Proc. VMCAI 2016, (c) Springer Program Analysis with Local Policy Iteration*

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Abstract. We present local policy iteration (LPI), a new algorithm for deriving numerical invariants that combines the precision of max-policy iteration with the flexibility and scalability of conventional Kleene iterations. It is defined in the Configurable Program Analysis (CPA) framework, thus allowing inter-analysis communication.

LPI uses adjustable-block encoding in order to traverse loop-free program sections, possibly containing branching, without introducing extra abstraction. Our technique operates over any template linear constraint domain, including the interval and octagon domains; templates can also be derived from the program source.

The implementation is evaluated on a set of benchmarks from the International Competition on Software Verification (SV-COMP). It competes favorably with state-of-the-art analyzers.

1 Introduction

Program analysis by *abstract interpretation* [1] derives facts about the execution of programs that are always true regardless of the inputs. These facts are proved using *inductive invariants*, which satisfy both the initial condition and the transition relation, and thus always hold. Such invariants are found within an *abstract domain*, which specifies what properties of the program can be tracked. Classic abstract domains for numeric properties include [products of] intervals and octagons [2], both of which are instances of *template linear constraint domains* [3].

Consider classic abstract interpretation with intervals over the program int i=0; while(i < 1000000) i++; After the first instruction, the analyzer has a candidate invariant $i \in [0,0]$. Going through the loop body it gets $i \in [1,1]$, thus by least upper bound with the previous state [0,0] the new candidate invariant is $i \in [0,1]$. Subsequent Kleene iterations yield [0,2], [0,3] etc. In order to enforce the convergence within a reasonable time, a widening operator is used, which extrapolates this sequence to $[0, +\infty)$. Then, a narrowing iteration yields [0, 100000]. In this case, the invariant finally obtained is the best possible, but the same approach yields the suboptimal invariant $[0, +\infty)$ if an unrelated nested

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loop is added to the program: while(i<100000)while(unknown()) i++;. This happens because the candidate invariant obtained with widening is its own post-image under the nested loop, hence narrowing cannot shrink the invariant.

In general, widenings and narrowings are brittle: a small program change may result in a different analysis behavior. Their result is *non-monotone*: a locally more precise invariant at one point may result in a less precise one elsewhere.

Max-policy iteration In contrast, max-policy iteration [4] is guaranteed to compute the least *inductive* invariant in the given abstract domain.³ To compute the bound h of the invariant $i \leq h$ for the initial example above, it considers that h must satisfy $h = \max i'$ s.t. $(i' = 0) \lor (i' = i + 1 \land i < 1000000 \land i \leq h)$ and computes the least inductive solution of this equation by successively considering separate cases:

- (i) $h = (\max i' \text{ s.t. } i' = 0) = 0$, which is not inductive, since one can iterate from i = 0 to i = 1.
- (ii) $h = \max i'$ s.t. $i' = i + 1 \land i < 1000000 \land i \le h$, which has two solutions over $\mathbb{R} \cup \{\infty, -\infty\}$: $h = -\infty$ (representing unreachable state, discarded) and h = 1000000, which is finally inductive.

Earlier presentations of policy iteration solve a sequence of global convex optimization problems whose unknowns are the bounds (here h) at every program location. Further refinements [5] allowed restricting abstraction to a cut-set [6] of program locations (a set of program points such that the control-flow graph contains no cycle once these points are removed), through a combination with *satisfiability modulo theory* (SMT) solving. Nevertheless, a global view of the program was needed, hampering scalability and combinations with other analyses.

Contribution We present the new local-policy-iteration algorithm (LPI) for computing inductive invariants using policy iteration. Our implementation is integrated inside the open-source CPAchecker [7] framework for software verification and uses the maximization-modulo-theory solver νZ [8]. To the best of our knowledge, this is the first policy-iteration implementation that is capable of dealing with C code. We evaluate LPI and show its competitiveness with state-of-the-art analyzers using benchmarks from the International Competition on Software Verification (SV-COMP).

Our solution improves on earlier max-policy approaches:

(i) Scalability LPI constructs optimization queries that are at most of the size of the largest loop in the program. At every step we only solve the optimization problem necessary for deriving the *local* candidate invariant.

(ii) Ability to cooperate with other analyses LPI is defined within the Configurable Program Analysis (CPA) [9] framework, which is designed to allow easy inter-analysis collaboration. Expressing policy iteration as a fixpoint-propagation algorithm establishes a common ground with other approaches (lazy abstraction, bounded model checking) and allows communicating with other analyses.

³ It does not, however, necessarily output the strongest (potentially non-inductive) invariant in an abstract domain, which in general entails solving the halting problem.

(iii) **Precision** LPI uses adjustable-block encoding [10], and thus benefits from the precision offered by SMT solvers, effectively checking executions of loop-free program segments without the need for over-approximation. *Path focusing* [5] has the same advantage, but at the cost of pre-processing the control-flow graph, which significantly hinders inter-analysis communication.

Related Work Policy iteration is not as widely used as classic abstract interpretation and (bounded) model checking. Roux and Garoche [11] addressed a similar problem of embedding the policy-iteration procedure inside an abstract interpreter, however their work has a different focus (finding quadratic invariants on relatively small programs) and the policy-iteration algorithm remains fundamentally un-altered. The tool REAVER [12] also performs policy iteration, but focuses on efficiently dealing with logico-numerical abstract domains; it only operates on Lustre programs. The ability to apply policy iteration on strongly connected components one by one was (briefly) mentioned before [13]. Our paper takes the approach significantly further, as our value-determination problem is more succinct, we apply the principle of locality to the policy-improvement phase, and we formulate policy iteration as a classic fixpoint-iteration algorithm, enabling communication with other analyses. Finally, it is possible to express the search for an inductive invariant as a nonlinear constraint solving problem [14] or as a quantifier elimination problem [15], but both these approaches scale poorly.

