Tight Cutoffs for Guarded Protocols with Fairness

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Abstract. Guarded protocols were introduced in a seminal paper by Emerson and Kahlon (2000), and describe systems of processes whose transitions are enabled or disabled depending on the existence of other processes in certain local states. We study parameterized model checking and synthesis of guarded protocols, both aiming at formal correctness arguments for systems with any number of processes. Cutoff results reduce reasoning about systems with an arbitrary number of processes to systems of a determined, fixed size. Our work stems from the observation that existing cutoff results for guarded protocols i) are restricted to closed systems, and ii) are of limited use for liveness properties because reductions do not preserve fairness. We close these gaps and obtain new cutoff results for open systems with liveness properties under fairness assumptions. Furthermore, we obtain cutoffs for the detection of global and local deadlocks, which are of paramount importance in synthesis. Finally, we prove tightness or asymptotic tightness for the new cutoffs.

1 Introduction

Concurrent hardware and software systems are notoriously hard to get correct. Formal methods like model checking or synthesis can be used to guarantee correctness, but the state explosion problem prevents us from using such methods for systems with a large number of components. Furthermore, correctness properties are often expected to hold for an *arbitrary* number of components. Both problems can be solved by *parameterized* model checking and synthesis approaches, which give correctness guarantees for systems with any number of components without considering every possible system instance explicitly.

While parameterized model checking (PMC) is undecidable in general [25], there exist a number of methods that decide the problem for specific classes of systems [12, 14, 16], as well as semi-decision procedures that are successful in many interesting cases [9, 18, 21]. In this paper, we consider the *cutoff* method that can guarantee properties of systems of arbitrary size by considering only systems of up to a certain fixed size, thus providing a decision procedure for PMC if components are finite-state.

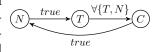
We consider systems that are composed of an arbitrary number of processes, each an instance of a process template from a given, finite set. Process templates can be viewed as synchronization skeletons [11], i.e., program abstractions that

suppress information not necessary for synchronization. In our system model, processes communicate by guarded updates, where guards are statements about other processes that are interpreted either conjunctively ("every other process satisfies the guard") or disjunctively ("there exists a process that satisfies the guard"). Conjunctive guards can model atomic sections or locks, disjunctive guards can model token-passing or to some extent pairwise rendezvous (cf. [13]).

This class of systems has been studied by Emerson and Kahlon [12], and cutoffs that depend on the size of process templates are known for specifications of
the form $\forall \bar{p}. \, \Phi(\bar{p})$, where $\Phi(\bar{p})$ is an LTL\X property over the local states of one
or more processes \bar{p} . Note that this does not allow us to specify fairness assumptions, for two reasons: i) to specify fairness, additional atomic propositions for
enabledness and scheduling of processes are needed, and ii) specifications with
global fairness assumptions are of the form $(\forall \bar{p}. \, fair(\bar{p})) \to (\forall \bar{p}. \, \Phi(\bar{p}))$. Because
neither is supported by [12], the existing cutoffs are of limited use for reasoning
about liveness properties.

Emerson and Kahlon [12] mentioned this limitation and illustrated it using the process template on the figure on the right. Transitions from the initial state

N to the "trying" state T, and from the critical state C to N are always possible, while the transition from T to C is only possible if no other process is in C. The existing cutoff results can be used



to prove safety properties like mutual exclusion for systems composed of arbitrarily many copies of this template. However, they cannot be used to prove starvation-freedom properties like $\forall p$. A $\mathsf{G}(T_p \to \mathsf{F}\,C_p)$, stating that every process p that enters its local state T_p will eventually enter state C_p , because without fairness of scheduling the property does not hold.

Also, Emerson and Kahlon [12] consider only closed systems. Therefore, in this example, processes always try to enter C. In contrast, in open systems the transition to T might be a reaction to a corresponding input from the environment that makes entering C necessary. While it is possible to convert an open system to a closed system that is equivalent under LTL properties, this comes at the cost of a blow-up.

Motivation. Our work is inspired by applications in parameterized synthesis [17], where the goal is to automatically construct process templates such that a given specification is satisfied in systems with an arbitrary number of components. In this setting, one generally considers open systems that interact with an uncontrollable environment, and most specifications contain liveness properties that cannot be guaranteed without fairness assumptions. Also, one is in general interested in synthesizing deadlock-free systems. Cutoffs are essential for parameterized synthesis, and we will show in Sect. 4 how size-dependent cutoffs can be integrated into the parameterized synthesis approach.

Contributions.

 We show that existing cutoffs for model checking of LTL\X properties are in general not sufficient for systems with fairness assumptions, and provide new cutoffs for this case.

- We improve some of the existing cutoff results, and give separate cutoffs for the problem of deadlock detection, which is closely related to fairness.
- We prove tightness or asymptotical tightness for all of our cutoffs, showing that smaller cutoffs cannot exist with respect to the parameters we consider.

Moreover, all of our cutoffs directly support *open systems*, where each process may communicate with an adversarial environment. This makes the blow-up incurred by translation to an equivalent closed system unnecessary. The results presented here are based on a more detailed preliminary version of this paper [4].

2 Related Work

As mentioned, we extend the results of Emerson and Kahlon [12] who study PMC of guarded protocols, but do not support fairness assumptions, nor provide cutoffs for deadlock detection. In [13] they extended their work to systems with limited forms of guards and broadcasts, and also proved undecidability of PMC of conjunctive guarded protocols wrt. LTL (including X), and undecidability wrt. LTL\X for systems with both conjunctive and disjunctive guards.

Bouajjani et al. [7] study parameterized model checking of resource allocation systems (RASs). Such systems have a bounded number of resources, each owned by at most one process at any time. Processes are pushdown automata, and can request resources with high or normal priority. RASs are similar to conjunctive guarded protocols in that certain transitions are disabled unless a processes has a certain resource. RASs without priorities and with processes being finite state automata can be converted to conjunctive guarded protocols (at the price of blow up), but not vice versa. The authors study parameterized model checking wrt. LTL\X properties on arbitrary or on strong-fair runs, and (local or global) deadlock detection. The proof structure resembles that of [12], as does ours.

German and Sistla [16] considered global deadlocks and strong fairness properties for systems with pairwise rendezvous communication in a clique. Emerson and Kahlon [13] have shown that disjunctive guard systems can be reduced to such pairwise rendezvous systems. However, German and Sistla [16] do not provide cutoffs, nor do they consider local deadlocks, and their specifications can talk about one process only. Aminof et al. [3] have recently extended these results to more general topologies, and have shown that for some decidable PMC problems there are no cutoffs, even in cliques.

Emerson and Namjoshi provide cutoffs for systems that pass a valueless token in a ring [14], which is essentially resource allocation of a single resource with a specific allocation scheme. Their results have been extended to more general topologies [2, 10]. All of these results consider fairness of token passing in the sense that every process receives the token infinitely often.

Many of the decidability results above have recently been surveyed by Bloem et al [6]. In addition, there are many methods based on semi-algorithms.

"Dynamic cutoff" approaches [1, 18] support larger classes of systems, and try to find cutoffs for a concrete system and specification. These methods can find smaller cutoffs than those that are statically determined for a whole class of systems and specifications, but are currently limited to safety properties. The invisible invariants method [23] tries to find invariants in small systems, and applies a specialized cutoff result to prove correctness of all instances, including an extension to liveness properties [15].

Finally, there are methods that work completely without cutoffs, like regular model checking [8], network invariants [19, 21, 26], and counter abstraction [24]. They are in general incomplete, but may provide decision procedures for certain classes of systems and specifications, and support liveness to some extent.

3 Preliminaries

3.1 System Model

We consider systems $A||B^n$, usually written $(A,B)^{(1,n)}$, consisting of one copy of a process template A and n copies of a process template B, in an interleaving parallel composition. We distinguish objects that belong to different templates by indexing them with the template. E.g., for process template $U \in \{A, B\}$, Q_U is the set of states of U. For this section, fix two disjoint finite sets Q_A , Q_B as sets of states of process templates A and B, and a positive integer n.

Processes. A process template is a transition system $U = (Q, \mathsf{init}, \Sigma, \delta)$ with

- -Q is a finite set of states including the initial state init,
- $-\Sigma$ is a finite input alphabet,
- $-\delta: Q \times \Sigma \times \mathcal{P}(Q_A \cup Q_B) \times Q$ is a guarded transition relation.

A process template is *closed* if $\Sigma = \emptyset$, and otherwise *open*.

We define the size |U| of a process template $U \in \{A, B\}$ as $|Q_U|$. A copy of template U will be called a U-process. Different B-processes are distinguished by subscript, i.e., for $i \in [1..n]$, B_i is the ith copy of B, and q_{B_i} is a state of B_i . A state of the A-process is denoted by q_A .

For the rest of this subsection, fix templates A and B. We assume that $\Sigma_A \cap \Sigma_B = \emptyset$. We will also write p for a process in $\{A, B_1, \dots, B_n\}$, unless p is specified explicitly.

Disjunctive and Conjunctive Systems. In a system $(A, B)^{(1,n)}$, consider global state $s = (q_A, q_{B_1}, \dots, q_{B_n})$ and global input $e = (\sigma_A, \sigma_{B_1}, \dots, \sigma_{B_n})$. We also write s(p) for q_p , and e(p) for σ_p . A local transition $(q_p, \sigma_p, g, q'_p) \in \delta_U$ of p is enabled for s and e if its guard g is satisfied for p in s, written $(s, p) \models g$. Disjunctive and conjunctive systems are distinguished by the interpretation of guards:

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In disjunctive systems: (s, p) \models g iff \exists p' \in \{A, B_1, \dots, B_n\} \setminus \{p\} : q_{p'} \in g.
In conjunctive systems: (s, p) \models g iff \forall p' \in \{A, B_1, \dots, B_n\} \setminus \{p\} : q_{p'} \in g.
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Note that we check containment in the guard (disjunctively or conjunctively) only for local states of processes different from p. A process is enabled for s and e if at least one of its transitions is enabled for s and e, otherwise it is disabled.

Like Emerson and Kahlon [12], we assume that in conjunctive systems init_A and init_B are contained in all guards, i.e., they act as neutral states. Furthermore, we call a conjunctive system 1-conjunctive if every guard is of the form $(Q_A \cup Q_B) \setminus \{q\}$ for some $q \in Q_A \cup Q_B$.

Then, $(A, B)^{(1,n)}$ is defined as the transition system $(S, \mathsf{init}_S, E, \Delta)$ with

- set of global states $S = (Q_A) \times (Q_B)^n$,
- global initial state $init_S = (init_A, init_B, \dots, init_B),$
- set of global inputs $E = (\Sigma_A) \times (\Sigma_B)^n$,
- and global transition relation $\Delta \subseteq S \times E \times S$ with $(s, e, s') \in \Delta$ iff
 - i) $s = (q_A, q_{B_1}, \dots, q_{B_n}),$
 - ii) $e = (\sigma_A, \sigma_{B_1}, \dots, \sigma_{B_n})$, and
 - iii) s' is obtained from s by replacing one local state q_p with a new local state q'_p , where p is a U-process with local transition $(q_p, \sigma_p, g, q'_p) \in \delta_U$ and $(s, p) \models g$.

We say that a system $(A, B)^{(1,n)}$ is of type (A, B). It is called a *conjunctive system* if guards are interpreted conjunctively, and a *disjunctive system* if guards are interpreted disjunctively. A system is *closed* if all of its templates are closed. We often denote the set $\{B_1, ..., B_n\}$ as \mathcal{B} .

Runs. A configuration of a system is a triple (s,e,p), where $s \in S$, $e \in E$, and p is either a system process, or the special symbol \bot . A path of a system is a configuration sequence $x=(s_1,e_1,p_1),(s_2,e_2,p_2),\ldots$ such that for all m<|x| there is a transition $(s_m,e_m,s_{m+1})\in \Delta$ based on a local transition of process p_m . We say that process p_m moves at moment m. Configuration (s,e,\bot) appears iff all processes are disabled for s and e. Also, for every p and m<|x|: either $e_{m+1}(p)=e_m(p)$ or process p moves at moment m. That is, the environment keeps input to each process unchanged until the process can read it. p

A system run is a maximal path starting in the initial state. Runs are either infinite, or they end in a configuration (s, e, \bot) . We say that a run is *initializing* if every process that moves infinitely often also visits its init infinitely often.

