ON COMPUTATION OF THE FIRST BAUES–WIRSCHING COHOMOLOGY OF A FREELY-GENERATED SMALL CATEGORY

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Abstract. The Baues–Wirsching cohomology is one of the cohomologies of a small category. Our aim is to describe the first Baues–Wirsching cohomology of the small category generated by a finite quiver freely. We consider the case where the coefficient is a natural system obtained by the composition of a functor and the target functor. We give an algorithm to obtain generators of the vector space of inner derivations. It is known that there exists a surjection from the vector space of derivations of the small category to the first Baues–Wirsching cohomology whose kernel is the vector space of inner derivations.

1. Introduction

Baues and Wirsching [1] introduced a cohomology of a small category, which is called nowadays the Baues–Wirsching cohomology. It is known that the Baues–Wirsching cohomology is a generalization of some cohomologies; e.g., the cohomology of a group G with coefficients in a left G-module, the singular cohomology of the classifying space of a small category with coefficients in a field, and so on. Let k be a field and D a natural system on a small category $\mathcal{C}$; that is, a functor from the category of factorizations in $\mathcal{C}$ to the category k-Mod of left k-modules. The n-th Baues–Wirsching cohomology of $\mathcal{C}$ with coefficients in D is denoted by $\mathrm{H}^n_{BW}(\mathcal{C}, D)$. For an equivalence $\phi : \mathcal{C} \to \mathcal{C}'$ of small categories and a natural system D on $\mathcal{C}$, Baues and Wirsching showed that the k-liner map $\tilde{\phi} : \mathrm{H}^n_{BW}(\mathcal{C}, D) \to \mathrm{H}^n_{BW}(\mathcal{C}', \phi^* D)$ induced by ϕ is an isomorphism for $n \in \mathbb{Z}$. The Baues–Wirsching cohomology is an invariant for the equivalence of small categories in this sense.

Assume that $\mathcal{C}$ is freely generated by a quiver and that $D = \check{D} \circ t$ is the composition of $\check{D}$ and the target functor t. In this case, it is known that $\mathrm{H}^n_{BW}(\mathcal{C}, D)$ vanishes for $n \geq 2$ and that $\mathrm{H}^0_{BW}(\mathcal{C}, D)$ is isomorphic to the limit $\lim_{\mathcal{C}} \check{D}$. Therefore, we focus on the first cohomology $\mathrm{H}^1_{BW}(\mathcal{C}, D)$. Let $k\mathcal{C}$ be the category algebra of $\mathcal{C}$, i.e. the algebra whose basis is a morphism of $\mathcal{C}$ and whose multiplication is the composition of morphisms (if the morphisms are not composable, then the multiplication is zero). Since $\mathcal{C}$ is generated by Q, the category algebra is the path algebra kQ. Define the functor $\pi_{\mathcal{C}}$ from $k\mathcal{C}$-Mod to the category $k\text{-Mod}^{\mathcal{C}}$ of functors from $\mathcal{C}$ to k-Mod as follows: $\pi_{\mathcal{C}}$ maps an object M in $k\mathcal{C}$-Mod to the functor which maps $x \in \mathrm{ob}(\mathcal{C})$ to $\mathrm{id}_x \cdot M$ and which maps $u \in \mathrm{mor}(\mathcal{C})$ to the left multiplicative map of u; and $\pi_{\mathcal{C}}$ maps a morphism f in $k\mathcal{C}$-Mod to the natural transformation $\{f|_{\mathrm{id}_x \cdot M}\}_{x \in \mathrm{ob}(\mathcal{C})}$. Since the set of objects in $\mathcal{C}$ is finite, $\pi_{\mathcal{C}}$ is an equivalence of categories. (See [2].) Our algorithm introduced in this article computes the first cohomology $\mathrm{H}^1_{BW}(\mathcal{C}, \pi_{\mathcal{C}}(N) \circ t)$ for a left $k\mathcal{C}$-module N.

The authors give a description of the first Baues–Wirsching cohomology in the case where $\mathcal{C}$ is a B_2-free poset [3]. The algorithm in this paper is a generalization of the idea of the special case.

Key words and phrases. Finite quivers; path algebras; category algebras; inner derivations; Gaussian elimination.

This article is organized as follows: In Section 2.1, we define some notation. In Section 2.2, we give algorithms. In Section 3, we show our main result. We calculate the first Baues–Wirsching cohomology for some examples in Section 4.

2. Definition

2.1. **Definition of the first Baues–Wirsching cohomology.**

We define some notation on the first Baues–Wirsching cohomology in this section.

