Large Independent Sets in Triangle-Free Planar Graphs*

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Abstract

Every triangle-free planar graph on n vertices has an independent set of size at least $(n + 1)/3$, and this lower bound is tight. We give an algorithm that, given a triangle-free planar graph G on n vertices and an integer $k \geq 0$, decides whether G has an independent set of size at least $(n + k)/3$, in time $2^{O(\sqrt{k})}n$. Thus, the problem is fixed-parameter tractable when parameterized by k. Furthermore, as a corollary of the result used to prove the correctness of the algorithm, we show that there exists $\varepsilon > 0$ such that every planar graph of girth at least five on n vertices has an independent set of size at least $n/(3 - \varepsilon)$.

Keywords. Planar graphs, independent set, fixed-parameter tractability, treewidth.

AMS subject classifcations: 68Q25, 68W05, 68R10.

1   Introduction

Every planar graph is 4-colorable by the deep Four-Colour-Theorem, whose proof was first announced by Appel and Haken in 1976 [2] and later simplified by Robertson, Sanders, Seymour and Thomas [33]. As a corollary, every planar graph on n vertices has an independent set of size at least $n/4$. The proof by Robertson et al. [33] comes with a quadratic-time algorithm to find a valid coloring of G with 4 colors, which can be used to find such an independent set. Yet, determining the size of a maximum independent set is NP-complete in planar graphs (even if they are triangle-free [23]). This motivates a search for an efficient algorithm that decides, for a fixed parameter $k \geq 0$ and an input n-vertex planar graph G, whether G has an independent set of size at least $(n + k)/4$.

The problem—which we call PLANAR INDEPENDENT SET ABOVE TIGHT LOWER BOUND, or PLANAR INDEPENDENT SET-ATLB for short—has received a lot of attention, although there has been essentially no progress. In fact, the question of whether PLANAR INDEPENDENT SET-ATLB is fixed-parameter tractable has been raised several times: first by Niedermeier [31], later by Bodlaender et al. [3], Mahajan et al. [24], by Sikdar [36], by Mnich [25], and by Crowston et al. [5]. Then the problem was raised again in June 2012 as a "tough customer", at WorKer 2012 [15]. We remark that until now, there is not even a polynomial-time algorithm known for the case of $k = 1$, and finding such an algorithm has been an open problem for more than 30 years. Yet, the existence of such an algorithm for $k = 1$ is certainly a necessary condition for the fixed-parameter tractability of PLANAR INDEPENDENT SET-ATLB. The lower bound of $n/4$ on the size of a maximum independent set is tight for an infinite family of planar graphs: for example, take a set of copies of K_4 or C_8^2 (the complement of the cube) or the icosahedron, and connect these copies arbitrarily in a planar way.

In this paper, we resolve the analogous problem for triangle-free planar graphs. By a theorem of Grötzsch [17], every triangle-free planar graph is 3-colorable, and thus admits an independent set that contains at least one-third of its vertices. Such a coloring, and thus also an independent set, can be found in linear time [9]. Later, Steinberg and Tovey [38] showed that every triangle-free planar graph contains a *non-uniform* 3-coloring, where one color class is guaranteed to contain one vertex more than

the other two color classes. Thus, any n-vertex triangle-free planar graph contains an independent set of size at least $(n+1)/3$, when $n \geq 3$. On the other hand, Jones [21] found triangle-free planar graphs on n vertices (for any $n \geq 2$ such that $n \equiv 2 \pmod 3$) with maximum independent sets of size exactly $(n+1)/3$; see Figure 1. This motivates a search for an efficient algorithm that decides, for a given
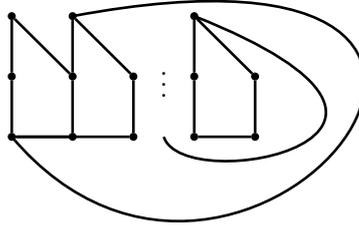

Figure 1: Triangle-free planar graphs on n vertices with maximum independent sets of size exactly $(n+1)/3$.

n-vertex triangle-free planar graph G and integer $k \geq 0$, whether G has an independent set of size at least $(n+k)/3$. In particular it was open whether for $k = 2$ there is a polynomial-time algorithm. Notice that it is non-trivial even to solve this problem in time $n^{O(k)}$, as a brute-force approach does not suffice.

As our main result, we show that the problem is fixed-parameter tractable when parameterized by k.

Theorem 1. *There is an algorithm that, given any n-vertex triangle-free planar graph G and any integer $k \geq 0$, in time $2^{O(\sqrt{k})} n$ decides whether G has an independent set of size at least $(n+k)/3$.*

Though many different techniques have been devised for solving optimization problems parameterized above lower bounds in fixed-parameter time, none of these techniques is applicable to our problem. Instead, our algorithm seems to be the first fixed-parameter algorithm for problems parameterized above guarantee that is based on the tree-width of a graph. The algorithm is an easy corollary of the following result.

Theorem 2. *There is a constant $c > 0$ such that every planar triangle-free graph on n vertices with tree-width $\geq t$ has an independent set of size $\geq \frac{n+ct^2}{3}$.*

According to Theorem 2, the algorithm of Theorem 1 is extremely simple. First, we test whether the treewidth of G is $O(\sqrt{k})$, using the linear-time constant factor approximation algorithm of Bodlaender et al. [4]. If that is the case, we find the largest independent set in G by dynamic programming in time $2^{O(\sqrt{k})} n$, see e.g. the book of Niedermeier [31]. Otherwise, we answer "yes". If we want to report the independent set of size $(n+k)/3$ when it exists, the algorithm becomes more involved, requiring an inspection of the proof of Theorem 2. We give the details in Section 7.