Given a system path $x = (s_1, e_1, p_1), (s_2, e_2, p_2), \ldots$ and a process p, the *local* path of p in x is the projection $x(p) = (s_1(p), e_1(p)), (s_2(p), e_2(p)), \ldots$ of x onto local states and inputs of p. Similarly define the projection on two processes p_1, p_2 denoted by $x(p_1, p_2)$.

Deadlocks and Fairness. A run is globally deadlocked if it is finite. An infinite run is locally deadlocked for process p if there exists m such that p is disabled for all $s_{m'}, e_{m'}$ with $m' \geq m$. A run is deadlocked if it is locally or globally deadlocked. A system has a (local/global) deadlock if it has a (locally/globally) deadlocked run. Note that absence of local deadlocks for all p implies absence of global deadlocks, but not the other way around.

¹ By only considering inputs that are actually processed, we approximate an action-based semantics. Paths that do not fulfill this requirement are not very interesting, since the environment can violate any interesting specification that involves input signals by manipulating them when the corresponding process is not allowed to move.

A run $(s_1, e_1, p_1), (s_2, e_2, p_2), ...$ is unconditionally-fair if every process moves infinitely often. A run is strong-fair if it is infinite and for every process p, if p is enabled infinitely often, then p moves infinitely often. We will discuss the role of deadlocks and fairness in synthesis in Sect. 4.

Remark 1. Why do we consider systems $A||B^n$? Emerson and Kahlon [12] showed how to generalize cutoffs for such systems to systems of the form $A^m||B^n$, and further to systems with an arbitrary number of process templates $U_1^{n_1}||\dots||U_m^{n_m}$. This generalization also works for our new results, except for the cutoffs for deadlock detection that are restricted to 1-conjunctive systems (see Section 5).

3.2 Specifications

Fix templates (A,B). We consider formulas in LTL\X, i.e., LTL without the next-time operator X. Let $h(A,B_{i_1},\ldots,B_{i_k})$ be an LTL\X formula over atomic propositions from $Q_A \cup \Sigma_A$ and indexed propositions from $(Q_B \cup \Sigma_B) \times \{i_1,\ldots,i_k\}$. For a system $(A,B)^{(1,n)}$ with $n \geq k$ and $i_j \in [1..n]$, satisfaction of A $h(A,B_{i_1},\ldots,B_{i_k})$ and E $h(A,B_{i_1},\ldots,B_{i_k})$ is defined in the usual way (see e.g. [5]).

Parameterized Specifications. A parameterized specification is a temporal logic formula with indexed atomic propositions and quantification over indices. We consider formulas of the forms $\forall i_1, \ldots, i_k$. A $h(A, B_{i_1}, \ldots, B_{i_k})$ and $\forall i_1, \ldots, i_k$. E $h(A, B_{i_1}, \ldots, B_{i_k})$. For given $n \geq k$,

$$(A,B)^{(1,n)} \models \forall i_1,...,i_k. \, \mathsf{A} \, h(A,B_{i_1},...,B_{i_k})$$

iff

$$(A,B)^{(1,n)} \models \bigwedge_{j_1 \neq ... \neq j_k \in [1..n]} \mathsf{A} \, h(A,B_{j_1},...,B_{j_k}).$$

By symmetry of guarded protocols, this is equivalent (cp. [12]) to $(A, B)^{(1,n)} \models A h(A, B_1, ..., B_k)$. The latter formula is denoted by $A h(A, B^{(k)})$, and we often use it instead of the original $\forall i_1, ..., i_k$. $A h(A, B_{i_1}, ..., B_{i_k})$. For formulas with path quantifier E, satisfaction is defined analogously, and equivalent to satisfaction of $E h(A, B^{(k)})$.

Specification of Fairness and Local Deadlocks. It is often convenient to express fairness assumptions and local deadlocks as parameterized specifications. To this end, define auxiliary atomic propositions move_p and en_p for every process p of system $(A,B)^{(1,n)}$. At moment m of a given run $(s_1,e_1,p_1),(s_2,e_2,p_2),\ldots$, let move_p be true whenever $p_m=p$, and let en_p be true if p is enabled for s_m,e_m . Note that we only allow the use of these propositions to define fairness, but not in general specifications. Then, an infinite run is

- local-deadlock-free if it satisfies $\forall p. \mathsf{GFen}_p$, abbreviated as $\Phi_{\neg dead}$,
- strong-fair if it satisfies $\forall p. \mathsf{GFen}_p \to \mathsf{GFmove}_p$, abbreviated as Φ_{strong} , and
- unconditionally-fair if it satisfies $\forall p. \mathsf{GF} \mathsf{move}_p$, abbreviated as Φ_{uncond} .

If fair is a fairness notion and $Ah(A, B^{(k)})$ a specification, then we write $A_{fair} h(A, B^{(k)})$ for $A(\Phi_{fair} \to h(A, B^{(k)}))$. Similarly, we write $E_{fair} h(A, B^{(k)})$ for $E(\Phi_{fair} \land h(A, B^{(k)}))$.

3.3 Model Checking and Synthesis Problems

For a given system $(A, B)^{(1,n)}$ and specification $h(A, B^{(k)})$ with $n \ge k$,

- the model checking problem is to decide whether $(A, B)^{(1,n)} \models A h(A, B^{(k)}),$
- the deadlock detection problem is to decide whether $(A, B)^{(1,n)}$ does not have global nor local deadlocks,
- the parameterized model checking problem (PMCP) is to decide whether $\forall m \geq n : (A, B)^{(1,m)} \models \mathsf{A}\,h(A, B^{(k)}),$ and
- the parameterized deadlock detection problem is to decide whether for all $m \geq n$, $(A, B)^{(1,m)}$ does not have global nor local deadlocks.

For a given number $n \in \mathbb{N}$ and specification $h(A, B^{(k)})$ with $n \geq k$,

- the template synthesis problem is to find process templates A, B such that $(A, B)^{(1,n)} \models A h(A, B^{(k)})$ and $(A, B)^{(1,n)}$ does not have global deadlocks.
- the bounded template synthesis problem for a pair of bounds $(b_A, b_B) \in \mathbb{N} \times \mathbb{N}$ is to solve the template synthesis problem with $|A| \leq b_A$ and $|B| \leq b_B$.
- the parameterized template synthesis problem is to find process templates A, B such that $\forall m \geq n : (A, B)^{(1,m)} \models \mathsf{A}\,h(A, B^{(k)})$ and $(A, B)^{(1,m)}$ does not have global deadlocks.

These definitions can be flavored with different notions of fairness (and similarly for the E path quantifier). In the next section we clarify the problems studied.

4 Reduction Method and Challenges

We show how to use existing cutoff results of Emerson and Kahlon [12] to reduce the PMCP to a standard model checking problem, and parameterized synthesis to template synthesis. We note the limitations of the existing results that are crucial in the context of synthesis.

Reduction by Cutoffs. A *cutoff* for a system type (A, B) and a specification Φ is a number $c \in \mathbb{N}$ such that:

$$\forall n \ge c : \left((A, B)^{(1,n)} \models \Phi \iff (A, B)^{(1,c)} \models \Phi \right).$$

Similarly, $c \in \mathbb{N}$ is a cutoff for (local/global) deadlock detection if $\forall n \geq c$: $(A, B)^{(1,n)}$ has a (local/global) deadlock iff $(A, B)^{(1,c)}$ has a (local/global) deadlock. For the systems and specifications presented in this paper, cutoffs can be computed from the size of process template B and the number k of copies of B mentioned in the specification, and are given as expressions like |B| + k + 1.

Remark 2. Our definition of a cutoff is different from that of Emerson and Kahlon [12], and instead similar to, e.g., Emerson and Namjoshi [14]. The reason is that we want the following property to hold for any (A, B) and Φ : if n_0 is the smallest number such that $\forall n \geq n_0$: $(A, B)^{(1,n)} \models \Phi$, then any $c < n_0$ is not a cutoff, any $c \geq n_0$ is a cutoff.

We call n_0 the *tight* cutoff. The definition in [12, page 2] requires that $\forall n \leq c.(A, B)^{(1,n)} \models \Phi$ if and only if $\forall n \geq 1 : (A, B)^{(1,n)} \models \Phi$, and thus allows stating $c < n_0$ as a cutoff if Φ does not hold for all n.

In model checking, a cutoff allows us to check whether any "big" system satisfies the specification by checking it in the cutoff system. As noted by Jacobs and Bloem [17], a similar reduction applies to the parameterized synthesis problem. For guarded protocols, we obtain the following semi-decision procedure for parameterized synthesis:

- 0. set initial bound (b_A, b_B) on size of process templates;
- 1. determine cutoff for (b_A, b_B) and Φ ;
- 2. solve bounded template synthesis problem for cutoff, size bound, and Φ ;
- 3. if successful return (A, B) else increase (b_A, b_B) and goto (1).

Existing Cutoff Results. Emerson and Kahlon [12] have shown:

Theorem 1 (Disjunctive Cutoff Theorem). For closed disjunctive systems, |B|+2 is a cutoff (\dagger) for formulas of the form $Ah(A,B^{(1)})$ and $Eh(A,B^{(1)})$, and for global deadlock detection.

Theorem 2 (Conjunctive Cutoff Theorem). For closed conjunctive systems, 2|B| is a cutoff $^{(\dagger)}$ for formulas of the form Ah(A) and Eh(A), and for global deadlock detection. For formulas of the form $Ah(B^{(1)})$ and $Eh(B^{(1)})$, 2|B|+1 is a cutoff.

Remark 3. $^{(\dagger)}$ Note that Emerson and Kahlon [12] proved these results for a different definition of a cutoff (see Remark 2). Their results also hold for our definition, except possibly for global deadlocks. For the latter case to hold with the new cutoff definition, one also needs to prove the direction "global deadlock in the cutoff system implies global deadlock in a large system" (later called Monotonicity Lemma). In Sect. 6.3 and 7.3 we prove these lemmas for the case of general deadlock (global or local).

Challenge: Open Systems. For any open system S there exists a closed system S' such that S and S' cannot be distinguished by LTL specifications (cp. Manna and Pnueli [22]). Thus, one approach to PMC for open systems is to use a translation between open and closed systems, and then use the existing cutoff results for closed systems.

While such an approach works in theory, it might not be feasible in practice: since cutoffs depend on the size of process templates, and the translation blows up the process template, it also blows up the cutoffs. Thus, cutoffs that directly support open systems are important.

Challenge: Liveness and Deadlocks under Fairness. We are interested in cutoff results that support liveness properties. In general, we would like to consider only runs where all processes move infinitely often, i.e., use the unconditional fairness assumption $\forall p$. GF move_p. However, this would mean that we accept all systems that always go into a local deadlock, since then the assumption is violated. This is especially undesirable in synthesis, because the synthesizer usually tries to violate the assumptions in order to satisfy the specification. To avoid this, we require the absence of local deadlocks under the strong

fairness assumption $\forall p.(\mathsf{GFen}_p \to \mathsf{GFmove}_p)$. Since strong fairness and absence of local deadlocks imply unconditional fairness, we can then use the latter as an assumption for the original specification.

In summary, for a parameterized specification Φ , we consider satisfaction of

"all runs are infinite"
$$\wedge A_{strong} \Phi_{\neg dead} \wedge A_{uncond} \Phi$$
.

This is equivalent to "all runs are infinite" $\land A_{strong}(\Phi_{\neg dead} \land \Phi)$, but by considering the form above we can separate the tasks of deadlock detection and of model checking LTL\X-properties, and obtain modular cutoffs.

In the following, we present cutoffs for problems of the forms (i) $A_{uncond} \Phi$, (ii) $A_{strong} \Phi_{\neg dead}$ and no global deadlocks (and the variants with E path quantifier).

5 New Cutoff Results

We present new cutoff results that extend Theorems 1 and 2, summarized in the table below. We distinguish between disjunctive and conjunctive systems, nonfair and fair executions, as well as between the satisfaction of LTL\X properties $h(A, B^{(k)})$ and the existence of deadlocks. All results hold for open systems, and for both path quantifiers A and E. Cutoffs depend on the size of process template B and the number $k \geq 1$ of B-processes a property talks about:

	$h(A, B^{(k)})$	deadlock detection	$h(A, B^{(k)})$	deadlock detection
	no fairness	no fairness	uncond. fairness	strong fairness
Disjunctive	B + k + 1	2 B - 1	2 B + k - 1	2 B - 1
Conjunctive	k+1	$2 B - 2 \ (*)$	$k + 1 \; (*)$	2 B - 2 (*)

Results marked with a (*) are for a restricted class of systems: For conjunctive systems with fairness, we require infinite runs to be initializing, i.e., all non-deadlocked processes return to init infinitely often.² Additionally, the cutoffs for deadlock detection in conjunctive systems only support 1-conjunctive systems. The reason for this restriction will be explained in Remark 4.