Let P and Q be finite sets, s and t maps from Q to P. We call the set Q equipped with the triple $(P; s, t)$ a *finite quiver*. We call an element of P a *vertex* and call an element of Q an *arrow*. An arrow $f \in Q$ such that $s(f) = a$ and $t(f) = b$ is denoted by $f : a \to b$. We call a sequence $f_1 \cdots f_l$ of arrows a *path of length* l if $s(f_i) = t(f_{i+1})$ for all i. A path $f_1 \cdots f_l$ such that $t(f_1) = s(f_l)$ is called a *cycle*. We say that a quiver Q is *acyclic* if Q has no cycle. Let Q' be a subset of Q and P' a subset of P. We call the set Q' equipped with the triple $(P'; s|_{Q'}, t|_{Q'})$ a *subquiver* of Q if $s(Q')$ and $t(Q')$ are subsets of P'.

Let Q be a finite quiver. The category defined in the following manner is called the *small category freely generated by* Q:

- the set of objects is the set of vertices of Q;
- a morphism from x to y is a path from x to y;
- the identity id_x is the path from x to x of length 0; and
- if $s(f) = t(g)$, then the composition of morphisms f and g is the concatenation of paths f and g.

Let $\mathcal{C}$ be a small category freely generated by Q. The category $\mathcal{F}(\mathcal{C})$ defined in the following manner is called the *category of factorizations in* $\mathcal{C}$:

- the objects are morphisms in $\mathcal{C}$;
- a morphism from α to β is a pair (u, v) of morphisms in $\mathcal{C}$ such that $\beta = u \circ \alpha \circ v$; and
- the composition of (u', v') and (u, v) is defined by $(u', v') \circ (u, v) = (u' \circ u, v \circ v')$.

A covariant functor from $\mathcal{F}(\mathcal{C})$ to k-Mod is called a *natural system* on a small category $\mathcal{C}$. Let D be a natural system on the small category $\mathcal{C}$. For $\alpha \in \mathrm{ob}(\mathcal{F}(\mathcal{C}))$, D_α denotes the k-module corresponding to α. For a pair (u, v) of composable morphisms, we define u_* and v^* by

$$u_* = D(u, \mathrm{id}_{s(v)}) : D_v \to D_{u \circ v},$$
$$v^* = D(\mathrm{id}_{t(u)}, v) : D_u \to D_{u \circ v}.$$

Let $d : \mathrm{mor}(\mathcal{C}) \to \prod_{\varphi \in \mathrm{mor}(\mathcal{C})} D_\varphi$ be a map such that $d(f) \in D_f$ for each $f \in \mathrm{mor}(\mathcal{C})$. We call d a *derivation* from $\mathcal{C}$ to D if $d(f \circ g) = f_*(dg) + g^*(df)$ for each pair (f, g) of composable morphisms. We define $\mathrm{Der}(\mathcal{C}, D)$ to be the k-vector space of derivations from $\mathcal{C}$ to D. We call d an *inner derivation* from $\mathcal{C}$ to D if there exists an element $(n_x)_{x \in \mathrm{ob}(\mathcal{C})} \in \prod_{x \in \mathrm{ob}(\mathcal{C})} D_{\mathrm{id}_x}$ such that $d(f) = f_*(n_{s(f)}) - f^*(n_{t(f)})$ for each $f \in \mathrm{mor}(\mathcal{C})$. We define $\mathrm{Ider}(\mathcal{C}, D)$ to be the k-vector space of inner derivations from $\mathcal{C}$ to D. The first Baues–Wirsching cohomology $\mathrm{H}^1_{BW}(\mathcal{C}, D)$ is the quotient space $\mathrm{Der}(\mathcal{C}, D)/\,\mathrm{Ider}(\mathcal{C}, D)$.

Remark 2.1. Let Q be a quiver, $\mathcal{C}$ a small category freely generated by Q, N a $k\mathcal{C}$-module, t the target functor, and $\tilde{D}$ the natural system $\pi_{\mathcal{C}}(N) \circ t$. For a pair (u, v) of composable morphisms, u_* (*resp.* v^*) maps $m \in \tilde{D}_v = \mathrm{id}_{t(v)} \cdot N$ (*resp.* $n \in \tilde{D}_u = \mathrm{id}_{t(u)} \cdot N$) to $u \cdot m \in \tilde{D}_{u \circ v} = \mathrm{id}_{t(u)} \cdot N$ (*resp.* $n \in \tilde{D}_{u \circ v} = \mathrm{id}_{t(u)} \cdot N$).

2.2. **Definition of algorithms.** In this section, we give algorithms to obtain generators of $\mathrm{Ider}(\mathcal{C}, D)$.

