# The trace monoids in the queue monoid and in the direct product of two free monoids 

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#### Abstract

We prove that a trace monoid embeds into the queue monoid if and only if it embeds into the direct product of two free monoids. We also give a decidable characterization of these trace monoids.


## 1 Introduction

Trace monoids model the behavior of concurrent systems whose concurrency is governed by the use of joint resources. They were introduced into computer science by Mazurkiewicz in his study of Petri nets [10]. Since then, much work has been invested on their structure, see 4] for comprehensive surveys. A basic fact about trace monoids is that they can be embedded into the direct product of free monoids [1]. Since the proof of this fact is constructive, an upper bound for the number of factors needed in such a free product is immediate (it is the number $\alpha$ of cliques needed to cover the dependence alphabet). If the dependence alphabet is a path on $n$ vertices, than this upper bound equals the exact number, namely $n-1$. But there are cases where the exact number is considerably smaller (the examples are from [3]:

- If the independence alphabet is the disjoint union of two copies of $C_{4}$ (the cycle on four vertices), then $\alpha=4$, but 3 factors suffice.
- If the independence alphabet is the disjoint union of $n$ copies of $K_{k}$ (the complete graph on $k$ vertices), then $\alpha=k^{n}$, but $k$ factors suffice.

The strongest result in this respect is due to Kunc [7]: Given a $C_{3}$ - and $C_{4}$-free dependence alphabet and a natural number $k$, it is decidable whether the trace monoid embeds into the direct product of $k$ free monoids. In this paper, we extend this positive result to all dependence alphabets, but only for the case $k=2$. More precisely, we give a complete and decidable characterization of all independence alphabets whose generated trace monoid embeds into the direct product of two free monoids.

Queue monoids, another class of monoids, have been introduced recently [56]. They model the behavior of a single fifo-queue. Intuitively, the basic actions (i.e., generators of the monoid) are the action of writing the letter $a$ into the queue (denoted $a$ ) and reading the letter $a$ from the queue (denoted $\bar{a}$ ). Sequences of actions are equivalent if they induce the same state change on any queue. For instance, writing a symbol into the queue and reading another symbol from the other end of the queue are two actions that can be permuted without changing the overall behavior, symbolically: $a \bar{b} \equiv \bar{b} a$. But there are also more complex equivalences that can be understood as "conditional commutativity", e.g., $a b \bar{b} \equiv a \bar{b} b$. The unconditional commutations allow to embed the direct product of two free monoids into the queue monoid [6]. In [6], it is conjectured that the monoid $\mathbb{N}^{3}$ cannot be embedded into the queue monoid. Note that these two monoids are special trace monoids and that any trace monoid embedding into the direct product of two free monoids consequently embeds into the queue monoid. In this paper, we prove the conjecture from [6] and characterize, more generally, the class of trace monoids that embed into the queue monoid.

In summary, this paper characterized two classes of trace monoids defined by their embedability into $\{a, b\}^{*} \times\{c, d\}^{*}$ and into the queue monoid, respectively. As it turns out, these two classes are the same, i.e., a trace monoid embeds into the direct product of two free monoids if and only if it embeds into the queue monoid, and this property is decidable.

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## 2 Preliminaries and main result

### 2.1 The trace monoid

Trace monoids are meant to model the behavior of concurrent systems whose concurrency is governed by the use of joint resources. Here, we take a slightly more abstract view and say that two actions are independent if they use disjoint resources. More formally, an independence alphabet is a pair $(\Gamma, I)$ consisting of a countable (i.e., finite or of size $\left.\aleph_{0}\right)$ set $\Gamma$ and an irreflexive and symmetric relation $I \subseteq \Gamma^{2}$ called the independence relation. By $D=\Gamma^{2} \backslash I$, we denote the complementary dependence relation.

An independence alphabet $(\Gamma, I)$ induces a trace monoid as follows: Let $\equiv_{I}$ denote the least congruence on the free monoid $\Gamma^{*}$ with $a b \equiv_{I} b a$ for all pairs $(a, b) \in I$. Then the trace monoid associated with $(\Gamma, I)$ is the quotient $\mathbb{M}(\Gamma, I)=\Gamma^{*} / \equiv_{I}$, the equivalence class containing $u \in \Gamma^{*}$ is denoted $[u]_{I}$.

Thus the defining equations of the trace monoid are the equations $a b \equiv_{I} b a$ for some pairs of letters $(a, b)$.

We will only need very basic properties of the trace monoid $\mathbb{M}(\Gamma, I)$, namely the following:
Proposition 2.1. Let $(\Gamma, I)$ be an independence alphabet.
(1) Let $\Gamma=\bigcup_{i \in I} C_{i}$ with $D=\bigcup_{i \in I} C_{i} \times C_{i}$. Then the trace monoid $\mathbb{M}\left(\Gamma, \Gamma^{2} \backslash I\right)$ embeds into the monoid

$$
\prod_{i \in I}\{a, b\}^{*},
$$

i.e., into a direct product of free monoids [1].
(2) The trace monoid $\mathbb{M}(\Gamma, I)$ is cancellative, i.e., uvw $\equiv_{I} u v^{\prime} w$ implies $v \equiv_{I} v^{\prime}$ for all words $u, v, v^{\prime}, w \in \Gamma^{*}$.