As a by-product of the proof of Theorem 2, we also obtain the following result which is of independent interest.

Theorem 3. *There exists a constant $\varepsilon > 0$ such that every triangle-free planar graph on n vertices and without separating 4-cycles has an independent set of size at least $\frac{n}{3-\varepsilon}$.*

Let us note an obvious consequence.

Corollary 4. *There exists a constant $\varepsilon > 0$ such that every planar graph of girth at least 5 on n vertices has an independent set of size at least $\frac{n}{3-\varepsilon}$.*

A well-known graph parameter giving a lower bound for the independence number of a graph G is the *fractional chromatic number* $\chi_f(G)$, see [35] for a definition and other properties. Here, let us only recall that $\alpha(G) \geq V(G)/\chi_f(G)$. The fractional chromatic number of planar graphs of girth at least 8 is at most $5/2$ (Dvořák et al. [13]), implying that for girth 8, we can set $\varepsilon = 1/2$. Not much is known about fractional chromatic number of graphs of girth between 5 and 7, but Theorem 3 and Corollary 4 make the following conjectures plausible.

Conjecture 1. *There exists a constant $\varepsilon > 0$ such that every triangle-free planar graph without separating 4-cycles has fractional chromatic number at most $3 - \varepsilon$.*

Conjecture 2. *There exists a constant $\varepsilon > 0$ such that every planar graph of girth ≥ 5 has fractional chromatic number at most $3 - \varepsilon$.*

Note that if Conjecture 2 holds, then $\varepsilon \leq 1/4$ by a construction of Pirnazar and Ullman [32]. Furthermore, the girth assumption cannot be reduced to 4 (or replaced by assuming odd girth at least 5) because of the construction in Fig. 1. Although Conjecture 1 appears to be stronger than Conjecture 2, 4-faces are usually easy to deal with in coloring arguments (by identifying a pair of opposite vertices), and thus we believe the two conjectures to be equivalent.

Let us also remark that the assumption of Corollary 4 that G is planar can be relaxed, since every graph on n vertices embeddable in a surface of genus g can be planarized by removing $O(\sqrt{gn})$ vertices [8].

Corollary 5. *There exist constants $\varepsilon, c > 0$ such that every graph of girth ≥ 5 and genus $\leq g$ on n vertices has an independent set of size $\geq \frac{n}{3-\varepsilon}(1 - c\sqrt{g/n})$.*

1.1 Organization and proof outline.

In Section 2, we recall some basic facts about tree-width, especially as relates to planar graphs. In Section 3, we review some results on classes of graphs with bounded expansion. In Section 4, we use these results to show that in every planar graph, we can remove a small fraction of vertices so that the resulting graph contains a large set of vertices that are pairwise far apart (Section 4). In Section 5, we apply coloring theory developed by Dvořák, Král' and Thomas to such a large set S of far apart vertices, to obtain a 3-coloring of the graph with further constraints in the neighborhoods of vertices of S which guarantee the existence of a large independent set. In Section 6, we combine the results and give proofs of our theorems. Finally, Section 7 describes an algorithm to find an independent set of size at least $(n + k)/3$ when the algorithm of Theorem 1 answers that such a set exists.

1.2 Related Work

From the combinatorial side, the study of lower bounds on the independence number in triangle-free graphs has a long history. Every n-vertex triangle-free graph has an independent set of size $\Omega(\sqrt{n \log n})$, and this bound is tight [22]. Staton [37] proved that every subcubic n-vertex triangle-free graph G satisfies $\alpha(G) \geq \frac{5n}{14}$. Furthermore, Heckman and Thomas [20] showed that if G is additionally planar, then $\alpha(G) \geq \frac{3n}{8}$.

From the algorithmic side, the studying the complexity of maximization problems parameterized above polynomial-time computable lower bounds is an active area of research. Since the influental survey by Mahajan et al. [24], research in this area has led to development of many new algorithmic techniques for fixed-parameter algorithms: algebraic methods [1, 5], probabilistic methods [18, 19], combinatorial methods [6, 26], and methods based on linear programming [7].

1.3 Preliminaries

Throughout, we consider graphs that are finite, undirected and loopless, and do not have parallel edges unless explicitly stated otherwise. For a graph G, let $V(G)$ denote its vertex set and $E(G)$ its set of edges. The degree of a vertex $v \in V(G)$ is the number $\deg_G(v)$ of edges that are incident to it. A graph is *planar* if it admits an embedding in the plane such that no two edges cross; a *plane graph* is an embedding of a planar graph without any edge crossings.

2 Treewidth of Planar Graphs

A *tree decomposition* of a graph G is a pair $(T, \mathcal{B})$, where $\mathcal{B}$ is a set of subsets of $V(G)$ (called the *bags* of the decomposition) with $V(G) = \bigcup_{B \in \mathcal{B}} B$, and T is a tree with vertex set $\mathcal{B}$, such that for each edge $uv \in E(G)$, there exists $B \in \mathcal{B}$ containing both u and v, and for every $v \in V(G)$, the set $\{B \in \mathcal{B} : v \in B\}$ induces a connected subtree of T. The width of the decomposition is the size of its largest bag minus one. The *tree-width* of a graph G, denoted by $\mathsf{tw}(G)$, is the minimum width of its tree decompositions. Note that the tree-width of a subgraph of G is at most as large as the tree-width of G.