Let Q be a finite quiver, and P the set of vertices of Q. For subsets Q_1, Q_3 of Q and a subset $\hat{P}$ of P, we define the set $H(\hat{P}; Q, Q_1, Q_3)$ to be

$$\left\{ h \in Q_3 \;\middle|\; \begin{array}{l} t(h) \in \hat{P}. \\ hp \text{ is not a cycle in } Q \text{ for any path } p \text{ in } Q_1. \end{array} \right\}.$$

For subsets Q_1, Q_2 of Q and $h \in H(\hat{P}; Q, Q_1, Q_3)$, we define the set $G(Q_1, Q_2; h)$ to be

$$\left\{ g \in Q_2 \;\middle|\; \begin{array}{l} \text{There exists a cycle in } Q_1 \cup Q_2 \cup \{ h \} \\ \text{which contains } g \text{ and } h. \end{array} \right\}.$$

Algorithm 2.2.

Input: a finite quiver Q.
Output: $((a_i)_{i=1}^{l}; (b_i)_{i=1}^{m}; (f_1)_{i=1}^{l}; (g_i)_{i=1}^{n}; (h_i)_{i=1}^{r})$.
Procedure:

1. Let P be the set of vertices of Q.
2. Let $\check{P} = \emptyset$, $\hat{P} = P$, $Q_1 = \emptyset$, $Q_2 = \emptyset$, $Q_3 = Q$.
3. While $H(\hat{P}; Q, Q_1, Q_3) \neq \emptyset$, do the following:
   (a) Choose an element $h \in H(\hat{P}; Q, Q_1, Q_3)$.
   (b) Let $Q' = ((Q_1 \cup Q_2) \setminus G(Q_1, Q_2; h)) \cup \{ h \}$.
   (c) Let $\bar{Q}$ be a maximal acyclic subquiver of Q including Q'.
   (d) Let $\check{P} = \{ a \in P \mid \exists f \in \bar{Q} \text{ such that } t(f) = a. \}$.
   (e) Let $\hat{P} = P \setminus \check{P}$.
   (f) For each $a \in \check{P}$, choose $f_a \in \bar{Q}$ so that $t(f_a) = a$.
   (g) Let $Q_1 = \{ f_a \mid a \in \check{P} \}$, $Q_2 = Q' \setminus Q_1$, and $Q_3 = Q \setminus Q'$.
4. Let $l = |\check{P}|$. For $i = 1, \ldots, l$, do the following:
   (a) Choose a vertex $x \in \check{P}$ such that there exists no arrow in Q_1 whose source is x.
   (b) Let $a_i = x$.
   (c) For $\alpha \in Q_1$ so that $t(\alpha) = x$, let $f_i = \alpha$.
   (d) Let $\check{P} = \check{P} \setminus \{ x \}$, and $Q_1 = Q_1 \setminus \{ \alpha \}$.
5. Let $\{ b_1, \ldots, b_m \} = \hat{P}$.
6. Let $\{ g_1, \ldots, g_n \} = Q_2$.
7. Let $\{ h_1, \ldots, h_r \} = Q_3$.

Remark 2.3. In Step 3 in Algorithm 2.2, $|H(\hat{P}; Q, Q_1, Q_3)|$ strictly decreases since $|\hat{P}|$ decreases in each step. Hence Step 3 is a finite procedure.

Remark 2.4. Let $((a_i)_{i=1}^{l}; (b_i)_{i=1}^{m}; (f_1)_{i=1}^{l}; (g_i)_{i=1}^{n}; (h_i)_{i=1}^{r})$ be an output of Algorithm 2.2. Let

$$\begin{aligned} \check{P} &= \{ a_1, \ldots, a_l \}, \\ \hat{P} &= \{ b_1, \ldots, b_m \}, \\ Q_1 &= \{ f_1, \ldots, f_l \}, \\ Q_2 &= \{ g_1, \ldots, g_n \}, \text{and} \\ Q_3 &= \{ h_1, \ldots, h_r \}. \end{aligned}$$

The set $\check{P} \coprod \hat{P}$ is decomposition of P. The set $Q_1 \coprod Q_2 \coprod Q_3$ is also decomposition of Q. By Step 4 in Algorithm 2.2, a_i corresponds to the target of f_i for $i = 1, \ldots, l$. Hence if there exists a path from a_j to a_i or a path from b_j to a_i in Q_1, then the path is unique. Since the quiver $Q_1 \cup Q_2$ is a maximal acyclic subquiver of Q, we can regard $\check{P}$ as a poset. Moreover, if $a_j \leq a_i$ in the poset $\check{P}$, then the inequality

$i \leq j$ holds. If Q is a finite acyclic quiver, then Q_3 is the empty set. By Step 3 in Algorithm 2.2, for h_i so that $t(h_i) \in \hat{P}$, there exists a path p in Q_1 such that $h_i p$ is a cycle in Q.

Algorithm 2.5.