In this paper, we will often use graph-theoretic terms to speak about an independence alphabet $(\Gamma, I)$ - where we identify $I$ with the set of edges $\{a, b\}$ for $(a, b) \in I$. In other words, we think of $(\Gamma, I)$ as a symmetric and loop-free graph. We will also take the liberty to write $(C, I)$ for the subgraph of $(\Gamma, I)$ induced by $C \subseteq \Gamma$. We call a connected component $C$ of $(\Gamma, I)$ nontrivial if it is not an isolated vertex. The connected component $C$ is bipartite if $I \cap C^{2} \subseteq\left(C_{1} \times C_{2}\right) \cup\left(C_{2} \times C_{1}\right)$ for some partition $C_{1} \uplus C_{2}$ of $C$. It is complete bipartite if $I \cap C^{2}=\left(C_{1} \times C_{2}\right) \cup\left(C_{2} \times C_{1}\right)$. Finally, an independence alphabet $(\Gamma, I)$ is $P_{4}$-free if no induced subgraph is isomorphic to $P_{4}$, i.e., if there are no four distinct vertices $a, b, c, d$ with $(a, b),(b, c),(c, d) \in I$ and $(b, d),(d, a),(a, c) \in D$.

Using this graph theoretic language, the sets $C_{i}$ in Proposition 2.1(1) form a covering of $(\Gamma, D)$ by cliques. It follows that the trace monoid $\mathbb{M}(\Gamma, I)$ can be embedded into the direct product of two free monoids whenever $(\Gamma, D)$ has a clique covering with two cliques. But the existence of a clique cover with two cliques is not necessary for such an embedding. As an example, consider the independence alphabet $(\Gamma, I)$ with $\Gamma=\left\{a_{i}, b_{i} \mid 0 \leq i<n\right\}$ and $I=\left\{\left(a_{i}, b_{i}\right),\left(b_{i}, a_{i}\right) \mid 0 \leq i<n\right\}$ (where $n \in \mathbb{N} \cup\left\{\aleph_{0}\right\}$ ). Then $D=\left\{\left(a_{i}, b_{j}\right),\left(b_{j}, a_{i}\right) \mid 0 \leq i, j<n, i \neq j\right\}$. Hence $|D|+n=n^{2}$ and all cliques in $(\Gamma, D)$ contain at most 2 elements. Our main result shows that, nevertheless, the trace monoid $\mathbb{M}(\Gamma, I)$ embeds into the direct product of two copies of $\{a, b\}^{*}$.

### 2.2 The queue monoid

The queue monoid models the behavior of a fifo-queue whose entries come from a set $A$. Consequently, the state of a valid queue is an element from $A^{*}$. In order to have a defined result even if a read action fails, we add the error state $\perp$. The basic actions are writing of the symbol $a \in A$ $\underline{i n t o}$ the queue (denoted $a$ ) and reading the symbol $a \in A$ from the queue (denoted $\bar{a}$ ). Formally, $\bar{A}$ is a disjoint copy of $A$ whose elements are denoted $\bar{a}$. Furthermore, we set $\Sigma=A \cup \bar{A}$. Hence,
the free monoid $\Sigma^{*}$ is the set of sequences of basic actions and it acts on the set $A^{*} \cup\{\perp\}$ by way of the function $:\left(A^{*} \cup\{\perp\}\right) \times \Sigma^{*} \rightarrow A^{*} \cup\{\perp\}$, which is defined as follows:

$$
q \cdot \varepsilon=q \quad q \cdot a u=q a . u \quad q \cdot \bar{a} u=\left\{\begin{array}{ll}
q^{\prime} \cdot u & \text { if } q=a q^{\prime} \\
\perp & \text { otherwise }
\end{array} \quad \perp \cdot u=\perp\right.
$$

for $q \in A^{*}, a \in A$, and $u \in \Sigma^{*}$.
Definition 2.2. Two words $u, v \in \Sigma^{*}$ are equivalent if $q . u=q . v$ for all queues $q \in A^{*}$. In that case, we write $u \equiv v$. The equivalence class wrt. $\equiv$ containing the word $u$ is denoted $[u]$.

Since $\equiv$ is a congruence on the free monoid $\Sigma^{*}$, we can define the quotient monoid $Q_{A}=\Sigma^{*} / \equiv$ that is called the monoid of queue actions or queue monoid for short.

Note that two queue monoids are not isomorphic if the generating sets have different size. But, for any generating set $A$, the queue monoid $Q_{A}$ embeds into $Q_{\{a, b\}}$ [6, Cor. 5.5] (the proof in [6] can easily be extended to infinite sets $A$ ). Since this paper is concerned with submonoids of $Q_{A}$, the concrete size of $A$ does not matter. Hence we will simply write $Q$ for $Q_{A}$, no matter what the set $A$ is.

Theorem 2.3 ([6, Theorem 4.3]). The equivalence relation $\equiv$ is the least congruence on the free monoid $\Sigma^{*}$ satisfying the following for all $a, b, c \in A$ :

$$
\begin{aligned}
a \bar{b} & \equiv \bar{b} a \text { if } a \neq b \\
a \bar{b} \bar{c} & \equiv \bar{b} a \bar{c} \\
a b \bar{c} & \equiv a \bar{c} b
\end{aligned}
$$

The second and third of these equations generalize nicely to words:
Lemma 2.4 ([6, Corollary 3.6]). Let $u, v, w \in A^{*}$.

- If $|u| \leq|w|$, then $u \overline{v w} \equiv \bar{v} u \bar{w}$.
- If $|u| \geq|w|$, then $u v \bar{w} \equiv u \bar{w} v$.

