Beyond Gödel's Theorem: Turing Nonrigidity Revisited

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In [4] it was argued that incomputability in nature dictates a mathematical model (due to Turing [39]) based on mechanical processes relative to appropriate abstractions of incomputable phenomena. (One should refer to that paper for background historical and technical detail.) For instance, it is Turing definability rather than the more familiar notions of provability, and completeness of axiomatic theories, which are more relevant to an analysis of the scope of scientific understanding in the real world; while recent results concerning Turing invariance and nonrigidity have both negative and positive consequences for science as a means to knowledge. Turing nonrigidity (see [6]) may reinforce scepticism about a narrow perspective based on scientific observation, as modelled by Turing computable processes: but the proliferation of invariant substructures of the Turing universe (see, for example, Cooper [5], Nies, Shore and Slaman [27] or Odifreddi [28]) can be viewed as reflecting negatively on the more radical postmodernist and (post-) structuralist views of the roles of culture and language in relation to science (cf. Gross and Levitt [17]) – objective reality does exist. Such comments can be framed in terms of qualifications to the Duhem-Quine thesis ("Any statement can be held true come what may, if we make drastic enough adjustments elsewhere in the system.", Quine [33], p. 43), and have obvious consequences for related empiricist and pragmatist views of the world (cf. again Quine, p. 44, "Physical objects are conceptually imported into the situation as convenient intermediaries – not by definition in terms of experience, but simply as irreducible points comparable, epistemologically, to the gods of Homer.") Of course, the theory itself does indicate difficulties in substantiating the Turing model, but, if not overstretched (viz. the ubiquitous Gödel's [15], [16] Theorem) such asymptotic representations can be useful and productive adjuncts to subjective intuition. For instance, unlike in mathematics where

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small variations in axioms can lead to fundamentally different theories, Turing nonrigidity and known countable automorphism bases indicate that although diverse basic assumptions about the real world, related to culture or religion, for example, are inevitable (perhaps even necessary), relative to the Turing model there is a convergence at higher levels of the informational structure suggested by relative rigidity of substructures.

The purpose of this note is to describe how, at a more basic level, the *material* Universe can be modelled according to the underlying structure of its information content. Modern information theory since Shannon [35], linking physical interactions and particular kinds of information transference, provides the background. See [8] for a more detailed discussion. *Some* such mathematical framework is essential to attempts to reduce the ad hoc arbitrariness of cosmological theory by situating it within a logically coherent Universe (see for instance Davies [9], p. 68: "On the philosophical side, there is an urgent need for these speculations to be placed in the context of a theory of mathematics"). In developing such a model, the concept of *immanence* potentially extends not just to the actions of natural laws, but to the process of formation of those laws. (Cf. Penrose's description of his 'strong determinism', [30], pp. 106–107, and his comment that "Like Einstein and his hidden-variable followers, I believe strongly that it is the purpose of physics to provide an *objective* description of reality.")

If one abstracts from the Universe its information content, structured via the basic (computable, at least in the sense of Solomonoff [38]) fundamental laws of nature, one obtains a particular (partial) manifestation of the Turing universe (usually investigated via the Turing degree structure), within which vague questions attain a precise analogue of quite plausible validity. Of course, the usefulness of such a model will depend on a clearer idea of *which* substructure of the Turing universe is being materialised. One can then get theoretical analogues to such questions as:

- How is the Universe capable of determining observed reality?

- How is the observed ambiguity at the quantum level reconciled with classical reality? And how can one explain nonlocal quantum phenomena (such as that that implied by Bell's [1] Theorem and the Einstein-Podolsky-Rosen [10] paradox)?

- Can one add to theoretical projections concerning the Universe under extreme conditions, such as near the Big Bang 'singularity'?

Basic to this is a discussion (see [8]) of how incomputability in nature arises in finite time from ostensibly computable structures, as evidenced by chaotic natural phenomena, graphic examples such as the Julia and Mandelbrot sets of how how incomputability arises mathematically from the iteration of simple rules, and, perhaps most significantly, by the discovery by Pour-El and Richards [31], [32] of a general class of differential equations giving rise to noncomputable solutions from computable initial data. It must be the case that a Universe low in information content, in which the essential character of its laws is defined according to an immanently determined logic, is incapable of defining anything unambiguously. Of course, the Turing model obviously requires some assumption such as the well-foundedness of the hierarchy of subatomic particles, but not necessarily discreteness of time and space.

The theoretical key to the first of the above questions is then the automorphism group of the Turing universe, encapsulating the range and interrelationships of the parallel worlds allowed by the underlying model. Given that information content in nature must first evolve via the quantum level, with the local theory of the Turing model immediately relevant, one needs to know what the global context dictates as regards local invariance, relative rigidity, automorphism bases, and structure of the computably enumerable Turing degrees. It may well be that a deeper understanding of the arcane mysteries of the Turing structure of computably enumerable objects (see, for example, Soare [37] or Lerman [25]) has the potential to dissipate the impression of arbitrariness attaching to the details of subatomic structure, although the possibility that other noncanonical levels of the arithmetical hierarchy may be relevant cannot be excluded.