2 Background

We represent a program P as a control flow automaton (CFA) (nodes, X, edges), where nodes is a set of control states, and $X = \{x_1, \ldots, x_n\}$ are the variables of P. Each edge $e \in edges$ is a tuple $(A, \tau(X, X'), B)$, where A and B are nodes, and $\tau(X, X')$ is a transition relation: a formula defining the semantics of a transition over the set of input variables X and fresh output variables X'. A concrete state of the program P is a map $X \to \mathbb{Q}$ from variables to rationals⁴. A set C of concrete states is represented using a first-order formula ϕ with free variables from X, such that for all $c \in C$ we have $c \models \phi$.

Template Linear Constraint Domains A template linear constraint is a linear inequality $t \cdot X \leq b$ where t is a vector of constants (template), and b is an unknown. A template linear constraint domain [3] (TCD) is an abstract domain defined by a matrix of coefficients a_{ij} , which determines what template linear constraints are expressible within the domain: each row t of the matrix is a template (the word "template" also refers to the symbolic product $t \cdot X$, e.g. i + 2j). An abstract state in a TCD is defined by a vector (d_1, \ldots, d_m) and represents the set $\bigwedge_{i=1}^m t_i \cdot X \leq d_i$ of concrete states. The d_i 's range over extended rationals $(\mathbb{R} \cup \{\infty, -\infty\})$, where positive infinity represents unbounded templates and negative infinity represents unreachable abstract states. The domain of products of intervals is one instance of TCD, where the templates are $\pm x_i \leq c_i$ for program variables x_i . The domain of octagons [2] is another, with templates $\pm x_i \pm x_j$ and $\pm x_i$. Any template linear constraint domain is a subset of the domain of convex polyhedra [16].

⁴ We support integers as well, as explained in Sec. 4.



Fig. 1: Running example – C program and the corresponding CFA

The strongest abstract postcondition in a TCD is defined by optimization: maximizing all templates subject to the constraints introduced by the semantics of the transition and the previous abstract state. For the edge $e = (A, \tau(X, X'), B)$, previous abstract state $D = (d_1, \ldots, d_m)$, and a set $\{t_1, \ldots, t_m\}$ of templates, the output abstract state is $D' = (d'_1, \ldots, d'_m)$ with

$$d'_i = (\max t_i \cdot X' \text{ s.t. } \bigwedge_i t_i \cdot X \leq d_i \wedge \tau(X, X'))$$

For example, for the abstract state $i \leq 0 \land j \leq 0$ under the transition $i' = i + 1 \land i \leq 10$ the new abstract state is $i \leq d^i \land y \leq d^j$, where $d^i = \max i'$ s.t. $i \leq 0 \land j \leq 0 \land i' = i + 1 \land i < 10 \land j' = j$ and d^j is the result of maximizing j' subject to the same constraints. This gets simplified to $i \leq 1 \land j \leq 0$.

Kleene iterations in a TCD (known as *value iterations*) may fail to converge in finite time, thus the use of *widenings*, which result in hard-to-control imprecision. **Policy Iteration** Policy iteration addresses the convergence problem of *value-iteration* algorithms by operating on an equation system that an inductive invariant has to satisfy. Consider the running example shown in Fig. 1. Suppose we analyze this program with the templates $\{i, j\}$, and look for the least inductive invariant $D = (d_A^i, d_A^j, d_B^i, d_B^j)$ that satisfies the following for all possible executions of the program $(x_N$ denotes the value of the variable x at the node N):

$$i_A \leq d_A^i \wedge i_B \leq d_B^i \wedge j_A \leq d_A^j \wedge j_B \leq d_B^j$$

To find it, we solve for the smallest D that satisfies the *fixpoint equation [system]* for the running example, stating that the set of abstract states represented by D is equal to its strongest postcondition within the abstract domain:

$$\begin{split} d^i_A &= \sup i' \text{ s. t. } (i' = 0 \land j' = 0) \\ &\lor (i \leq d^i_A \land j \leq d^j_A \land i < 10 \land i' = i + 1 \land j' = j) \lor \bot \\ d^j_A &= \sup j' \text{ s. t. } (i' = 0 \land j' = 0) \\ &\lor (i \leq d^i_A \land j \leq d^j_A \land i < 10 \land i' = i + 1 \land j' = j) \lor \bot \\ d^i_B &= \sup i' \text{ s. t. } (\neg (i < 10) \land i \leq d^i_A \land j' \leq d^j_A \land i' = i) \\ &\lor (i \leq d^i_B \land j \leq d^j_B \land j < 10 \land j' = j + 1 \land i' = i) \lor \bot \\ d^j_B &= \sup j' \text{ s. t. } (\neg (i < 10) \land i \leq d^i_A \land j' \leq d^j_A \land i' = i) \\ &\lor (i \leq d^i_B \land j \leq d^j_B \land j < 10 \land j' = j + 1 \land i' = i) \lor \bot \\ d^j_B &= \sup j' \text{ s. t. } (\neg (i < 10) \land i \leq d^i_A \land j' \leq d^j_A \land i' = i) \\ &\lor (i \leq d^i_B \land j \leq d^j_B \land j < 10 \land j' = j + 1 \land i' = i) \lor \bot \end{split}$$

Note the equation structure: (i) Disjunctions represent non-deterministic choice for a new value. (ii) The argument \perp is added to all disjunctions, representing

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infeasible choice, corresponding to the bound value $-\infty$. (iii) Supremum is taken because the bound must be higher than all the possible options, and it has to be $-\infty$ if no choice is feasible.

A simplified equation system with each disjunction replaced by one of its arguments is called a *policy*. The least solution of the whole equation system is the least solution of at least one policy (obtained by taking the solution, and picking one argument for each disjunction, such that the solution remains unchanged). Policy iteration finds the least tuple of unknowns (d's) satisfying the fixpoint equation by iterating over possible policies, and finding a solution for each one.

