# Reasoning about embedded dependencies using inclusion dependencies 

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#### Abstract

The implication problem for the class of embedded dependencies is undecidable. However, this does not imply lackness of a proof procedure as exemplified by the chase algorithm. In this paper we present a complete axiomatization of embedded dependencies that is based on the chase and uses inclusion dependencies and implicit existential quantification in the intermediate steps of deductions.


Keywords: axiomatization, chase, implication problem, dependence logic, embedded dependency, tuple generating dependency, equality generating dependency, inclusion dependency

## 1 Introduction

Embedded dependencies generalize the concept database dependencies within the framework of first-order logic. Their implication is undecidable but however recursively enumerable, thus enabling complete axiomatizations. A standard example of such a proof procedure is the chase that was invented in the late 1970s [12], and then soon extended to equality and tuple generating dependencies [3]. In this paper we present an axiomatization for the class of embedded dependencies that simulates the chase at the logical level using inclusion dependencies. In particular, completeness of the rules is obtained by constructing deductions in which all the intermediate steps are inclusion dependencies, except for the first and the last step. These inclusion dependencies consist of attributes of which some are new, i.e., such that they are not allowed to appear at any earlier stage of the deduction.

As a background example, consider the combined class of functional and inclusion dependencies. It is well known that the corresponding implication problem is undecidable, lacking hence finite axiomatization 415. In these situations, one strategy has been to search for axiomatizations within a more general class of dependencies, and partly for this reason many different dependency notions were introduced in the 1980s. For instance, a textbook on dependency theory from 1991 considers more than 80 different dependency classes 6]. In [7] Mitchell proposed another strategy by presenting an axiomatization of functional and inclusion dependencies using a notion of new attributes which are to be thought of as
implicitly existentially quantified. In this paper we take an analogous approach, and present an axiomatization for embedded dependencies where new attributes correspond to new values obtained from an associated chasing sequence. These attributes are implicitly existentially quantified in the sense of team semantics, that is, a semantic framework that has teams, i.e., sets of assignments, as its underlying concept [8. Team semantics is compositionally applicable to logics that extend first-order logic with various database dependencies 9|10. In this setting, inclusion logic, i.e., first-order logic with additional inclusion dependencies, captures positive greatest fixed-point logic and hence all PTIME recognizable classes of finite, ordered models 1112|13. Therefore, inclusion dependencies with new attributes can be thought of as existentially quantified inclusion logic formulae which in turn translate into greatest fixed-point logic. Moreover, all existentially quantified dependencies that appear in deductions translate into existential second-order logic. This may in part enable succinct intermediate steps in contrast to axiomatic systems that simulate the chase by composing first-order definable dependencies.

The methods described in this paper generalize the axiomatization of conditional independence and inclusion dependencies presented in 14 . It is also worth noting that extending relations with new attributes reminds of algebraic dependencies, that are, typed embedded dependencies defined in algebraic terms. The complete axiomatization of algebraic dependencies presented in 15 involves also an extension schema that introduces new copies of attributes.

## 2 Preliminaries

For two sets $A$ and $B$, we write $A B$ to denote their union, and for two sequences $\boldsymbol{a} \boldsymbol{b}$, we write $\boldsymbol{a} \boldsymbol{b}$ to denote their concatenation. For a sequence $\boldsymbol{a}=\left(a_{1}, \ldots, a_{n}\right)$ and a mapping $f$, we write $f(\boldsymbol{a})$ for $\left(f\left(a_{1}\right), \ldots, f\left(a_{n}\right)\right)$. We denote by id the identity function and by $\mathrm{pr}_{i}$ the function that maps a sequence to its $i$ th projection. For a function $f$ and $A \subseteq \operatorname{Dom}(f)$, we write $\left.f\right|_{A}$ for the restriction of $f$ to $A$, and for a set of mappings $F$, we write $\left.F\right|_{A}$ for $\left\{\left.f\right|_{A}: f \in F\right\}$.

We start by fixing two countably infinite sets Val and Att, the first denoting possible values of relations and the second attributes. For notational convenience, we will assume that $\mathrm{Val}=\mathrm{Att}$. For $R \subseteq$ Att, a tuple over $R$ is a mapping $R \rightarrow \mathrm{Val}$, and a relation over $R$ is a set of tuples over $R$. We may sometimes write $r[R]$ to denote that $r$ is a relation over $R$. Values of a relation $r$ over $R$ are denoted by $\operatorname{Val}(r)$, i.e., $\operatorname{Val}(r):=\{t(A): t \in r, A \in R\}$. Let $f$ be a valuation, i.e., a mapping $\mathrm{Val} \rightarrow \mathrm{Val}$. Then for a tuple $t$, we write $f(t):=f \circ t$, and for a relation $r, f(r):=\{f(t): t \in r\}$. A valuation $f$ embeds a relation $r$ (a tuple $t$ ) to $r^{\prime}$ if $f(r) \subseteq r^{\prime}(f(t) \in r)$. Since we are usually interested only valuations of a relation, we say that $f: \operatorname{Val}(r) \rightarrow \mathrm{Val}$ is a valuation on $r$. For a valuation $f$ on $r$, we say that $g$ is an extension of $f$ to another relation $r^{\prime}$ if $g$ is a valuation on $r^{\prime}$ such that it agrees with $f$ on values of $\operatorname{Val}(r) \cap \operatorname{Val}\left(r^{\prime}\right)$.

Embedded dependencies (ed's) can be written using first-order logic in the following way.

Definition 1 (Embedded dependency). Embedded dependency is a firstorder sentence of the form

$$
\forall x_{1}, \ldots, x_{n}\left(\phi\left(x_{1}, \ldots, x_{n}\right) \rightarrow \exists z_{1} \ldots \exists z_{k} \psi\left(y_{1}, \ldots, y_{m}\right)\right)
$$

where $\left\{z_{1}, \ldots, z_{k}\right\}=\left\{y_{1}, \ldots, y_{m}\right\} \backslash\left\{x_{1}, \ldots, x_{n}\right\}$ and

- $\phi$ is a (possibly empty) conjunction of relational atoms using all of the variables $x_{1}, \ldots, x_{n}$;
$-\psi$ is a conjunction of relational and equality atoms using all of the variables $z_{1}, \ldots, z_{k}$;
- there are no equality atoms in $\psi$ involving existentially quantified variables.

If at most one relation symbol occurs in an ed, then we say that the ed is unirelational, and otherwise it is multirelational. An ed is called typed if there is an assignment of variables to column positions such that variables in relation atoms occur only in their assigned position, and each equality atom involves a pair of variables assigned to the same position. Otherwise we say that an ed is untyped. If $\psi$ contains only one atom, then we say that the ed is singlehead, and otherwise it is multi-head. A single-head ed where $\psi$ is an equality is called an equality generating dependency (egd). If $\psi$ is a conjunction of relational atoms, then the ed is called a tuple generating dependency (tgd). For notational simplicity, we restrict attention to unirelational ed's. It is easy to se that any ed is equivalent to a set of tgd's and egd's, and hence we restrict attention to ed's that belong to either of these subclasses.

The following alternative tableau presentation for egd's and tgd's are used in this paper.
Definition 2. Let $T$ and $T^{\prime}$ be finite relations over $R$, and $x, y \in \operatorname{Val}(T)$. Then $(T, x=y)$ and $\left(T, T^{\prime}\right)$ are an egd and a tgd over $R$, respectively, with the below satisfaction relation for a relation $r$ over $S \supseteq R$ :
$-r \models(T, x=y) \Leftrightarrow$ for all valuations $f$ such that $\left.f(T) \subseteq r\right|_{R}$, it holds that $f(x)=f(y)$.
$-r \models\left(T, T^{\prime}\right) \Leftrightarrow$ for all valuations $f$ on $T$ such that $\left.f(T) \subseteq r\right|_{R}$, there is an extension $g$ of $f$ to $T^{\prime}$ such that $\left.g\left(T^{\prime}\right) \subseteq r\right|_{R}$.

Sometimes we write $\sigma[R]$ to denote that $\sigma$ is a dependency over $R$. If $T$ or $T^{\prime}$ is a singleton, then we may omit the set braces in the notation, e.g., write $(T, t)$ instead of $(T,\{t\})$.

We also extend valuations to dependencies. For an egd $\sigma=(T, x=y)$ we write $\operatorname{Val}(\sigma)=\operatorname{Val}(T)$, and for a $\operatorname{tgd} \tau=\left(T, T^{\prime}\right)$ we write $\operatorname{Val}(\sigma)=\operatorname{Val}(T) \cup$ $\operatorname{Val}\left(T^{\prime}\right)$. Moreover, if $f$ is a valuation, then $f(\sigma)=(f(T), f(x)=f(y))$ and $f(\tau)=\left(f(T), f\left(T^{\prime}\right)\right)$.