Determining tree-width exactly is NP-hard; however, Bodlaender et al. [4] proved the following.

Theorem 6. *There exists an algorithm that for an input n-vertex graph G and integer $p \geq 1$, in time $2^{O(p)}n$ either outputs that the treewidth of G is larger than p, or gives a tree decomposition of G of width at most $5p + 4$.*

Let G be a graph. Suppose that $G = G_1' \cup G_2'$ and that $G_1 \supseteq G_1'$ and $G_2 \supseteq G_2'$ are graphs such that $V(G_1' \cap G_2')$ induces a clique both in G_1 and in G_2. In this situation, we say that G is a *clique-sum* of G_1 and G_2. That is, G is obtained from G_1 and G_2 by identifying corresponding vertices of cliques of the same size, and possibly removing edges afterwards. We need the following well-known fact.

Proposition 7. *If G is a clique-sum of G_1 and G_2, then $\mathsf{tw}(G) \leq \max(\mathsf{tw}(G_1), \mathsf{tw}(G_2))$.*

Robertson, Seymour and Thomas [34] gave a bound on the tree-width of planar graphs.

Proposition 8 ([34]). *A planar graph with n vertices has tree-width at most $6\sqrt{n} + 1$.*

We need to extend this bound to a related class of graphs.

Lemma 9. *Let G_0 be a triangle-free plane graph with n vertices. Let G be the graph obtained from G_0 by adding the edges uw and vx for each 4-face $uvwx$. Then G has tree-width at most $41\sqrt{n}$.*

Proof. By Euler's formula, since G_0 is triangle-free, it has at most n faces. Let R be the set of 4-faces of G_0. Let G_1 be the graph obtained from G_0 by adding a vertex v_f adjacent to all vertices of f for each $f \in R$. Note that G_1 is planar and $|V(G_1)| \leq 2n$, and thus by Proposition 8, G_1 has a tree decomposition $(T, \mathcal{B}_1)$ with bags of size at most $6\sqrt{2n} + 2$. Let $\mathcal{B}$ be obtained from $\mathcal{B}_1$ by replacing all appearances of v_f by $V(f)$. Observe that $(T, \mathcal{B})$ is a tree decomposition of G with bags of size at most $24\sqrt{2n} + 8 \leq 41\sqrt{n} + 1$. Therefore, G has tree-width at most $41\sqrt{n}$. $\square$

Let G be a plane graph. A subgraph G_0 of G is *4-swept* if G_0 has no separating 4-cycles and every face of G_0 which is not a face of G has length 4. Let $s(G)$ denote the maximum number of vertices of a 4-swept subgraph of G.

Lemma 10. *Every plane graph G has tree-width at most $41\sqrt{s(G)}$.*

Proof. Let $G_0, \ldots, G_{m-1}$ and $H_1, \ldots, H_m$ be subgraphs of G obtained as follows. We set $G_0 = G$. Suppose that G_{i-1} was already constructed, for some $i \geq 1$. If G_{i-1} contains no separating 4-cycle, then let $m = i$ and $H_m = G_{i-1}$. Otherwise, let C_i be a separating 4-cycle in G_{i-1} bounding as small open disk Λ_i as possible. Let H_i be the subgraph of G_{i-1} drawn in the closure of Λ_i, and let G_i be the subgraph of G_{i-1} drawn in the complement of Λ_i.

Observe that $H_1, \ldots, H_m$ are 4-swept subgraphs of G, that $G = H_1 \cup \ldots \cup H_m$, and that for $1 \leq i < j \leq m$, the intersection of H_i and H_j is a part of a boundary of a 4-face both in H_i and H_j. For $1 \leq i \leq m$, let H_i' be the graph obtained from H_i by adding the edges uw and vx for each 4-face $uvwx$. Then G is a clique-sum of $H_1', \ldots, H_m'$, and $\mathsf{tw}(H_i) \leq 41\sqrt{|V(H_i)|} \leq 41\sqrt{s(G)}$ for $1 \leq i \leq m$. By Proposition 7, we conclude that G has tree-width at most $41\sqrt{s(G)}$. $\square$

3 Classes of Graphs with Bounded Expansion

In this section, we survey results on classes of graphs with bounded expansion that we need in the paper. Let G be a graph and let $r \geq 0$ be an integer. Let us recall that a graph H is an *r-shallow minor* of G if H can be obtained from a subgraph of G by contracting vertex-disjoint subgraphs of radii at most r and deleting the resulting loops and parallel edges. Following Nešetřil and Ossona de Mendez [27], we denote by $\nabla_r(G)$ the maximum of $|E(G')|/|V(G')|$ over all r-shallow minors G' of G. Thus, $\nabla_0(G)$ is the maximum of $|E(G')|/|V(G')|$ taken over all subgraphs G' of G. Since every subgraph of a graph G has a vertex of degree at most $2\nabla_0(G)$, we see that G has an (acyclic) orientation with maximum in-degree at most $2\nabla_0(G)$.