For program semantics consisting of linear assignments and possibly nondeterministic guards it is possible to find a fixpoint of each policy using one linear programming step. This is based on the result that for a monotone and concave function⁵ f and x_0 such that $f(x_0) > x_0$, the least fixpoint of f greater than x_0 can be computed in a single step⁶.

It is possible to solve the global equation system by solving all (exponentially many) policies one by one. Instead, *policy iteration* [4] computes solutions for a sequence of policies; each solution is guaranteed to be less than the least solution of the original equation system, and the solutions form an ascending sequence. The iteration starts with the policy having least possible value (\perp for each disjunction, the solution is $-\infty$ assignment to all unknowns), and eventually terminates when a solution of the original equation system (an inductive invariant) is found. The termination is guaranteed as there is only a finite number of solutions.

For each policy the algorithm finds a global value: the least fixpoint in the template constraints domain of the reduced equation system. For instance, in the running example, for the policy $d_A^i = \sup i'$ s. t. $i' = 0 \land j' = 0$ (only one unknown is shown for brevity) the global value is $d_A^i = 0$. This step is called value determination. After the global value is computed the algorithm checks whether the policy can be *improved*: that is, whether we can find another policy that will yield a larger value than the previously obtained global value. In the running example we want to test the following policy for the possibility of improvement:

 $d_A^i = \sup i'$ s. t. $(i \le d_A^i \land j \le d_A^i \land i < 10 \land i' = i + 1 \land j' = j)$

We do so by computing the *local value*: substituting the unknown (d_A^i) on the right hand side with the value from the previously obtained global value, and checking whether the result is greater than the previously obtained bound. In our example we get the local value $d_A^i = 1$, which is indeed an improvement over $d_A^i = 0$ (*policy-improvement* step). After the policy is selected, we go back to the value-determination step, obtaining $d_A^i = 10$, and we repeat the process until convergence (reaching a step where no policy can be further improved).

Under the assumption that the operations on the edges can be expressed as conjunctions of linear (in)equalities, it can be shown [4] that: (i) The valuedetermination step can be performed with linear programming. (ii) The resulting

 $^{^{5}}$ Order-concave in the presence of multiple templates, see [4] for detailed discussion

⁶ Over rationals, we discuss the applicability to integers in Sec. 4

value is an under-approximation of the least inductive invariant. (iii) Each policy is selected at most once and the final fixed point yields the least inductive invariant in the domain.

Example 1 (Policy-Iteration Trace on the Running Example). We solve for the unknowns $(d_A^i, d_A^j, d_B^i, d_B^j)$, defining a (global) abstract value v.

In our example, disjunctions arise from multiple incoming edges to a single node, hence a policy is defined by a choice of an incoming edge per node per template, or \perp if no such choice is feasible. We represent a policy symbolically as a 4-tuple of predecessor nodes (or \perp), as there are two nodes, with two policies to be chosen per node. The order corresponds to the order of the tuple of the unknowns. The initial policy s is $(\perp, \perp, \perp, \perp)$. The trace on the example is:

- 1. Policy improvement: $s = (I, I, \bot, \bot)$, obtained with a local value $(0, 0, -\infty, -\infty)$.
- 2. Value determination: corresponds to the initial condition $v = (0, 0, -\infty, -\infty)$.
- 3. Policy improvement: $s = (A, I, \bot, \bot)$, selecting the looping edge, local value is $(1, 0, -\infty, -\infty)$.
- 4. Value determination: accelerates the loop convergence to $v = (10, 0, -\infty, -\infty)$.
- 5. Policy improvement: s = (A, I, A, A), with a local value (10, 0, 10, 0) finally there is a feasible policy for the templates associated with the node B.
- 6. Value determination: does not affect the result v = (10, 0, 10, 0).
- 7. Policy improvement: select the second looping edge: s = (A, I, A, B) obtaining a local value (10, 0, 10, 1).
- 8. Value determination: accelerate the second loop to v = (10, 0, 10, 10).
- 9. Finally, the policy cannot be improved any further and we terminate.

On this example we could have obtained the same result by Kleene iteration, but in general the latter might fail to converge within finite time. The usual workaround is to use heuristic widening, with possible and hard-to-control imprecision. Our value-determination step can be seen as a widening that provides an under-approximation to the least fixed point.

Each policy improvement requires at least four (small) linear programming (LP) queries, and each value determination requires one (rather large) LP query.

Path Focusing and Large-Block Encoding In traditional abstract interpretation and policy iteration, the obtained invariant is expressed as an abstract state at each CFA node. This can lead to a significant loss in precision: for instance, since most abstract domains only express convex properties, it is impossible to express $|x| \ge 1$, which is necessary to prove this assertion: if $(abs(x) \ge 1)$ assert(x != 0);

This loss can be recovered by reducing the number of "intermediate" abstract states by allowing more expressive formulas associated with edges. Formally, two consecutive edges $(A, \tau_1(X, X'), B)$ and $(B, \tau_2(X, X'), C)$, with no other edges incoming or outgoing to B can be merged into one edge $(A, \tau_1(X, \hat{X}) \land$ $\tau_2(\hat{X}, X'), C)$. Similarly, two parallel edges $(A, \tau_1(X, X'), B)$ and $(A, \tau_2(X, X'), B)$, with no other edges incoming to B can be replaced by a new edge $(A, \tau_1(X, X'), B)$, $\tau_2(X, X'), B)$. For a well-structured CFA, repeating this transformation in a fixpoint manner (until no more edges can be merged) will lead to a new CFA where the only remaining nodes are loop heads.

Such a transformation was shown to increase both precision and performance for model checking [17]. Adjustable block encoding [10] gets the same advantages without the need for CFA pre-processing. Independently, the approach was applied with the same result to Kleene iterations [18] and to max-policy iterations [5]. In fact, the CFA in Fig. 1 was already reduced in this manner for the ease of demonstration.