Input: $((a_i)_{i=1}^{l}; (b_i)_{i=1}^{m}; (f_1)_{i=1}^{l}; (g_i)_{i=1}^{n}; (h_i)_{i=1}^{r})$.
Output: (V, W).
Procedure:

(1) Let $Q_1 = \{ f_1, \dots, f_l \}$.
(2) (We define elements $v_{i,j}$ in the path algebra kQ.) For $j = 1, \dots, l$, do the following:
  (a) For $i = 1, \dots, l$, let $v_{i,j} = 0$.
  (b) Let $v_{j,j} = \mathrm{id}_{a_j}$
  (c) For $i = 1, \dots, n$, do the following:
    (i) Let $v_{l+i,j} = 0$.
    (ii) If there exists a path p from a_j to $t(g_i)$ in Q_1, then let $v_{l+i,j} = v_{l+i,j} + p$.
    (iii) If there exists a path p from a_j to $s(g_i)$ in Q_1, then let $v_{l+i,j} = v_{l+i,j} - g_i p$.
  (d) For $i = 1, \dots, r$, do the following:
    (i) Let $v_{l+n+i,j} = 0$.
    (ii) If there exists a path p from a_j to $t(h_i)$ in Q_1, then let $v_{l+n+i,j} = v_{l+n+i,j} + p$.
    (iii) If there exists a path p from a_j to $s(h_i)$ in Q_1, then let $v_{l+n+i,j} = v_{l+n+i,j} - h_i p$.
(3) Let $V = (v_{i,j})_{1 \leq i \leq l+n+r,\ 1 \leq j \leq l}$.
(4) (We define elements $w_{i,j}$ in the path algebra kQ.) For $j = 1, \dots, m$, do the following:
  (a) For $i = 1, \dots, l$, let $w_{i,j} = 0$.
  (b) For $i = 1, \dots, n$, do the following:
    (i) Let $w_{l+i,j} = 0$.
    (ii) If there exists a path p from b_j to $t(g_i)$ in Q_1, then let $w_{l+i,j} = w_{l+i,j} + p$.
    (iii) If there exists a path p from b_j to $s(g_i)$ in Q_1, then let $w_{l+i,j} = w_{l+i,j} - g_i p$.
  (c) For $i = 1, \dots, r$, do the following:
    (i) Let $w_{l+n+i,j} = 0$.
    (ii) If there exists a path p from b_j to $t(h_i)$ in Q_1, then let $w_{l+n+i,j} = w_{l+n+i,j} + p$.
    (iii) If there exists a path p from b_j to $s(h_i)$ in Q_1, then let $w_{l+n+i,j} = w_{l+n+i,j} - h_i p$.
(5) Let $W = (w_{i,j})_{1 \leq i \leq l+n+r,\ 1 \leq j \leq m}$.

Remark 2.6. Let (V, W) be the output of Algorithm 2.5 for some input. The matrix $(v_{i,j})_{1 \leq i \leq l,\ 1 \leq j \leq l}$ is the identity matrix, i.e., the diagonal matrix whose entries one $(\mathrm{id}_{a_1}, \dots, \mathrm{id}_{a_l})$. The matrix $(w_{i,j})_{1 \leq i \leq l,\ 1 \leq j \leq m}$ is the zero matrix.

3. Our main result

We show our main result in this section. Our main result computes the first Baues–Wirsching cohomology via the column echelon matrix obtained by our algorithm.

Let Q be a finite quiver, $\mathcal{C}$ a small category freely generated by Q. Fix a left $k\mathcal{C}$-module N, and consider the natural system $\tilde{D} = \pi_{\mathcal{C}}(N) \circ t$.

Let $T = ((a_i)_{i=1}^{l}; (b_i)_{i=1}^{m}; (f_1)_{i=1}^{l}; (g_i)_{i=1}^{n}; (h_i)_{i=1}^{r})$ be the output of Algorithm 2.2 for Q. We define the k-vector space A_1, A_2, and A_3 by

$$A_1 = \bigoplus_{i=1}^{l} \tilde{D}_{f_i}, \qquad A_2 = \bigoplus_{i=1}^{n} \tilde{D}_{g_i}, \text{ and } \qquad A_3 = \bigoplus_{i=1}^{r} \tilde{D}_{h_i}.$$

Let (V, W) be the output of Algorithm 2.5 for T. Let v_j and w_j be the j-th column vector of V and W, respectively. The vectors v_j and w_j are elements of $\bigoplus_{i=1}^{l+n+r} k\mathcal{C}$. We define the k-vector spaces $\bar{V}$ and $\bar{W}$ by

$$\bar{V} = \left\langle v_j n_{a_j} \,\middle|\, n_{a_j} \in \mathrm{id}_{a_j} \cdot N, 1 \le j \le l \right\rangle,$$
$$\bar{W} = \left\langle w_j n_{b_j} \,\middle|\, n_{b_j} \in \mathrm{id}_{b_j} \cdot N, 1 \le j \le m \right\rangle.$$

Theorem 3.1. *The first Baues–Wirsching cohomology* $\mathrm{H}^1_{BW}(\mathcal{C}, \tilde{D})$ *is isomorphic to*

$$(A_1 \oplus A_2 \oplus A_3)/(\bar{V} + \bar{W})$$

as k-vector spaces.