All cutoffs in the table are tight – no smaller cutoff can exist for this class of systems and properties – except for the case of deadlock detection in disjunctive systems without fairness. There, the cutoff is asymptotically tight, i.e., it must increase linearly with the size of the process template.

Proof Structure

To justify the entries in the table, we first recapitulate the proof structure of the original Theorems 1 and 2. The proofs are based on two lemmas, Monotonicity

² This assumption is in the same flavor as the restriction that init_A and init_B appear in all conjunctive guards. Intuitively, the additional restriction makes sense since conjunctive systems model shared resources, and everybody who takes a resource should eventually release it.

and Bounding. We give some basic proof ideas of the lemmas from [12] and mention extensions to the cases with fairness and deadlock detection. For cases where this extension is not easy, we will introduce additional proof techniques and explain how to use them in Sections 6 and 7. Note that we only consider properties of the form $h(A, B^{(1)})$ — the proof ideas extend to general properties $h(A, B^{(k)})$ without difficulty. Similarly, in most cases the proof ideas extend to open systems without major difficulties — mainly because when we construct a simulating run, we have the freedom to choose the input that is needed. Only for the case of deadlock detection we have to handle open systems explicitly.

1) Monotonicity lemma: if a behavior is possible in a system with $n \in \mathbb{N}$ copies of B, then it is also possible in a system with one additional process:

$$(A,B)^{(1,n)} \models \mathsf{E}\, h(A,B^{(1)}) \implies (A,B)^{(1,n+1)} \models \mathsf{E}\, h(A,B^{(1)}),$$

and if a deadlock is possible in $(A, B)^{(1,n)}$, then it is possible in $(A, B)^{(1,n+1)}$.

Proof ideas. The lemma is easy to prove for properties $\mathsf{E}\,h(A,B^{(1)})$ in both disjunctive and conjunctive systems, by letting the additional process stay in its initial state init_B forever (cp. [12]). This cannot disable transitions with disjunctive guards, as these check for existence of a local state in another process (and we do not remove any processes), and it cannot disable conjunctive guards since they contain init_B by assumption. However, this construction violates fairness, since the new process never moves. This can be resolved in the disjunctive case by letting the additional process mimic all transitions of an existing process. But in general this does not work in conjunctive systems (due to the non-reflexive interpretation of guards). For this case and for deadlock detection, the proof is not trivial and may only work for $n \geq c$, for some lower bound $c \in \mathbb{N}$ (see Sect. 6, 7).

2) Bounding lemma: for a number $c \in \mathbb{N}$, a behavior is possible in a system with c copies of B if it is possible in a system with $n \geq c$ copies of process B:

$$(A,B)^{(1,c)} \models \mathsf{E}\, h(A,B^{(1)}) \iff (A,B)^{(1,n)} \models \mathsf{E}\, h(A,B^{(1)}),$$

and a deadlock is possible in $(A, B)^{(1,c)}$ if it is possible in $(A, B)^{(1,n)}$.

Proof ideas. For disjunctive systems, the main difficulty is that removing processes might falsify guards of the local transitions of A or B_1 in a given run (see Sect. 6). For conjunctive systems, removing processes from a run is easy for the case of infinite runs, since a transition that was enabled before cannot become disabled. Here, the difficulty is in preserving deadlocks, because removing processes may enable processes that were deadlocked before (Sect. 7).

6 Proof Techniques for Disjunctive Systems

6.1 LTL\X Properties without Fairness: Existing Constructions

We revisit the main technique of the original proof of Theorem 1 [12]. It constructs an infinite run y of $(A, B)^{(1,c)}$ with $y \models h(A, B^{(1)})$, based on an infinite

run x of $(A, B)^{(1,n)}$ with n > c and $x \models h(A, B^{(1)})$. The idea is to copy local runs x(A) and $x(B_1)$ into y, and construct runs of other processes in a way that enables all transitions along x(A) and $x(B_1)$. The latter is achieved with the flooding construction.

Flooding Construction [12]. Given a run $x = (s_1, e_1, p_1), (s_2, e_2, p_2) \dots$ of $(A, B)^{(1,n)}$, let $\mathsf{Visited}_{\mathcal{B}}(x)$ be the set of all local states visited by B-processes in x, i.e., $\mathsf{Visited}_{\mathcal{B}}(x) = \{q \in Q_B \mid \exists m \exists i. \ s_m(B_i) = q\}$.

For every $q \in \mathsf{Visited}_{\mathcal{B}}(x)$ there is a local run of $(A, B)^{(1,n)}$, say $x(B_i)$, that visits q first, say at moment m_q . Then, saying that process B_{i_q} of $(A, B)^{(1,c)}$ floods q means:

$$y(B_{i_q}) = x(B_i)[1:m_q](q)^{\omega}.$$

In words: the run $y(B_{i_q})$ is the same as $x(B_i)$ until moment m_q , and after that the process never moves.

The construction achieves the following. If we copy local runs of A and B_1 from x to y, and in y for every $q \in \mathsf{Visited}_{\mathcal{B}}(x)$ introduce one process that floods q, then: if in x at some moment m there is a process in state q', then in y at moment m there will also be a process (different from A and B_1) in state q'. Thus, every transition of A and B_1 , which is enabled at moment m in x, will also be enabled in y.

Proof idea of the bounding lemma. The lemma for disjunctive systems without fairness can be proved by copying local runs x(A) and $x(B_1)$, and flooding all states in $\mathsf{Visited}_{\mathcal{B}}(x)$. To ensure that at least one process moves infinitely often in y, we copy one additional (infinite) local run from x. Finally, it may happen that the resulting collection of local runs violates the interleaving semantics requirement. To resolve this, we add stuttering steps into local runs whenever two or more processes move at the same time, and we remove global stuttering steps in y. Since the only difference between $x(A, B_1)$ and $y(A, B_1)$ are stuttering steps, y and x satisfy the same $\mathsf{LTL} \backslash \mathsf{X}$ -properties $h(A, B^{(1)})$. Since $|\mathsf{Visited}_{\mathcal{B}}(x)| \leq |B|$, we need at most 1 + |B| + 1 copies of B in $(A, B)^{(1,c)}$.

6.2 LTL\X Properties with Fairness: New Constructions

The flooding construction does not preserve fairness, and also cannot be used to construct deadlocked runs since it does not preserve disabledness of transitions of processes A or B_1 . For these cases, we provide new proof constructions.

Consider the proof task of the bounding lemma for disjunctive systems with fairness: given an unconditionally fair run x of $(A, B)^{(1,n)}$ with $x \models h(A, B^{(1)})$, we want to construct an unconditionally fair run y of $(A, B)^{(1,c)}$ with $y \models h(A, B^{(1)})$. In contrast to unfair systems, we need to ensure that all processes move infinitely often in y. The insight is that after a finite time all processes will start looping around some set Visited^{inf} of states. We construct a run y that mimics this. To this end, we introduce two constructions. Flooding with evacuation is similar to flooding, but instead of keeping processes in their flooding states forever it evacuates the processes into Visited^{inf}. Fair extension lets all processes move infinitely often without leaving Visited^{inf}.

Flooding with Evacuation. Given a subset $\mathcal{F} \subseteq \mathcal{B}$ and an infinite run x = $(s_1, e_1, p_1) \dots$ of $(A, B)^{(1,n)}$, define

Visited
$$_{\mathcal{F}}^{inf}(x) = \{q \mid \exists \text{ infinitely many } m: s_m(B_i) = q \text{ for some } B_i \in \mathcal{F} \}$$
 (1)

Visited_F^{fin}(x) = {q |
$$\exists$$
 only finitely many $m: s_m(B_i) = q$ for some $B_i \in \mathcal{F}$ } (2)

Let $q \in \mathsf{Visited}_{\mathcal{F}}^{\mathit{fin}}(x)$. In run x there is a moment f_q when q is reached for the first time by some process from \mathcal{F} , denoted B_{first_q} . Also, in run x there is a moment l_q such that: $s_{l_q}(B_{\mathsf{last}_q}) = q$ for some process $B_{\mathsf{last}_q} \in \mathcal{F}$, and $s_t(B_i) \neq q$ for all $B_i \in \mathcal{F}$, $t > l_q$ —i.e., when some process from \mathcal{F} is in state q for the last time in x. Then, saying that process B_{i_q} of $(A, B)^{(1,c)}$ floods $q \in \mathsf{Visited}_{\mathcal{F}}^{fn}(x)$ and then evacuates into Visited $_{\mathcal{F}}^{inf}(x)$ means:

$$y(B_{i_q}) = x(B_{\mathsf{first}_q})[1:f_q] + (q)^{(l_q - f_q + 1)} + x(B_{\mathsf{last}_q})[l_q:m] + (q')^{\omega},$$

where q' is the state in $\mathsf{Visited}_{\mathcal{F}}^{inf}(x)$ that $x(B_{\mathsf{last}_q})$ reaches first, at some moment $m \geq l_q$. In words, process B_{i_q} mimics process B_{first_q} until it reaches q, then does nothing until process B_{last_q} starts leaving q, then it mimics B_{last_q} until it reaches Visited $_{\mathcal{F}}^{inf}(x)$.

The construction ensures: if we copy local runs of all processes not in \mathcal{F} from x to y, then all transitions of y are enabled. This is because: for any process p of $(A,B)^{(1,c)}$ that takes a transition in y at any moment, the set of states visible to process p is a superset of the set of states visible to the original process in $(A,B)^{(1,n)}$ whose transitions process p copies.

Fair Extension. Here, we consider a path x that is the postfix of an unconditionally fair run x' of $(A,B)^{(1,n)}$, starting from the moment where no local states from $\mathsf{Visited}^{fin}_{\mathcal{B}}(x')$ are visited anymore. We construct a corresponding unconditionally-fair path y of $(A, B)^{(1,c)}$, where no local states from Visited^{fin}_B(x')

Formally, let $n \geq 2|B|$, and x an unconditionally-fair path of $(A,B)^{(1,n)}$ such that Visited $_{\mathcal{B}}^{fin}(x) = \emptyset$. Let $c \geq 2|B|$, and s'_1 a state of $(A, B)^{(1,c)}$ with

- $\begin{array}{l} -\ s_1'(A_1)=s_1(A_1),\ s_1'(B_1)=s_1(B_1)\\ -\ \text{for every}\ q\in \mathsf{Visited}_{B_2..B_n}^{inf}(x)\backslash \mathsf{Visited}_{B_1}^{inf}(x),\ \text{there are two processes}\ B_{i_q},B_{i_q'} \end{array}$ of $(A, B)^{(1,c)}$ that start in q, i.e., $s'_1(B_{i_q}) = s'_1(B_{i'_q}) = q$
- for every $q \in \mathsf{Visited}_{B_2..B_n}^{inf}(x) \cap \mathsf{Visited}_{B_1}^{inf}(x)$, there is one process B_{i_q} of $(A,B)^{(1,c)}$ that starts in q
- for some $q^* \in \mathsf{Visited}_{B_2...B_n}^{inf}(x) \cap \mathsf{Visited}_{B_1}^{inf}(x)$, there is one additional process of $(A,B)^{(1,c)}$, different from any in the above, called $B_{i'_{q^*}}$, that starts in q^* .
- any other process B_i of $(A, B)^{(1,c)}$ starts in some state of Visited $_{B_2...B_n}^{inf}(x)$.

Note that if $\mathsf{Visited}_{B_2...B_n}^{inf}(x) \cap \mathsf{Visited}_{B_1}^{inf}(x) = \emptyset$, then the third and fourth prerequisites are trivially satisfied.