On the reduced CFA the number of possible policies associated with a single edge becomes exponential, and explicitly iterating over them is no longer feasible. Instead, the path focusing approach uses a *satisfiability modulo theory* (SMT) solver to select an improved policy.

Configurable Program Analysis CPA [9] is a framework for expressing algorithms performing program analysis. It uses a generic fixpoint-computation algorithm, which is *configured* by a given analysis. We formulate LPI as a CPA.

The CPA framework makes no assumptions on the performed analysis, thus many analyses were successfully expressed and implemented within it, such as bounded model checking, abstract interpretation and k-induction (note that an analysis defined within the framework is also referred to as a CPA).

Each CPA configures the fixpoint algorithm by providing an *initial abstract* state, a transfer relation (specifying how to produce successors), a merge operator (specifying whether and how to merge abstract states), and a stop operator (specifying whether a newly produced abstract state is covered). The algorithm keeps a set of reached abstract states and a list of "frontier" abstract states, and at each step produces successor states from the frontier states using the transfer relation, and then tries to merge the new states with existing states using the merge operator. If a new state is covered by the set of reached states according to the stop operator, it is discarded, otherwise it is added to the set of reached states and the list of frontier states. We show the CPA algorithm as Alg. 1.

3 Local Policy Iteration (LPI)

The running example presented in the background (Ex. 1) has four value-determination steps and five policy-improvement steps. Each policy-improvement step corresponds to at most #policies \times #templates \times #nodes LP queries, and each value-determination step requires solving an LP problem with at least #policies \times #templates \times #nodes variables. Most of these queries are redundant, as the updates propagate only *locally* through the CFA: there is no need to re-compute the policy if no new information is available.

We develop a new policy-iteration-based algorithm, based on the principle of *locality*, which aims to address the scalability issues and the problem of communicating invariants with other analyses. We call it *local policy iteration* or LPI. To make it scalable, we consider the structure of a CFA being analyzed, and we aim to exploit its *sparsity*.

A large majority of (non-recursive) programs are well-structured: they consist of statements and possibly nested loops. Consider checking a program P

Algorithm 1 CPA Algorithm (taken from [9])

1: Input: a CPA (D, transfer-relation, merge, stop), an initial abstract state $e_0 \in E$
(let E denote the set of elements of the semi-lattice of D)
2: Output : a set of reachable abstract states
3: Variables: a set reached of elements of E , a set waitlist of elements of E
4: waitlist $\leftarrow \{e_0\}$
5: reached $\leftarrow \{e_0\}$
6: while waitlist $\neq \emptyset$ do
7: Pop e from waitlist
8: for all $e' \in \text{transfer-relation}(e)$ do
9: for all $e'' \in$ reached do
10: \triangleright Combine with existing abstract state
11: $e_{\text{new}} \leftarrow \text{merge}(e', e'')$
12: if $e_{\text{new}} \neq e''$ then
13: waitlist \leftarrow (waitlist \cup { e_{new} }) \ { e'' }
14: reached \leftarrow (reached \cup { e_{new} }) \ { e'' }
15: \triangleright Whether e' is already covered by existing states
16: if \neg stop(e' , reached) then
17: waitlist \leftarrow waitlist $\cup \{e'\}$
18: reached \leftarrow reached $\cup \{e'\}$
19: return reached

against an error property E. If P has no loops, it can be converted into a single formula $\Psi(X')$, and an SMT solver can be queried for the satisfiability of $\Psi(X') \wedge E(X')$, obtaining either a counter-example or a proof of unreachability of E. However, in the presence of loops, representing all concrete states reachable by a program as a formula over concrete states in a decidable first-order logic is impossible, and abstraction is required. For example, bounded model checkers unroll the loop, lazy-abstraction-based approaches partially unroll the loop and use the predicates from Craig interpolants to "cover" future unrollings, and abstract interpretation relies on abstraction within an abstract domain.

In LPI, we use the value-determination step to "close" the loop and compute the fixpoint value for the given policy. Multiple iterations through the loop might be necessary to find the optimal policy and reach the global fixpoint. In the presence of nested loops, the process is repeated in a fixpoint manner: we "close" the inner loop, "close" the outer loop with the new information from the inner loop available, and repeat the process until convergence. Each iteration selects a new policy, thus the number of possible iterations is bounded.

Formally, we state LPI as a Configurable Program Analysis (CPA), which requires defining the lattice of abstract states, the transfer relation, the merge operator, and the stop operator. The CPA for LPI is intended to be used in combination with other CPAs such as a CPA for tracking location information (the program counter), and thus does not need to keep track of this information itself. To avoid losing precision, we do not express the invariant as an abstract state at every node: instead the transfer relation operates on formulas and we only perform over-approximation at certain *abstraction points* (which correspond

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to loop heads in a well-structured CFA). This approach is inspired by adjustableblock encoding [10], which performs the same operation for predicate abstraction. One difference to path focusing [18] is that we still traverse intermediate nodes, which facilitates inter-analysis communication.

We introduce two lattices: *abstracted states* (not to be confused with *abstract states* in general: both intermediate and abstracted states are *abstract*) for states associated with abstraction points (which can only express abstract states in the template constraints domain) and *intermediate states* for all others (which can express arbitrary concrete state spaces using decidable SMT formulas).

An *abstracted state* is an element of a template constraints domain with meta-information added to record the *policy* being used.

Definition 1 (Abstracted State). An abstracted state is a mapping from the externally given set T of templates to tuples (d, policy, backpointer), where $d \in \mathbb{R}$ is a bound for the associated template t (the represented property is $t \cdot X \leq d$), policy is a formula representing the policy that was used for deriving d (policy has to be monotone and concave, and in particular contain no disjunctions), and backpointer is an abstracted state that is a starting point for the policy (base case is an empty mapping).