Proof. According to Baues and Wirsching [1], if $\mathcal{C}$ is freely generated by $S \subset \mathrm{mor}(\mathcal{C})$, then we can identify $\mathrm{Der}(\mathcal{C}, D)$ with $\prod_{\alpha \in S} D_\alpha$. Via the identification, $\mathrm{Ider}(\mathcal{C}, D)$ is the k-vector space

$$\left\{ (\alpha_*(n_{s(\alpha)}) - \alpha^*(n_{t(\alpha)}))_\alpha \in \prod_{\alpha \in S} D_\alpha \,\middle|\, (n_x)_x \in \prod_{x \in \mathrm{ob}(\mathcal{C})} D_{\mathrm{id}_x} \right\}.$$

Let

$$Q = \{\, f_i \mid 1 \le i \le l \,\} \cup \{\, g_i \mid 1 \le i \le n \,\} \cup \{\, h_i \mid 1 \le i \le r \,\}, \text{ and}$$
$$P = \{\, a_i \mid 1 \le i \le l \,\} \cup \{\, b_i \mid 1 \le i \le m \,\}.$$

It follows that $\mathrm{Der}(\mathcal{C}, \tilde{D}) \cong A_1 \oplus A_2 \oplus A_3$. Hence $\mathrm{Ider}(\mathcal{C}, \tilde{D})$ is isomorphic to the k-vector space

$$B = \left\{ (\alpha n_{s(\alpha)} - n_{t(\alpha)})_{\alpha \in Q} \in A \,\middle|\, (n_x)_x \in \bigoplus_{x \in P} \mathrm{id}_x \cdot N \right\}.$$

For $x \in P$ and $m \in \mathrm{id}_x \cdot N$, we define $r_x m = (r_{x,\alpha} m)_{\alpha \in Q} \in A$ by

$$r_{x,\alpha} m = \begin{cases} -\alpha m & (s(\alpha) = x) \\ m & (t(\alpha) = x) \\ 0 & (\text{otherwise}). \end{cases}$$

It is clear that the k-vector space B is equal to

$$\langle r_x m \,|\, x \in P, m \in \mathrm{id}_x \cdot N \rangle.$$

For $j = 1, \ldots, l$ and $n_{a_j} \in \mathrm{id}_{a_j} \cdot N$, we define $\overline{r_{a_j}} n_{a_j}$ to be $r_{a_j} n_{a_j} + \sum_{k=1}^{i-1} \overline{r_{a_k}} f_k n_{a_j}$. For $j = 1, \ldots, m$ and $n_{b_j} \in \mathrm{id}_{b_j} \cdot N$, we define $\overline{r_{b_j}} n_{b_j}$ to be $r_{b_j} n_{b_j} + \sum_{k=1}^{l} \overline{r_{a_k}} f_k n_{b_j}$. It follows from the direct calculation that $\overline{r_{a_j}} n_{a_j}$ and $\overline{r_{b_j}} n_{b_j}$ are equal to $v_j n_{a_j}$ and $w_j n_{b_j}$, respectively. Hence we have Theorem 3.1. $\square$

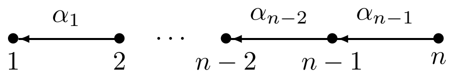

FIGURE 1. The quiver in Example 4.1.

4. Some examples

In this section, we apply our algorithm to some examples of finite quivers to calculate the first Baues–Wirsching cohomology. First we apply our algorithm to some quivers whose set of vertices is a B_2-free poset, which is discussed in [3].

Example 4.1. Let $P_n = \{ 1, \dots, n \}$. Define α_i to be an arrow from $i+1$ to i. Let $Q_n = \{ \alpha_i \mid i = 1, \dots, n-1 \}$. The quiver Q_n is a chain in Figure 1. An output of Algorithm 2.2 for Q_n is

$$\begin{aligned} a_i &= i \text{ for } i = 1, \dots, n-1, \\ b_1 &= n, \text{ and} \\ f_i &= \alpha_i \text{ for } i = 1, \dots, n-1. \end{aligned}$$

Consider the small category $\mathcal{C}_n$ generated by Q_n. An output of Algorithm 2.5 is

$$\begin{aligned} v_j &= \bigoplus_{k=1}^{n-1} \delta_{j,k} \operatorname{id}_{a_k} \text{ for } j = 1, \dots, n-1, \text{ and} \\ w_1 &= 0^{\oplus(n-1)}. \end{aligned}$$