A class $\mathcal{G}$ of graphs has *bounded expansion* if there exist constants $c_0, c_1, \ldots$ such that $\nabla_r(G) \leq c_r$ for every $G \in \mathcal{G}$ and $r \geq 0$. Many natural classes of sparse graphs have bounded expansion. Here, we only need that the class of planar graphs has bounded expansion; we refer to the book by Nešetřil and Ossona de Mendez [30] for more information on the topic.

Let D be a directed graph, and let D' be a directed graph obtained from D by adding, for every pair of vertices $x, y \in V(D)$,

- the edge xy if D has no edge from x to y and there exists a vertex $z \in V(D)$ such that D has an edge oriented from x to z and an edge oriented from z to y (*transitivity*), and

- either the edge xy or the edge yx if x is not adjacent to y and there exists a vertex z such that D has an edge oriented from x to z and an edge oriented from y to z (*fraternality*).

We call D' an *oriented augmentation* of D.

Let G be a graph. We construct a sequence D_0, ..., D_ℓ of directed graphs as follows. Let D_0 be an orientation of G with maximum in-degree at most $2\nabla_0(G)$. For $1 \le i \le \ell$, let D_i be an oriented augmentation of D_{i-1}, in that the orientations of the edges added according to the fraternality rule are chosen so that the subgraph of D_i formed by these edges has maximum in-degree at most $2\nabla_0(G_i)$, where G_i is the underlying undirected graph of D_i. This is possible, because G_i itself has an orientation with maximum in-degree at most $2\nabla_0(G_i)$. We say that D_ℓ is an ℓ-*th oriented augmentation* of G. We use the following important property of classes of graphs with bounded expansion [27].

Proposition 11. *Let $\mathcal{G}$ be a class of graphs and let $\ell \ge 0$ be an integer. If $\mathcal{G}$ has bounded expansion, then there exists an integer $m_d \ge 0$ such that all ℓ-th oriented augmentations of graphs from $\mathcal{G}$ have maximum indegree at most m_d.*

4 Large Scattered Sets

For an integer $d \ge 1$ and a graph G, a vertex set $Q \subseteq V(G)$ is d-*scattered* if the distance between any two vertices of Q in G is greater than d. In this section we discuss graph classes whose members admit large scattered subsets, possibly after removal of a small number of vertices.

For integers $d, m, N \ge 1$ and $r \ge 0$, a graph G is (d, r, m, N)-*wide* if for any set $S \subseteq V(G)$ of size at least N, there exists a set $Q \subseteq S$ of size at least m and a set $X \subseteq V(G)$ of size at most r such that Q is d-scattered in $G - X$. For integers $d \ge 1$ and $r \ge 0$, a class of graphs $\mathcal{G}$ is (d, r)-*wide* if for every m, there exists N such that every graph in $\mathcal{G}$ is (d, r, m, N)-wide. A class of graphs $\mathcal{G}$ is *uniformly quasi-wide* if for every integer $d \ge 1$ there exists an integer $r \ge 0$ such that $\mathcal{G}$ is (d, r)-wide. By results of Nešetřil and Ossona de Mendez [29], every class of graphs with bounded expansion is uniformly quasi-wide (in fact, they prove a stronger result concerning nowhere-dense graph classes).

We need a variant of wideness where the value of N is linear in m. Note that there exist classes of graphs with bounded expansion which do not have this property—for instance, consider the class containing the graphs H_a for each integer $a > 0$, where H_a is the disjoint union of a stars with a rays. With $S = V(H_a)$, the largest possible 2-scattered set after removal of r vertices has size $ra + (a - r) = \Theta(\sqrt{|S|})$. However, note that we can obtain a d-scattered set of size $a^2 = |S| - a$ at the cost of removing a vertices, which suggests the variant of wideness where the set of removed vertices is small relatively to the size of the d-scattered set.

For integers $d, t, K \ge 1$, a graph G is (d, t, K)-*fat* if for any set $S \subseteq V(G)$ there exists a set $Q \subseteq S$ of size at least $|S|/K$ and a set $X \subseteq V(G) \setminus Q$ of size at most $|Q|/t$ such that Q is d-scattered in $G - X$. A class of graphs $\mathcal{G}$ is *fat* if for all integers $d, t \ge 1$, there exists K such that every graph in $\mathcal{G}$ is (d, t, K)-fat.

We are going to show that every class of graphs with bounded expansion is fat. However, first we need to derive the following auxiliary result.

Lemma 12. *Let $c, t \ge 1$ be integers, and let $K_0 = c^{2c}(2c + 2)^{c+1}t^c$. Let G be a bipartite graph with parts S and Z. If all vertices in S have degree at most c, then there exist $Q \subseteq S$ and $X \subseteq Z$ such that $|Q| \ge |S|/K_0$, $|X| \le |Q|/t$ and for all distinct $u, v \in Q$, all common neighbors of u and v belong to X.*

Proof. For $0 \le i \le c$, let $d_i = \dfrac{K_0}{c^{2i+1}(2c + 2)^{i+1}t^i}$. Let Z_0 consist of vertices in Z of degree at least d_0 and for $1 \le i \le c$, let Z_i consist of the vertices in Z of degree at least d_i, but less than d_{i-1}. Since all vertices of S have degree at most c, for each $v \in S$ there exists $i \in \{0, \dots, c\}$ such that v has no neighbors in Z_i. By the pigeonhole principle, there exists $i \in \{0, \dots, c\}$ and a set $B \subseteq S$ of size at least $|S|/(c + 1)$ such that no vertex of B has a neigbor in Z_i. We set $X = Z_0 \cup \dots \cup Z_{i-1}$ and we choose Q as an inclusion-wise maximal subset of B such that no two vertices of Q have a common neighbor in $Z \setminus X$.