The preorder on abstracted states is defined by component-wise comparison of bounds associated with respective templates (lack of a bound corresponds to an unbounded template). The concretization is given by the conjunction of represented template linear constraints, disregarding policy and backpointer metainformation. For example, an abstracted state $\{x : (10, _, _)\}$ (underscores represent meta-information irrelevant to the example) concretizes to $\{c \mid c[x] \le 10\}$, and the initial abstracted state $\{\}$ concretizes to all concrete states.

Intermediate states represent reachable state-spaces using formulas directly, again with meta-information added to record the "used" policy.

Definition 2 (Intermediate State). An intermediate state is a tuple (a_0, ϕ) , where a_0 is a starting abstracted state, and $\phi(X, X')$ is a formula over a set of input variables X and output variables X'.

The preorder on intermediate states is defined by syntactic comparison only: states with identical starting states and identical formulas are deemed equal, and incomparable otherwise. The concretization is given by satisfiable assignments to X' subject to $\phi(X, X')$ and the constraints derived from a_0 applied to input variables X. For example, an intermediate state $(\{x : (10, _, _)\}, x' = x + 1)$ concretizes to the set $\{c \mid c[x] \leq 11\}$ of concrete states.

Abstraction (Alg. 2) is the conversion of an intermediate state (a_0, ϕ) to an abstracted state, by maximizing all templates $t \in T$ subject to constraints introduced by a_0 and ϕ , and obtaining a backpointer and a policy from the produced model \mathcal{M} . This amounts to selecting the appropriate disjuncts in each disjunction of ϕ . To do so, we annotate ϕ with marking variables: each disjunction $\tau_1 \vee \tau_2$ in ϕ is replaced by $(m \wedge \tau_1) \vee (\neg m \wedge \tau_2)$ where m is a fresh propositional variable. A policy associated to a bound is then identified by the values of the marking

Algorithm 2 LPI Abstraction

1: **Input:** intermediate state (a_0, ϕ) , set T of templates 2: Output: generated abstracted state new 3: $new \leftarrow empty$ abstracted state 4: for all template $t \in T$ do 5: $\phi \leftarrow \phi$ with disjunctions annotated using a set of marking variables M 6: ▷ Maximize subject to the constraints introduced by the formula 7: \triangleright and the starting abstracted state. 8: $d \leftarrow \max t \cdot X'$ subject to $\phi(X, X') \wedge a_0$ $\mathcal{M} \leftarrow \text{model}$ at the optimal 9: \triangleright Replace marking variables M in $\hat{\phi}$ with their value from the model \mathcal{M} , 10: 11: \triangleright generating a concave formula that represents the policy. 12:Policy $\psi \leftarrow \phi[M/\mathcal{M}]$ 13: $new[t] \leftarrow (d, \psi, a_0)$ 14: return new

variables at the optimum (subject to the constraints introduced by ϕ and a_0), and is obtained by replacing the marking variables in ϕ with their values from \mathcal{M} . Thus the abstraction operation effectively performs the *policy-improvement* operation for the given node, as only the policies which are feasible with respect to the current candidate invariant (given by previous abstracted state) are selected.

Example 2 (LPI Propagation and Abstraction). Let us start with an abstracted state $a = \{x : (100, _, _)\}$ (which concretizes to $\{c \mid c[x] \le 100\}$, underscores stand for some policy and some starting abstracted state) and a set $\{x\}$ of templates.

After traversing a section of code if $(x \le 10)x += 1$; else x = 0; we get an intermediate state (a, ϕ) with $\phi = (x \le 10 \land x' = x + 1 \lor x > 10 \land x' = 0)$ and a backpointer to the starting *abstracted state a*. Suppose in our example the given C code fragment ends with a loop head. Then we use *abstraction* (Alg. 2) to convert the intermediate state (a, ϕ) into a new abstracted state.

Firstly, we annotate ϕ with marking variables, which are used to identify the selected policy, obtaining $x \leq 10 \wedge x' = x + 1 \wedge m_1 \vee x > 10 \wedge x' = 0 \wedge \neg m_1$. Afterwards, we optimize the obtained formula (together with the constraints from the starting abstracted state a) for the highest values of templates. This amounts to a single OPT-SMT query:

 $\sup x'$ s.t. $x \le 100 \land (x \le 10 \land x' = x + 1 \land m_1 \lor x > 10 \land x' = 0 \land \neg m_1)$

The query is satisfiable with a maximum of 11, and an SMT model \mathcal{M} : { $x': 11, m_1: \text{true}, x: 10$ }. Replacing the marking variable m_1 in ϕ with its value in \mathcal{M} gives us a disjunction-free formula $x \leq 10 \wedge x' = x + 1$, which we store as a *policy*. Finally, the newly created abstracted state is { $x: (11, x \leq 10 \wedge x' = x + 1, a)$ }.

The local value-determination step (Alg. 3) computes the least fixpoint for the chosen policy across the entire strongly connected component where the current node n lies. The algorithm starts with a map *influencing* from nodes to

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abstracted states, which is generated by transitively following policy backpointers, and converting the resulting set of abstracted states to a map⁷. From this map, we generate a global optimization problem, where the set of fresh variables $d_{n_i}^t$ represents the maximal value a template t can obtain at the node n_i using the policies selected. Variable $d_{n_i}^t$ is made equal to the namespaced⁸ output value of the policy $\psi(X, X')$ chosen for t at n_i (line 13). For each policy ψ and the associated backpointer a_0 , we constrain the input variables of ψ using a set of variables $d_{n_0}^{t_0}$ representing bounds at the node n_0 associated with a_0 (line 16). This set of "input constraints" for value determination results in a quadratic number of constraints in terms of the number of selected policies. Finally, for each template t we maximize for d_n^t (line 20), which is the maximum possible value for t at node n under the current policy, and we record the bound in the generated abstracted state (line 21), keeping the old policy and backpointer.