The fair extension extends state s'_1 of $(A, B)^{(1,c)}$ to an unconditionally-fair path $y = (s'_1, e'_1, p'_1) \dots$ with $y(A_1, B_1) = x(A_1, B_1)$ as follows:

- (a) $y(A_1) = x(A_1), y(B_1) = x(B_1)$
- (b) for every $q \in \mathsf{Visited}_{B_2..B_n}^{inf}(x) \setminus \mathsf{Visited}_{B_1}^{inf}(x)$: in run x there is $B_i \in \{B_2..B_n\}$ that starts in q and visits it infinitely often. Let B_{i_q} and $B_{i'_q}$ of $(A,B)^{(1,c)}$ mimic B_i in turns: first B_{i_q} mimics B_i until it reaches q, then $B_{i'_q}$ mimics B_i until it reaches q, and so on.
- (c) arrange states of Visited $_{B_2..B_n}^{inf}(x) \cap \text{Visited}_{B_1}^{inf}(x)$ in some order (q^*, q_1, \ldots, q_l) . The processes $B_{i'_{q^*}}, B_{i_{q^*}}, B_{i_{q_1}}, \ldots, B_{i_{q_l}}$ behave as follows. Start with $B_{i'_{q^*}}$: when B_1 enters q^* in y, it carries $^3B_{i'_{q^*}}$ from q^* to q_1 , then carries $B_{i_{q_1}}$ from q_1 to q_2, \ldots , then carries $B_{i_{q_l}}$ from q_l to q^* , then carries $B_{i_{q^*}}$ from q_1 to q_2 , then carries $B_{i_{q_1}}$ from q_2 to q_3 , and so on.
- (d) any other B_i of $(A, B)^{(1,c)}$, starting in $q \in \mathsf{Visited}_{B_2...B_n}^{inf}(x)$, mimics B_{i_q} .

Note that parts (b) and (c) of the constrution ensure that there is always at least one process in every state from $\mathsf{Visited}_{B_2..B_n}^{inf}(x)$. This ensures that the guards of all transitions of the construction are satisfied. Excluding processes in (d), the fair extension uses up to 2|B| copies of B.

Proof idea of the bounding lemma. Let c = 2|B|. Given an unconditionally-fair run x of $(A, B)^{(1,n)}$ we construct an unconditionally-fair run y of the cutoff system $(A, B)^{(1,c)}$ such that $y(A, B_1)$ is stuttering equivalent to $x(A, B_1)$.

Note that in x there is a moment m such that all local states that are visited after m are in $\mathsf{Visited}_{\mathcal{B}}^{inf}(x)$.

The construction has two phases. In the first phase, we apply flooding for states in Visited_B^{inf}(x), and flooding with evacuation for states in Visited_B^{fin}(x):

- (a) $y(A) = x(A), y(B_1) = x(B_1)$
- (b) for every $q \in \mathsf{Visited}_{B_2..B_n}^{inf}(x) \setminus \mathsf{Visited}_{B_1}^{inf}(x)$, devote two processes of $(A,B)^{(1,c)}$ that flood q
- (c) for some $q^* \in \mathsf{Visited}_{B_2..B_n}^{inf}(x) \cap \mathsf{Visited}_{B_1}^{inf}(x)$, devote one process of $(A,B)^{(1,c)}$ that floods q^*
- (d) for every $q \in \mathsf{Visited}_{B_2..B_n}^{fin}(x)$, devote one process of $(A,B)^{(1,c)}$ that floods q and evacuates into $\mathsf{Visited}_{B_2..B_n}^{inf}(x)$
- (e) let other processes (if any) mimic process B_1

The phase ensures that at moment m in y, there are no processes in Visited^{fin}_{\mathcal{B}}(x), and all the pre-requisites of the fair extension are satisfied.

The second phase applies the fair extension, and then establishes the interleaving semantics as in the bounding lemma in the non-fair case. The overall construction uses up to 2|B| copies of B.

³ "Process B_1 starting at moment m carries process B_i from q to q'" means: process B_i mimics the transitions of B_1 starting at moment m at q until B_1 first reaches q'.

⁴ A careful reader may notice that if $|\mathsf{Visited}_{B_1}^{inf}(x)| = 1$ and $|\mathsf{Visited}_{B_2...B_n}^{inf}(x)| = |B|$, then the construction uses 2|B| + 1 copies of B. But one can slightly modify the construction for this special case, and remove process $B_{i',\star}$ from the pre-requisites.

6.3 Detection of Local and Global Deadlocks: New Constructions

Monotonicity Lemmas. The lemma for deadlock detection, for fair and unfair cases, is proven for $n \geq |B| + 1$. In the case of local deadlocks, process B_{n+1} mimics a process that moves infinitely often in x. In the case of global deadlocks, by pigeon hole principle, in the global deadlock state there is a state q with at least two processes in it—let process B_{n+1} mimic a process that deadlocks in q.

Bounding Lemmas. For the case of global deadlocks, fairness does not affect the proof of the bounding lemma. The insight is to divide deadlocked local states into two disjoint sets, dead_1 and dead_2 , as follows. Given a globally deadlocked run x of $(A, B)^{(1,n)}$, for every $q \in \mathsf{dead}_1$, there is a process of $(A, B)^{(1,n)}$ deadlocked in q with input i, that has an outgoing transition guarded " $\exists q$ " – hence, adding one more process into q would unlock the process. In contrast, $q \in \mathsf{dead}_2$ if any process deadlocked in q stays deadlocked after adding more processes into q. Let us denote the set of B-processes deadlocked in dead_1 by \mathcal{D}_1 . Finally, abuse the definition in Eq. 2 and denote by $\mathsf{Visited}_{\mathcal{B}\backslash\mathcal{D}_1}^{fin}(x)$ the set of states that are visited by B-processes not in \mathcal{D}_1 before reaching a deadlocked state.

Given a globally deadlocked run x of $(A, B)^{(1,n)}$ with $n \ge 2|B| - 1$, we construct a globally deadlocked run y of $(A, B)^{(1,c)}$ with c = 2|B| - 1 as follows:

- copy from x into y the local runs of processes in $\mathcal{D}_1 \cup \{A\}$
- flood every state of dead₂
- for every $q \in \mathsf{Visited}_{\mathcal{B} \setminus \mathcal{D}_1}^{fin}(x)$, flood q and evacuate into dead_2 .

The construction ensures: (1) for any moment and any process in y, the set of local states that are visible to the process includes all the states that were visible to the corresponding process in $(A, B)^{(1,n)}$ whose transitions we copy; (2) in y, there is a moment when all processes deadlock in $\mathsf{dead}_1 \cup \mathsf{dead}_2$.

For the case of local deadlocks, the construction is similar but slightly more involved, and needs to distinguish between unfair and fair cases. In the unfair case, we also copy the behaviour of an infinitely moving process. In the strong-fair case, we continue the runs of non-deadlocked processes with the fair extension.

7 Proof Techniques for Conjunctive Systems

7.1 LTL\X Properties without Fairness: Existing Constructions

Recall that the Monotonicity lemma is proven by keeping the additional process in the initial state. To prove the bounding lemma, Emerson and Kahlon [12] suggest to simply copy the local runs x(A) and $x(B_1)$ into y. In addition, we may need one more process that moves infinitely often to ensure that an infinite run of $(A, B)^{(1,n)}$ will result in an infinite run of $(A, B)^{(1,c)}$. All transitions of copied processes will be enabled because removing processes from a conjunctive system cannot disable a transition that was enabled before.

7.2 LTL\X Properties with Fairness: New Constructions

The proof of the Bounding lemma is the same as in the non-fair case, noting that if the original run is unconditional-fair, then so will be the resulting run.

Proving the Monotonicity lemma is more difficult, since the fair extension construction from disjunctive systems does not work for conjunctive systems – if an additional process mimics the transitions of an existing process then it disables transitions of the form $q \stackrel{\text{"} \forall \neg q^{"}}{\to} q'$ or $q \stackrel{\text{"} \forall \neg q^{"}}{\to} q'$. Hence, we add the restriction of initializing runs, which allows us to construct a fair run as follows. The additional process B_{n+1} "shares" a local run $x(B_i)$ with an existing process B_i of $(A, B)^{(1,n+1)}$: one process stutters in init_B while the other makes transitions from $x(B_i)$, and whenever $x(B_i)$ enters init_B (this happens infinitely often), the roles are reversed. Since this changes the behavior of B_i , B_i should not be mentioned in the formula, i.e., we need $n \geq 2$ for a formula $h(A, B^{(1)})$.

7.3 Detection of Local and Global Deadlocks: New Constructions

Monotonicity lemmas for both fair and unfair cases are proven by keeping process B_{n+1} in the initial state, and copying the runs of deadlocked processes. If the run of $(A, B)^{(1,n)}$ is globally deadlocked, then process B_{n+1} may keep moving in the constructed run, i.e., it may only be locally deadlocked. In case of a local deadlock in $(A, B)^{(1,n)}$, distinguish two cases: there is an infinitely moving B-process, or all B-processes are deadlocked (and thus A moves infinitely often). In the latter case, use the same construction as in the global deadlock case (the correctness argument uses the fact that systems are 1-conjunctive, runs are initializing, and there is only one process of type A). In the former case, copy the original run, and let B_{n+1} share a local run with an infinitely moving B-process.

Bounding lemma (no fairness). In the case of global deadlock detection, Emerson and Kahlon [12] suggest to copy a subset of the original local runs. For every local state q that is present in the final state of the run, we need at most two local runs that end in this state. In the case of local deadlocks, our construction uses the fact that systems are 1-conjunctive. In 1-conjunctive systems, if a process is deadlocked, then there is a set of states DeadGuards that all need to be populated by other processes in order to disable all transitions of the deadlocked process. Thus, the construction copies: (i) the local run of a deadlocked process, (ii) for each $q \in DeadGuards$, the local run of a process that is in q at the moment of the deadlock, and (iii) the local run of an infinitely moving process.

Bounding lemma (strong fairness). We use a construction that is similar to that of properties under fairness for disjunctive systems (Sect. 6.2): in the setup phase, we populate some "safe" set of states with processes, and then we extend the runs of non-deadlocked processes to satisfy strong fairness, while ensuring that deadlocked processes never get enabled.

Let $c = 2|Q_B \setminus \{\text{init}_B\}|$. Let $x = (s_1, e_1, p_1) \dots$ be a locally deadlocked strong-fair initializing run of $(A, B)^{(1,n)}$ with n > c. We construct a locally deadlocked strong-fair initializing run y of $(A, B)^{(1,c)}$.

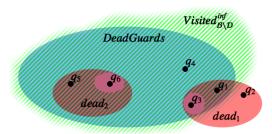


Fig. 1: Bounding lemma (strong fairness): Venn diagram for dead_1 , dead_2 , $\mathsf{Dead}\mathsf{Guards}$, $\mathsf{Visited}^{inf}_{\mathcal{B}\backslash\mathcal{D}}(x)$. States $q_1,...,q_6$ are to illustrate that the corresponding sets may be non-empty. E.g., in x, a process may be deadlocked in $q_1 \in (\mathsf{Dead}\mathsf{Guards} \cap \mathsf{dead}_1 \cap \mathsf{Visited}^{inf}_{\mathcal{B}\backslash\mathcal{D}}(x))$, and another process in $q_3 \in \mathsf{dead}_1 \cap \mathsf{Dead}\mathsf{Guards} \backslash \mathsf{Visited}^{inf}_{\mathcal{B}\backslash\mathcal{D}}(x)$.

Let $\mathcal{D} \subseteq \mathcal{B}$ be the set of deadlocked B-processes in x. Let d be the moment in x starting from which every process in \mathcal{D} is deadlocked. Let $\mathsf{dead}(x)$ be the set of states in which processes \mathcal{D} of $(A,B)^{(1,n)}$ are deadlocked. Let $\mathsf{dead}_2(x) \subseteq \mathsf{dead}(x)$ be the set of deadlocked states such that: for every $q \in \mathsf{dead}_2(x)$, there is a process $B_i \in \mathcal{D}$ with $s_d(B_i) = q$ and that for input $e_{\geq d}(B_i)$ has a transition guarded with " $\forall \neg q$ ". Thus, a process in q is deadlocked with $e_d(B_i)$ only if there is another process in q in every moment $\geq d$. Let $\mathsf{dead}_1(x) = \mathsf{dead}(x) \setminus \mathsf{dead}_2(x)$. Define DeadGuards to be the set

 $\{ q \mid \exists B_i \in \mathcal{D} \text{ with a transition guarded "} \forall \neg q \text{" in } (s_d(B_i), e_d(B_i)) \}.$

Figure 1 illustrates properties of sets DeadGuards, $dead_1$, $dead_2$, $Visited_{\mathcal{B}\backslash\mathcal{D}}^{inf}(x)$. In the **setup phase**, we copy from x into y:

- the local run of A;
- for every $q \in \mathsf{dead}_1$, the local run of one process deadlocked in q;
- for every $q \in \mathsf{dead}_2$, the local runs of two⁵ processes deadlocked in q;
- for every $q \in DeadGuards \setminus dead$, the local run of a process that reaches q after moment d.
- Finally, we keep one B-process in $init_B$ until moment d.