Example 1. Consider the relation $r$ and the tgd's $\sigma_{1}:=\left(\left\{t, t^{\prime}\right\},\{u\}\right)$ and $\sigma_{2}:=$ ( $\left\{t, t^{\prime}\right\},\left\{v, v^{\prime}\right\}$ ) obtained from Fig. $1^{1}$ We notice that there are two valuations

[^0]on $\left\{t, t^{\prime}\right\}$ that embed $\left\{t, t^{\prime}\right\}$ to $r$, namely $f:=\{(x, 0),(y, 1),(z, 2)\}$ and $g:=$ $\{(x, 3),(y, 0),(z, 1)\}$. Then $r \models \sigma_{1}$ since $f$ and $g$ embed $u$ into $r$, witnessed by tuples $s_{2}$ and $s_{3}$, respectively. We also notice that $r \not \vDash \sigma_{2}$ since, although $f \cup\{(a, 3)\}$ embeds $\left\{v, v^{\prime}\right\}$ into $r$, no extension of $g$ does the same.
\[

\left.r=$$
\begin{array}{c|ccc|} 
& A & B & C \\
\hline s_{0} & 0 & 1 & 2 \\
\hline s_{1} & 3 & 0 & 1 \\
\hline s_{2} & 2 & 3 & 0 \\
\hline s_{3} & 1 & 4 & 3
\end{array}
$$ \quad \quad \sigma_{1}=$$
\begin{array}{c|ccc|}
\hline t & A & B & C \\
\hline & x & y & z \\
t^{\prime} & & x & y \\
\hline u & z & & x
\end{array}
$$ \quad \sigma_{2}=$$
\begin{gathered}
\\
\hline t^{\prime}
\end{gathered}
$$ \right\rvert\, $$
\begin{array}{lll}
A & B & C \\
\hline v & x & y \\
\hline & z & a \\
v^{\prime} & & \\
\hline
\end{array}
$$
\]

Fig. 1.

Next we define inclusion dependencies which are examples of possibly untyped tgd's.
Definition 3 (Inclusion dependency). Let $A_{1}, \ldots, A_{n}$ and $B_{1}, \ldots, B_{n}$ be (not necessarily distinct) tuples of attributes. Then $A_{1} \ldots A_{n} \subseteq B_{1} \ldots B_{n}$ is an inclusion dependency (ind) over $R=\left\{A_{i}, B_{i}: i=1, \ldots, n\right\}$ with the following semantic rule for a relation $r$ over $S \supseteq R$ :

$$
r \models A_{1} \ldots A_{n} \subseteq B_{1} \ldots B_{n} \Leftrightarrow \forall s \in r \exists s^{\prime} \in r \forall i=1, \ldots, n: s\left(A_{i}\right)=s^{\prime}\left(B_{i}\right)
$$

The axiomatization presented in the next section involves inclusion dependencies that introduce new attributes. These attributes are here interpreted as existentially quantified in lax team semantics sense 9:

$$
\begin{equation*}
r \models \exists A \phi \Leftrightarrow r[f / A] \models \phi \text { for some } f: r \rightarrow \mathcal{P}(\mathrm{Val}) \backslash\{\emptyset\}, \tag{1}
\end{equation*}
$$

where $r[f / A]:=\{t(x / A): x \in f(A)\}$ and $t(x / A)$ is the mapping that agrees with $t$ everywhere except that it maps $A$ to $x$. Interestingly, inclusion logic formulae with this concept of existential quantification can be characterized with positive greatest fixed-point logic formulae (see Theorem 15 in [11]).

## 3 Axiomatization

In this section we present an axiomatization for the class of all embedded dependencies. The axiomatization contains an identity rule and three rules for the chase. We also involve conjunction in the language and therefore incorporate its usual introduction and elimination rules in the definition. Regarding the equalities that appear in the rules, note that both $A B \subseteq A A$ and $A B \subseteq B B$ indicate that the values of $A$ and $B$ coincide in each row. Therefore, we use $A=B$ to denote ind's of either form. For a $\operatorname{tgd}$ (an egd) $\sigma$, we say that $x \in \operatorname{Val}(\sigma)$ is distinct if it appears at most once as a value in $\sigma$. Namely,

- for a $\operatorname{tgd} \sigma=\left(T, T^{\prime}\right)[R], x$ is distinct if for all $t, t^{\prime} \in T \cup T^{\prime}$ and $A, B \in R$, if $t(A)=x=t^{\prime}(B)$, then $t=t^{\prime}$ and $B=B^{\prime}$;
- for an egd $\sigma=(T, y=z)[R], x$ is distinct if $x \notin\{y, z\}$ and for all $t, t^{\prime} \in T$ and $A, B \in R$, if $t(A)=x=t^{\prime}(B)$, then $t=t^{\prime}$ and $B=B^{\prime}$.

Lastly, note that in the following rules we assume that values can appear as attributes and vice versa.

Definition 4. In addition to the below rules we adopt the usual introduction and elimination rules for conjunction. In the last three rules, we assume that $\boldsymbol{A}$ is a sequence listing the attributes of $R$.

## EE Equality Exchange:

$$
\text { if } A=B \wedge \sigma \text {, then } \tau
$$

where $\sigma$ is an ind and $\tau$ is obtained from $\sigma$ by replacing any number of occurrences of $A$ by $B$ and any number of occurrences of $B$ by $A$.
CS Chase Start:

$$
\left(T^{*}, \mathrm{id}\right)[R S] \wedge \bigwedge_{t \in T} t(\boldsymbol{A}) \subseteq \boldsymbol{A}
$$

where $T=\left.T^{*}\right|_{R}, S=\operatorname{Val}(T)$ consists of new attributes, and $R$ consists of distinct values.
CR Chase Rule:

$$
\begin{gathered}
\text { tgd: } \quad \text { if }\left(T, T^{\prime}\right)[R] \wedge \bigwedge_{t \in T} f \circ t(\boldsymbol{A}) \subseteq \boldsymbol{A}, \text { then } \bigwedge_{t^{\prime} \in T^{\prime}} f \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}, \\
\text { egd: } \quad \text { if }(T, x=y)[R] \wedge \bigwedge_{t \in T} f \circ t(\boldsymbol{A}) \subseteq \boldsymbol{A}, \text { then } f(x)=f(y),
\end{gathered}
$$

where tgd: $f$ is a valuation that it is 1-1 on $\operatorname{Val}\left(T^{\prime}\right) \backslash \operatorname{Val}(T)$, and $f(x)$ is a new attribute for $x \in \operatorname{Val}\left(T^{\prime}\right) \backslash \operatorname{Val}(T)$.
CT Chase Termination:

$$
\begin{gathered}
\text { tgd: } \quad \text { if }\left(T^{*}, \mathrm{id}\right)[R S] \wedge \bigwedge_{t^{\prime} \in T^{\prime}} u \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}, \text { then }\left(T, T^{\prime}\right)[R], \\
\text { egd: } \quad \text { if }\left(T^{*}, \mathrm{id}\right)[R S] \wedge x=y, \text { then }(T, x=y)[R],
\end{gathered}
$$

where $T=\left.T^{*}\right|_{R}, S=\operatorname{Val}(T)$, and $\operatorname{Val}\left(\left.T^{*}\right|_{S}\right)$ consists of distinct values. Moreover, tgd: $u$ is a mapping $\operatorname{Val}\left(T^{\prime}\right) \rightarrow$ Att that is the identity on $\operatorname{Val}(T) \cap$ $\operatorname{Val}\left(T^{\prime}\right)$, and egd: $x, y \in \operatorname{Val}(T)$.

For a dependency $\sigma$ over $R$, we let $\operatorname{Att}(\sigma):=R$, and for a set of dependencies $\Sigma$, we let $\operatorname{Att}(\Sigma):=\bigcup_{\sigma \in \Sigma} \operatorname{Att}(\sigma)$.

Definition 5. A deduction from $\Sigma$ is a sequence $\left(\sigma_{1}, \ldots, \sigma_{n}\right)$ such that:

1. Each $\sigma_{i}$ is either an element of $\Sigma$, an instance of [CS], or follows from one or more formulae of $\left\{\sigma_{1}, \ldots, \sigma_{i-1}\right\}$ by one of the rules presented above.
2. For each $A \in \operatorname{Att}\left(\sigma_{i}\right)$, if $A$ is new in $\sigma_{i}$, then $A \notin \operatorname{Att}\left(\Sigma \cup\left\{\sigma_{1}, \ldots, \sigma_{i-1}\right\}\right)$, and otherwise $A \in \operatorname{Att}\left(\Sigma \cup\left\{\sigma_{1}, \ldots, \sigma_{i-1}\right\}\right)$.