For a $k\mathcal{C}_n$-module N,

$$\begin{aligned} A_1 &= \bigoplus_{k=1}^{n-1} \operatorname{id}_{a_k} \cdot N, \\ A_2 &= 0, \\ A_3 &= 0, \\ \bar{V} &= \bigoplus_{k=1}^{n-1} \operatorname{id}_{a_k} \cdot N, \text{ and} \\ \bar{W} &= 0. \end{aligned}$$

By Theorem 3.1, we have

$$\mathrm{H}^1_{BW}(\mathcal{C}_n, \pi_{\mathcal{C}_n}(N) \circ t) = 0.$$

Example 4.2. Let $P_n = \{ 1, \dots, n \} \cup \{ x = 0 \}$. Define α_i to be an arrow from i to x. Let $Q_n = \{ \alpha_i \mid i = 1, \dots, n \}$. The quiver Q_n is a quiver such that the targets of each arrow is x. See Figure 2. An output of Algorithm 2.2 for Q_n is

$$\begin{aligned} a_1 &= x, \\ b_i &= i \text{ for } i = 1, \dots, n, \\ f_1 &= \alpha_n, \text{ and} \\ g_i &= \alpha_i \text{ for } i = 1, \dots, n-1. \end{aligned}$$

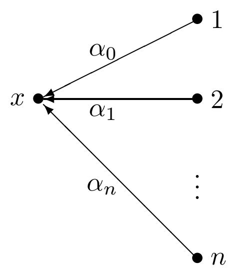

Figure 2. The quiver in Example 4.2.

Consider the small category $\mathcal{C}_n$ generated by Q_n. An output of Algorithm 2.5 is

$$\begin{aligned} v_1 &= (\mathrm{id}_{a_1})^{\oplus n}, \\ w_j &= 0 \oplus \left(\bigoplus_{k=1}^{n-1} (-\delta_{j,k} g_k) \right) \text{ for } j = 1, \dots, n, \text{ and} \\ w_n &= 0 \oplus \left(f_1^{\oplus(n-1)} \right). \end{aligned}$$

For a $k\mathcal{C}_n$-module N,

$$\begin{aligned} A_1 &= \mathrm{id}_{a_1} \cdot N, \\ A_2 &= \bigoplus_{k=1}^{n-1} \mathrm{id}_{a_1} \cdot N, \\ A_3 &= 0, \\ \bar{V} &= \left\langle m^{\oplus n} \,\middle|\, m \in \mathrm{id}_{a_1} \cdot N \right\rangle, \text{ and} \\ \bar{W} &= \left(\bigoplus_{k=1}^{n-1} g_k \cdot N \right) + \left\langle f_1^{\oplus(n-1)} n_{b_3} \,\middle|\, n_{b_3} \in \mathrm{id}_{b_3} \cdot N \right\rangle. \end{aligned}$$

By Theorem 3.1, we have

$$\mathrm{H}^1_{BW}(\mathcal{C}, \pi_{\mathcal{C}_n}(N) \circ t) \cong (A_1 \oplus A_2)/(\bar{V} + \bar{W}).$$

Moreover, if $N = k\mathcal{C}_n$, then

$$\begin{aligned} &\mathrm{H}^1_{BW}(\mathcal{C}_n, \pi_{\mathcal{C}_n}(k\mathcal{C}_n) \circ t) \\ &\cong \left(\bigoplus_{k=1}^{n-1} (\langle \mathrm{id}_{a_1}, f_1 \rangle + \langle g_j \mid j \neq k \rangle) \right) \Big/ \left\langle f_1^{\oplus(n-1)} \right\rangle. \end{aligned}$$

Example 4.3. Let $P_n = \{\, x_j = (0, j) \mid j \in \mathbb{Z}/n\mathbb{Z} \,\} \cup \{\, y_j = (1, j) \mid j \in \mathbb{Z}/n\mathbb{Z} \,\}$. Define α_j and β_j to be arrows from y_j and y_{j-1} to x_j, respectively. Let $Q_n = \{\, \alpha_j \mid j \in \mathbb{Z}/n\mathbb{Z} \,\} \cup \{\, \beta_j \mid j \in \mathbb{Z}/n\mathbb{Z} \,\}$. The quiver Q_n is a zigzag circle in Figure 3. An output of Algorithm 2.2 for Q_n is

$$\begin{aligned} a_j &= x_j \text{ for } j = 1, \dots, n, \\ b_j &= y_j \text{ for } j = 1, \dots, n, \\ f_j &= \alpha_j \text{ for } j = 1, \dots, n, \text{ and} \\ g_j &= \beta_j \text{ for } j = 1, \dots, n. \end{aligned}$$

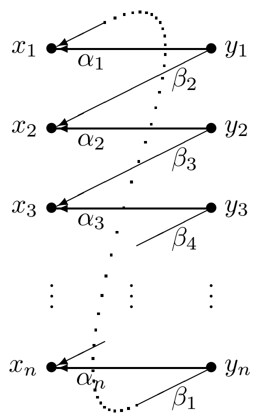

FIGURE 3. The quiver in Example 4.3.