The setup phase ensures: in every state $q \in \mathsf{dead}$, there is at least one process deadlocked in q at moment d in y. Now we need to ensure that the non-deadlocked processes in $DeadGuards \setminus \mathsf{dead}$ and init_B move infinitely often, which is done using the looping extension described bellow.

Order arbitrarily $DeadGuards \setminus dead = (q_1, ..., q_k) \subseteq Visited_{\mathcal{B}\setminus\mathcal{D}}^{inf}(x)$. Let $\mathcal{P} \subseteq \{B_1, ..., B_c\}$ be the non-deadlocked processes of $(A, B)^{(1,c)}$ that we moved into $(q_1, ..., q_k) \cup \{init_B\}$ in the setup phase. Note that $|\mathcal{P}| = |(q_1, ..., q_k)| + 1$.

⁵ Strictly speaking, in x we might not have two deadlocked processes in a state in $dead_2$ – one process may be deadlocked, others enter and exit the state infinitely often. In such case, there is always a non-deadlocked process in the state. Then, copy the local run of such infinitely moving process until it enters the deadlocked state, and then deadlock it by providing the same input as the deadlocked process receives.

The **looping phase** is: set i = 1, and repeat infinitely the following.

- let $B_{\mathsf{init}} \in \mathcal{P}$ be the process of $(A, B)^{(1,c)}$ that is currently in init_B , and $B_{q_i} \in \mathcal{P}$ be the process of $(A, B)^{(1,c)}$ that is currently in q_i
- let $\tilde{B}_{q_i} \in \mathsf{Visited}_{\mathcal{B}\setminus\mathcal{D}}^{inf}(x)$ be a process of $(A,B)^{(1,n)}$ that visits q_i and init_B infinitely often. Let B_{init} of $(A,B)^{(1,c)}$ copy transitions of \tilde{B}_{q_i} on some path $\mathsf{init}_B \to \ldots \to q_i$, then let B_{q_i} copy transitions of \tilde{B}_{q_i} on some path $q_i \to \ldots \to \mathsf{init}_B$. For copying we consider only the paths of \tilde{B}_{q_i} that happen after moment d.
- $-i=i\oplus 1$

Remark 4. In 1-conjunctive systems, the set DeadGuards is "static", i.e., there is always at least one process in each state of DeadGuards starting from the moment of the deadlock. In contrast, in general conjunctive systems where guards can overlap, there is no such set. However, there is a similar set of sets of states, such that one state from each set always needs to be populated to ensure the deadlock.

8 Conclusion

We have extended the cutoff results for guarded protocols of Emerson and Kahlon [12] to support local deadlock detection, fairness assumptions, and open systems. In particular, our results imply decidability of the parameterized model checking problem for this class of systems and specifications, which to the best of our knowledge was unknown before. Furthermore, the cutoff results can easily be integrated into the parameterized synthesis approach [17, 20].

Since conjunctive guards can model atomic sections and read-write locks, and disjunctive guards can model pairwise rendezvous (for some classes of specifications, cp. [13]), our results apply to a wide spectrum of systems models. But the expressivity of the model comes at a high cost: cutoffs are linear in the size of a process, and are shown to be tight (with respect to this parameter). For conjunctive systems, our new results are restricted to systems with 1-conjunctive guards, effectively only allowing to model a single shared resource. We conjecture that our proof methods can be extended to systems with more general conjunctive guards, at the price of even bigger cutoffs. We leave this extension and the question of finding cutoffs that are independent of the size of processes for future research.

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A Additional Definitions and Notation

For a global state s of system $(A, B)^{(1,n)}$ and a local state q (of template A or B), we write $q \in s$ as shorthand for $\exists p \in \{A, B_1, ..., B_n\} s(p) = q$.

For a sequence $x = x_1, x_2, \ldots$ denote the subsequence between the *i*th and *j*th element of the sequence as $x[i:j] = x_i, \ldots, x_j$.

By $q_i \stackrel{e:g}{\to} q_j$ denote a process transition from q_i to q_j for input e and guarded by guard g. We skip the input e and guard g if they are not important or can be inferred from the context.

Given system state s, let Set(s) be the set $\{q \mid \exists p : s(p) = q\}$.

B Cutoffs for Disjunctive Systems

B.1 Disjunctive Systems without Fairness

Lemma 1 (Monotonicity: Disj, Properties, Unfair). For disjunctive systems:

$$\forall n \ge 1$$
:
 $(A,B)^{(1,n)} \models \mathsf{E} \, h(A,B_1) \ \Rightarrow \ (A,B)^{(1,n+1)} \models \mathsf{E} \, h(A,B_1).$

Proof. Given run x of $(A, B)^{(1,n)}$ we construct a run y of $(A, B)^{(1,n+1)}$: copy x into y and keep the additional process in the initial state.

Lemma 2 (Bounding: Disj, Properties, Unfair). For disjunctive systems:

$$\forall n \ge |B| + 2: (A, B)^{(1,|B|+2)} \models \mathsf{E} \, h(A, B_1) \iff (A, B)^{(1,n)} \models \mathsf{E} \, h(A, B_1).$$

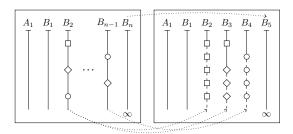
The proof is from [12, Lemma 4.1.2]. We recapitulate it to introduce the notion of "a process floods a state", destutter, *interleave*, and "process mimics another process" which are used in our proofs later.

Proof. Let c = |B| + 2 and $n \ge c$. Let $x = (s_1, e_1, p_1), (s_2, e_2, p_2)...$ be a run of $(A, B)^{(1,n)}$ that satisfies $\mathsf{E}\, h(A, B_1)$. We construct a run y of the cutoff system $(A, B)^{(1,c)}$ with $y(A, B_1) \simeq x(A, B_1)$.

Let $\mathsf{Visited}(x)$ be the set of all visited states by B-processes in run x: $\mathsf{Visited}(x) = \{q \mid \exists m \exists i : s_m(B_i) = q\}.$

Construct the run y of $(A, B)^{(1,c)}$ as follows:

- a. copy runs of A and B_1 from x to y: y(A) = x(A), $y(B_1) = x(B_1)$
- b. x is infinite, hence it has at least one infinitely moving process, denoted B_{∞} . Devote one unique process B_{∞} in $(A,B)^{(1,c)}$ that copies the behaviour of B_{∞} of $(A,B)^{(1,n)}$: $y(B_{\infty})=x(B_{\infty})$.
- c. for every $q \in \text{Visited}$ there is a process of $(A, B)^{(1,n)}$, denoted B_i , that visits q first, at moment denoted m_q . Then devote one unique process in $(A, B)^{(1,c)}$, denoted B_{i_q} , that floods q: set $y(B_{i_q}) = x(B_i)[1:m_q](q)^{\omega}$. In words: the run $y(B_{i_q})$ repeats exactly that of $x(B_i)$ till moment m_q , after which the process is never scheduled.
- d. let any other process B_i of $(A, B)^{(1,c)}$ not used in the previous steps (if any) mimic the behavior of B_1 of $(A, B)^{(1,c)}$: $y(B_i) = y(B_1)$.



The figure illustrates the construction. The correctness follows from the observation that any transition of any process at any moment m of y was done by some process in x at moment m and hence is enabled. Also note that if ≥ 2 processes transit simultaneously in y, then the guards of their transitions will be enabled even if both of them are removed from the state space. Note that it is possible that in y:

- more than one process transits at the same moment. Then, *interleave* the transitions of such processes, namely arbitrarily sequentialize them.
- at some moment no processes move. Then remove elements of the run y the resulting run is denoted destutter(y).

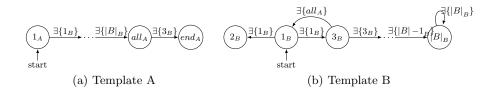
This construction uses $|Visited| + 2 \le |B| + 2$ copies of B (ignoring case (d)).

Tightness 1 (Disj, Props, Unfair). The cutoff in Lemma 2 is tight, i.e., for any k there exist process templates (A, B) with |B| = k and $LTL \setminus X$ formula $h(A, B_1)$ such that:

$$(A,B)^{(1,|B|+2)} \models \mathsf{E}\,h(A,B_1) \ \ and \ \ (A,B)^{(1,|B|+1)} \not\models \mathsf{E}\,h(A,B_1).$$

Proof. The idea of the proof relies on the subtleties of the definition of a run: it is infinite (thus not globally deadlocked), and in each step of a run exactly one process moves.

Consider the templates in the figure below and let $\mathsf{E}\,h(A,B_1) = \mathsf{E}(\mathsf{F}\,3_{B_1} \wedge \mathsf{F}\,\mathsf{G}(2_{B_1} \wedge end_A))$. In words: there exists a run in a system where process B_1 visits 3_B and process B_1 with A eventually always stay in 2_B and end_A .



We need one process in every state of B to enable the transitions of A to all_A . Only when A in all_A , B_1 can move $3_B \to 1_B$, and then at some point to 2_B . After B_1 moves $3_B \to 1_B$, A moves $all_A \to end_A$ which requires process $B_{i\neq 1}$ in 3_B . Finally, to make the run infinite there should be at least two processes in $|B|_B$.

Lemma 3 (Monotonicity: Disj, Deadlocks, Unfair). For disjunctive systems:

$$\forall n > |B| + 1 : (A,B)^{(1,n)}$$
 has a deadlock $\Rightarrow (A,B)^{(1,n+1)}$ has a deadlock

Proof. Given a deadlocked run x of $(A,B)^{(1,n)}$ we build a deadlocked run of $(A,B)^{(1,n+1)}$. If the run x is locally deadlocked, then it has at least one infinitely moving process, thus let the additional process mimic that process. If the run x is globally deadlocked run, then due to n > |B| in some state there are at least two processes deadlocked. Thus, let the new process mimic a process deadlocked in that state – the run constructed will also be globally deadlocked.

Lemma 4 (Bounding: Disj, Deadlocks, Unfair). For disjunctive systems:

- with c = |B| + 2 and any n > c:

$$(A,B)^{(1,c)}$$
 has a local deadlock $\Leftarrow (A,B)^{(1,n)}$ has a local deadlock

- with c = 2|B| - 1 and any n > c

$$(A,B)^{(1,c)}$$
 has a global deadlock $\Leftarrow (A,B)^{(1,n)}$ has a global deadlock

- with c = 2|B| - 1 and any n > c:

$$(A,B)^{(1,c)}$$
 has a deadlock $\Leftarrow (A,B)^{(1,n)}$ has a deadlock

Proof. Given a (globally or locally) deadlocked run of $(A, B)^{(1,n)}$ we construct (globally or locally) deadlocked run of $(A, B)^{(1,c)}$, where c depends on the nature of the given run. We do this using the construction template.

Let $\mathcal{B} = \{B_1, ..., B_n\}$. The template depends on set $\mathcal{C} \subseteq \{B_1, ..., B_c\}$:

- a. set y(A) = x(A)
- b. for every $B_i \in \mathcal{C}$, set $y(B_i) = x(B_i)$
- c. for every $q \in \mathsf{Visited}_{\mathcal{BC}}^{inf}(x)$, devote one process of $(A,B)^{(1,c)}$ that floods q
- d. for every $q \in \mathsf{Visited}_{\mathcal{BC}}^{\widehat{fin}}(x)$, devote one process of $(A,B)^{(1,c)}$ that floods q and then evacuates into $\mathsf{Visited}_{\mathcal{BC}}^{inf}(x)$
- e. let other processes (if any) mimic some process from (c)

1) Local Deadlock. We distinguish three cases:

- 1a) A deadlocks, B_1 moves infinitely often
- 1b) A moves infinitely often, B_1 deadlocks
- 1c) A neither deadlocks nor moves infinitely often, B_1 deadlocks, B_2 moves infinitely often.

1a: "A deadlocks, B_1 moves infinitely often".