Let $\pi: \Sigma^{*} \rightarrow A^{*}$ be the homomorphism defined by $\pi(a)=a$ and $\pi(\bar{a})=\varepsilon$ for all $a \in A$. Similarly, define the homomorphism $\bar{\pi}: \Sigma^{*} \rightarrow A^{*}$ by $\pi(a)=\varepsilon$ and $\pi(\bar{a})=a$ for all $a \in A$. Then, from Theorem 2.3. we immediately get

$$
u \equiv v \Longrightarrow \pi(u)=\pi(v) \text { and } \bar{\pi}(u)=\bar{\pi}(v)
$$

for all words $u, v \in \Sigma^{*}$. Hence the homomorphisms $\pi$ and $\bar{\pi}$ define homomorphisms from $Q$ to $A^{*}$ by $[u] \mapsto \pi(u)$ and $[u] \mapsto \bar{\pi}(u)$. The words $\pi(u)$ and $\bar{\pi}(u)$ are called the positive and negative projection of $u$ (or $[u]$ ).

Ordering the equations from Theorem 2.3 from left to right, we obtain a semi-Thue system. This semi-Thue system is confluent and terminating. Hence any equivalence class of $\equiv$ has a unique normal form. To describe these normal forms, we write $\left\langle a_{1} a_{1} \ldots a_{n}, \overline{b_{1} b_{2}} \ldots \overline{b_{n}}\right\rangle$ for $a_{1} \overline{b_{1}} a_{2} \overline{b_{2}} \ldots a_{n} \overline{\overline{b_{n}}}$ (where $n \in \mathbb{N}$ and $a_{i}, b_{i} \in A$ for all $1 \leq i \leq n$ ). Then a word $u \in \Sigma^{*}$ is in normal form iff there are three words $u_{1}, u_{2}, u_{3} \in A^{*}$ with $u=\overline{u_{1}}\left\langle u_{2}, \overline{u_{2}}\right\rangle u_{3}$. We write $\operatorname{nf}(u)$ for the unique word from the equivalence class $[u$ ] in normal form. Furthermore, the mixed or central part of the word $\operatorname{nf}(u)$, i.e., the word $u_{2}$ with $\operatorname{nf}(u)=\overline{u_{1}}\left\langle u_{2}, \overline{u_{2}}\right\rangle u_{3}$ is denoted $\mu(u)$.

The importance of this word $\mu(u)$ is described by the following observation: Let $u, v \in \Sigma^{*}$. Then the following are equivalent:

1. $u \equiv v$
2. $\operatorname{nf}(u)=\operatorname{nf}(v)$
3. $\pi(u)=\pi(v), \bar{\pi}(u)=\bar{\pi}(v)$, and $\mu(u)=\mu(v)$

Next, we describe the normal form of the product of two words in normal form. For this, we need the concept of the overlap of two words: Let $u, v \in A^{*}$. Then the overlap of $u$ and $v$ is the longest word $x$ that is both, a suffix of $u$ and a prefix of $v$. We write $\operatorname{ol}(u, v)$ for this overlap.
Theorem 2.5 ([6, Theorem 5.5]). Let $u, v \in A^{*}$. Then $\operatorname{nf}(u v)=\bar{s}\langle\mu(u v), \overline{\mu(u v)}\rangle t$ with

$$
\begin{aligned}
\mu(u v) & =\operatorname{ol}(\mu(u) \bar{\pi}(v), \pi(u) \mu(v)) \\
s \mu(u v) & =\bar{\pi}(u v) \text { and } \\
\mu(u v) t & =\pi(u v)
\end{aligned}
$$

In the following lemma we describe the normal form of the $n$-th power of an element of the queue monoid $Q$. This will turn out useful in the following considerations.
Lemma 2.6. Let $u \in A^{*}$. Then for every $n \geq 1$ we have

$$
\mu\left(u^{n}\right)=\operatorname{ol}\left(\mu(u) \bar{\pi}(u)^{n-1}, \pi(u)^{n-1} \mu(u)\right) .
$$

Proof. We prove the lemma by induction on $n$. The statement is obvious for $n=1$.
Let $n>1$ and assume that the statement holds for every $i<n$. Then by the induction hypothesis

$$
\mu\left(u^{n-1}\right)=\operatorname{ol}\left(\mu(u) \bar{\pi}(u)^{n-2}, \pi(u)^{n-2} \mu(u)\right)
$$

Now set

$$
s=\mathrm{ol}\left(\mu\left(u^{n-1}\right) \bar{\pi}(u), \pi\left(u^{n-1}\right) \mu(u)\right)
$$

such that $\mu\left(u^{n}\right)=\mu\left(u^{n-1} u\right)=s$ by Theorem 2.5. It remains to be shown that $s$ is the overlap of the words $\mu(u) \bar{\pi}(u)^{n-1}$ and $\pi(u)^{n-1} \mu(u)$. To simplify notation, let $s^{\prime}$ denote this overlap, i.e., set

$$
s^{\prime}=\operatorname{ol}\left(\mu(u) \bar{\pi}(u)^{n-1}, \pi(u)^{n-1} \mu(u)\right)
$$