We say that $\sigma$ is provable from $\Sigma$, written $\Sigma \vdash \sigma$, if there is a deduction $\left(\sigma_{1}, \ldots, \sigma_{n}\right)$ from $\Sigma$ with $\sigma=\sigma_{n}$ and such that no attributes in $\sigma$ are new in $\sigma_{1}, \ldots, \sigma_{n}$.

We will also use the following rules that are derivable from [EE]:

ES Equality Symmetry:

$$
\text { if } A=B, \text { then } B=A
$$

ET Equality Transitivity:

$$
\text { if } A=B \wedge B=C, \text { then } A=C
$$

One may find the chase rules slightly convoluted at first sight. However, the ideas behind the rules are relatively simple as illustrated in the following examples.

Example 2 (Chase Start). Let $\sigma_{0}:=\left(\left\{t_{0}, t_{1}\right\},\left\{u_{0}\right\}\right)[R S]$ be as in Figure 2, for $R:=\{A, B, C\}$ and $S:=\{x, y, z\}$. Then

Fig. 2.

$$
\tau:=\sigma_{0} \wedge x y z \subseteq A B C \wedge x y \subseteq B C
$$

is an instance of [CS]. Here $x, y, z$ are interpreted either as values or as new attributes. By the latter we intuitively mean that any relation $r[A B C]$ can be extended to some $r^{\prime}[A B C x y z]$ such that $r^{\prime} \models \tau$. For instance, one can define $r^{\prime}:=q(r)$ where $q$ is the following SPJR query

$$
A B C \bowtie\left(\pi_{x y z}\left(\sigma_{x y=B C}\left(\rho_{x y z / A B C}(A B C) \bowtie A B C\right)\right)\right)
$$

where $\sigma$ refers to (S)election, $\pi$ to (P)rojection, $\bowtie$ to (J)oin, and $\rho$ to (R)ename operator. Then $q(r)$ is a relation over $R S$ such that its restriction to $x y z$ lists all $a b c$ for which there exist $s, s^{\prime} \in r$ such that $s(A B C)=a b c$ and $s^{\prime}(B C)=a b$. Let $\sigma_{1}=\left(\left\{t_{0}, t_{1}\right\},\left\{u_{1}\right\}\right)[R]$ be as in Figure 2. Now,

$$
r \models \sigma_{1} \Leftrightarrow q(r) \models z x \subseteq A C .
$$

Hence proving $\Sigma \models \sigma_{1}$ reduces to showing that $\Sigma \cup\{\tau\} \models z x \subseteq A C$.

Example 3 (Chase Rule). Assume

$$
\begin{equation*}
\sigma_{2} \wedge x y z \subseteq A B C \wedge x y \subseteq B C \tag{2}
\end{equation*}
$$

where $\sigma_{2}=\left(\left\{t_{0}, t_{1}\right\},\left\{u_{2}, u_{3}\right\}\right)[R]$ is as in Fig. 2 for $R:=\{A, B, C\}$. Then, interpreting $f$ as id, one can derive with one application of [CR]

$$
\begin{equation*}
z v \subseteq A C \wedge v z \subseteq A C \tag{3}
\end{equation*}
$$

from (2). Note that in (3) $v$ is interpreted as a new attribute, and the idea is that any relation $r[R]$ satisfying (2) and with $v \notin R$ can be extended to a relation $r^{\prime}[R \cup\{v\}]$ satisfying (3) by introducing suitable values for $v$.

Example 4 (Chase Termination). Assume

$$
\begin{equation*}
\sigma_{0} \wedge z x \subseteq A C \tag{4}
\end{equation*}
$$

where $\sigma_{0}=\left(\left\{t_{0}, t_{1}\right\},\left\{u_{0}\right\}\right)[R S]$ is as in Fig. 2, for $R:=\{A, B, C\}$ and $S:=$ $\{x, y, z\}$. Then, letting $u=\mathrm{id}$, one can derive $\sigma_{1}$ as in Fig. 2 from (4) with one application of [CT].

## 4 Soundness Theorem

In this section we show that the axiomatization presented in the previous section is sound. First note that the next lemma follows from the definitions of egd's, tgd's and ind's.

Lemma 1. Let $\sigma$ be a dependency over $R$, and let $r$ and $r^{\prime}$ be relations over supersets of $R$ and with $\left.r\right|_{R}=\left.r^{\prime}\right|_{R}$. Then $r \models \sigma \Leftrightarrow r^{\prime} \models \sigma$.

Then we prove the following lemma which implies soundness of the axioms. For attribute sets $R, R^{\prime}$ with $R \subseteq R^{\prime}$ and a relation $r$ over $R$, we say that a relation $r^{\prime}$ over $R^{\prime}$ is an extension of $r$ to $R^{\prime}$ if $\left.r^{\prime}\right|_{R}=r$. Recall from equation 1 that exactly such extensions are used in the existential quantification of lax team semantics.

Lemma 2. Let $r$ be a relation over $\operatorname{Att}(\Sigma)$ such that $r \models \Sigma$, and let $\left(\sigma_{1}, \ldots, \sigma_{n}\right)$ be a deduction from $\Sigma$. Then there exists an extension $r^{\prime}$ of $r$ to $\operatorname{Att}(\Sigma \cup$ $\left.\left\{\sigma_{1}, \ldots, \sigma_{n}\right\}\right)$ such that $r^{\prime} \models \Sigma \cup\left\{\sigma_{1}, \ldots, \sigma_{n}\right\}$.

Proof. We prove the claim by induction on $n$. We denote by $R_{n}$ the set $\operatorname{Att}(\Sigma \cup$ $\left.\left\{\sigma_{1}, \ldots, \sigma_{n}\right\}\right)$. Assuming the claim for $n-1$, we first find an extension $r_{n-1}$ of $r$ to $R_{n-1}$ such that $r_{n-1} \models \Sigma \cup\left\{\sigma_{1}, \ldots, \sigma_{n-1}\right\}$. If $\sigma_{n}$ is obtained by an application of a conjunction or some ind rule, then it is easy to see that we may choose $r_{n}:=r_{n-1}$. Hence, it suffices to consider the cases where $\sigma_{n}$ is obtained by using one of the chase rules. Due to Lemma 11 it suffices to find an extension $r_{n}$ of $r_{n-1}$ to $R_{n}$ such that $r_{n} \models \sigma_{n}$. In the following cases, $\boldsymbol{A}$ denotes a sequence listing the attributes of $R \subseteq R_{n-1}$.

Case [CS]. Assume that $\sigma_{n}$ is obtained by $[\mathrm{CS}]$ and is of the form

$$
\left(T^{*}, \mathrm{id}\right)[R S] \wedge \bigwedge_{t \in T} t(\boldsymbol{A}) \subseteq \boldsymbol{A}
$$

where $T=\left.T^{*}\right|_{R}, S=\operatorname{Val}(T)$ consists of new attributes and $R$ of distinct values. Let $r_{n}:=r_{n-1} \bowtie r$ be an extension of $r_{n-1}$ to $R_{n}=R_{n-1} S$, where

$$
r:=\left\{h: h \text { is a valuation on } T \text { such that }\left.h(T) \subseteq r_{n-1}\right|_{R}\right\}
$$

We claim that $r_{n} \models \sigma_{n}$. Consider the first conjunct of $\sigma_{n}$, and let $h$ be a valuation on $T^{*}$ such that $\left.h\left(T^{*}\right) \subseteq r_{n}\right|_{R S}$. Then $\left.h\right|_{S}$ is is a valuation on $T$ such that $\left.h(T) \subseteq r_{n}\right|_{R}=\left.r_{n-1}\right|_{R}$, i.e., $\left.h\right|_{S}=\left.t_{0}\right|_{S}$ for some $t_{0} \in r_{n}$. Since $R$ consists of distinct values and thus $R \cap \operatorname{Dom}(h)=\emptyset$, we may define $h^{\prime}$ as an extension of $h$ with $A \mapsto t_{0}(A)$, for $A \in R$. Then $\left.h^{\prime}\right|_{R S}=\left.\left.t_{0}\right|_{R S} \in r_{n}\right|_{R S}$, and therefore $r_{n} \models\left(T^{*}, \mathrm{id}\right)[R S]$.

Consider then $t(\boldsymbol{A}) \subseteq \boldsymbol{A}$, for $t \in T$, and let $t_{0} \in r_{n}$. By the definition, $\left.t_{0}\right|_{S}=h$ for some valuation $h$ on $T$ such that $\left.h(T) \subseteq r_{n}\right|_{R}$, and hence we obtain that $t_{0} \circ t(\boldsymbol{A})=h \circ t(\boldsymbol{A})=t_{1}(\boldsymbol{A})$ for some $t_{1} \in r_{n}$. Therefore, $r_{n} \models t(\boldsymbol{A}) \subseteq \boldsymbol{A}$.