Let c = |B| + 1, and $C = \{B_1\}$. Note that $\mathsf{Visited}_{B_2..B_n}^{inf}(x) \neq \emptyset$. The resulting construction uses $|\mathsf{Visited}_{B_2..B_n}^{fin}(x)| + |\mathsf{Visited}_{B_2..B_n}^{inf}(x)| + 1 \leq |B| + 1$ copies of B.

1b: "A moves infinitely often, B_1 deadlocks".

Let c = |B| + 1, and $C = \{B_1\}$. Let q_{\perp} be the state in which B_1 deadlocks. Instantiate the construction template.

Process B_1 of $(A, B)^{(1,c)}$ is deadlocked in y starting from some moment d, because any state it sees (in $\mathsf{Visited}_{A,B_2..B_n}^{inf}(x)$) was also seen by B_1 in $(A, B)^{(1,n)}$ in x at some moment $d' \geq d$ (note that d' may be not the same moment as d).

1c: "A neither deadlocks nor moves infinitely often, B_1 deadlocks, B_2 moves infinitely often".

Instantiate the construction template with c = |B| + 2 and $C = \{B_1, B_2\}$.

Finally, |B| + 2 is a (possibly not tight) cutoff for local deadlock detection problem.

2) Global Deadlock. Let $x = (s_1, e_1, p_1)...(s_d, e_d, \bot)$ be a globally deadlocked run of $(A, B)^{(1,n)}$ with $n \ge c$.

Let us abuse the definition of Visited^{inf}_{\mathcal{F}}(x) and Visited^{finf}_{\mathcal{F}}(x), in Eq. 1 and 2 resp., and adapt it to the case of finite runs. To this end, given a finite run $x = (s_1, e_1, p_1)...(s_d, e_d, \bot)$, extend it to the infinite sequence $(s_1, e_1, p_1)...(s_d, e_d, \bot)^{\omega}$, and apply the definition of Visited^{inf}_{\mathcal{F}}(x) and Visited^{finf}_{\mathcal{F}}(x) to the sequence.

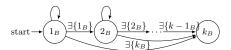
Let \mathcal{D}_1 be the set of processes deadlocked in unique states: $\forall p \in \mathcal{D}_1 /\exists p' \neq p$: $s_d(p') = s_d(p)$. Instantiate the construction template with $\mathcal{C} = \mathcal{D}_1$ and c = 2|B| - 1.

3) Deadlocks. As the cutoff for the deadlock detection problem we take the largest cutoff in (1)-(2), namely, 2|B|-1, but it may be not tight – finding the tight cutoffs for local deadlock and for deadlock detection problems is an open problem.

Tightness 2 (Disj, Deadlocks, Unfair). The cutoff c = 2|B| - 1 for deadlock detection in disjunctive systems is asymptotically optimal but possibly not tight, i.e.: for any k there are templates (A, B) with |B| = k such that:

$$(A,B)^{(1,|B|-1)}$$
 does not have a deadlock, but $(A,B)^{(1,|B|)}$ does.

Proof. The figure below illustrates templates (A, B) to prove the asymptotical optimality of cutoff 2|B|-1 for deadlock detection problem. Template A is any that never deadlocks. The system has a local deadlock only when there are at least |B| copies of B, which is a constant factor of 2|B|-1.



B.2 Disjunctive Systems with Fairness

Lemma 5 (Monotonicity: Disj, Props, Fair). For disjunctive systems:

$$\forall n \ge 1:$$

$$(A,B)^{(1,n)} \models \mathsf{E}_{uncond} \, h(A,B_1) \implies (A,B)^{(1,n+1)} \models \mathsf{E}_{uncond} \, h(A,B_1),$$

⁶ 2|B|-1 copies is enough, because: $\mathsf{Visited}_{\mathcal{BC}}^{fin}(x) \cap \mathsf{Visited}_{\mathcal{BC}}^{inf}(x) = \emptyset$, $\mathsf{Visited}_{\mathcal{BC}}^{inf}(x) \cap \mathsf{Visited}_{\mathcal{BC}}^{inf}(x) = \emptyset$, and if $\mathsf{Visited}_{\mathcal{BC}}^{fin}(x) \neq \emptyset$, then $\mathsf{Visited}_{\mathcal{BC}}^{inf}(x) \neq \emptyset$.

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Proof. In run x of $(A, B)^{(1,n)}$ with $n \ge 1$ all processes move infinitely often. Hence let the run y of $(A, B)^{(1,n+1)}$ copy x, and let the new process mimic an infinitely moving B process of $(A, B)^{(1,n)}$.

Lemma 6 (Bounding: Disj, Props, Fair). For disjunctive systems:

$$\forall n > 2|B|:$$

$$(A,B)^{(1,2|B|)} \models \mathsf{E}_{uncond} \, h(A,B_1) \qquad \Longleftrightarrow \qquad (A,B)^{(1,n)} \models \mathsf{E}_{uncond} \, h(A,B_1),$$

The proof was given in the main text, in Section 6.2.

Tightness 3 (Disj, Props, Fair). The cutoff in Lemma 6 is tight, i.e., for any k there exist process templates (A, B) with |B| = k and $LTL \setminus X$ formula $h(A, B_1)$ such that:

$$(A,B)^{(1,2|B|)} \models \mathsf{E}\,h(A,B_1) \ \ and \ \ (A,B)^{(1,2|B|-1)} \not\models \mathsf{E}\,h(A,B_1).$$

The proof was described in the main text, in Section 6.2.

Proof. Consider process templates A, B in the figure below and property Etrue. \Box

$$\operatorname{start} \longrightarrow \underbrace{\begin{pmatrix} 1_A \end{pmatrix} \exists \{1_B\} }_{\text{(a)}} \cdots \exists \{k_B\} \underbrace{\begin{pmatrix} all_A \end{pmatrix}}_{\text{(a)}} \operatorname{start} \longrightarrow \underbrace{\begin{pmatrix} 1_B \end{pmatrix} \exists \{1_B\} \\ 2_B \end{pmatrix}}_{\text{(b)}} \exists \{2_B\} \cdots \exists \{k-1_B\} \underbrace{\begin{pmatrix} all_A \end{pmatrix} }_{\text{(b)}} \operatorname{Template B}$$

Lemma 7 (Monotonicity: Disj, Deadlocks, Fair). For disjunctive systems, on strong-fair or finite runs:

$$\forall n \geq |B| + 1 : (A, B)^{(1,n)} \text{ has a deadlock } \Rightarrow (A, B)^{(1,n+1)} \text{ has a deadlock}$$

Proof. See proof of Lemma 3.

Lemma 8 (Bounding: Disj, Deadlocks, Fair). For disjunctive systems, on strong-fair or finite runs:

- with
$$c = 2|B| - 1$$
 and any $n > c$:

$$(A,B)^{(1,c)}$$
 has a local deadlock $\Leftarrow (A,B)^{(1,n)}$ has a local deadlock

$$- with c = 2|B| - 1 and any n > c$$

$$(A,B)^{(1,c)}$$
 has a global deadlock $\Leftarrow (A,B)^{(1,n)}$ has a global deadlock

- with
$$c = 2|B| - 1$$
 and any $n > c$:

$$(A,B)^{(1,c)}$$
 has a deadlock $\Leftarrow (A,B)^{(1,n)}$ has a deadlock

Proof. If $(A,B)^{(1,n)}$ has a global deadlock, then the fairness does not influence the cutoff, and the proof from Lemma 4, case "Global Deadlocks", applies giving the cutoff 2|B|-1. Hence below consider only the case of local deadlocks.

Given a strong-fair deadlocked run x of $(A, B)^{(1,n)}$, we first construct a strong-fair deadlocked run y of $(A,B)^{(1,c)}$ with c=2|B| and then argue that c can be reduced to 2|B|-1. The construction is similar to that in Lemma 4 – the differences originate from the need to infinitely move non deadlocked processes.

Let $dead_{<2}(x)$ be the set of deadlocked states in the run x that are only deadlocked if there is no other process in the same state, and let \mathcal{D}_1 be the set of processes deadlocked in the run x in $dead_{<2}(x)$. Let $dead_2(x)$ be the set of states that are deadlocked in the run x even if there is another process in the same state.

Notes:

- $|\mathcal{D}_1| = |\mathsf{dead}_{<2}(x)| \le |B|$
- $\operatorname{\mathsf{dead}}_{<2}(x) \cap \operatorname{\mathsf{dead}}_2(x) = \emptyset$
- Visited $_{\mathcal{B},\mathcal{D}_1}^{fin}(x) \cap \mathsf{dead}_{\leq 2}(x) \neq \emptyset$ is possible, because a state from Visited $_{\mathcal{B},\mathcal{D}_1}^{fin}(x)$ can first be visited by a process in $\mathcal{B}\setminus\mathcal{D}_1$, and later deadlocked by the process in \mathcal{D}_1 . - $\mathsf{dead}_2(x)\subseteq\mathsf{Visited}_{\mathcal{B}\setminus\mathcal{D}_1}^{inf}(x)$, and hence $\mathsf{Visited}_{\mathcal{B}\setminus\mathcal{D}_1}^{fin}(x)\cap\mathsf{dead}_2(x)=\emptyset$.

The construction has two phases. The first phase:

- a. for every $p \in \{A\} \cup \mathcal{D}_1$, set y(p) = x(p)
- b. for every $q \in \mathsf{dead}_2(x)$, devote one process of $(A, B)^{(1,c)}$ that floods it
- c. for every $q \in \mathsf{Visited}_{\mathcal{B},\mathcal{D}_1}^{inf}(x) \backslash \mathsf{dead}_2(x)$, devote two processes of $(A,B)^{(1,c)}$ that flood
- d. for every $q \in \mathsf{Visited}_{\mathcal{B}\backslash\mathcal{D}_1}^{\mathit{fin}}(x),$ devote one process of $(A,B)^{(1,c)}$ that floods it and then evacuates into $\mathsf{Visited}_{\mathcal{B} \setminus \mathcal{D}_1}^{inf}(x)$
- e. let other processes (if any) mimic some process from (c)

After this phase all B processes will be in $\mathsf{Visited}^{inf}_{\mathcal{B}\backslash\mathcal{D}_1}(x) \cup \mathsf{dead}_{<2}(x)$. The second phase applies to processes in $\mathsf{Visited}^{inf}_{\mathcal{B}\backslash\mathcal{D}_1}(x)\backslash\mathsf{dead}_2(x)$ the fair extension⁷. How many processes does the construction use? Note that the sets $\mathsf{dead}_{<2}(x) \cup$ Visited $_{\mathcal{B}\backslash\mathcal{D}_1}^{fin}(x)$, dead $_2(x)$, Visited $_{\mathcal{B}\backslash\mathcal{D}_1}^{inf}(x)\backslash \mathsf{dead}_2(x)$ are disjoint, thus:

$$|\mathsf{Visited}^{fin}_{\mathcal{B}\backslash\mathcal{D}_1}(x)| + |\mathsf{dead}_{<2}(x)| + |\mathsf{dead}_2(x)| + 2|\mathsf{Visited}^{inf}_{\mathcal{B}\backslash\mathcal{D}_1}(x)\backslash\mathsf{dead}_2(x)| \leq (3)$$

$$2|\mathsf{Visited}^{\mathit{fin}}_{\mathcal{B}\backslash\mathcal{D}_1}(x) \cup \mathsf{dead}_{<2}(x)| + |\mathsf{dead}_2(x)| + 2|\mathsf{Visited}^{\mathit{inf}}_{\mathcal{B}\backslash\mathcal{D}_1}(x) \setminus \mathsf{dead}_2(x)| \leq \qquad (4)$$

$$|B| + |\mathsf{Visited}_{\mathcal{B}\mathcal{D}_1}^{\mathit{fin}}(x) \cup \mathsf{dead}_{\leq 2}(x)| + |\mathsf{Visited}_{\mathcal{B}\mathcal{D}_1}^{\mathit{inf}}(x) \setminus \mathsf{dead}_2(x)| \leq 2|B|$$

Let us reduce the estimate to $\leq 2|B|-1$:

- assume that $dead_2(x) = \emptyset$ (otherwise, Eq.3 and the sets disjointness give 2|B|-1)
- assume that $\mathsf{Visited}_{\mathcal{B},\mathcal{D}_1}^{fin}(x) \neq \emptyset$ (the other case together with eq.4, the sets disjointness, and the first item gives 2|B|-1)

 $^{^{7}}$ The fair extension requires run x to be unconditionally-fair, but here we have a run in which all processes that are not deadlocked move infinitely often. To adapt the construction to this case: copy local runs of processes $\{A\} \cup \mathcal{D}_1$, and do not extend local runs of processes that are in state in $dead_2$.