First, let us estimate the size of Q. For each vertex $v \in B$, its neighbors either belong to X or have degree less than d_i, and thus at most cd_i vertices are at distance exactly two from v in $G - X$. Since Q

is maximal, each vertex of $B \setminus Q$ is at distance two from a vertex of Q in $G - X$, and thus

$$|Q| \geq \frac{|S|/(c+1)}{1 + cd_i} \geq \frac{|S|/(c+1)}{2cd_i} \geq \frac{|S|}{2c(c+1)d_0} = \frac{|S|}{K_0} \ .$$

Note that we use the fact that $cd_i \geq cd_c = 1$.

If $i = 0$, then $X = \emptyset$, and thus Q and X satisfy the conclusions of the lemma. If $i \geq 1$, then we need to estimate the size of X. Note that each vertex of X has degree at least d_{i-1}, and thus $|X|d_{i-1} \leq |E(G)| \leq c|S|$. Consequently,

$$\frac{|Q|}{|X|} \geq \frac{\frac{|S|}{2c(c+1)d_i}}{c|S|/d_{i-1}} = \frac{d_{i-1}}{2c^2(c+1)d_i} = t \ . \qquad \square$$

For a path P, let $\ell(P)$ denote its length (the number of its edges). We say that a path P with directed edges is *reduced* if either $\ell(P) = 1$, or $\ell(P) = 2$ and both of its edges are directed away from the middle vertex of P.

Lemma 13. *Every class of graphs with bounded expansion is fat.*

Proof. Let $\mathcal{G}$ be a class of graphs with bounded expansion and consider fixed integers $d, t \geq 1$. Let $\mathcal{G}_d$ be the class of d-th oriented augmentations of graphs in $\mathcal{G}$, and let m_d be the smallest integer such that the maximum in-degree of every graph in $\mathcal{G}_d$ is at most m_d, which exists by Proposition 11. Let $K_0 = m_d^{2m_d}(2m_d + 2)^{m_d+1}t_d^m$ and let $K = (2m_d + 1)K_0$. We will show that every $G \in \mathcal{G}$ is (d, t, K)-fat.

Consider a set $S \subseteq V(G)$. Let G_d be a d-th oriented augmentation of G and let G'_d be the underlying undirected graph of G_d. Since G_d has maximum in-degree at most m_d, it follows that the maximum average degree of G'_d is at most $2m_d$, and thus G'_d has a proper coloring by at most $2m_d + 1$ colors. By considering the intersections of the color classes of this coloring with S, we conclude that there exists a set $S_0 \subseteq S$ of size at least $|S|/(2m_d + 1)$ which is independent in G'_d. Let Z be the set of in-neighbors of vertices of S_0 in G_d. Let H be the bipartite graph with parts S_0 and Z such that sz is an edge of H if and only if $s \in S_0$, $z \in Z$ and G_d contains an edge directed from z to s. Let Q and X be the sets obtained by applying Lemma 12 to H. Note that $|Q| \geq |S_0|/K_0 \geq |S|/K$ and $|X| \leq |Q|/t$ as required.

It remains to show that Q is d-scattered in $G - X$. Suppose that there exists a path $P_0 \subseteq G - X$ of length at most d between two vertices $u, v \in Q$. For $1 \leq i \leq d - 1$, let G_i denote the intermediate i-th oriented augmentation of G obtained during the construction of G_d. For $1 \leq i \leq d$, let P_i be a path between u and v in the underlying undirected graph of G_i such that $V(P_i) \subseteq V(P_0)$ and P_i is as short as possible. Note that $\ell(P_i) \leq \ell(P_{i-1})$, and if $\ell(P_i) = \ell(P_{i-1})$, then P_i (taken with the orientation of its edges as in G_i) is reduced. Since the length of P_0 is at most d, we conclude that P_d with the orientation as in G_d is reduced. Since Q is an independent set in G'_d, it follows that $\ell(P_d) \neq 1$. Therefore, $\ell(P_d) = 2$ and the middle vertex x of P_d is a common in-neighbor of u and v in G_d. However, this implies that x belongs to X, contrary to the assumption that P is disjoint with X. $\qquad \square$

5 Colorings and Independent Sets

Let us now turn our attention back to independent sets. As we mentioned before, if G is a 3-colorable graph on n vertices, then G has an independent set of size at least $n/3$. This bound can be improved when some vertices in the coloring have monochromatic neighborhood, since such vertices can be moved to two different color classes.