By Slaman and Woodin [36], it is known that the computably enumerable Turing degrees form an automorphism base for the global structure. The latter has a countably infinite automorphism group (see [6]), providing a theoretical explanation for the observed quantum ambiguity. The form of the noninvariance attached to subatomic individual states suggested by the Heisenberg [21] Uncertainty Principle (but not necessarily to non-unary relations, such as those derived from the weak and the strong nuclear forces) would lead one to expect many local relations, but not singletons, rigid relative to the global model. The former is borne out by Cooper [5], Jockusch and Shore [23], Nerode and Shore [26] and Nies, Shore and Slaman [27], for instance, providing a rich source of subatomic structure, but there appear to be no invariant computably enumerable singletons other than $\mathbf{0}$ and $\mathbf{0}'$. It is worth noting that the relationship between invariance and definability is not well understood, so it may be that there are well-defined elements of the quantum environment which cannot be theoretically captured in a framework derived (inductively or otherwise) from scientific observation.

The consolidation of quantum entities into classically observable objects has an immediate parallel in the actions of the algebraic operations on the Turing degrees and the invariance of $\mathbf{0}'$ (see [5]). The world we observe already encapsulates a high level of concealed information content. It is Slaman and Woodin's proof of the relative rigidity of the Turing universe above $\mathbf{0}''$ which provides the theoretical counterpart of the classically observable universe. The phase transition from quantum ambiguity to an immanently defined reality will involve an accretion of information content concerning the way in which contingencies retrospectively select a consistent invariant history, which will lift the phenomena in question to the cone above $\mathbf{0}''$. It is this process which also provides a theoretical explanation of EPR in terms of the higher order logical structure of the Universe, and hence gives substance to Bohm's (see [2]) concept of globally originating laws of cause and effect, the explanations in terms of decoherence due to Gell-Mann and Hartle [14], Omnès [29] and others, and the orderly retreat from

Laplacian determinism. In this context (providing not just 'explanation' but the 'understanding' looked for in Feynman [12]), the unsatisfactory elements of the various ontological interpretations of quantum theory (including the 'many worlds' interpretation of Everett [11], and its variants), and their disagreements, originating with their attempts (or lack of them) to place classical reality within a universe of quantum uncertainty, are potentially removable. It is worth mentioning here that although the cone above $\mathbf{0}'$ is not an automorphism base (in the proof [6] of nonrigidity, it is relatively straightforward to make the presentation $*$ of the nontrivial automorphism low, so that the automorphism itself will leave unchanged everything above $\mathbf{0}'$), one can construct a Turing automorphism which witnesses the existence of a noninvariant atomic jump class, so that relative rigidity of the cone above $\mathbf{0}'$ fails.

Finally, depending on the extent to which the Turing model is found to be helpful, one may use it to obtain mathematical projections of the behaviour of the Universe near the Hawking and Penrose [19] 'initial singularity'. In doing this, consideration of the extent to which the global theory is realised at the lower levels of the Turing model appropriate to the Universe at different stages of its development becomes crucial. The mathematical singularity within the Big Bang (and any final 'Big Crunch') can only be extrapolated from a context in which gravity and the laws of general relatively derive from an extra-Universal source, whereas in the Turing model such relations derive their authority from the invariance of (first order) arithmetically defined relations on the cone above $\mathbf{0}''$ (Slaman and Woodin [36]). But it is known that at increasingly local levels globally definable relations lose their Turing invariance (for instance, Cooper [5], [4], lowness is defined globally but is noninvariant within the structure of the computably enumerable degrees). The scenario suggested close to the Big Bang, and tacitly accepted by Hawking (see Hartle and Hawking [18]) in more recent work concerning the 'No-Boundary' model, is one in which, travelling backwards in time, there occurs a great homogenisation of information content which is likely to be incapable of uniquely defining the familiar features of the material universe. Not only do quantum effects come into play, they increasingly predominate, and eventually change out of all recognition. This is in a theoretical framework in close agreement with Hawking's assumption of a combination of the many worlds interpretation of quantum theory (as opposed to the Copenhagen Interpretation) and (see [13]) Feynman's 'sum-over-histories' as the preferred approach to a quantum description of the early Universe. Information content is not necessarily destroyed, but is increasingly randomised, necessitating appropriate theoretical counterparts. (Note, Kucera [24], that the 2-random Turing degrees avoid the cone above any nonzero degree below $\mathbf{0}'$.) The fact that the random degrees form an automorphism base for the Turing universe suggests that despite the huge apparent chaos near the Big Bang, and lack of structured information content, the latter may still be sufficient to deterministically decide the form of the Universe at other points in time. Hence, assuming Penrose's *cosmic censorship conjecture* ('Nature abhors a naked singularity', [20], p. 21), one obtains a theoretical framework capable of reinstating meaningful cyclic models of the Universe, in which the force of any rebound associated with the 'singularities' is directly related to the proximity of the dissolution of invariant gravity (presumably not within the Planck interval of 10^{-43} seconds), which in turn depends on the size of the Universe and hence how much information content is involved. [2]

The resulting answer to the question: "If the laws of physics could break down at the beginning of the universe, why couldn't they break down anywhere?" ([20], p. 76), while not being precisely that envisaged by Hawking himself in his article, is a particularly satisfying one.

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[2] The mystery of what lies at the base of this hierarchy of Turing computable and Turing definable relations remains. But strange attractors (see, for example [22]) provide concrete examples of Turing definability arising from simple computational relationships. There is no surprise in this primitive emergence of definability to one familiar with the role of Sacks' [34] Splitting Theorem in the proof (see [3]) of Turing definability of the various levels of the arithmetical hierarchy.

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