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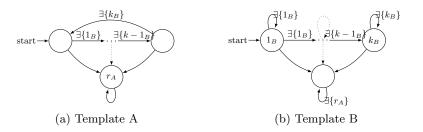
- hence, the construction in step (d) evacuates the process in $q \in \mathsf{Visited}_{\mathcal{B}\backslash\mathcal{D}_1}^{fin}(x)$ into $\mathsf{Visited}_{\mathcal{B}\backslash\mathcal{D}_1}^{inf}(x)\backslash\mathsf{dead}_2(x)$. Hence modify step (c) of the construction and for q devote a single process of $(A,B)^{(1,c)}$ that floods it. This will give $\leq 2|B|-1$.

This concludes the proof.

Tightness 4 (Disj, Deadlocks, Fair). The cutoff c = 2|B| - 1 for deadlock detection in disjunctive systems on strong-fair or finite runs is tight, i.e.: for any k there are templates (A, B) with |B| = k such that:

$$(A,B)^{(1,2|B|-2)}$$
 does not have a deadlock, but $(A,B)^{(1,2|B|-1)}$ does.

Proof. The figure below shows process templates (A, B) such that any system $(A, B)^{(1,n)}$ with $n \leq 2|B| - 2$ does not deadlock on strong-fair runs, but larger systems do. \square



C Cutoffs for Conjunctive Systems

C.1 Conjunctive Systems without Fairness

Lemma 9 (Monotonicity: Conj. Props, Unfair). For conjunctive systems,

$$\forall n \ge 1: (A,B)^{(1,n)} \models \mathsf{E} \, h(A,B_1) \ \Rightarrow \ (A,B)^{(1,n+1)} \models \mathsf{E} \, h(A,B_1).$$

Proof. Let the new process stutter in init state.

Lemma 10 (Bounding: Conj, Props, Unfair). For conjunctive systems,

$$\forall n \geq 2: (A,B)^{(1,2)} \models \mathsf{E} \, h(A,B_1) \iff (A,B)^{(1,n)} \models \mathsf{E} \, h(A,B_1).$$

Proof. The proof is inspired by the first part of the proof of [12, Lemma 5.2].

Let $x = (s_1, e_1, p_1)(s_2, e_2, p_2) \dots$ be a run of $(A, B)^{(1,n)}$. Note that by the semantics of conjunctive guards, the transitions along any local run of x will also be enabled in any system $(A, B)^{(1,c)}$ with $c \leq n$, where the processes exhibit a subset of the local runs of x. Thus, we obtain a run of $(A, B)^{(1,c)}$ by copying a subset of the local runs of x, and removing elements of the new global run where all processes stutter.

Then, based on an infinite run x of the original system, we construct an infinite run y of the cutoff system. Let y(A) = x(A) and $y(B_1) = x(B_1)$. The second copy of

template B in $(A, B)^{(1,2)}$ is needed to ensure that the run y is infinite, i.e., at least one process moves infinitely often. If both x(A) and $x(B_1)$ eventually deadlock, then there exists a process B_i of $(A, B)^{(1,n)}$ that makes infinitely many moves, and we set $y(B_2) = x(B_i)$. Otherwise, we set $y(B_2) = x(B_2)$.

Tightness 5 (Conj, Props, Unfair). The cutoff c = 2 is tight for parameterized model checking of properties $\mathsf{E}\,h(A,B_1)$ in the 1-conjunctive systems, i.e., there is a system type (A,B) and property $Eh(A,B_1)$ which is not satisfied by $(A,B)^{(1,1)}$ but is by $(A,B)^{(1,2)}$.

Proof. The figure below shows templates (A, B), $\mathsf{E}\,h(A, B_1) = \mathsf{E}\,\mathsf{F}\,b$. An infinite run that satisfies the formula needs one copy of B that stays in the initial state, and one that moves into b.



Lemma 11 (Monotonicity: Conj, Deadlocks, Unfair). For conjunctive systems:

$$\forall n \geq 1 : (A,B)^{(1,n)} \text{ has a deadlock } \Rightarrow (A,B)^{(1,n+1)} \text{ has a deadlock}$$

Proof. Given a deadlocked run x of $(A,B)^{(1,n)}$, we construct a deadlocked run of $(A,B)^{(1,n+1)}$. Let y copy run x, and keep the new process in init. If x is globally deadlocked and d is the moment when the deadlock happens in x, then schedule the new process arbitrarily after moment d.

Lemma 12 (Bounding: 1-Conj, Deadlocks, Unfair). For 1-conjunctive systems:

- with $c = 2|Q_B \setminus \{\text{init}\}|$ and any $n > c^{-8}$
 - $(A,B)^{(1,c)}$ has a global deadlock $\Leftarrow (A,B)^{(1,n)}$ has a global deadlock
- with $c = |Q_B \setminus \{\text{init}\}| + 2$ and any n > c:
 - $(A,B)^{(1,c)}$ has a local deadlock $\Leftarrow (A,B)^{(1,n)}$ has a local deadlock
- with $c = 2|Q_B \setminus \{\text{init}\}|$ and any n > c:

$$(A,B)^{(1,c)}$$
 has a deadlock $\Leftarrow (A,B)^{(1,n)}$ has a deadlock

Proof. The proof is inspired by the second part of the proof of [12, Lemma 5.2], but in addition to global we consider local deadlocks.

Global Deadlocks. Let $c=2|Q_B\setminus\{\mathsf{init}\}|$). Let run $x=(s_1,e_1,p_1)\dots(s_d,e_d,\bot)$ of $(A,B)^{(1,n)}$ with n>c be globally deadlocked. We construct a globally deadlocked run y in $(A,B)^{(1,c)}$:

⁸ This statement also applies to systems without restriction to 1-conjunctive guards.

- a. for every $q \in Set(s_d) \setminus \{init\}$:
 - if s_d has two processes in state q, then devote two processes of $(A, B)^{(1,c)}$ that mimic the behaviour of the two of $(A, B)^{(1,n)}$ correspondingly
 - otherwise, s_d has only one process in state q, then devote one process of $(A, B)^{(1,c)}$ that mimics the process of $(A, B)^{(1,n)}$
- b. for any process of $(A, B)^{(1,c)}$ not used in the construction (if any): let it mimic an arbitrary *B*-process of $(A, B)^{(1,n)}$ not used in the construction (including (b))

The construction uses (if ignore (b)) $\leq 2|Q_B \setminus \{\text{init}\}|$ processes B. Note that the proof does not assume that the system is 1-conjunctive.

Local Deadlocks. Let $c = |Q_B \setminus \{\text{init}\}| + 2$. Let run $x = (s_1, e_1, p_1) \dots$ of $(A, B)^{(1,n)}$ with n > c be locally deadlocked. We will construct a run y of $(A, B)^{(1,c)}$ where at least one process deadlocks and exactly one process moves infinitely often.

Wlog. we distinguish three cases:

- 1. A moves infinitely often in x, and B_1 deadlocks
- 2. A deadlocks, and B_1 moves infinitely often
- 3. A neither deadlocks nor moves infinitely often, B_1 deadlocks, B_2 moves infinitely often
- **1.** "A moves infinitely often in x, and B_1 deadlocks".

Let q_{\perp}, e_{\perp} be the deadlocked state and input of B_1 in x, and let d be the moment from which B_1 is deadlocked.

Let $DeadGuards = \{q_1, \ldots, q_k\}$ be the set of states such that for every $q_i \in DeadGuards$ there is an outgoing transitions from q_{\perp} with e_{\perp} guarded " $\forall \neg q_i$ ", and assume $DeadGuards \neq \emptyset$ (if it is empty, then we keep every process in init until someone reaches q_{\perp} and then schedule the rest arbitrarily). (Recall that $q_i \in Q_B \cup Q_A$).

The construction is:

- a. $y(A) = x(A), y(B_1) = x(B_1)$
- b. for each $q \in DeadGuards$, at moment d in x there is a process p_q in state q. If $p_q \in \{B_1, ..., B_n\}$, then let one process of $(A, B)^{(1,c)}$ mimic it till moment d, and then stutter in q.
- c. let other processes of $(A, B)^{(1,c)}$ (if any) stay in init.

The construction uses (if ignore (c)) $\leq |Q_B \setminus \{\text{init}\}| + 1 \text{ processes } B$.

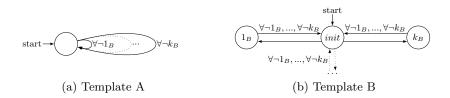
Note: the assumption of 1-conjunctive systems implies that, in order to deadlock B_1 , we need a process in each state in BlockGuards. This implies that having a process in each state of BlockGuards does not disable any A's transition after moment d.

- **2.** "A deadlocks, and B_1 moves infinitely often": use the construction from (1).
- **3.** "A neither deadlocks nor moves infinitely often, B_1 deadlocks, B_2 moves infinitely often". Use the construction from (1), and additionally: $y(B_2) = x(B_2)$. Thus, the construction uses (if ignore (c)) $\leq |Q_B \setminus \{\text{init}\}| + 2$ processes B.

Deadlocks. Take the higher value among the cases considered above $c = 2|Q_B\{\{\text{init}\}\}|$: if x is locally deadlocked then the monotonicity lemma ensures that there is a deadlocked run in $(A, B)^{(1,c)}$.

Tightness 6 (1-Conj, Deadlocks, Unfair). The cutoff c = 2|B| - 2 is tight for parameterized deadlock detection in the 1-conjunctive systems, i.e., for any k there is a system type (A, B) with |B| = k such that there is a deadlock in $(A, B)^{(1,2|B|-2)}$, but not in $(A, B)^{(1,2|B|-3)}$.

Proof. The figure below provides templates (A, B) that proves the observation. In the figure the edge with $\forall \neg b_1, \dots, \forall \neg b_k$ denotes edges with guards $\forall \neg b_1, \dots, \forall \neg b_k$. To get the global deadlock we need at least two processes in each $b_i \in \{b_1, \dots, b_k\}$. Note that the system does not have local deadlocks.



C.2 Conjunctive Systems with Fairness

In this section, subscript i in path quantifiers, E_i and A_i , denotes the quantification over initializing runs.

Lemma 13 (Monotonicity: Conj, Props, Fair). For unconditionally-fair initializing runs of conjunctive systems:

$$\forall n \geq 2:$$

$$(A,B)^{(1,n)} \models \mathsf{E}_{uncond,i} \, h(A,B_1) \ \Rightarrow \ (A,B)^{(1,n+1)} \models \mathsf{E}_{uncond,i} \, h(A,B_1).$$

Proof. Given a unconditionally-fair initializing run x of $(A, B)^{(1,n)}$, we construct a unconditionally-fair initializing run y in $(A, B)^{(1,n+1)}$, with one additional process p. First, copy all local runs of all processes of $(A, B)^{(1,n)}$ from the run x into y. Then, let process p' stutter in init until some other process $p \neq B_1$ enters init. Then, exchange the roles of processes p' and p: let p stutter in init, while p' takes the transitions of p from the original run, until it enters init. And so on. In this way, we continue to interleave the run between p' and p, and obtain a unconditionally-fair initializing run for all processes, with $y(A, B_1) = x(A, B_1)$. Thus, if $(A, B)^{(1,n)} \models \mathsf{E} h(A, B_1)$, then $(A, B)^{(1,n+1)} \models \mathsf{E} h(A, B_1)$.

Lemma 14 (Bounding: Conj, Props, Fair). For unconditionally-fair initializing runs of conjunctive systems:

$$\forall n \geq 1:$$
 $(A,B)^{(1,1)} \models \mathsf{E}_{uncond} \, h(A,B_1) \iff (A,B)^{(1,n)} \models \mathsf{E}_{uncond} \, h(A,B_1)$

Proof. Given an unconditionally-fair [initializing] run x of $(A, B)^{(1,n)}$ with n > c construct an unconditionally-fair [initializing] run y in the cutoff system $(A, B)^{(1,1)}$: copy the local runs of processes A, B_1 .

Tightness 7 (1-Conj, Props, Fair). The cutoff c=2 is tight for parameterized model checking of $\mathsf{E}\,h(A,B_1)$ on unconditionally-fair initializing runs in 1-conjunctive systems, i.e., there is a system type (A,B) and property $\mathsf{E}\,h(A,B_1)$ which is satisfied by $(A,B)^{(1,1)}$ but not by $(A,B)^{(1,2)}$.