Note that $s$ is a suffix of $\mu\left(u^{n-1}\right) \bar{\pi}(u)$. Since $\mu\left(u^{n-1}\right)$ is a suffix of $\mu(u) \bar{\pi}(u)^{n-2}$, it follows that $s$ is a suffix of $\mu(u) \bar{\pi}(u)^{n-1}$. By its very definition, $s$ is also a prefix of $\pi\left(u^{n-1}\right) \mu(u)$. Since $s^{\prime}$ is the longest word that is both, a suffix of $\mu(u) \pi(u)^{n-1}$ and a prefix of $\pi(u)^{n-1} \mu(u)$, it follows that $\left|s^{\prime}\right| \geq|s|$. Since $s=\operatorname{ol}\left(\mu\left(u^{n-1}\right) \bar{\pi}(u), \pi\left(u^{n-1}\right) \mu(u)\right)$, we get $|s|-\left|\mu\left(u^{n-1}\right)\right| \leq|\bar{\pi}(u)|$, i.e., $|s| \leq\left|\mu\left(u^{n-1}\right) \bar{\pi}(u)\right|$. Since both, $s^{\prime}$ and $\mu\left(u^{n-1}\right) \bar{\pi}(u)$ are suffixes of $\mu(u)(\bar{\pi}(u))^{n-1}$, it follows that $s^{\prime}$ is a suffix of $\mu\left(u^{n-1}\right) \bar{\pi}(u)$. Since it is also a prefix of $\pi(u)^{n-1} \mu(u)$, we get $|s| \geq\left|s^{\prime}\right|$. Hence we showed $|s|=\left|s^{\prime}\right|$. Consequently, $s$ and $s^{\prime}$ are prefixes of $\pi\left(u^{n-1}\right) \mu(u)$ of the same length and therefore $s=s^{\prime}$.

### 2.3 The main result

The results of this paper are summarised in the following theorem. It characterizes those trace monoids that can be embedded into the queue monoid as well as those that embed into the direct product of two free monoids. In particular, these two classes of trace monoids are the same. And, in addition, given a finite independence alphabet, it is decidable whether the generated trace monoid falls into this class.
Theorem 2.7. Let $(\Gamma, I)$ be a countable independence alphabet. Then the following are equivalent:
(1) The trace monoid $\mathbb{M}(\Gamma, I)$ embeds into the queue monoid $Q$.
(2) The trace monoid $\mathbb{M}(\Gamma, I)$ embeds into the direct product $\{a, b\}^{*} \times\{c, d\}^{*}$ of two free monoids.
(3) One of the following conditions hold:
(3.a) All nodes in $(\Gamma, I)$ have degree $\leq 1$.
(3.b) The independence alphabet $(\Gamma, I)$ has only one non-trivial connected component and this component is complete bipartite.
The implication "(2) implies (1)" follows immediately from [5, Prop 8.2] since there, we showed that $\{a, b\}^{*} \times\{c, d\}^{*}$ embeds into the queue monoid $Q$. In the following section, we present embeddings of $\mathbb{M}(\Gamma, I)$ whenever $(\Gamma, I)$ satisfies condition (3). The main work here is concerend with independence alphabets satisfying (3.a). The subsequent section shows that any trace monoid that embeds into the queue monoid satisfies condition (3). Technically, this proof is much harder than the first one.

## 3 (3) implies (2) in Theorem 2.7

Let $(\Gamma, I)$ be an independence alphabet satisfying (3.a) or (3.b) of Theorem 2.7. We will prove that $\mathbb{M}(\Gamma, I)$ embeds into the direct product of two free monoids (Lemma 3.1).

Lemma 3.1. Let ( $\Gamma, I$ ) be an (at most countably infinite) independence alphabet such that all nodes in $(\Gamma, I)$ have degree $\leq 1$. Then $\mathbb{M}(\Gamma, I)$ embeds into the direct product of two countably infinite free monoids.

Proof. Consider the independence alphabet $(\Sigma, I)$ with $\Sigma=\left\{a_{i}, b_{i} \mid i \in \mathbb{N}\right\}$ and

$$
I=\left\{\left(a_{i}, b_{i}\right),\left(b_{i}, a_{i}\right) \mid i \in \mathbb{N}\right\} .
$$

Then $(\Gamma, I)$ can be seen as a sub-alphabet of $(\Sigma, I)$ so that $\mathbb{M}(\Gamma, I)$ embeds into $\mathbb{M}(\Sigma, I)$.
We embed $\mathbb{M}(\Sigma, I)$ into the direct product

$$
M=\left\{c_{i} \mid i \in \mathbb{N}\right\} \times\left\{d_{i} \mid i \in \mathbb{N}\right\} .
$$

Note that in this monoid $\left(c_{i}, d_{i}\right)$ and $\left(c_{i}, d_{i} d_{i}\right)$ commute. Hence there is a homomorphism $\eta: \mathbb{M}(\Sigma, I) \rightarrow$ $M$ with $\eta\left(a_{i}\right)=\left(c_{i}, d_{i}\right)$ and $\eta\left(b_{i}\right)=\left(c_{i}, d_{i} d_{i}\right)$ for all $i \in \mathbb{N}$.