Case [CR]. Assume that $\sigma_{n}$ is of the form (i) $\bigwedge_{t^{\prime} \in T^{\prime}} f \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}$ or (ii) $f(x)=f(y)$, and is obtained by [CR] from
(i) $\left(T, T^{\prime}\right)[R] \wedge \wedge_{t \in T} f \circ t(\boldsymbol{A}) \subseteq \boldsymbol{A}$,
(ii) $(T, x=y)[R] \wedge \bigwedge_{t \in T} f \circ t(\boldsymbol{A}) \subseteq \boldsymbol{A}$,
where in case (ii) $f$ is a valuation on $T \cup T^{\prime}$ such that it is $1-1$ on $S:=\operatorname{Val}\left(T^{\prime}\right) \backslash$ $\operatorname{Val}(T)$ and $f(x)$ is a new attribute for $x \in S$. Let $s \in r_{n-1}$. Since $r_{n-1} \models$ $\bigwedge_{t \in T} f \circ t(\boldsymbol{A}) \subseteq \boldsymbol{A}$, we first obtain that $\left.s \circ f(T) \subseteq r_{n-1}\right|_{R}$.
(i) Since $r_{n-1} \models\left(T, T^{\prime}\right)[R]$ we find a mapping $g: S \rightarrow$ Val such that $h\left(T^{\prime}\right) \subseteq$ $\left.r_{n-1}\right|_{R}$, for $h=g \cup(s \circ f)$. Since $f$ is 1-1 on $S$, we can now define $r_{n}$ as the relation obtained from $r_{n-1}$ by extending each $s \in r_{n-1}$ with $f(x) \mapsto g(x)$ for $x \in S$. Then for each $s \in r_{n},\left.s \circ f\left(T^{\prime}\right) \subseteq r_{n}\right|_{R}$, and hence we obtain that $r_{n} \models \bigwedge_{t^{\prime} \in T^{\prime}} f \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}$.
(ii) It suffices to show that $r_{n-1} \models f(x)=f(y)$. Since $s \circ f(x)=s \circ f(y)$ by $r_{n-1} \models(T, x=y)[R]$, this follows immediately.

Case [CT]. Assume that $\sigma_{n}$ is of the form (i) $\left(T, T^{\prime}\right)[R]$ or (ii) $(T, x=y)[R]$ and is obtained by $[\mathrm{CT}]$ from
(i) $\left(T^{*}, \mathrm{id}\right)[R S] \wedge \bigwedge_{t^{\prime} \in T^{\prime}} u \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}$, where $u$ is a mapping $\operatorname{Val}\left(T^{\prime}\right) \rightarrow \operatorname{Att}$ that is the identity on $\operatorname{Val}(T) \cap \operatorname{Val}\left(T^{\prime}\right)$,
(ii) $\left(T^{*}, \mathrm{id}\right)[R S] \wedge x=y$, where $x, y \in \operatorname{Val}(T)$.

Moreover, in both cases $T=\left.T^{*}\right|_{R}, S=\mathrm{Val}(T)$, and $\operatorname{Val}\left(\left.T^{*}\right|_{S}\right)$ consists of distinct values. It suffices to show that $r_{n-1} \models \sigma_{n}$, so let $h$ be a valuation on $T$ such that $\left.h(T) \subseteq r_{n-1}\right|_{R}$. Since $\operatorname{Val}\left(\left.T^{*}\right|_{S}\right)$ consists of disctinct values, $h$ can be extended
to a valuation $h^{\prime}$ on $T^{*}$ such that $\left.h^{\prime}\left(T^{*}\right) \subseteq r_{n-1}\right|_{R S}$. Since $r_{n-1} \models\left(T^{*}, \mathrm{id}\right)[R S]$, there is an extension $h^{\prime \prime}$ of $h^{\prime}$ to attributes in $R$ such that $\left.\left.h^{\prime \prime}\right|_{R S} \in r_{n-1}\right|_{R S}$. Hence, we obtain that $\left.\left.h\right|_{S} \in r_{n-1}\right|_{S}$. Let then $s \in r_{n-1}$ be such that it agrees with $h$ on $S$.
(i) Since $r_{n-1} \models \bigwedge_{t^{\prime} \in T^{\prime}} u \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}$, we obtain that $\left.s \circ u\left(T^{\prime}\right) \subseteq r_{n-1}\right|_{R}$. Moreover, we notice that $s \circ u=h$ on $\operatorname{Val}(T) \cap \operatorname{Val}\left(T^{\prime}\right)$.
(ii) Since $r_{n-1} \models x=y$, we obtain that $s(x)=s(y)$. Then $h(x)=h(y)$ since $x, y \in S$.

Hence, in both cases we obtain that $r_{n-1} \models \sigma_{n}$. This concludes the [CT] case and the proof.

Using the previous lemma, soundness of the rules follows.
Theorem 1. Let $\Sigma \cup\{\sigma\}$ be a finite set of egd's and tgd's over $R$. Then $\Sigma \models \sigma$ if $\Sigma \vdash \sigma$.
Proof. Let $r$ be a relation such that $r \models \Sigma$, and assume that $\left(\sigma_{1}, \ldots, \sigma_{n}\right)$ is a deduction from $\Sigma$ where $\sigma=\sigma_{n}$ contains no attributes that appear as new in $\sigma_{1}, \ldots, \sigma_{n}$. If $R^{\prime}:=\operatorname{Att}\left(\Sigma \cup\left\{\sigma_{1}, \ldots, \sigma_{n}\right\}\right)$, then by Lemma2 2 we find an extension $r^{\prime}$ of $\left.r\right|_{R}$ to $R^{\prime}$ such that $r^{\prime} \models \sigma$. Then using Lemma we obtain that $r \models \sigma$.

## 5 Chase Revisited

In this section we define the chase for the class of egd's and tgd's. The chase algorithm was generalized to typed egd's and tgd's in [3], and here we present the chase using notation similar to that in [16. First let us assume, for notational convenience, that there is a total, well-founded order $<$ on the set Val, e.g., $x_{1}<x_{2}<x_{3}<\ldots$ for $\mathrm{Val}=\left\{x_{1}, x_{2}, x_{3}, \ldots\right\}$. Let $\Sigma \cup\{\sigma\}$ be a set of egd's and tgd's over $R$. A chasing sequence of $\sigma$ over $\Sigma$ is a (possibly infinite) sequence $\sigma_{0}, \sigma_{1}, \ldots, \sigma_{n}, \ldots$ where $\sigma_{0}=\sigma$, and $\sigma_{n+1}$ is obtained from $\sigma_{n}$, with $T:=\operatorname{pr}_{1}\left(\sigma_{n}\right)$, according to either of the following rules.

Let $\tau \in \Sigma$ be of the form $(S, x=y)$, and suppose that there is a valuation $f$ on $S$ such that $f(S) \subseteq T$ but $f(x) \neq f(y)$. Then $\tau$ (and $f$ ) can be applied to $\sigma_{n}$ as follows:

- egd rule: Let $\sigma_{n+1}:=g\left(\sigma_{n}\right)$ where $g: \mathrm{Val} \rightarrow \mathrm{Val}$ is the identity everywhere except that it maps $f(y)$ to $f(x)$ if $f(x)<f(y)$, and $f(x)$ to $f(y)$ if $f(y)<$ $f(x)$.
Let $\tau \in \Sigma$ be of the form $\left(S, S^{\prime}\right)$, and suppose that there is a valuation $f$ on $S$ such that $f(S) \subseteq T$, but there exists no extension $f^{\prime}$ of $f$ to $S^{\prime}$ such that $f\left(S^{\prime}\right) \subseteq T$. Then $\tau$ can be applied to $\sigma_{n}$ as follows:
- $\operatorname{tg} \boldsymbol{d}$ rule: List all $f_{1}, \ldots, f_{n}$ that have the above property, and for each $f_{i}$ choose a distinct extension to $S^{\prime}$, i.e., an extension $f_{i}^{\prime}$ to $S^{\prime}$ such that each variable in $\operatorname{Val}\left(S^{\prime}\right) \backslash \operatorname{Val}(S)$ is assigned a distinct new value greater than any value in $\operatorname{Val}\left(\sigma_{0}\right) \cup \ldots \cup \operatorname{Val}\left(\sigma_{n}\right)$. Moreover, no new value is assigned by two $f_{i}^{\prime}, f_{j}^{\prime}$ where $i \neq j$. Then we let $\sigma_{n+1}:\left(T \cup f_{1}^{\prime}\left(S^{\prime}\right) \cup \ldots \cup f_{m}^{\prime}\left(S^{\prime}\right), \operatorname{pr}_{2}\left(\sigma_{n}\right)\right)$.