Proof. The figure below shows
$$(A, B)$$
, $\mathsf{E}\,h(A, B_1) = \mathsf{E}\,\mathsf{FG}(b_{init} \to a_1)$.

Lemma 15 (Monotonicity: Conj, Deadlocks, Fair). For 1-conjunctive systems on strong fair initializing or finite runs:

$$\forall n \geq 1 : (A,B)^{(1,n)}$$
 has a deadlock $\Rightarrow (A,B)^{(1,n+1)}$ has a deadlock

Proof. Let x be a globally deadlocked or locally deadlocked strong-fair initializing run of $(A,B)^{(1,n)}$. We will build a globally deadlocked or locally deadlocked strong-fair initializing run of $(A,B)^{(1,n+1)}$.

If x is finite, then y is the copy of x, and the new process stays in init_B until every process become deadlocked, and then is scheduled arbitrarily. Note that y constructed this way may be locally deadlocked rather than globally deadlocked as x is.

Now consider the case when x is locally deadlocked strong-fair initializing.

Let \mathcal{D} be the set of deadlocked B-processes in x, and d be the moment when the processes become deadlocked.

Consider the case Visited $_{\mathcal{B},\mathcal{D}}^{inf}(x) \neq \emptyset$: copy x into y, and let the new process B_{n+1} wait in init_B and interleave the roles with a process B that moves infinitely often in x, similarly to as described in the proof of Lemma 13.

Consider the case $\mathsf{Visited}^{inf}_{\mathcal{B}\backslash\mathcal{D}}(x) = \emptyset$: every B process of $(A,B)^{(1,n)}$ is deadlocked and thus $\mathcal{D} = \mathcal{B}$. Define

$$DeadGuards = \{ q \mid \exists P \in \mathcal{D} \text{ with a transition guarded "} \forall \neg q \text{" in } (s_d(P), e_d(P)) \}.$$

Note that $Q_A \cap DeadGuards = \emptyset$, because A visits infinitely often init_A and we consider 1-conjunctive systems. Hence, copy x into y, and let the new process B_{n+1} wait in init_B until every process $B_1, ..., B_n$ become deadlocked, and then schedule B_{n+1} arbitrarily.

Lemma 16 (Bounding: 1-Conj, Deadlocks, Fair). For 1-conjunctive systems on strong-fair initializing or finite runs:

- with
$$c = 2|Q_B \setminus \{\text{init}\}|$$
 and any $n > c$:

$$(A,B)^{(1,c)}$$
 has a global deadlock $\Leftarrow (A,B)^{(1,n)}$ has a global deadlock

- with
$$c = 2|Q_B \setminus \{\text{init}\}| + 1$$
 and any $n > c$ (when $|Q_B| > 2$):

$$(A,B)^{(1,c)}$$
 has a local deadlock $\Leftarrow (A,B)^{(1,n)}$ has a local deadlock

- with
$$c = 2|Q_B \setminus \{\text{init}\}|$$
 and any $n > c$:

$$(A,B)^{(1,c)}$$
 has a deadlock $\Leftarrow (A,B)^{(1,n)}$ has a deadlock

Proof. Global Deadlocks. $c = 2|Q_B \setminus \{\text{init}_B\}|$, see Lemma 12, the fairness does not matter on finite runs.

Local Deadlocks. Let $c = 2|Q_B \setminus \{\text{init}_B\}|$. Let $x = (s_1, e_1, p_1) \dots$ be a locally deadlocked strong-fair intitializing run of $(A,B)^{(1,n)}$ with n>c. We construct a locally deadlocked strong-fair initializing run y of $(A, B)^{(1,c)}$.

Let \mathcal{D} be the set of deadlocked processes in x. Let d be the moment in x starting from which every process in \mathcal{D} is deadlocked.

Let dead(x) be the set of states in which processes \mathcal{D} of $(A, B)^{(1,n)}$ are deadlocked. Let $dead_2(x) \subseteq dead(x)$ be the set of deadlocked states such that: for every $q \in$ $dead_2(x)$, there is a process $P \in \mathcal{D}$ with $s_d(P) = q$ and that for input $e_{>d}(P)$ has a transition guarded with " $\forall \neg q$ ". Thus, a process in q is deadlocked with $e_d(P)$ only if there is another process in q in every moment $\geq d$.

Let $dead_1(x) = dead(x) \setminus dead_2(x)$. I.e., for any $q \in dead_1(x)$, there is a process P of $(A, B)^{(1,n)}$ which is deadlocked in $s_d(P) = q$ with input $e_d(P)$, and no transitions from q with input $e_d(P)$ are guarded with " $\forall \neg q$ ".

 $DeadGuards = \{ q \mid \exists P \in \mathcal{D} \text{ with a transition guarded "} \forall \neg q \text{" in } (s_d(P), e_d(P)) \}.$

We illustrate properties of sets DeadGuards, $dead_1$, $dead_2$, $Visited_{BD}^{inf}(x)$ in Fig. 2. Let us assume $DeadGuards \neq \emptyset$ – the other case is straightforward. The construction has two phases, the setup and the looping. The setup phase is:

- a. y(A) = x(A)
- for every $q \in \mathsf{dead}_1$: devote one process of $(A,B)^{(1,c)}$ that copies a process of $(A,B)^{(1,n)}$ deadlocked in q
- c. for every $q \in \mathsf{dead}_2 \setminus \mathsf{Visited}^{inf}_{\mathcal{BD}}(x)$: devote two processes of $(A,B)^{(1,c)}$ that copy
- the behaviour of two processes of $(A, B)^{(1,n)}$ that deadlock in q d. for every $q \in \mathsf{dead}_2 \cap \mathsf{Visited}^{inf}_{\mathcal{B}\backslash \mathcal{D}}(x)$: in x, there is a process, $B_q^{inf} \in \mathcal{B}\backslash \mathcal{D}$, that visits q infinitely often, and there is a process, $B_q^{\perp} \in \mathsf{dead}_2$, deadlocked in q. Then: 1. devote one process of $(A,B)^{(1,c)}$ that copies the behaviour of B_q^{\perp}

 - 2. devote one process of $(A,B)^{(1,c)}$ that copies the behaviour of B_q^{inf} until it reaches q at a moment after d, and then provide the same input as B_q^{\perp} receives at moment d. This will deadlock the process.
- e. for every $q \in DeadGuards \setminus \mathsf{dead}$: note that $q \in \mathsf{Visited}_{\mathcal{B}\backslash\mathcal{D}}^{inf}(x)$ and, thus, there is a process, $B_q^{inf} \in \mathcal{B} \backslash \mathcal{D}$, that visits q infinitely often. Devote one process of $(A,B)^{(1,c)}$ that copies the behaviour of B_q^{inf} until it reaches q at a moment after d f. if $DeadGuards \backslash \text{dead} \neq \emptyset$ or $A \in \mathcal{D}$, then devote one process that stays in init_B.
- The process will be used in the looping phase to ensure that the run y is infinite, and that every process of $(A, B)^{(1,c)}$ used in (e) moves infinitely often (and thus yis strong-fair).
- let any other process of $(A, B)^{(1,c)}$ (if any) copy behaviour of a process of $(A, B)^{(1,n)}$ that was not used in the construction so far (including this step)

The setup phase ensures: in every state $q \in \mathsf{dead}$, there is at least one process deadlocked in q at moment d in y. Now we need to ensure that the non-deadlocked processes described in steps (e) and (f) move infinitely often.

The looping phase is applied to processes in (e) and (f) only⁹. Order arbitrarily $DeadGuards \land dead = (q_1, ..., q_k) \subseteq Visited_{\mathcal{B}\backslash\mathcal{D}}^{inf}(x)$. Note that $init_B \notin (q_1, ..., q_k)$. Let \mathcal{P} be the set of processes of $(A, B)^{(1,c)}$ used in steps (e) or (f). Note that $|\mathcal{P}| = |(q_1, ..., q_k)| + 1$.

⁹ If there are no such processes, then the setup phase produces the sought run y.

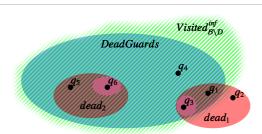


Fig. 2: Venn diagram for sets DeadGuards, $dead_1$, $dead_2$, $Visited_{BD}^{inf}(x)$:

- (q_1) dead $_1 \cap DeadGuards \cap \mathsf{Visited}^{inf}_{\mathcal{B}\setminus\mathcal{D}}(x) \neq \emptyset$ is possible: in x, there is a process deadlocked in state q_1 , there is a non-deadlocked process that visits q_1 infinitely often, and there is a process deadlocked in a state $q \neq q_1$ with a transition guarded " $\forall \neg q_1$ "
- (q_3) dead₁ \cap $DeadGuards \setminus Visited_{\mathcal{B} \setminus \mathcal{D}}^{inf}(x) \neq \emptyset$ is possible: similarly to q_1 , except that no non-deadlocked processes visit q_3 infinitely often
- (q_2) dead₁\(\text{Visited}_{\mathcal{B}\mathcal{D}}^{inf}(x) \cup DeadGuards\) $\neq \emptyset$ is possible: in x, there is a process deadlocked in state q_2 , no other processes visit q_2 infinitely often, and no processes are deadlocked with a transition guarded " $\forall \neg q_2$ "
- (q_4) DeadGuards \ dead $\neq \emptyset$ is possible: there is a process deadlocked in a state $q \neq q_4$ with a transition guarded " $\forall \neg q_4$ "
- (q_5) dead₂ \cap Visited $_{\mathcal{BD}}^{inf}(x) \cap DeadGuards \neq \emptyset$ is possible: there is at least one process deadlocked in q_5 with a transition guarded " $\forall \neg q_5$ ", and some non-deadlocked process visits q_5 infinitely often (this process does not deadlock in q_5 , because in q_5 it receives an input different from that of the deadlocked processes)
- (q_6) dead₂ $\cap DeadGuards \setminus Visited_{\mathcal{B} \setminus \mathcal{D}}^{inf}(x) \neq \emptyset$ is possible: similarly to q_5 , except no non-deadlocked processes visit q_6 infinitely often

The looping phase is: set i = 1, and repeat infinitely the following.

- let $P_{\mathsf{init}} \in \mathcal{P}$ be the process that is currently in init_B , and $P_{q_i} \in \mathcal{P}$ in q_i let $B_{q_i} \in \mathsf{Visited}_{\mathcal{B},\mathcal{D}}^{inf}(x)$ be a process of $(A,B)^{(1,n)}$ that visits q_i and init_B infinitely often. Let P_{init} of $(A,B)^{(1,c)}$ copy transitions of B_{q_i} on some path $\mathsf{init}_B \to \ldots \to g_i$, then let P_{g_i} copy transitions of B_{q_i} on some path $g_i \to \ldots \to \mathsf{init}_B$. For copying we consider only the paths of B_{q_i} that happen after moment d.
- $-i=i\oplus 1$

The number of copies of B that the construction uses in the worst case is (if ignore (g), assume $Q_B > 2$, $DeadGuards \setminus dead = \emptyset$, and $A \in \mathcal{D}$):

$$1_{(f)} + 2|\mathsf{dead}_2|_{(c),(d)} + |\mathsf{dead}_1|_{(b)} \le 2|Q_B \setminus \{\mathsf{init}_B\}| + 1.$$

Deadlocks. The largest value of c among those for "Local Deadlocks" and for "Global Deadlocks" can be used as the sought value of c for the case of general deadlocks. But

it will not be the smallest one. In the proof of the case "Local Deadlocks", in the setup phase, item (e) can be modified for the case when $A \in \mathcal{D}$: since we do not need to ensure that y is infinite, we avoid allocating a process in state init_B . For a given locally deadlocked strong-fair run, the setup phase may produce the globally deadlocked run, but that is allright for the case of general deadlocks. With this note, for the general case $c = 2|Q_B \setminus \{\mathsf{init}_B\}|$.

Tightness 8 (1-Conj, Deadlocks, Fair). The cutoff c = 2|B| - 2 is tight for deadlock detection on strong-fair initializing or finite runs in the 1-conjunctive systems, i.e., for any k > 2 there is a system type (A, B) with |B| = k such that there is a strong-fair initializing deadlocked run in $(A, B)^{(1,2|B|-2)}$, but not in $(A, B)^{(1,2|B|-3)}$.

Proof. Consider the same templates as in Observation 6.