To show that this homomorphism is injective, we use lexicographic normal forms. So let $\sqsubseteq$ be a linear order on $\Sigma$ with $a_{i} \sqsubset b_{i}$ for all $i \in \mathbb{N}$. Now let $u \in \Sigma^{*}$ be in lexicographic normal form wrt. $\sqsubseteq$. Then the word $u$ has the form

$$
u=a_{i_{1}}^{k_{1}} b_{i_{1}}^{\ell_{1}} a_{i_{2}}^{k_{2}} b_{i_{2}}^{\ell_{2}} \cdots a_{i_{s}}^{k_{s}} b_{i_{s}}^{\ell_{s}}
$$

where $i_{a} \in \mathbb{N}, k_{a}+\ell_{a}>0$ for all $1 \leq a \leq s$ and $i_{a} \neq i_{a+1}$ for all $1 \leq a<s$. The image of $u$ equals

$$
\eta(u)=\left(\begin{array}{cccc}
c_{i_{1}}^{k_{1}+\ell_{1}} & c_{i_{2}}^{k_{2}+\ell_{2}} & \cdots & c_{i_{s}}^{k_{s}+\ell_{s}} \\
d_{i_{1}}^{k_{1}+2 \ell_{1}} & d_{i_{2}}^{k_{2}+2 \ell_{2}} & \cdots & d_{i_{s}^{s}+2 \ell_{s}}^{s}
\end{array}\right) .
$$

Next let also $v$ be a word in lexicographic normal form:

$$
v=a_{j_{1}}^{m_{1}} b_{j_{1}}^{n_{1}} a_{j_{2}}^{m_{2}} b_{j_{2}}^{n_{2}} \cdots a_{j_{t}}^{m_{t}} b_{j_{t}}^{n_{t}}
$$

where $j_{a} \in \mathbb{N}, m_{a}+n_{a}>0$ for all $1 \leq a \leq t$ and $j_{a} \neq j_{a+1}$ for all $1 \leq a<t$. The image of $u^{\prime}$ equals

$$
\eta(v)=\left(\begin{array}{cccc}
c_{j_{1}}^{m_{1}+n_{1}} & c_{j_{2}}^{m_{2}+n_{2}} & \cdots & c_{j_{t}+n_{t}}^{m_{t}} \\
d_{j_{1}}^{m_{1}}+2 n_{1} & d_{j_{2}}^{m_{2}+2_{2}} & \cdots & d_{j_{t}}^{m_{t}+2_{t}}
\end{array}\right) .
$$

Suppose $\eta(u)=\eta(v)$. Since all the exponents of $c_{i}$ and $d_{i}$ in the expressions for $\eta(u)$ and for $\eta(v)$ are positive and consecutive $c_{i}$ and $d_{i}$ have distinct indices, we obtain $s=t, i_{a}=j_{a}, k_{a}+\ell_{a}=m_{a}+n_{a}$ and $k_{a}+2 \ell_{a}=m_{a}+2 n_{a}$ for all $1 \leq a \leq s$. Hence $k_{a}=m_{a}$ and $\ell_{a}=n_{a}$ for all $1 \leq a \leq s$ and therefore $u=v$. Hence $\eta$ embeds $\mathbb{M}(\Sigma, I)$ into $M$ and we get

$$
\mathbb{M}(\Gamma, I) \hookrightarrow \mathbb{M}(\Sigma, I) \hookrightarrow M
$$

Theorem 3.2. Let $(\Gamma, I)$ be an independence alphabet such that one of the following conditions holds:

1. all nodes in $(\Gamma, I)$ have degree $\leq 1$ or
2. $(\Gamma, I)$ has only one non-trivial connected component and this component is complete bipartite

Then $M(\Gamma, I)$ embeds into $\{a, b\}^{*} \times\{c, d\}^{*}$.

Proof. Let $(\Gamma, I)$ be such that the first condition holds, i.e., all nodes in $(\Gamma, I)$ have degree $\leq 1$. Then by Lemma 3.1 there is an embedding of $M(\Gamma, I)$ into a direct product of two countably infinite free monoids.

Now let $(\Gamma, I)$ be such that the second condition holds, i.e., $(\Gamma, I)$ has only one non-trivial connected component and this component is complete bipartite. In other words, $\Gamma=\Gamma_{1} \uplus \Gamma_{2} \uplus \Gamma_{3}$ with $I=\Gamma_{1} \times \Gamma_{2} \cup \Gamma_{2} \times \Gamma_{1}$. Then the corresponding dependence alphabet ( $\Gamma, D$ ) can be covered by the two cliques induced by $\Gamma_{1} \cup \Gamma_{3}$ and $\Gamma_{2} \cup \Gamma_{3}$. Consequently, [2, Corollary 1.4.5 (General Embedding Theorem), p. 26] implies that $M(\Gamma, I)$ is a submonoid of a direct product of two countably infinite free monoids.

Note that the countably infinite free monoid $\left\{a_{i} \mid i \in \mathbb{N}\right\}^{*}$ embeds into $\{a, b\}^{*}$ via $a_{i} \mapsto a^{i} b$. Hence, in any case, $\mathbb{M}(\Gamma, I)$ embeds into $\{a, b\}^{*} \times\{c, d\}^{*}$.

## 4 (1) implies (3) in Theorem 2.7

Definition 4.1. Let $(\Gamma, I)$ be an independence alphabet and $\eta: \mathbb{M}(\Gamma, I) \hookrightarrow Q$ be an embedding. We partition $\Gamma$ into sets $\Gamma_{+}, \Gamma_{-}$, and $\Gamma_{ \pm}$according to the emptiness of the projections of $\eta(a)$ :
$-a \in \Gamma_{+}$iff $\pi(\eta(a)) \neq \varepsilon$ and $\bar{\pi}(\eta(a))=\varepsilon$.
$-a \in \Gamma_{-}$iff $\pi(\eta(a))=\varepsilon$ and $\bar{\pi}(\eta(a)) \neq \varepsilon$.
$-a \in \Gamma_{ \pm}$iff $\pi(\eta(a)) \neq \varepsilon$ and $\bar{\pi}(\eta(a)) \neq \varepsilon$.
We will prove the following:
$-\left(\Gamma_{+} \cup \Gamma_{-}, I\right)$ is complete bipartite (Proposition 4.2).