Construction of a chasing sequence is restricted with the following two conditions:
(i) Whenever an egd is applied, it is applied repeatedly until it is no longer applicable.
(ii) No dependency is starved, i.e., each dependency that is applicable infinitely many times is applied infinitely many times.

Let $\overline{(\Sigma, \sigma)}=\sigma_{0}, \sigma_{1}, \ldots$ be a chasing sequence of $\sigma$ over $\Sigma$. Due to the possibility of applying egd's, a chasing sequence may not be monotone with respect to $\subseteq$. Hence, depending on whether $\sigma$ is a tgd or an egd, we define
$-\operatorname{egd}: \operatorname{chase} \overline{(\Sigma, \sigma)}:=\left(T^{1}, x=y\right)$,
$-\operatorname{tgd}:$ chase $\overline{(\Sigma, \sigma)}:=\left(T^{1}, T^{2}\right)$,
where $T^{i}:=\left\{u: \exists m \forall n \geq m\left(u \in \operatorname{pr}_{i}\left(\sigma_{n}\right)\right)\right\}$ and $x=y$ is $\operatorname{pr}_{2}\left(\sigma_{n}\right)$ for $n \in \mathbb{N}$ such that $\operatorname{pr}_{2}\left(\sigma_{n}\right)=\operatorname{pr}_{2}\left(\sigma_{m}\right)$ for all $m \geq n$. Note that "newer" values introduced by the tgd rule are always greater than the "older" ones, and values may only be replaced with smaller ones. Hence, no value can change infinitely often, and therefore chase $\overline{(\Sigma, \sigma)}$ is always well defined and non-empty.

We also associate each chasing sequence with the following descending valuations $\rho_{n}$, for $n \geq 0$. We let $\rho_{0}=\mathrm{id}, \rho_{n+1}=g \circ \rho_{n}$ if $\sigma_{n+1}$ is obtained by an application of the egd rule where $\sigma_{n+1}=g\left(\sigma_{n}\right)$, and $\rho_{n+1}=\mathrm{id} \circ \rho_{n}$ otherwise. We then define $\rho(x)=\lim _{n \rightarrow \infty} \rho_{n}(x)$, i.e., $\rho(x)=\rho_{n}(x)$ if $n \in \mathbb{N}$ such that $\rho_{m}(x)=\rho_{n}(x)$ for all $m \geq n$. Then we obtain that

$$
\operatorname{chase} \overline{(\Sigma, \sigma)}=\bigcup_{n=0}^{\infty} \rho\left(\sigma_{n}\right)
$$

A dependency $\tau$ is trivial if

- $\tau$ is of the form $(T, x=x)$, or
$-\tau$ is of the form $\left(T, T^{\prime}\right)$ and there is a valuation $f$ on $T^{\prime}$ such that $f$ is the identity on $\operatorname{Val}(T) \cap \operatorname{Val}\left(T^{\prime}\right)$ and $f\left(T^{\prime}\right) \subseteq T$.

It is well-known that the chase algorithm captures unrestricted implication of dependencies. The proof of the following proposition is hence located in Appendix.

Proposition 1. Let $\Sigma \cup\{\sigma\}$ be a set of egd's and tgd's over $R$. Then the following are equivalent:
(i) $\Sigma \models \sigma$,
(ii) there is a chasing sequence $\overline{(\Sigma, \sigma)}=\sigma_{0}, \sigma_{1}, \ldots$ of $\sigma$ over $\Sigma$ such that chase $\overline{(\Sigma, \sigma)}$ is trivial,
(iii) there is a chasing sequence $\overline{(\Sigma, \sigma)}=\sigma_{0}, \sigma_{1}, \ldots$ of $\sigma$ over $\Sigma$ such that $\sigma_{n}$ is trivial, for some $n$.

## 6 Completeness Theorem

In this section we show that the rules presented in Definition 4 are complete for the implication problem of embedded dependencies. Let us first illustrate the use of the axioms in the following simple example.
Example 5. Consider the implication problem $\left\{\sigma, \sigma^{\prime}\right\} \models \tau$ where $\sigma, \sigma^{\prime}, \tau$ are illustrated in Fig. 3, e.g., $\sigma=(T, t)$ where $T$ consists of the top two rows of $\sigma$ and $t$ is the bottom row. Note that $\sigma$ and $\tau$ are embedded multivalued dependencies of the form $A \rightarrow B \mid C$ and $A \rightarrow B \mid C D$, respectively, and $\sigma^{\prime}$ is a functional dependency of the form $C \rightarrow D$. It is easy to see that the implication holds, and

$$
\sigma=\left|\begin{array}{cccc}
A & B & C & D \\
\hline a_{0} & b_{0} & c_{0} & d_{0} \\
a_{0} & b_{1} & c_{1} & d_{1} \\
\hline a_{0} & b_{0} & c_{1} & d_{2}
\end{array}\right|
$$

$$
\sigma^{\prime}=\left|\begin{array}{cccc}
A & B & C & D \\
\hline a_{0} & b_{0} & c_{0} & d_{0} \\
a_{1} & b_{1} & c_{0} & d_{1} \\
\hline & d_{0} & = & d_{1}
\end{array}\right|
$$

$$
\tau=\left|\begin{array}{cccc}
A & B & C & D \\
\hline a_{0} & b_{0} & c_{0} & d_{0} \\
a_{0} & b_{1} & c_{1} & d_{1} \\
\hline a_{0} & b_{0} & c_{1} & d_{1}
\end{array}\right|
$$

Fig. 3.
this can be also verified by a chasing sequence $\tau_{0}, \tau_{1}, \tau_{2}$ of $\tau$ over $\left\{\sigma, \sigma^{\prime}\right\}$ where $\tau_{2}$ is trivial. In the chasing sequence, $\tau_{0}=\tau$ and $\tau_{1}$ is the result of applying $\sigma$ to $\tau_{0}$. For this, note that there exists two valuations on $T$ that embed $T$ to $\operatorname{pr}_{1}\left(\tau_{0}\right)$ but has no extension that embeds $t$ into $\operatorname{pr}_{1}\left(\tau_{0}\right)$. These valuations are the identity and the function $f$ that swaps the values of the top and bottom row of $T$. Then $\tau_{1}$ is obtained by adding to $\operatorname{pr}_{1}\left(\tau_{0}\right) \mathrm{id}^{*}(t)$ and $f^{*}(t)$ where $\mathrm{id}^{*}$ and $f^{*}$ are distinct extensions of id and $f$ to $t$, e.g., id ${ }^{*}=\mathrm{id}$ also on $d_{2}$ and $f^{*}$ maps $d_{2}$ to $d_{3}$. Also, $\tau_{2}$ is the result of applying $\sigma^{\prime}$ to $\tau_{1}$ two times, i.e., $\tau_{2}$ is obtained from $\tau_{1}$ by replacing $d_{3}$ with $d_{0}$ and $d_{2}$ with $d_{1}$. Clearly $\tau_{2}$ is trivial, and hence we obtain the claim by Proposition 1

$$
\tau_{0}=\left|\begin{array}{cccc}
A & B & C & D \\
a_{0} & b_{0} & c_{0} & d_{0} \\
a_{0} & b_{1} & c_{1} & d_{1} \\
\hline a_{0} & b_{0} & c_{1} & d_{1}
\end{array}\right| \quad \tau_{1}=\left|\begin{array}{cccc}
A & B & C & D \\
\hline a_{0} & b_{0} & c_{0} & d_{0} \\
a_{0} & b_{1} & c_{1} & d_{1} \\
\hdashline a_{0} & b_{0} & c_{1} & \bar{d}_{2} \\
a_{0} & b_{1} & c_{0} & d_{3} \\
\hline a_{0} & b_{0} & c_{1} & d_{1}
\end{array}\right| \quad \tau_{2}=\left|\begin{array}{cccc}
A & B & C & D \\
\hline a_{0} & b_{0} & c_{0} & d_{0} \\
a_{0} & b_{1} & c_{1} & d_{1} \\
a_{0} & b_{0} & c_{1} & d_{1} \\
a_{0} & b_{1} & c_{0} & d_{0} \\
\hline a_{0} & b_{0} & c_{1} & d_{1}
\end{array}\right|
$$

Fig. 4.