Lemma 14. *Let G be a graph on n vertices and let $Q, X \subseteq V(G)$ be disjoint sets. If Q is an independent set of G and $G - X$ has a 3-coloring φ such that the neighborhood of each vertex in Q is monochromatic, then $\alpha(G) \geq \frac{n - |X| + |Q|}{3}$.*

Proof. For $i \in \{1, 2, 3\}$, let Q_i be the set of vertices of Q such that φ assigns the color i to their neighbors, and let Q_0 be the set of vertices of Q that are isolated in $G - X$. Hence, Q is the disjoint union of Q_0, Q_1, Q_2 and Q_3. Let V_i be the set of vertices of $G - X - Q$ to which φ assigns the color i. Let $S_i = V_i \cup Q_0 \cup Q_{i+1} \cup Q_{i+2}$, where $Q_4 = Q_1$ and $Q_5 = Q_2$, and note that S_i is an independent set in G. We have $|S_1| + |S_2| + |S_3| = (|V_1| + |V_2| + |V_3|) + 2(|Q_1| + |Q_2| + |Q_3|) + 3|Q_0| = (n - |X| - |Q|) + 2|Q| + |Q_0| \geq n - |X| + |Q|$. Therefore, at least one of S_1, S_2 and S_3 has size at least $\frac{n - |X| + |Q|}{3}$. $\qquad \square$

For a plane graph G, let $F(G)$ denote the set of faces of G. Consider a cycle $C \subset G$. Removing C splits the plane into two open connected subsets, the bounded one is called the *open interior* of C. The *closed interior* of C is the closure of the open interior of C. The cycle C is *separating* if both the open interior of C and the complement of the closed interior of C contain a vertex of G. The following result of Dvořák, Král' and Thomas [11] is our main tool for obtaining colorings with monochromatic neighborhoods.

Proposition 15 ([11]). *There exists an integer $D_0 \geq 0$ such that for any triangle-free plane graph G without separating 4-cycles, for any sets $Q_1 \subseteq V(G)$ of vertices of degree at most 4 and $Q_2 \subseteq F(G)$ of 4-faces such that the elements of $Q_1 \cup Q_2$ have pairwise distance at least D_0, and for any 3-coloring ψ of the boundaries of the faces in Q_2, there exists a 3-coloring φ of G such that the neighborhood of each vertex in Q_1 is monochromatic and the pattern of the coloring on each face in Q_2 is the same as in the coloring ψ.*

In the statement, two colorings have the same *pattern* on a subgraph F if they differ on F only by a permutation of colors.

We use the following result by Gimbel and Thomassen [16].

Proposition 16 ([16]). *Let G be a triangle-free planar graph and let $C = v_1 v_2 \ldots$ be an induced cycle of length at most 6 in G. If a 3-coloring ψ of C does not extend to a 3-coloring of G, then $|C| = 6$ and $\psi(v_1) = \psi(v_4) \neq \psi(v_2) = \psi(v_5) \neq \psi(v_3) = \psi(v_6) \neq \psi(v_1)$.*

Corollary 17. *Let G be a triangle-free planar graph. For any vertex $v \in V(G)$ of degree at most 3, there exists a 3-coloring of G such that the neighborhood of v is monochromatic.*

Proof. Let $v_1, \ldots, v_t$ be the neighbors of v in the cyclic order around v. The claim holds by Grötzsch' theorem if $t \leq 1$, and thus assume that $2 \leq t \leq 3$. Let G' be the graph obtained from G by removing v and adding new vertices $u_1, \ldots, u_t$ and the edges of the cycle $C = v_1 u_1 v_2 u_2 \ldots v_t u_t v_1$. Note that C is an induced cycle, since G is triangle-free. Let $\psi(v_1) = \ldots = \psi(v_t) = 1$ and $\psi(u_1) = \ldots = \psi(u_t) = 2$. By Proposition 16, ψ extends to a 3-coloring φ of G'. By setting $\varphi(v) = 2$, we extend φ to a 3-coloring of G such that the neighborhood of v is monochromatic. $\qquad\square$

We need a variation of Proposition 15 which allows some separating 4-cycles. Let us recall that a subgraph G_0 of a plane graph G is *4-swept* if G_0 has no separating 4-cycles and every face of G_0 which is not a face of G has length 4.

Lemma 18. *There exists an integer $D_1 \geq 1$ such that the following holds for any triangle-free plane graph G and any 4-swept subgraph G_0 of G. Let $X, Q \subseteq V(G_0)$ be disjoint sets such that each vertex of Q has degree at most 4 in G_0. If Q is D_1-scattered in $G_0 - X$, then $G - X$ has a 3-coloring such that at least $|Q| - 6|X|$ vertices have monochromatic neighborhood and form an independent set.*

Proof. Let $D_1 = D_0 + 4$, where D_0 is the constant of Proposition 15.

Let R be the set of 4-faces of G_0 which are not faces of G. Let Z be the set of vertices $z \in Q$ such that there exists a face $x_1 z x_2 v \in R$ such that x_1 and x_2 belong to X. Let H be the graph (possibly with parallel edges) with vertex set X such that two vertices x_1 and x_2 are adjacent if there exists a face $x_1 v_1 x_2 v_2 \in R$, for some $v_1, v_2 \in V(G_0)$. Since G_0 has no separating 4-cycles, we conclude that either G_0 is isomorphic to $K_{2,3}$ and $|X| \geq 2$, or H has no parallel edges. Since H is a planar graph, it follows that $|E(H)| \leq 3|X|$. Note that $|Z| \leq 2|E(H)| \leq 6|X|$.