- Every node $a \in \Gamma_{ \pm}$has degree $\leq 1$ (Corollary 4.11 which is the most difficult part of the proof).
- Any letter from $\Gamma_{+} \cup \Gamma_{-}$is connected to any edge (Proposition 4.4).
- The graph $(\Gamma, I)$ is $P_{4}$-free (Proposition 4.13).

At the end of this section, we infer that the independence alphabet $(\Gamma, I)$ has the required property from Theorem 2.7 (3).

### 4.1 The set $\Gamma_{+} \cup \Gamma_{-}$induces a complete bipartite subgraph of $(\Gamma, I)$

Proposition 4.2. Let $(\Gamma, I)$ be an independence alphabet, let $\eta: \mathbb{M}(\Gamma, I) \hookrightarrow Q$ be an embedding . Then $\left(\Gamma_{+}, I\right)$ and $\left(\Gamma_{-}, I\right)$ are discrete and $\left(\Gamma_{+} \cup \Gamma_{-}, I\right)$ is complete bipartite.

Proof. We first show that $\left(\Gamma_{+}, I\right)$ is discrete.
Towards a contradiction, suppose there are $a, b \in \Gamma_{+}$with $(a, b) \in I$. Let $u=\pi(\eta(a))$ and $v=\pi(\eta(b))$. Since $\pi \circ \eta: \mathbb{M}(\Gamma, I) \rightarrow A^{*}$ is a homomorphism and since $[a b]_{I}=[b a]_{I}$, we get $u v=v u$. Hence $u$ and $v$ have a common root, i.e., there is a word $p$ and there are $i, j>0$ with $u=p^{i}$ and $v=p^{j}$. Hence

$$
\pi\left(\eta(a)^{j}\right)=u^{j}=v^{i}=\pi\left(\eta(b)^{i}\right)
$$

Clearly, we also have

$$
\bar{\pi}\left(\eta(a)^{j}\right)=\varepsilon=\bar{\pi}\left(\eta(b)^{i}\right)
$$

Hence

$$
\eta(a)^{j}=\left[u^{j}\right]=\left[v^{i}\right]=\eta(b)^{i} .
$$

Since $\eta$ is injective, this implies $a^{i} \equiv_{I} b^{j}$ and therefore $a=b$, contradicting $(a, b) \in I$. Hence, there are no $a, b \in \Gamma_{+}$with $(a, b) \in I$, i.e., $\left(\Gamma_{+}, I\right)$ is discrete.

Symmetrically, also $\left(\Gamma_{-}, I\right)$ is discrete.

It remains to be shown that $(a, b) \in I$ for any $a \in \Gamma_{+}$and $b \in \Gamma_{-}$. So let $a \in \Gamma_{+}$and $b \in \Gamma_{-}$. Then there are words $u, v \in A^{*}$ with $\eta(a)=[u]$ and $\eta(b)=[\bar{v}]$ (note that $u$ and $v$ are nonempty since $\eta$ is an injection). We have the following:

$$
\begin{array}{rlr}
\eta\left(a b b^{|u|}\right) & =\left[u \overline{v v}{ }^{|u|}\right] \\
& =\left[\bar{v} u \bar{v}^{|u|}\right] \\
& =\eta\left(b a b^{|u|}\right) & \quad \text { by Lemma 2.4] since }|u| \leq\left|v^{|u|}\right|
\end{array}
$$

Since $\eta$ is injective, this implies $a b b^{|u|} \equiv \equiv_{I} b a b^{|u|}$ and therefore $a b \equiv_{I} b a$. Now $(a, b) \in I$ follows from $a \neq b$.

### 4.2 Nodes from $\Gamma_{+} \cup \Gamma_{-}$are connected to any edge

Lemma 4.3. Let $u, v, w \in \Sigma^{+}$such that $\bar{\pi}(u)=\varepsilon$, vw $\equiv w v$ and $v \neq w$. Then there exist vectors $\vec{x}=\left(x_{u}, x_{v}, x_{w}\right)$ and $\vec{y}=\left(y_{u}, y_{v}, y_{w}\right)$ in $\mathbb{N}^{3}$ such that $x_{v}+x_{w} \neq 0$ and

$$
\begin{equation*}
u^{x_{u}} v^{x_{v}} u w^{x_{w}} \equiv u^{y_{u}} w^{y_{w}} u v^{y_{v}} . \tag{1}
\end{equation*}
$$

(Note that the two sides of this equation differ in particular in the order of the words $v$ and $w$. )
Proof. Since $v w \equiv w v$, there exist primitive words $p$ and $q$ and natural numbers $a_{v}, a_{w}, b_{v}, b_{w}$ satisfying the following:

$$
\begin{array}{ll}
\pi(v)=p^{a_{v}} & \pi(w)=p^{a_{w}} \\
\bar{\pi}(v)=q^{b_{v}} & \bar{\pi}(w)=q^{b_{w}}
\end{array}
$$

Since $v, w \neq \varepsilon$, we get $a_{v}+b_{v} \neq 0 \neq a_{w}+b_{w}$.
We first show that there are natural numbers $x_{v}, x_{w}, y_{v}, y_{w}$ (not all zero) that satisfy the following system of linear equations.