This procedure can now be simulated with our axioms as follows. First, with one application of [CS] we derive

$$
(T, \mathrm{id})[R S] \wedge a_{0} b_{0} c_{0} d_{0} \subseteq A B C D \wedge a_{0} b_{1} c_{1} d_{1} \subseteq A B C D
$$

where $T=\left\{t, t^{\prime}\right\}, R=\{A, B, C, D\}$, and $S=\left\{a_{0}, b_{0}, b_{1}, c_{0}, c_{1}, d_{0}, d_{1}\right\}$ is a set of values that are interpreted as new attributes. Here $t(x)$ and $t^{\prime}(x)$, for $x \in S$, and $A, B, C, D$ are interpreted as distinct values. $(T, t)[R S]$ is illustrated in Fig. 5 where all the distinct values are hidden. Now with one application of [CR],


Fig. 5. $(T$, id $)[R S]$
letting $f=$ id, we derive $a_{0} b_{0} c_{1} d_{2} \subseteq A B C D$ from

$$
\begin{equation*}
\sigma \wedge a_{0} b_{0} c_{0} d_{0} \subseteq A B C D \wedge a_{0} b_{1} c_{1} d_{1} \subseteq A B C D \tag{5}
\end{equation*}
$$

Note that in this step, $d_{2}$ is interpreted as a new attribute. Let then $f$ be the valuation that is the identity on $a_{0}, b_{0}, b_{1}, d_{1}$, and otherwise maps $a_{1} \mapsto a_{0}$, $c_{0} \mapsto c_{1}$, and $d_{0} \mapsto d_{2}$. We notice that $f\left(a_{0} b_{0} c_{0} d_{0}\right)=a_{0} b_{0} c_{1} d_{2}$ and $f\left(a_{1} b_{1} c_{0} d_{1}\right)=$ $a_{0} b_{1} c_{1} d_{1}$. Hence, we may derive with one application of [CR] $f\left(d_{0}\right)=f\left(d_{1}\right)$, i.e., $d_{2}=d_{1}$ from

$$
\sigma^{\prime} \wedge f\left(a_{0} b_{0} c_{0} d_{0}\right) \subseteq A B C D \wedge f\left(a_{1} b_{1} c_{0} d_{1}\right) \subseteq A B C D
$$

Then we apply [EE] and derive $a_{0} b_{0} c_{1} d_{1} \subseteq A B C D$ from

$$
d_{2}=d_{1} \wedge a_{0} b_{0} c_{1} d_{2} \subseteq A B C D
$$

Finally, we may apply $[\mathrm{CT}]$ and derive $\tau$ from $(T, \mathrm{id})[R S] \wedge a_{0} b_{0} c_{1} d_{1} \subseteq A B C D$.
The following lemma shows that the above technique extends to all chasing sequences. The proof is straightforward and located in Appendix.

Lemma 3. Let $\overline{(\Sigma, \sigma)}=\sigma_{0}, \sigma_{1}, \ldots$ be a chasing sequence of $\sigma$ over $\Sigma$, where $\Sigma \cup\{\sigma\}$ is a finite set of egd's and tgd's over $R$, let $\boldsymbol{A}$ be a sequence listing the attributes of $R$, let $T:=\operatorname{pr}_{1}(\sigma)$ and $T_{i}:=\operatorname{pr}_{1}\left(\sigma_{i}\right)$, and let $n \in \mathbb{N}$. Then there exists a deduction from $\Sigma$, with attributes from $R \cup \bigcup_{i \in \mathbb{N}} \operatorname{Val}\left(T_{i}\right)$, listing the following dependencies:
(i) $\left(T^{*}, \mathrm{id}\right)[R S]$ where $\left.T^{*}\right|_{R}=T, S=\operatorname{Val}(T)$, and $\left.T^{*}\right|_{S}$ consists of distinct values,
(ii) $f(x)=f(y)$, for each application of $(S, x=y)$ and $f$ to $\sigma_{m}$, for $m<n$,
(iii) $t(\boldsymbol{A}) \subseteq \boldsymbol{A}$, for $t \in T_{m}$ where $m \leq n$.

With the lemma, we can now show completeness.
Theorem 2. Let $\Sigma \cup\{\sigma\}$ be a finite set of egd's and tgd's over $R$. Then $\Sigma \models$ $\sigma \Leftrightarrow \Sigma \vdash \sigma$.

Proof. Assume that $\Sigma \models \sigma$, and let $\boldsymbol{A}$ be a sequence listing $R$. Then by Proposition 1 there is a chasing sequence $\overline{(\Sigma, \sigma)}=\sigma_{0}, \sigma_{1}, \ldots$ of $\sigma$ over $\Sigma$ such that $\sigma_{n}$ is trivial for some $n$. Let $D=\left(\tau_{1}, \ldots, \tau_{l}\right)$ be a deduction from $\Sigma$ obtained by Lemma 3, and let $T:=\operatorname{pr}_{1}(\sigma)$ and $T_{i}:=\operatorname{pr}_{1}\left(\sigma_{i}\right)$.

Assume first that $\sigma$ is an egd of the form $(T, x=y)$. Then $\sigma_{n}$ is $\left(T_{n}, z=z\right)$ where $z=\rho_{n}(x)=\rho_{n}(y)$. Now, either $\rho_{i+1}(x)$ is $\rho_{i}(x)$, or the equality $\rho_{i+1}(x)=$ $\rho_{i}(x)$ (or its reverse) is listed in $D$ by item (ii). Hence, using repeatedly [ES,ET] we may further on derive $z=x$. Since $z=y$ is derivable analogously, we therefore obtain $x=y$ by [ES,ET]. Then with one application of [CT], we derive $(T, x=y)$ from $\left(T^{*}, \mathrm{id}\right)[R S] \wedge x=y$ where $\left.T^{*}\right|_{R}=T$. Note that the $\left(T^{*}, \mathrm{id}\right)[R S]$ of the correct form is listed in $D$ by item (i) of Lemma 3.

Assume then that $\sigma$ is a tgd of the form $\left(T, T^{\prime}\right)$, and let $T_{i}^{\prime}:=\operatorname{pr}_{2}\left(\sigma_{i}\right)$. Then $\sigma_{n}$ is $\left(T_{n}, T_{n}^{\prime}\right)$, and there is a valuation $f$ on $T_{n}^{\prime}$ such that $f$ is the identity on $\operatorname{Val}\left(T_{n}\right) \cap \operatorname{Val}\left(T_{n}^{\prime}\right)$ and $f\left(T_{n}^{\prime}\right) \subseteq T_{n}$. Let $t^{\prime} \in T^{\prime}$. Then $\rho_{n} \circ t^{\prime} \in T_{n}^{\prime}$ and by item (iii) of Lemma 3] we obtain that $f \circ \rho_{n} \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}$ is listed in $D$. For $A \in R$, we have then two cases :

- If $t^{\prime}(A) \in \operatorname{Val}\left(T^{\prime}\right) \cap \operatorname{Val}(T)$, then we first notice that $f \circ \rho_{n} \circ t^{\prime}(A)$ is $\rho_{n} \circ$ $t^{\prime}(A)$ since $\rho_{n} \circ t^{\prime}(A) \in \operatorname{Val}\left(T_{n}^{\prime}\right) \cap \operatorname{Val}\left(T_{n}\right)$. Also we notice that the equality $\rho_{n} \circ t^{\prime}(A)=t^{\prime}(A)$ can be derived analogously to the egd case.
- If $t^{\prime}(A) \in \operatorname{Val}\left(T^{\prime}\right) \backslash \operatorname{Val}(T)$, then $f \circ \rho_{n} \circ t^{\prime}(A)=f \circ t^{\prime}(A)$ since by the definition of the chase $\rho_{n}$ is the identity on $\operatorname{Val}\left(T^{\prime}\right) \backslash \operatorname{Val}(T)$.

Now, letting $f^{*}$ be the mapping $\operatorname{Val}\left(T^{\prime}\right) \rightarrow$ Att which is the identity on $\operatorname{Val}\left(T^{\prime}\right) \cap$ $\operatorname{Val}(T)$ and agrees with $f$ on $\operatorname{Val}\left(T^{\prime}\right) \backslash \operatorname{Val}(T)$, we can by the previous reasoning and using repeatedly $[\mathrm{EE}]$ derive $f^{*} \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}$ from $f \circ \rho_{n} \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}$. Finally, we can then with one application of [CT] derive $\left(T, T^{\prime}\right)$ from

$$
\left(T^{*}, \mathrm{id}\right)[R S] \wedge \bigwedge_{t^{\prime} \in T^{\prime}} f^{*} \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}
$$

## 7 Typed dependencies

Consider then the class of typed embedded dependencies. In this setting [CS] and [CT] can be replaced with rules that involve only embedded join dependencies (ejd's) and inclusion dependencies. We define ejd's over tuples of attributes as follows.

