' \cap S_2(v_0, v_4)| \geq 2$.

Case b. S_3 does induce a P_3. Without loss of generality, we assume that $u_4u_0, u_0u_1 \in E$. Let $C_1 = C \setminus \{v_0, v_2\} \cup \{u_0, u_2\}$. Note that $S_3^1 \cap S_1 = \emptyset$. Since $u_1, u_3 \notin S_3^1$, we have $|S_3^1 \cap S_2| \geq 2$ by the minimality of C. If $|S_3^1 \cap S_2(v_0, v_1)| \geq 2$ or $|S_3^1 \cap S_2(v_1, v_2)| \geq 2$, then K_5 would arise. If $p \in S_3^1 \cap S_2(v_0, v_1)$ and $q \in S_3^1 \cap S_2(v_1, v_2)$, then u_1 is complete to $\{p, q\}$ by Observation 1 (2). Also, $pu_0 \in E$ by definition of p and thus $\{u_0, u_1, v_0, v_1, p\}$ induces a K_5. Therefore, $S_2(v_3, v_4) \neq \emptyset$. Now we repeat the argument for $C_4 = C \setminus \{v_0, v_3\} \cup \{u_0, u_3\}$ and obtain that $S_2(v_1, v_2) \neq \emptyset$. So, $S_2(v_4, v_0) = S_2(v_0, v_1) = \emptyset$. Let $p \in S_3^1 \cap S_2(v_3, v_4)$ and $q \in S_3^4 \cap S_2(v_1, v_2)$. Let $C_2 = C \setminus \{v_1, v_3\} \cup \{u_1, u_3\}$ and $C_3 = C \setminus \{v_2, v_4\} \cup \{u_2, u_4\}$. Note that $|S_3^2 \cap S_2| \geq 2$ and $|S_3^2 \cap S_2| \geq 2$. Since $S_2(v_0, v_4) = \emptyset$ and $|S_3^2 \cap S_2(v_2, v_1)| \leq 1$, $S_3^2 \cap S_2(v_2, v_3) \neq \emptyset$. Let $r \in S_3^2 \cap S_2(v_2, v_3)$. By definition of r, we have r is complete to $\{u_1, u_3\}$. Similarly, $S_3^3 \cap S_2(v_2, v_3) \neq \emptyset$. If $r \in S_3^3 \cap S_2(v_2, v_3)$, then r is complete to $\{u_2, u_4\}$. So, $u_0u_1ru_4 = C_4$. Hence, there exists $r' \neq r$ such that $r' \in S_3^3 \cap S_2(v_2, v_3)$. Thus, $S_2(v_3, v_4) = \{p\}$, $S_2(v_3, v_2) = \{r, r'\}$, and $S_2(v_2, v_1) = \{q\}$. Now $p \in S_3^3$ and $q \in S_3^2$, i.e., p (respectively q) is complete to $\{u_2, u_4\}$ (respectively $\{u_1, u_3\}$). By the fact that $ru_3 \in E$ and Observation 1 (2), we have u_3 is complete to $\{p, r, r'\}$ and thus $\{u_3, v_3, p, r, r'\}$ induces a K_5.

Case c. S_3 is isomorphic to $P_3 + P_2$. Without loss of generality, we assume that $u_0u_1, u_1u_2, u_3u_4 \in E$. Let $C_i = C \setminus \{v_i\} \cup \{u_i\}$ for each i. By the minimality of C, we have $S_3^i \cap S_2 \neq \emptyset$ for each $i \neq 1$. Let $r \in S_3^3$ and $s \in S_3^4$. If $r \in S_2(v_1, v_2)$ and $s \in S_2(v_0, v_1)$, then $u_4sru_3 = C_4$. If $r \in S_2(v_0, v_4)$ and $s \in S_2(v_2, v_3)$, let $t \in S_3^2 \cap S_2$. Note that $t \in S_2(v_3, v_4)$. By Observation 1 (2), we have t is complete to $\{u_3, u_4\}$ and so $\{u_3, u_4, v_3, v_4, t\} = K_5$. The remaining two cases are symmetric and we may assume that $r \in S_2(v_0, v_4)$ and $s \in S_2(v_0, v_1)$. Let $t \in S_3^0 \cap S_2$. If $t \in S_2(v_1, v_2)$, then s is complete to $\{u_0, u_1\}$ by Observation 1 (2). Hence, $\{u_0, u_1, v_0, v_1, s\} = K_5$. So, $t \in S_2(v_3, v_4)$. Then u_4 is complete to $\{r, t\}$. Since G is K_5-free, $tu_3 \in E$ and thus $tru_3v_3 = C_4$.