The local-value-determination algorithm is almost identical to max-strategy evaluation [5], except for two changes: we only add potentially relevant constraints from the "closed" loop (found by traversing backpointers associated with policies), and we maximize objectives one by one, not for their sum (which avoids special casing infinities, and enables optimizations outlined in Sec. 4). Unlike classic policy iteration, we only run local value determination after merges on loop heads, because in other cases the value obtained by abstraction is the same as the value which could be obtained by value determination.

Formulation as a CPA The *initial state* is the abstracted state $\{\}$ (empty map), representing \top of the template constraints domain. The *stop operator* checks whether a newly created abstracted state is covered by one of the existing abstracted states using the preorder described above. The *transfer relation* finds the successor state for a given CFA edge. It operates only on intermediate state ($a_0, true$). Then, the transfer-relation operator runs symbolic execution: the successor of an intermediate state ($a, \phi(X, X')$) under the edge ($A, \tau(X, X'), B$) is the intermediate state ($a, \phi'(X, X')$) with $\phi'(X, X') \equiv \exists \hat{X}.\phi(X, \hat{X}) \wedge \tau(\hat{X}, X')$. If the successor node is a loop head, then *abstraction* (Alg. 2) is performed on the resulting state.

The *merge operator* has two operation modes, depending on whether we are dealing with abstracted states or with intermediate states.

For two abstracted states, we perform the join: for each template, we pick the largest bound out of the two possible, and we keep the corresponding policy and the backpointer. If the merge "closes" the loop (that is, we merge at the loop head, and one of the updated policies has a backpointer to a state inside the loop), we find the map *influencing* by recursively following the backpointers of the joined state, and run *local value determination* (Alg. 3). For two intermediate states (a_1, ϕ_1) and (a_2, ϕ_2) with a_1 identical to a_2 the merge operator returns the disjunction $(a_1, \phi_1 \lor \phi_2)$. Otherwise, we keep the states separate.

⁷ The are no collisions as abstracted states are joined at nodes.

 $^{^{8}}$ Name spacing means creating fresh copies by attaching a certain prefix to variable names.

Algorithm 3 Local Value Determination

1: Input: node n, map influencing from nodes to abstracted states, set T of templates 2: Output: generated abstracted state new 3: constraints $\leftarrow \emptyset$ 4: for all node $n_i \in influencing$ do 5:state $s \leftarrow influencing[n_i]$ 6: for all template $t \in s$ do 7: (bound d, policy ψ , backpointer a_0) $\leftarrow s[t]$ Generate a unique string *namespace* 8: 9: \triangleright Prefix all variables in ψ . 10: $\triangleright X'_{namespace}, X_{namespace}$ is a set of namespaced output/input variables for ψ . constraints \leftarrow constraints $\cup \{\psi[X/X_{namespace}]|X'/X'_{namespace}]\}$ 11: $d_{n}^{t} \leftarrow \text{fresh variable (upper bound on } t \text{ at } n)$ 12:constraints \leftarrow constraints $\cup \left\{ d_{n_i}^t = t \cdot X'_{namespace} \right\}$ 13: $n_0 \leftarrow \text{location} \text{ associated with } a_0$ 14:15:for all $t_0 \in a_0$ do 16:constraints \leftarrow constraints $\cup \{t_0 \cdot X_{namespace} \leq d_{n_0}^{t_0}\}$ 17: $new \leftarrow$ empty abstracted state 18: for all templates $t \in T$ do 19: $(d_0, \psi, a_0) \leftarrow influencing[n]$ $d \leftarrow \max d_n^t$ subject to constraints 20: 21: $new[t] \leftarrow (d, \psi, a_0)$ 22: return new

The local-value-determination problem only contains the constraints resulting from policies of the abstracted states associated with nodes in the current loop. This optimization does not affect the invariant as only the nodes dominating the loop head can change it. Of those, only the invariants of the nodes reachable from the loop head can be affected by the computation: i.e., the strongly connected component of n.

Properties of LPI

Soundness LPI, like any configurable program analysis, terminates when no more updates can be performed, and newly produced abstract states are subsumed (in the preorder defined by the lattice) by the already discovered ones. Thus, it is an inductive invariant: the produced abstract states satisfy the initial condition and all successor states are subsumed by the existing invariant. Hence the obtained invariant is sound.

Termination An infinite sequence of produced *abstract* states must contain infinitely many *abstracted* states, as they are associated with loop heads. However, each subsequent abstraction on the same node must choose a different policy to obtain a successively higher value, but the number of policies is finite. An infinite sequence is thus impossible, hence termination.

Optimality In the absence of integers, LPI terminates with the same invariant as classical policy iteration with SMT [5]. The outline of the proof is that LPI can be seen as an efficient oracle for selecting the next policy to update (note

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that policies selected by LPI are always *feasible* with respect to the current invariant candidate). Skipping value-determination steps when they have no effect, and attempting to include only relevant constraints in the value-determination problem do not alter the values of obtained fixed points.