$$
\left.\begin{array}{rl}
a_{v} x_{v} & =a_{w} y_{w}  \tag{2}\\
a_{w} x_{w} & =a_{v} y_{v} \\
b_{v} x_{v}+b_{w} x_{w} & =b_{w} y_{w}+b_{v} y_{v}
\end{array}\right\}
$$

If $a_{v}=0$, then set $x_{v}=y_{v}=1$ and $x_{w}=y_{w}=0$. Symmetrically, if $a_{w}=0$, we set $x_{v}=y_{v}=0$ and $x_{w}=y_{w}=1$. If $a_{v} b_{w}=a_{w} b_{v}$, then set $x_{v}=y_{v}=a_{w}+b_{w}>0$ and $x_{w}=y_{w}=a_{v}+b_{v}>0$.

Now consider the case $a_{v} \neq 0 \neq a_{w}$ and $a_{v} b_{w} \neq a_{w} b_{v}$. The system (2) has a nontrivial solution over the field $\mathbb{Q}$. Consequently, there are integers $x_{v}, x_{w}, y_{v}, y_{w}$ (not all zero) satisfying these equations. We show $x_{v}>0 \Longleftrightarrow x_{w}>0$ : First note that $x_{v} \neq 0$ iff $y_{w} \neq 0$ and $x_{w} \neq 0$ iff $y_{v} \neq 0$. Since not all of the integers $x_{v}, x_{w}, y_{v}, y_{w}$ are zero, we get $x_{v} \neq 0$ or $x_{w} \neq 0$. Furthermore, since we have a solution, we get

$$
y_{w}=\frac{a_{v}}{a_{w}} x_{v} \text { and } y_{v}=\frac{a_{w}}{a_{v}} x_{w}
$$

Substituting these into the third equation yields

$$
\left(b_{v}-b_{w} \frac{a_{v}}{a_{w}}\right) \cdot x_{v}=\left(b_{v} \frac{a_{w}}{a_{v}}-b_{w}\right) \cdot x_{w}=\left(b_{v}-b_{w} \frac{a_{v}}{a_{w}}\right) \cdot \frac{a_{w}}{a_{v}} \cdot x_{w} .
$$

From $a_{v} b_{w} \neq a_{w} b_{v}$, we get $b_{v}-b_{w} \frac{a_{v}}{a_{w}} \neq 0$. Hence $x_{v}=\frac{a_{w}}{a_{v}} \cdot x_{w}$ and therefore $a_{v} x_{v}=a_{w} x_{w}$ follow. Now $a_{v}, a_{w}>0$ imply $x_{v}>0 \Longleftrightarrow x_{w}>0$. Consequently, all of $x_{v}, x_{w}, y_{v}, y_{w}$ are non-negative or all are non-positive. Hence $\left|x_{v}\right|,\left|x_{w}\right|,\left|y_{v}\right|,\left|y_{w}\right|$ is a solution to the system (2) in natural numbers as required.

From now on, let $x_{v}, x_{w}, y_{v}, y_{w} \in \mathbb{N}$ be a nontrivial solution of the system (22). Furthermore, let $x_{u}=y_{u} \in \mathbb{N}$ such that $\left|\bar{\pi}\left(v^{x_{v}} u w^{x_{w}}\right)\right| \leq|u| \cdot x_{u}=\left|u^{x_{u}}\right|$. Then we have the following:

$$
\begin{array}{rlr}
u^{x_{u}} v^{x_{v}} u w^{x_{w}} & \equiv u^{x_{u}} \overline{\bar{\pi}\left(v^{x_{v}} u w^{x_{w}}\right)} \pi\left(v^{x_{v}} u w^{x_{w}}\right) & \\
& =u^{x_{u}} \bar{q}^{b_{v} x_{v}+b_{w} x_{w}} p^{a_{v} x_{v}} u p^{a_{w} x_{w}} & \\
& =u^{y_{u}} \bar{q}^{b_{w} y_{w}+b_{v} y_{v}} p^{a_{w} y_{w}} u p^{a_{v} y_{v}} & \\
& =u^{y_{u}} \bar{\pi}\left(w^{y_{w}} u v^{y_{v}}\right) & \pi\left(w^{y_{w}} u v^{y_{v}}\right) \\
& \equiv u^{y_{u}} w^{y_{w}} u v^{y_{v}} &
\end{array}
$$

Thus, we found the vectors $\vec{x}$ and $\vec{y}$ satisfying Equation (11) with $x_{v}+x_{w} \neq 0$.
Proposition 4.4. Let $(\Gamma, I)$ be an independence alphabet and let $\eta: \mathbb{M}(\Gamma, I) \hookrightarrow Q$ be an embedding. Let $a \in \Gamma_{+} \cup \Gamma_{-}$and $b, c \in \Gamma$ with $(b, c) \in I$. Then $(a, b) \in I$ or $(a, c) \in I$.

Proof. If $a \in\{b, c\}$, we get $(a, b) \in I$ or $(a, c) \in I$ from $(b, c) \in I$. So assume $a \notin\{b, c\}$. There are words $u, v, w \in \Sigma^{+}$with $\eta(a)=[u], \eta(b)=[v]$, and $\eta(c)=[w]$. Since $(b, c) \in I$, we get $[v w]=\eta(b c)=\eta(c b)=[w v]$ and therefore $v w \equiv w v$. Furthermore, $[v]=\eta(b) \neq \eta(c)=[w]$ since $\eta$ is injective and since $b \neq c$ follows from $(b, c) \in I$. Hence in particular $v \neq w$.