Consider the small category $\mathcal{C}_n$ generated by Q_n. An output of Algorithm 2.5 is

$$v_j = (\bigoplus_{k=1}^{n} \delta_{j,k} \operatorname{id}_{a_k}) \oplus (\bigoplus_{k=1}^{n} \delta_{j,k} \operatorname{id}_{a_k}) \text{ for } j = 1, \dots, n, \text{ and}$$

$$w_j = (0^{\oplus n}) \oplus (\bigoplus_{k=1}^{n} (\delta_{j,k} f_k - \delta_{j+1,k} g_k)) \text{ for } j = 1, \dots, n.$$

For a $k\mathcal{C}_n$-module N,

$$\begin{aligned}
A_1 &= \bigoplus_{k=1}^{n} \operatorname{id}_{a_k} \cdot N, \\
A_2 &= \bigoplus_{k=1}^{n} \operatorname{id}_{a_k} \cdot N, \\
A_3 &= 0, \\
\bar{V} &= \left\langle v_j n_{a_j} \,\middle|\, n_{a_j} \in \operatorname{id}_{a_j} \cdot N, j = 1, \dots, n \right\rangle, \text{ and} \\
\bar{W} &= \left\langle w_j n_{b_j} \,\middle|\, n_{b_j} \in \operatorname{id}_{b_j} \cdot N, j = 1, \dots, n \right\rangle.
\end{aligned}$$

By Theorem 3.1, we have

$$\mathrm{H}^1_{BW}(\mathcal{C}_n, \pi_{\mathcal{C}_n}(N) \circ t) \cong (A_1 \oplus A_2)/(\bar{V} + \bar{W}).$$

Moreover, if $N = k\mathcal{C}_n$, then

$$\mathrm{H}^1_{BW}(\mathcal{C}_n, \pi_{\mathcal{C}_n}(k\mathcal{C}_n) \circ t) \cong \bigoplus_{k=0}^{n-1} \langle \operatorname{id}_{a_k}, f_k \rangle .$$

Next we consider examples which are not posets.

Example 4.4. Let $P_n = \mathbb{Z}/n\mathbb{Z}$. Define α_j to be an arrow from $j+1$ to j. Let $Q_n = \{ \alpha_j \mid j \in \mathbb{Z}/n\mathbb{Z} \}$. The quiver Q_n is a circle in Figure 4. An output of Algorithm 2.2 for Q_n is

$$\begin{aligned}
a_j &= j \text{ for } j = 1, \dots, n-1, \\
b_1 &= n, \\
f_j &= \alpha_j \text{ for } j = 1, \dots, n-1, \text{ and} \\
h_1 &= \alpha_n.
\end{aligned}$$

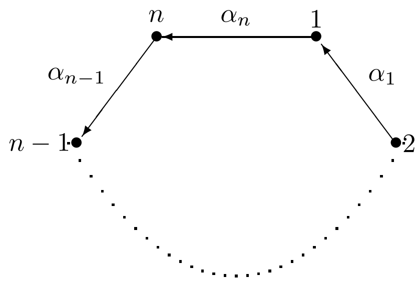

FIGURE 4. The quiver in Example 4.4.

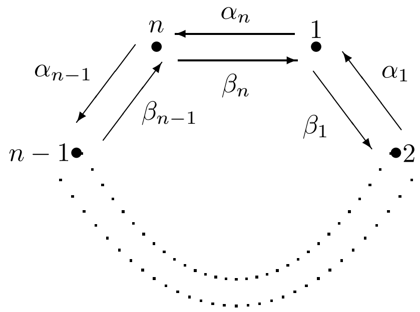

FIGURE 5. The quiver in Example 4.5.

Consider the small category $\mathcal{C}_n$ generated by Q_n. An output of Algorithm 2.5 is

$$\begin{aligned} v_j &= \left(\bigoplus_{k=1}^{n-1} \delta_{j,k} \operatorname{id}_{a_k}\right) \oplus (-h_1 f_1 \cdots f_{j-1}) \text{ for } j = 1, \dots, n, \\ w_1 &= 0^{\oplus(n-1)} \oplus (\operatorname{id}_{b_1} - h_1 f_1 \cdots f_{n-1}). \end{aligned}$$

For a $k\mathcal{C}_n$-module N,

$$\begin{aligned} A_1 &= \bigoplus_{k=1}^{n-1} \operatorname{id}_{a_k} \cdot N, \\ A_2 &= 0, \\ A_3 &= \operatorname{id}_{b_1} \cdot N, \\ \bar{V} &= \left\langle v_j n_{a_j} \,\middle|\, n_{a_j} \in \operatorname{id}_{a_j} \cdot N, j = 1, \dots, n-1 \right\rangle, \text{ and} \\ \bar{W} &= \langle w_1 n_{b_1} \,|\, n_{b_1} \in \operatorname{id}_{b_1} \cdot N \rangle . \end{aligned}$$