Example 3 (LPI Trace on the Running Example). We revisit the running example (Fig. 1) with LPI:

- 1. We start with the empty abstracted state $a_0 \equiv \{\}$.
- 2. Transfer relation under the edge (I, ϕ_1, A) produces the new intermediate state $(a_0, i' = 0 \land j' = 0)$ associated with A. As A is a loop head, we perform an abstraction to obtain the abstracted state $a_1 \equiv \{i : (0, _, a_0), j : (0, _, a_0)\}$ (corresponding to $i \leq 0 \land j \leq 0$) [2 linear programming problems].
- 3. Transfer relation explores the edge (A, ϕ_2, A) and produces the intermediate state $(a_1, i \leq 0 \land j' \leq 0 \land i' = i + 1)$. Again we perform an abstraction, obtaining the abstracted state $a_2 \equiv \{i : (1, ..., a_1), j : (0, ..., a_1)\}$ [2 LP problems].
- 4. The merge operator on node A merges the new state a_2 with the previous state a_1 , yielding the abstracted state $a_3 \equiv \{i : (1, _, a_1), j : (0, _, a_0)\}$. Value determination "closes" the loop, producing $a_4 \equiv \{i : (10, _, a_1), j : (0, _, a_0)\}$. [1 LP problem].
- 5. Transfer relation explores the edge (A, ϕ_3, B) and produces the intermediate state $(a_3, i' \leq 10 \land (\neg i' < 10) \land j' \leq 0)$, which is abstracted to $a_5 \equiv \{i : (10, \neg, a_4), j : (0, \neg, a_4)\}$ [2 LP problems].
- 6. The edge (B, ϕ_4, B) is explored, resulting in the intermediate state $(a_4, i' \leq 10 \land j \leq 0 \land j' = j + 1)$, which is abstracted into $a_6 \equiv \{i : (10, \neg, a_5), j : (1, \neg, a_5)\}$ [2 LP problems].
- 7. Value determination produces the state $a_7 \equiv \{i : (10, _, a_4), j : (10, _, a_5)\}$, and the exploration concludes. [1 LP problem].

Compared to the original algorithm there are two value-determination problems instead of four, both on considerably smaller scale. There are also only ten LP problems, compared to more than twenty in the original version. The improvement in performance is more than a fixed constant: if the number of independent loops in the running example was to increase from 2 to N, the increase in the analysis time of classic policy iteration would be quadratic, while LPI would scale linearly.

4 Extensions and Implementation Aspects

Template Synthesis The template constraints domain requires templates defined for the given program. In LPI, we can simulate the interval and octagon domains by synthesizing templates of the form $\pm x$, $\pm x \pm y$ for every numeric variable x, y in the program *alive* at the given program node. Moreover, the templates can be synthesized from error properties: e.g. for assert(x >= 2 * y) we could generate the templates $\pm (x - 2y)$.

We show the analysis time of LPI (excluding startup and parsing) in the interval-domain-mode vs. octagon-domain-mode in Fig. 2 (each data point corresponds to an analyzed program). The number of octagon templates is quadratic



Fig. 2: Octagon vs. Interval LPI Analysis Time (Dataset and Setup as in Sec. 5)

in terms of the number of interval templates, thus we expect a quadratic rise in analysis time, however in practice we observe a sub-quadratic increase.

This has motivated us to experiment with simulating a more expressive domain. We generate templates $\pm 2x \pm y$, $\pm x \pm y \pm z$, and even $\pm 2x \pm y \pm z$, for every possible combination of live variables x, y, z at the given program location. Using this new "rich" template generation strategy we achieve a significant precision improvement as shown by the number of verified programs in the legend of Fig. 3a.

Dealing With Integers Original publications on max-policy iteration in template constraints domain deal exclusively with reals, whereas C programs operate primarily on integers⁹. Excessively naive handling of integers leads to poor results: with an initial condition x = 0, $x \in [0, 4]$ is inductive for the transition system $x' = x + 1 \land x \neq 4$ in integers, but not in rationals, due to the possibility of the transition x = 3.5 to x = 4.5. An obvious workaround is to rewrite each strict inequality a < b into $a \le b - 1$: on this example, the transition becomes $x = x + 1 \land (x \le 3 \lor x \ge 5)$ and $x \in [0, 4]$ becomes inductive on rationals. However, to make use of data produced by an additional *congruence* analysis, we use optimization modulo theory with integer and real variables for abstraction, and mixed integer linear programming for value determination.

Unfortunately, linear relations over the integers are not *concave*, which is a requirement for the least fixpoint property of policy iteration. Thus the encoding described above may still result in an over-approximation. Consider the following program:

```
x=0; x_new=unknown();
while (2 * x_new == x+2) {
    x = x_new; x_new = unknown();
}
```

LPI terminates with a fixpoint $x \leq 2$, yet the least fixpoint is $x \leq 1$.

⁹ Previous work [19] deals with finding the exact interval invariants for programs involving integers, but only for a very restricted program semantics.

Congruence A congruence analysis which tracks whether a variable is even or odd can be run in parallel with LPI (a more general congruence analysis may be used, but we did not find the need for it on our examples). During the LPI abstraction step, the congruence information is conjoined to the formula being maximized, and the bounds from LPI are used for the congruence analysis.

This combination enhances the precision on our dataset (cf. Fig. 3a), and demonstrates the usefulness of expressing policy iteration as a typical fixpoint computation. Furthermore, it provides a strong motivation to use integer formulas for integer variables in programs, and not their rational relaxation.

Optimizations In Sec. 3 we describe the local value-determination algorithm which adds a quadratic number of constraints in terms of policies. In practice this is often prohibitively expensive. The quadratic blow-up results from the "input" constraints to each policy, which determine the bounds on the input variables. We propose multiple optimization heuristics which increase the performance.

As a motivation example, consider a long trace ending with an assignment $\mathbf{x} = \mathbf{1}$. If this trace is feasible and chosen as a policy for the template x, the output bound will be 1, regardless of the input. With that example in mind, consider the abstraction procedure from which we derive the bound d for the template t. Let $(_, \phi(X, X'))$ be the intermediate state used for the abstraction (Alg. 2). We check the satisfiability of $\phi(X, X') \wedge t \cdot X' > d$; if the result is unsatisfiable, then the bound of t is *input-independent*, that is, it is always d if the trace is feasible. Thus we do not add the *input constraints* for the associated policy in the value-determination stage. Also, when computing the map *influencing* from nodes to abstracted states for the value-determination problem, we do not follow the backpointers for input-independent policies, potentially drastically shrinking the resulting constraint set. Similarly, if none of the variables of the "input template" occur in the policy, the initial constraint is irrelevant and can be dropped.