Definition 6. Let $\boldsymbol{A}_{1}, \ldots, \boldsymbol{A}_{n}$ be tuples of attributes listing $R_{1}, \ldots, R_{n}$, respectively, and let $R:=\bigcup_{i=1}^{n} R_{i}$. Then $\bowtie\left(\boldsymbol{A}_{i}\right)_{i=1}^{n}$ is an embedded join dependency with the semantic rule

$$
-r \models \bowtie\left(\boldsymbol{A}_{i}\right)_{i=1}^{n} \text { if and only if }\left.r\right|_{\boldsymbol{R}}=\left.\left.r\right|_{\boldsymbol{R}_{1}} \bowtie \ldots \bowtie r\right|_{\boldsymbol{R}_{n}} .
$$

The two alternative rules for the chase are now the following. We call a relation typed if none of its values appears in two distinct columns.

CS* Chase Start*:

$$
\bigwedge_{t \in T} \boldsymbol{A} \subseteq t(\boldsymbol{A}) \wedge \bowtie(t(\boldsymbol{A}))_{t \in T} \wedge \bigwedge_{t \in T} t(\boldsymbol{A}) \subseteq \boldsymbol{A}
$$

where $T$ is a typed relation and $\operatorname{Val}(T)$ is a set of new attributes.
CT* Chase Termination*:

$$
\begin{aligned}
t g d & : \text { if } \bigwedge_{t \in T} \boldsymbol{A} \subseteq t(\boldsymbol{A}) \wedge \bowtie(t(\boldsymbol{A}))_{t \in T} \wedge \bigwedge_{t^{\prime} \in T^{\prime}} u \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}, \text { then }\left(T, T^{\prime}\right)[R], \\
& e g d: \text { if } \bigwedge_{t \in T} \boldsymbol{A} \subseteq t(\boldsymbol{A}) \wedge \bowtie(t(\boldsymbol{A}))_{t \in T} \wedge x=y, \text { then }(T, x=y)[R],
\end{aligned}
$$

where tgd: $u$ is a mapping $\operatorname{Val}\left(T^{\prime}\right) \rightarrow$ Att that is the identity on $\operatorname{Val}\left(T^{\prime}\right) \cap$ $\operatorname{Val}\left(T^{\prime}\right)$, and egd: $x, y \in \operatorname{Val}(T)$.

The first rule is sound for typed dependencies since, for arbitrary $r$ with $\operatorname{Dom}(r) \cap \operatorname{Val}(T)=\emptyset$, an instance of $\left[\mathrm{CS}^{*}\right]$ is satisfied by $r \bowtie q(r)$ where $q$ is the SPJR query

$$
\rho_{t_{1}(\boldsymbol{A}) / \boldsymbol{A}} \boldsymbol{A} \bowtie \ldots \bowtie \rho_{t_{n}(\boldsymbol{A}) / \boldsymbol{A}} \boldsymbol{A},
$$

where $\rho$ is the rename operator and $T=\left\{t_{1}, \ldots, t_{n}\right\}$. However, a counter example for soundness can be easily constructed for untyped dependencies. If $T$ and $r$ are the relations illustrated in Fig. 6, then no extension $r^{\prime}$ of $r$ to $\operatorname{Val}(T)$ satisfies $\wedge_{t \in T} t(A B) \subseteq A B$.

$$
T=\begin{array}{l|ll} 
& A & B \\
\hline t & x & y \\
t^{\prime} & y & x \\
\hline
\end{array} \quad r=\begin{array}{|l|l|l|}
\hline & A & B \\
\hline s & 0 & 1 \\
\hline
\end{array}
$$

Fig. 6.

Soundness of $\left[\mathrm{CT}^{*}\right]$ is obtained analogously to that of [CT]. Also, completeness is obtained by deriving exactly in the same way as in the general case, $\bigwedge_{t^{\prime} \in T^{\prime}} u \circ t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}$ (in the tgd case) or $x=y$ (in the egd case) from $\Lambda_{t \in T} t(\boldsymbol{A}) \subseteq \boldsymbol{A}$. Let us then write $\Sigma \vdash^{*} \sigma$ if $\sigma$ is deduced from $\Sigma$ in the sense of Definition 5 and using rules [EE, CS* $, \mathrm{CR}, \mathrm{CT}^{*}$ ] together with elimination and introduction of conjunction. Then we obtain the following theorem.

Theorem 3. Let $\Sigma \cup\{\sigma\}$ be a finite set of typed egd's and tgd's over $R$. Then $\Sigma \models \sigma \Leftrightarrow \Sigma \vdash^{*} \sigma$.

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## References

1. Aho, A.V., Beeri, C., Ullman, J.D.: The theory of joins in relational databases. ACM Trans. Database Syst. 4(3) (1979) 297-314
2. Maier, D., Mendelzon, A.O., Sagiv, Y.: Testing implications of data dependencies. ACM Trans. Database Syst. 4 (December 1979) 455-469
3. Beeri, C., Vardi, M.Y.: A proof procedure for data dependencies. J. ACM 31(4) (September 1984) 718-741
4. Chandra, A.K., Vardi, M.Y.: The implication problem for functional and inclusion dependencies is undecidable. SIAM Journal on Computing 14(3) (1985) 671-677
5. Mitchell, J.C.: The implication problem for functional and inclusion dependencies. Information and Control 56(3) (1983) 154-173
6. Thalheim, B.: Database schemes and databases. In: Dependencies in Relational Databases. Teubner-Texte zur Mathematik. Vieweg+Teubner Verlag (1991) 7-24
7. Mitchell, J.C.: Inference rules for functional and inclusion dependencies. In Fagin, R., Bernstein, P.A., eds.: PODS, ACM (1983) 58-69
8. Hodges, W.: Compositional Semantics for a Language of Imperfect Information. Journal of the Interest Group in Pure and Applied Logics 5 (4) (1997) 539-563
9. Galliani, P.: Inclusion and exclusion dependencies in team semantics: On some logics of imperfect information. Annals of Pure and Applied Logic 163(1) (2012) 68-84
10. Väänänen, J.: Dependence Logic. Cambridge University Press (2007)
11. Galliani, P., Hella, L.: Inclusion Logic and Fixed Point Logic. In Rocca, S.R.D., ed.: Computer Science Logic 2013 (CSL 2013). Volume 23 of Leibniz International Proceedings in Informatics (LIPIcs)., Dagstuhl, Germany, Schloss Dagstuhl-LeibnizZentrum fuer Informatik (2013) 281-295
12. Immerman, N.: Relational queries computable in polynomial time. Information and control 68(1) (1986) 86-104
13. Vardi, M.Y.: The complexity of relational query languages. In: Proceedings of the fourteenth annual ACM symposium on Theory of computing, ACM (1982) 137-146
14. Hannula, M., Kontinen, J.: A finite axiomatization of conditional independence and inclusion dependencies. In Beierle, C., Meghini, C., eds.: Foundations of Information and Knowledge Systems - 8th International Symposium, FoIKS 2014, Bordeaux, France, March 3-7, 2014. Proceedings. Volume 8367 of Lecture Notes in Computer Science., Springer (2014) 211-229
15. Yannakakis, M., Papadimitriou, C.H.: Algebraic dependencies. Journal of Computer and System Sciences 25(1) (1982) $2-41$
16. Abiteboul, S., Hull, R., Vianu, V.: Foundations of Databases. Addison-Wesley (1995)

## Appendix

Proof. (Proposition (1) Let $\Sigma \cup\{\sigma\}$ be a set of egd's and tgd's over $R$. The direction $(i i) \Rightarrow(i i i)$ is clear because it suffices to choose $\sigma_{n}$ such that all the relevant tuples and values remain fixed in $\sigma_{m}$ for $m \geq n$. We show $(i) \Rightarrow(i i)$ and $(i i i) \Rightarrow(i)$.
(i) $\Rightarrow(i i)$ : Assume that there is a chasing sequence $\overline{(\Sigma, \sigma)}=\frac{\sigma_{0}, \sigma_{1}, \ldots \text { of }}{(\Sigma, \sigma)}$. $\sigma$ over $\Sigma$ such that chase $\overline{(\Sigma, \sigma)}$ is non-trivial. We claim that chase $\overline{(\Sigma, \sigma)} \models \Sigma$ and chase $\overline{(\Sigma, \sigma)} \not \models \sigma$. Let $T_{n}$ denote $\operatorname{pr}_{1}\left(\sigma_{n}\right)$. Assume first that $(S, x=y) \in \Sigma$ and assume to the contrary that $f$ is a valuation such that $f(S) \subseteq$ chase $\overline{(\Sigma, \sigma)}$ but $f(x) \neq f(y)$. Then there exists $m \in \mathbb{N}$ such that $f(S) \in T_{n}$ for all $n \geq m$, contradicting the assumption that no dependency is starved in the chase.