Let $Q_0 = Q \setminus Z$. Let Q_1 be the set of vertices of Q_0 that are not incident with the faces of R. For each vertex $v \in Q_0 \setminus Q_1$, we choose one incident 4-face in R; let Q_2' be the set of these faces. By the choice of Z, each face in Q_2' is incident with exactly one vertex of $Q_0 \setminus Q_1$, and thus $|Q_2'| = |Q_0 \setminus Q_1|$. For each $f \in Q_2'$, let G_f be the subgraph of G drawn in the closure of f. By Euler's formula, there exists a vertex $v_f \in V(G_f) \setminus V(f)$ whose degree in G is at most 3. Let ψ_f be a 3-coloring of G_f such that the neighborhood of v_f is monochromatic, which exists by Corollary 17. Let $I = \{v_f : f \in Q_2'\}$.

Let G_1 be the graph obtained from $G_0 - X$ as follows. For each face $f \in Q_2'$ whose boundary intersects X, note that by the choice of Z, there exists a subpath P of the boundary walk of f such that $X \cap V(F)$ are exactly the internal vertices of P. Add to G_1 a path of length $|P|$, with the same endvertices as P and with new internal vertices of degree two.

Note that each face in Q_2' corresponds to a 4-face of G_1; let Q_2 be the set of such faces of G_1. Observe that the distance in G_1 between any two elements of $Q_1 \cup Q_2$ is at least D_0. Furthermore, for each $f \in Q_2'$, we can naturally interpret ψ_f as a coloring of the corresponding face of Q_2.

By Proposition 15, there is a 3-coloring φ_0 of G_1 such that the neighborhood of every vertex of Q_1 is monochromatic and the pattern of φ_0 on every face $f \in Q_2$ is the same as the pattern of ψ_f. By permuting the colors in the colorings $\psi_f : f \in Q_2$, we can assume that their restrictions to the boundaries of the faces in Q_2 are equal to the restriction of φ_0. For each 4-face $f \in R \setminus Q_2$, let ψ_f be a 3-coloring of the subgraph of G drawn in the closure of f matching φ_0 on f, which exists by Proposition 16.

Let φ be the union of φ_0 and the colorings ψ_f for all 4-faces f of G_0, restricted to $V(G) \setminus X$. Note that φ is a 3-coloring of $G - X$ such that all vertices in $Q_1 \cup I$ have monochromatic neighborhood in φ. The claim of this lemma holds, since $|Q_1 \cup I| = |Q_0|$. $\qquad\square$

6 Proofs

It is well known that planar triangle-free graphs have many vertices of degree at most 4.

Lemma 19. *Any n-vertex planar triangle-free graph G has at least $n/5$ vertices of degree at most 4.*

Proof. Let n_4 denote the number of vertices of G of degree at most 4. By Euler's formula, planar triangle-free graphs have average degree at most 4, and thus $5(n - n_4) \leq 4n$. The claim of the lemma follows. $\quad\square$

Let us recall that $s(G)$ denotes the maximum number of vertices of a 4-swept subgraph of a plane graph G. We can now state the result from which Theorems 2 and 3 will be derived.

Theorem 20. *There exists a constant $c > 0$ such that every plane triangle-free graph G on n vertices has an independent set of size at least $\frac{n + cs(G)}{3}$.*

Proof. Let D_1 be the distance from Lemma 18. By Lemma 13, there exists a constant K such that every planar graph is $(D_1, 14, K)$-fat. Let $c = \frac{1}{10K}$.

Let G_0 be a 4-swept subgraph of G such that $|V(G_0)| = s(G)$. Let S be the set of vertices of G_0 of degree at most 4; by Lemma 19, we have $|S| \geq s(G)/5$. Since G_0 is $(D_1, 14, K)$-fat, there exist sets $Q \subseteq S$ and $X \subseteq V(G_0) \setminus Q$ such that Q is D_1-scattered in $G_0 - X$, $|Q| \geq |S|/K \geq \frac{s(G)}{5K}$, and $|X| \leq |Q|/14$.

By Lemma 18, $G - X$ has a proper 3-coloring and an independent set Q' with $|Q'| \geq |Q| - 6|X|$ such that the neighborhood of each vertex in Q' is monochromatic. By Lemma 14, we have, as required,

$$\alpha(G) \geq \frac{n - |X| + |Q| - 6|X|}{3} \geq \frac{n + |Q|/2}{3} \geq \frac{n + \frac{s(G)}{10K}}{3} \ . \qquad\square$$

Proof of Theorem 3. Since G has no separating 4-cycles, we have $s(G) = n$. Therefore, $\alpha(G) \geq \frac{n + cn}{3} = \frac{n}{3/(1+c)}$ by Theorem 20, and we can set $\varepsilon = \frac{3c}{1+c}$. $\qquad\square$

Theorem 2 is a corollary of Theorem 20 and Lemma 10.

7 Finding A Large Independent Set

In this section we describe an algorithm to actually find an independent set of size at least $(n + k)/3$ in a given n-vertex triangle-free planar graph possessing such a set.

We will need to be able to locate separating 4-cycles efficiently.

Lemma 21. *There exists a linear-time algorithm that reports a separating 4-cycle in a given plane graph G, or decides that there is no separating 4-cycle.*

Proof. Note that given edges of a 4-cycle C in G, we can test whether C bounds a face in constant time (given a standard representation of the plane embedding of G), and thus we can decide whether C is separating.