Furthermore, we limit the size of the value-determination LP by merging some of the unknowns. This is equivalent to equating these variables, thus strengthening the constraints. The result thus under-approximates the fixed point of the selected policy. If it is less than the policy fixed point (not inductive with respect to the policy), we fall back to the normal value determination.

During *abstraction* on the intermediate state (a_0, ψ) , we may skip the optimization query based on a syntactic check: if we are optimizing for the template t, and none of the variables of t occur in ψ , we return the bound associated with $a_0[t]$.

Additionally, during maximization we add a redundant lemma to the set of constraints that specifies that the resultant value has to be strictly larger than the current bound. This significantly speeds up the maximization by shrinking the search space.

Iteration Order In our experiments, we have found performance to depend on the iteration order. Experimentally, we have determined a good iteration order to be the recursive iteration strategy using the weak topological ordering [20]. This is a strength of LPI: it blends into existing iteration strategies.

Unrolling We unroll loops up to depth 2, as some invariants can only be expressed in the template constraints domain in the presence of unrollings (e.g., invariants involving a variable whose initial value is set only inside the loop).

Abstraction Refinement for LPI As a template constraints domain can be configured by the number of templates present, it is a perfect candidate for refinement, as templates can be added to increase the precision of the analysis.

However, a full abstraction-refinement algorithm for LPI would be outside of the scope of this work, and thus to obtain the results we use a naive algorithm that iteratively tries progressively more precise and costly configurations until the program can be verified. The configurations we try are (in that order): (i) Intervals (ii) Octagons (iii) Previous + Unrolling (iv) Previous + Rich Templates $(\pm x \pm y \pm z)$ (v) Previous + Congruence Analysis.

5 Experiments

We have evaluated our tool on the benchmarks from the category "Loops" of the International Competition on Software Verification (SV-COMP'15) [21] consisting of 142 C programs, 93 of which are correct (the error property is unreachable). We have chosen this category for evaluation because its programs contain numerical assertions about variables modified in loops, whereas other categories of SV-COMP mostly involve variables with a small finite set of possible values that can be enumerated effectively. All experiments were performed with the same resources as in SV-COMP'15: an Intel Core i7-4770 quad-core CPU with 3.40 GHz, and limits of 15 GB RAM and 900 s CPU time per program. The tool is integrated inside the open-source verification framework CPAchecker [7], used configuration and detailed experimental results are available at http://lpi.metaworld.me.

We compare LPI (with abstraction refinement) with three tools representing different approaches to program analysis: **BLAST 2.7.3 (SV-COMP'15)** [22], which uses lazy abstraction, **PAGAI (git hash 254c2fc693)** [23], which uses abstract interpretation with path focusing, and **CPAchecker 1.3.10-svcomp15** (**SV-COMP'15)** [7], the winner of SV-COMP 2015 category "Overall", which uses an ensemble of different techniques: explicit value, k-induction, and lazy predicate abstraction. For LPI we use CPAchecker in version 1.4.10-lpi-vmcai16.

Because LPI is an incomplete approach, it can only produce safety proofs (no counter-examples). Thus in Table 1 we present the statistics on the number of safety proofs produced by different tools. The first five columns represent *differences* between approaches: the cell corresponding to the row A and a column B (read "A vs. B") displays the number of programs A could verify and B could not. In the column *Unique* we show the number of programs only the given tool could verify (out of the analyzers included in the comparison). The column *Verified* shows the total number of programs a tool could verify. The column *Incorrect* shows false positives: programs that contained a bug, yet were deemed correct by the tool — our current implementation unsoundly ignores integer overflows, as though the program used mathematical integers.¹⁰

¹⁰ It is possible to add sound overflow handling, as done in e.g. Astrée, to our approach, at the expense of extra engineering.

vs. PAGAI LPI BLAST CPAchecker Unique Verified Incorrect

PAGAI		4	13	15	1	52	1
LPI	13		20	20	7	61	1
BLAST	6	4		8	0	45	1
CPAchecker	21	17	21		12	58	2

Table 1: Number of verified programs of different tools



Each data point is an analyzed program, timeouts are excluded.

From this table we see that LPI verifies more examples than other tools can, including seven programs that others cannot.

Timing Results In Sec. 4 we have described the various possible configurations of LPI. As trying all possible combinations of features is exponential, tested configurations represent cumulative stacking of features. We present the timing comparison across those in the quantile plot in Fig. 3a, and in the legend we report the number of programs each configuration could verify. Each data point is an analyzed program, and the series are sorted separately for each configuration.

The quantile plot for timing comparison across different tools is shown in Fig. 3b. We have included two LPI configurations in the comparison: fastest (LPI-Intervals) and the most precise one (LPI-Refinement, switches to a more expensive strategy out of the ones in Fig. 3a if the program cannot be verified). From the plot we can see that LPI performance compares favorably with lazy abstraction, but that it is considerably outperformed by abstract interpretation. The initial difference in the analysis time between the CPACHECKER-based tools and others is due to JVM start-up time of about 2 seconds.

6 Conclusion and Future Work

We have demonstrated that LPI is a viable approach to program analysis, which can outperform state-of-the-art competitors either in precision (abstract interpretation), or both in precision and scalability (predicate abstraction). However, much work needs to be done to bring policy-iteration-based approaches to the level of maturity required for analyzing industrial-scale codebases, in particular:

- Sound handling of machine integers and floats, and overflow checking in particular. The only incorrect result given by LPI on the dataset was due to the unsound overflow handling. It is possible to check the obtained invariants for inductiveness using bitvectors or overflow checks.
- Template abstract domains are perfect candidates for *refinement*: dynamically adding templates during the analysis. Using counter-examples and refining the domain using CEGAR [24] approach is a promising research direction.

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