By Theorem 3.1, we have

$$\mathrm{H}^1_{BW}(\mathcal{C}_n, \pi_{\mathcal{C}_n}(N) \circ t) \cong (A_1 \oplus A_3)/(\bar{V} + \bar{W}).$$

Moreover, if $N = k\mathcal{C}_n$, then

$$\begin{aligned} &\mathrm{H}^1_{BW}(\mathcal{C}_n, \pi_{\mathcal{C}_n}(k\mathcal{C}_n) \circ t) \\ &\cong \langle \operatorname{id}_{a_0} \rangle + \langle h_1 f_1 \cdots f_{j-1} \,|\, j = 1, \dots, n-1 \rangle . \end{aligned}$$

Example 4.5. Let $P_n = \mathbb{Z}/n\mathbb{Z}$. Define α_j to be an arrow from $j+1$ to j, and β_j to be an arrow from j to $j+1$. Let $Q_n = \{\, \alpha_j \mid j \in \mathbb{Z}/n\mathbb{Z} \,\} \cup \{\, \beta_j \mid j \in \mathbb{Z}/n\mathbb{Z} \,\}$. The quiver Q_n is a circle in Figure 5. An output of Algorithm 2.2 for Q_n is

$$\begin{aligned}
a_j &= j \text{ for } j = 1, \ldots, n-1,\\
b_1 &= n,\\
f_j &= \alpha_j \text{ for } j = 1, \ldots, n-1,\\
g_1 &= \beta_n,\\
h_j &= \beta_j \text{ for } j = 1, \ldots, n-1, \text{ and}\\
h_n &= \alpha_n.
\end{aligned}$$

Consider the small category $\mathcal{C}_n$ generated by Q_n. We define $p_{i,j}$ in $k\mathcal{C}$ by

$$p_{i,j} = \begin{cases} f_i \cdots f_j & (\text{if } i < j+1)\\ \mathrm{id}_{a_i} & (\text{if } i = j+1)\\ 0 & (\text{if } i > j+1)\end{cases}.$$

An output of Algorithm 2.5 is

$$\begin{aligned}
v_{i,j} &= \delta_{i,j}\,\mathrm{id}_{a_j}\\
&\quad \text{for } i = 1, \ldots, n-1,\ j = 1, \ldots, n-1,\\
v_{n,j} &= p_{1,j-1}\\
&\quad \text{for } j = 1, \ldots, n-1,\\
v_{n+i,j} &= p_{i+1,j-1} - h_i p_{i,j-1}\\
&\quad \text{for } i = 1, \ldots, n-1,\ j = 1, \ldots, n-1,\\
v_{2n,j} &= -h_n p_{1,j-1}\\
&\quad \text{for } j = 1, \ldots, n-1,\\
w_{i,1} &= 0\\
&\quad \text{for } i = 1, \ldots, n-1,\\
w_{n,1} &= p_{1,n-1} - g_1,\\
w_{n+i,1} &= p_{i+1,n-1} - h_i p_{i,n-1}\\
&\quad \text{for } i = 1, \ldots, n-1, \text{ and}\\
w_{2n,1} &= \mathrm{id}_{b_1} - h_n p_{1,n-1}.
\end{aligned}$$

Let $v_j = \bigoplus_{i=1}^{2n} v_{i,j}$ for $j = 1, \ldots, n-1$, and $w_1 = \bigoplus_{i=1}^{2n} w_{i,1}$. For a $k\mathcal{C}_n$-module N,

$$\begin{aligned}
A_1 &= \bigoplus_{k=1}^{n-1} \mathrm{id}_{a_k} \cdot N,\\
A_2 &= \mathrm{id}_{a_1} \cdot N,\\
A_3 &= \Big(\bigoplus_{k=2}^{n-1} \mathrm{id}_{a_k} \cdot N\Big) \oplus (\mathrm{id}_{b_1} \cdot N) \oplus (\mathrm{id}_{b_1} \cdot N),\\
\bar{V} &= \left\langle v_j n_{a_j} \,\middle|\, n_{a_j} \in \mathrm{id}_{a_j} \cdot N, j = 1, \ldots, n-1 \right\rangle, \text{ and}\\
\bar{W} &= \langle w_1 n_{b_1} \mid n_{b_1} \in \mathrm{id}_{b_1} \cdot N \rangle.
\end{aligned}$$

By Theorem 3.1, we have

$$\mathrm{H}^1_{BW}(\mathcal{C}_n, \pi_{\mathcal{C}_n}(N) \circ t) \cong (A_1 \oplus A_2 \oplus A_3)/(\bar{V} + \bar{W}).$$

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