Case 6. $|S_3| = 6$. There are three possible configurations for S_3.

C **is of type (2,1,1,1,1)**. Let $S_3(v_0) = \{x, x'\}$, $S_3(v_1) = \{y\}$, $S_3(v_2) = \{r\}$, $S_3(v_3) = \{t\}$, $S_3(v_4) = \{z\}$. We may assume that $xy \notin E$. We also assume that $rt \notin E$ or $G = G_{2,2}$. Let $C' = C \setminus \{v_1, v_4\} \cup \{y, z\}$ be an induced C_5. As $xy \notin E$, we have $S_3' \cap (S_1 \cup S_2) \neq \emptyset$ by the minimality of C. Let $p \in S_3'$. Then p is complete to $\{y, z\}$. It is easy to check that $p \in (N(v_0) \cap (S_1 \cup S_2)) \cup S_2(v_2, v_3)$. If $p \in S_1(v_0)$, then $px \in E$ and so $pxv_1y = C_4$. If $p \in S_1(v_0, v_1)$, then the fact that $pyv_2v_3v_4x \neq P_6$ implies that $px \in E$. Thus $xz \in E$ or $pxv_4z = C_4$. Hence, $x'z \notin E$ and thus $x'p \notin E$. So, $x' \notin S_3'$. By symmetry, if $p \in S_2(v_0, v_4)$, then $x' \notin S_3'$. If $p \in S_2(v_2, v_3)$, then by symmetry we assume that $py \in E$. Since tv_3pyv_0x does not induce a P_6, we have $tp \in E$. Therefore p is the only vertex in $S_2(v_2, v_3)$ that is adjacent to y otherwise K_5 would occur. Thus there is also at most one vertex in $S_2(v_2, v_3)$ that is adjacent to z. Also, $ry \notin E$ otherwise $tv_3ryv_0x = P_6$. By symmetry, $zt \notin E$. Hence, $|S_3' \cap (S_1 \cup S_2)| \geq 3$ and $|S_3' \cap (S_1 \cup S_2)| \geq 4$ if $S_3' \cap (S_2(v_0, v_4) \cup S_2(v_0, v_1)) \neq \emptyset$ by the minimality of C. But now we either have a K_5 or contradicts (P9).

C **is of type (2,2,0,1,1)**. Let $S_3(v_0) = \{x, x'\}$, $S_3(v_1) = \{y, y'\}$, $S_3(v_3) = \{t\}$, $S_3(v_4) = \{w\}$. Note that $wt \notin E$ or $G = G_{2,2}$. We may assume that y is anti-complete to $\{x, x'\}$. Let $C' = C \setminus \{v_1, v_4\} \cup \{w, y\}$. Thus by the minimality of C we have $S_3' \cap (S_1 \cup S_2) \neq \emptyset$. Let $p \in S_3'$ and it is easy to check that $p \in S_2(v_0, v_1) \cup S_2(v_0, v_4) \cup S_2(v_2, v_3)$. Suppose first that $p \in S_2(v_0, v_1)$. Then p is complete to $\{y, w\}$. Note that $tp \notin E$ or $tpv_0v_4 = C_4$. Since $tv_4wpv_1y' \neq P_6$, we have $py' \in E$ and thus $\{v_0, v_1, y, y', p\}$ induces a K_5. Suppose now that $p \in S_2(v_0, v_4)$. Again, p is complete to $\{y, w\}$. Note that $tp \notin E$ or $tpyv_2 = C_4$. Then the fact that $tv_3wpyy' \neq P_6$ implies that $py' \in E$. Now $\{x, x', y, y, v_0, v_1, v_4, p\}$ induces a Hajos graph with one additional dominating vertex. Finally, assume that $p \in S_2(v_2, v_3)$. Then p is adjacent to exactly one of $\{y, w\}$. Suppose that $pw \in E$. Then $tp \notin E$ or $v_4wpt = C_4$. By G is K_5-free, w is not complete to $\{x, x'\}$, say $wx \notin E$, and hence $xp \notin E$ or $xv_4wp = C_4$. Now $tv_2pwv_0x = P_6$. Therefore, $py \in E$ and $pw \notin E$. We may assume that $xw \notin E$, and now $pv_2v_1xv_4w = P_6$.