Assume that $\left(S, S^{\prime}\right) \in \Sigma$, and assume that $f$ is a valuation such that $f(S) \subseteq$ chase $(\Sigma, \sigma)$, and let $m \in \mathbb{N}$ be such that $f(S) \subseteq T_{n}$ for all $n \geq m$. Then there is an extension $f^{\prime}$ of $f$ to $S^{\prime}$ such that $f^{\prime}\left(S^{\prime}\right) \subseteq T_{m^{\prime}}$ for some $m^{\prime} \geq m$, where we define $T_{n}^{\prime}:=\operatorname{pr}_{2}\left(\sigma_{n}\right)$. Note that $\rho_{n} \circ f^{\prime}(S) \subseteq T_{n}^{\prime}$ for all $n \geq m^{\prime}$, and hence there exists $m^{\prime \prime} \geq m^{\prime}$ such that $\rho_{m^{\prime \prime}} \circ f^{\prime}(S) \subseteq T_{n}^{\prime}$ for all $n \geq m^{\prime \prime}$.
 chase $\overline{(\Sigma, \sigma)} \models\left(S, S^{\prime}\right)$.

Finally, we show that chase $\overline{(\Sigma, \sigma)} \not \models \sigma$. Analogously to the previous case we find a valuation $\rho_{n}$ such that $\rho_{n}(T) \subseteq \operatorname{pr}_{1}(\operatorname{chase} \overline{(\Sigma, \sigma)})$. If $\sigma$ is of the form $(T, x=$ $y$ ), then we obtain that $\rho_{n}(x)=\rho_{n}(y)$ is $\operatorname{pr}_{2}(\operatorname{chase} \overline{(\Sigma, \sigma)})$. Since chase $\overline{(\Sigma, \sigma)}$ is non-trivial, $\rho_{n}(x)$ and $\rho_{n}(y)$ must be two distinct values. Hence, $\rho_{n}$ witnesses chase $\overline{(\Sigma, \sigma)} \not \models(T, x=y)$.

Assume then that $\sigma$ is of the form ( $\left(T, T^{\prime}\right)$. Then analogously $\rho_{n} \circ T \subseteq$ $\operatorname{pr}_{1}(\operatorname{chase} \overline{(\Sigma, \sigma)})$ and $\rho_{n} \circ T^{\prime}=\operatorname{pr}_{2}(\operatorname{chase}(\overline{\Sigma, \sigma)})$ for some $n \in \mathbb{N}$. Also note that by the construction $\rho_{n}$ is the identity on $\operatorname{Val}\left(T^{\prime}\right) \backslash \operatorname{Val}(T)$. Now, if there is an extension $h$ of $\left.\rho_{n}\right|_{\mathrm{Val}^{\prime}(T)}$ to $T^{\prime}$ such that $h\left(T^{\prime}\right) \subseteq \operatorname{pr}_{1}(\operatorname{chase} \overline{(\Sigma, \sigma)})$, then chase $\overline{(\Sigma, \sigma)}$ is trivial. Hence $\rho_{n}$ is a witness of chase $\overline{(\Sigma, \sigma)} \not \models \sigma$.
$($ iii $) \Rightarrow(i)$ : Let $\overline{(\Sigma, \sigma)}=\sigma_{0}, \sigma_{1}, \ldots$ be a chasing sequence of $\sigma$ over $\Sigma$, where $\sigma_{n}$ is trivial, and let $T_{i}$ (or $T$ ) denote $\mathrm{pr}_{1}\left(\sigma_{i}\right)\left(\right.$ or $\mathrm{pr}_{1}(\sigma)$ ). Assume that $r \models \Sigma$, and let $f$ be a valuation on $T$ to $r$. Using the chase construction rules and the assumption it is easy to show inductively that for all $n$ there is an extension $f_{n}$ of $f$ to $\cup_{i=0}^{n} T_{i}$ such that
(i) $f_{n}\left(T_{n}\right) \subseteq r$,
(ii) $f_{n} \circ \rho_{n}=f_{n}$.

Assume first that $\sigma$ is of the form $(T, x=y)$, and hence $\rho_{n}(x)=\rho_{n}(y)$. Then by the induction claim we obtain that $f(x)=f(y)$. Assume that $\sigma$ is of the form ( $T, T^{\prime}$ ), and let $h$ be a valuation such that $h\left(T_{n}^{\prime}\right) \subseteq T_{n}$ and $h$ is the identity on $\operatorname{Val}\left(T_{n}\right) \cap \operatorname{Val}\left(T_{n}^{\prime}\right)$. Then $f_{n} \circ h \circ \rho_{n}\left(T^{\prime}\right) \subseteq r$ where, by the induction claim, $f_{n} \circ h \circ \rho_{n}$ is $f$ on $\operatorname{Val}(T) \cap \operatorname{Val}\left(T^{\prime}\right)$. Hence, we obtain that $r \models \sigma$ in both cases. This concludes the proof.

Proof (Lemma [圄). W.l.o.g. we may assume that no attribute of $R$ appears as a value in the chasing sequence, i.e., $R \cap \bigcup_{i \in \mathbb{N}} \operatorname{Val}\left(\sigma_{i}\right)=\emptyset$. We show the claim by induction on $n$.

The base case. First it suffices to deduce by one application of [CS]

$$
\left(T^{*}, \mathrm{id}\right)[R S] \wedge \bigwedge_{t \in T} t(\boldsymbol{A}) \subseteq \boldsymbol{A}
$$

where $\left(T^{*}, \mathrm{id}\right)[R S]$ is of the form described in (i).
The inductive step. Assuming the claim for $n$, we next show the claim for $n+1$. Assume first that $\sigma_{n+1}$ is obtained from $\sigma_{n}$ by using the egd rule for $(S, x=y) \in \Sigma$ over a valuation $f$ on $S$ such that $f(S) \subseteq T_{n}$ and $f(x) \neq f(y)$. Then $T_{n+1}=g\left(T_{n}\right)$ where $g$ is the identity everywhere except that it maps, say $f(y)$ to $f(x)$. By the induction assumption, it now suffices to consider (ii) and (iii) only in the cases that associate with the construction of $\sigma_{n+1}$.
(ii) The equality $f(x)=f(y)$ can be derived with one application of $[\mathrm{CR}]$, since $f \circ s(\boldsymbol{A}) \subseteq \boldsymbol{A}$, for all $s \in S$, have been deduced by the assumption.
(iii) Let $t \in T_{n+1}$, and let $t^{\prime} \in T_{n}$ be such that $t=g \circ t^{\prime}$. If $f(y) \notin \operatorname{Val}\left(t^{\prime}\right)$, then $t(\boldsymbol{A}) \subseteq \boldsymbol{A}$ has been derived by the induction assumption. Otherwise, $t^{\prime}(A)=f(y)$ for some $A \in R$. Now using repeatedly [EE] to $f(y)=f(x)$ and $t^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}$, we obtain $t(\boldsymbol{A}) \subseteq \boldsymbol{A}$.

Assume then that $\sigma_{n+1}$ is obtained from $\sigma_{n}$ by using the $\operatorname{tgd}$ rule for $\left(S, S^{\prime}\right) \in$ $\Sigma$. W.l.o.g. we may assume that there is only one valuation $f$ on $S$ with the property that $f$ embeds $S$ to $T_{n}$, but no extension of $f$ to $S^{\prime}$ embeds $S^{\prime}$ to $T_{n}^{\prime}$. Let $f^{\prime}$ be the distinct extension associated with this step, i.e., $f^{\prime}$ is an extension of $f$ to $S^{\prime}$ such that each variable in $\operatorname{Val}\left(S^{\prime}\right) \backslash \operatorname{Val}(S)$ is assigned a distinct new value greater than any value appearing in $\sigma_{0}, \ldots, \sigma_{n}$. By the induction assumption, none of these new values appear in the deduction. Hence, by the assumption we may with one application of $[\mathrm{CR}]$ from $\left(S, S^{\prime}\right) \wedge \bigwedge_{s \in S} f^{\prime} \circ s(\boldsymbol{A})$ deduce $\bigwedge_{s^{\prime} \in S^{\prime}} f^{\prime} \circ s^{\prime}(\boldsymbol{A}) \subseteq \boldsymbol{A}$ where all the new values are interpreted as new attributes. Since $T_{n+1}=T_{n} \cup f^{\prime}\left(S^{\prime}\right)$, this concludes item (iii) and thus the proof.


[^0]:    ${ }^{1}$ In a tableau presentation of a dependency $\sigma$, the distinct values of $\sigma$ are sometimes denoted by blank cells.