Firstly, greedily find an ordering $v_1, \ldots, v_n$ of the vertices of G so that for $1 \leq i \leq n$, the vertex v_i is adjacent to at most 5 vertices in the set $\{v_1, \ldots, v_{i-1}\}$. Next, enumerate all 4-cycles $v_{i_1} v_{i_2} v_{i_3} v_{i_4}$ such that $i_2 < \max(i_1, i_3) < i_4$. This can be done in linear time, by first choosing v_{i_4}, then v_{i_1} and v_{i_3} among

at most 5 neighbors of v_{i_4} that precede it in the ordering, and finally v_{i_2} among at most 5 neighbors of $v_{\max(i_1,i_3)}$ that precede it in the ordering. Test each of these cycles, and if any is separating, report it.

Hence, we can now assume that if G contains a separating 4-cycle, then we can label its vertices as $v_{i_1}v_{i_2}v_{i_3}v_{i_4}$ so that $i_4 > i_2 > \max(i_1,i_3)$. Enumerate all triples (a,b,c) such that $a < b < c$ and $v_av_c, v_bv_c \in E(G)$ (note that there are at most $\frac{5}{2}n$ such triples) and sort the list of triples (compared lexicographically) in $O(n)$ time using radix sort. Group the triples according to the first two entries. For each such group $(a,b,c_1), (a,b,c_2), \ldots, (a,b,c_t)$, test all 4-cycles of form ac_ibc_j with $1 \le i \le \min(t,4)$, and if any is separating, report it. Note that if $t \ge 4$, one of the tested 4-cycles necessarily is separating. $\square$

Suppose that the algorithm of Theorem 1 decided that a planar triangle-free graph G with n vertices contains an independent set of size at least $(n+k)/3$. In case that G has tree-width $O(\sqrt{k})$, the algorithm for finding independence number of a graph with bounded tree-width can also easily return one of the largest independent sets.

Suppose that G has tree-width $\Omega(\sqrt{k})$, i.e., $s(G) = \Omega(k)$. In this case, we know that G contains an independent set of size at least $(n+k)/3$ by Theorem 20. Let us now go over the proof of Theorem 20 in order to obtain an algorithm to report this set.

Firstly, we need to find a 4-swept subgraph G_0 of G of size $\Omega(k)$. This can be accomplished by applying the procedure from the first paragraph of the proof of Lemma 10 to obtain 4-swept subgraphs $H_1, \ldots, H_m$ of G and choosing G_0 as the largest of these graphs. In order to obtain each of the 4-swept subgraphs, we need to locate a separating 4-cycle with minimal interior. One way to do so is to find an arbitrary separating 4-cycle using Lemma 21, then find an arbitrary separating 4-cycle in the graph drawn in its closed interior, etc., until a 4-cycle with no separating 4-cycles in its interior is found. In total, we can find G_0 in time $O(n^3)$.

Let D_1 be the distance from Lemma 18 and let K be the constant given by Lemma 13 such that every planar graph is $(D_1, 14, K)$-fat. Let S_0 be the set of vertices of G_0 of degree at most 4. Let S be an arbitrarily chosen subset of S_0 of size $2Kk$. We now proceed as in the proof of Lemmas 13 and 12 to obtain $Q \subseteq S$ and $X \subseteq V(G_0) \setminus Q$ such that Q is D_1-scattered in $G_0 - X$, $|Q| \ge |S|/K = 2k$, and $|X| \le |Q|/14$. Note that a D_1-th oriented augmentation of G_0 can be obtained in linear time; we refer to [28] for details. Observe that the rest of the steps described in the proofs of the lemmas can be performed in linear time as well.

Let Q' be the set of size $|Q| - 6|X|$ obtained as in Lemma 18, such that $G - X$ has a proper 3-coloring in that the neighborhood of each vertex in Q' is monochromatic. Note that $|Q'| \le |Q| \le |S| = O(k)$. Since all vertices of Q' have degree at most four in $G - X$, this condition amounts to fixing colors of $O(k)$ vertices of $G - X$. Hence, we can obtain such a 3-coloring of $G - X$ using an algorithm by Dvořák et al. [14] in time $2^{O(k)}n^2$. Finally, we report the largest of the three independent sets constructed as in Lemma 14.

Therefore, we can modify the algorithm of Theorem 1 to also report the large independent set if it exists, at the expense of increasing the its time complexity to $O(n^3) + 2^{O(k)}n^2$. Let us remark that this can be improved to $2^{O(k)}n^2$ by a more involved implementation of the first step, using a semidynamic data structure described by Dvořák et al. [10] to repeatedly find separating 4-cycles.

8 Discussion

We gave a fixed-parameter algorithm for finding an independent set of size at least $n/3 + k$ in triangle-free planar graphs on n vertices, for every integer $k \ge 0$. Let us remark that the subexponential dependence on k in the running time of our algorithm is optimal, under the Exponential Time Hypothesis (this follows from a reduction by Madhavan [23]).

Several intriguing questions remain. Does the problem admit a polynomial kernel? That is, can any triangle-free planar graph on n vertices be efficiently (in polynomial time) compressed to an equivalent graph G' on $k^{O(1)}$ vertices? Also, while we can decide the existence of the independent set in linear time (in n), we can only find such an independent set in quadratic time. Can this be improved?

Unfortunately, it is unlikely our techniques could be used for PLANAR INDEPENDENT SET-ATLB in general planar graphs. The analogue of Proposition 15 is false for general planar graphs, and there exist n-vertex planar graphs with largest independent set of size $n/4$ and arbitrarily large tree-width.

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