C **is of type (2,2,,1,0,1)**. Let $S_3(v_0) = \{x, x'\}$, $S_3(v_1) = \{y, y'\}$, $S_3(v_2) = \{t\}$, $S_3(v_4) = \{w\}$. We may assume that y is anti-complete to $\{x, x'\}$ and x is anti-complete to $\{y, y'\}$. Let $C' = C \setminus \{v_1, v_4\} \cup \{y, w\}$. By the minimality of C, we have $S_3' \cap (S_1 \cup S_2) \neq \emptyset$. Let $p \in S_3'$ and it is easy to check that $p \in S_2(v_0, v_1) \cup S_2(v_0, v_4) \cup S_2(v_2, v_3)$. Suppose first that $p \in S_2(v_0, v_1)$. Then p is complete to $\{y, w\}$. Now Since $pyv_2v_3v_4x(x') \neq P_6$, we have p is complete to $\{x, x'\}$ and thus $\{v_0, v_1, p, x, x'\}$ induces a K_5. Now suppose that $p \in S_2(v_0, v_4)$. Again, p is complete to $\{y, w\}$. Note that $tp \notin E$ or $pv_0v_1t = C_4$. If $py' \in E$, then $xp \in E$ or $xv_1y'pv_4$ and v_0 would induce a W_5. But now $xpy'v_1 = C_4$. Hence, $py' \notin E$. Now the fact that $tv_3v_4pyy' \neq P_6$ implies that $ty \in E$ or $ty' \in E$. If $ty \in E$, then $y'ytv_3v_4x = P_6$. Otherwise, $ty' \in E$. Then $px \notin E$ or $pxv_1y = C_4$. Now $ty'ypv_4x = P_6$. Thus there exist vertices $p, p' \in S_3' \cap S_2(v_2, v_3)$. Suppose that $py \in E$ and so $pw \in E$. Since $xv_4v_3pyy' \neq P_6$, we have $py' \in E$. Note that t is not complete to $\{y, y'\}$. Thus, $tp \in E$ as $tv_3py(y')v_0x \neq P_6$. Now $py(y')v_1t = C_4$. Hence, w is complete to $\{p, p'\}$. Now $\{y, y', v_0, v_1, v_2, p, p'v_3, w\}$ induces a $G_{3,1}$.

Case 7. $|S_3| = 7$. Suppose that $S_3(v_0) = \{x\}$, $S_3(v_1) = \{y\}$, $S_3(v_4) = \{z\}$, $S_3(v_2) = \{r, r'\}$ and $S_3(v_3) = \{t, t'\}$. We may assume that r is anti-complete to $\{t, t'\}$ and t is anti-complete to $\{r, r'\}$ or K_5 would occur. Let $C_r = C \setminus \{v_2\} \cup \{r\}$. Since $t, t' \notin S_3^r$ we have $|S_3^r \cap (S_1 \cup S_2)| \geq 2$ by minimality of C. Let p and p' be two vertices in the $S_3^r \cap (S_1 \cup S_2)$. Then r is complete to $\{p, p'\}$. It is routine to check that p and p' belong to $S_2(v_0, v_1) \cup S_2(v_3, v_4)$. First suppose that $\{p, p'\} \subseteq S_2(v_0, v_1)$. Let $C_t = C \setminus \{v_3\} \cup \{t\}$. Then there exist q and q' such that q and q' belong to $S_2(v_4, v_0)$ or $S_2(v_1, v_2)$. If $\{q, q'\} \subseteq S_2(v_4, v_0)$ or $\{q, q'\} \subseteq S_2(v_1, v_2)$ then $\{p, p'q, q', v_1\}$ would induce a K_5. Hence there must be the case that $q \in S_2(v_4, v_0)$ and $q' \in S_2(v_1, v_2)$. By definition of q and q', t is complete to $\{q, q'\}$, which contradicts (P10). Therefore, $S_2(v_3, v_4) \neq \emptyset$. Repeating the argument for C_t we have $S_2(v_1, v_2) \neq \emptyset$. So, $S_2(v_0, v_4) = S_2(v_0, v_1) = \emptyset$ and thus $p, p' \in S_2(v_3, v_4)$ and $q, q' \in S_2(v_1, v_2)$. Now Let $C_y = C \setminus \{v_1\} \cup \{y\}$ and $C_z = C \setminus \{v_4\} \cup \{z\}$. The same argument shows that $S_3^y \cap S_2(v_2, v_3) \neq \emptyset$ and $S_3^z \cap S_2(v_2, v_3) \neq \emptyset$. As $|S_2(v_1, v_2)| \geq 2$, we obtain that $S_2(v_2, v_3)$ contains only one vertex u. Thus $uy \in E$ and $uz \in E$. But now $uyv_0z = C_4$.

This completes the proof. □