We first consider the case $a \in \Gamma_{+}$, i.e., $\bar{\pi}(u)=\varepsilon$. From Lemma 4.3, we find natural numbers $x_{u}, x_{v}, x_{w}, y_{u}, y_{v}, y_{w}$ with $u^{x_{u}} v^{x_{v}} u w^{x_{w}} \equiv u^{y_{u}} w^{y_{w}} u v^{y_{v}}$ and $x_{v}+x_{w}+y_{v}+y_{w} \neq 0$. Consequently,

$$
\begin{aligned}
\eta\left(a^{x_{u}} b^{x_{v}} a c^{x_{w}}\right) & =\left[u^{x_{u}} v^{x_{v}} u w^{x_{w}}\right] \\
& =\left[u^{y_{u}} w^{y_{w}} u v^{y_{v}}\right] \\
& =\eta\left(a^{y_{u}} c^{y_{w}} a b^{y_{v}}\right) .
\end{aligned}
$$

Since $\eta$ is injective, this implies

$$
a^{x_{u}} b^{x_{v}} a c^{x_{w}} \equiv_{I} a^{y_{u}} c^{y_{w}} a b^{y_{v}}
$$

If $x_{v} \neq 0$, then $(a, b) \in I$. Similarly, if $x_{w} \neq 0$, then $(a, c) \in I$. This settles the case $\bar{\pi}(u)=\varepsilon$.
Now let $\pi(u)=\varepsilon$. By duality, Lemma 4.3 yields natural numbers $x_{u}, x_{v}, x_{w}, y_{u}, y_{v}, y_{w}$ with $x_{v}+x_{w}+y_{v}+y_{w} \neq 0$ and $v^{x_{v}} u w^{x_{w}} u^{x_{u}} \equiv w^{y_{w}} u v^{y_{v}} u^{y_{u}}$. Then we can derive $(a, b) \in I$ or $(a, c) \in I$ as above.

### 4.3 Nodes from $\Gamma_{ \pm}$have degree $\leq 1$

Let $a \in \Gamma_{ \pm}$. Then there are nonempty primitive words $p$ and $q$ with $\pi(\eta(a)) \in p^{+}$and $\bar{\pi}(\eta(a)) \in q^{+}$, i.e., $p$ and $q$ are the primitive roots of the two projections of $\eta(a)$. The proof of the fact that $a$ has at most one neighbor in $(\Gamma, I)$ distinguishes two cases: first, we handle the case that $p$ and $q$ are not conjugated (recall that $p$ and $q$ are conjugated iff there are words $g \in A^{*}$ and $h \in A^{+}$with $p=g h$ and $q=h g)$. The second case, namely that $p$ and $q$ are conjugated, turns out to be far more difficult.

## Non-conjugated roots

Proposition 4.5. Let $(\Gamma, I)$ be an independence alphabet and let $\eta: \mathbb{M}(\Gamma, I) \hookrightarrow Q$ be an embedding. Let furthermore $b \in \Gamma$ and $p, q \in A^{+}$be primitive with $p \nsim q$ such that

$$
\pi(\eta(b)) \in p^{+} \text {and } \bar{\pi}(\eta(b)) \in q^{+}
$$

Then there is at most one letter $a \in \Gamma$ with $(a, b) \in I$.

Proof. Towards a contradiction, suppose there are distinct letters $a$ and $c$ in $\Gamma$ with $(a, b),(b, c) \in I$. Let

$$
u=\operatorname{nf}\left(\eta\left([a b]_{I}\right)\right), v=\operatorname{nf}(\eta(b)), \text { and } w=\operatorname{nf}\left(\eta\left([b c]_{I}\right)\right)
$$

Since $(a, b) \in I$, we have $a b \equiv_{I} b a$ and therefore $\eta\left([a b]_{I}\right)=\eta\left([b a]_{I}\right)$. This implies $\pi(\eta(a)) \pi(\eta(b))=$ $\pi(\eta(b)) \pi(\eta(a))$, i.e., the two words $\pi(\eta(a))$ and $\pi(\eta(b))$ commute in the free monoid. Since $\pi(\beta(b)) \in p^{+}$and $p$ is primitive, this implies $\pi(\eta(a)) \in p^{*}$ and therefore $\pi(u)=\pi(\eta(a)) \pi(\eta(b)) \in$ $p^{+}$. Similarly, $\bar{\pi}(u) \in q^{+}$as well as $\pi(w) \in p^{+}$and $\bar{\pi}(w) \in q^{+}$. Hence there are positive natural numbers $a_{u}, a_{v}, a_{w}, b_{u}, b_{v}, b_{w}$ such that the following hold:

$$
\begin{array}{lll}
\pi(u)=p^{a_{u}} & \pi(v)=p^{a_{v}} & \pi(w)=p^{a_{w}} \\
\bar{\pi}(u)=q^{b_{u}} & \bar{\pi}(v)=q^{b_{v}} & \bar{\pi}(w)=q^{b_{w}}
\end{array}
$$

First we prove that there exist vectors $\vec{x}=\left(x_{u}, x_{v}, x_{w}\right) \in \mathbb{N}^{3}$ and $\vec{y}=\left(y_{u}, y_{v}, y_{w}\right) \in \mathbb{N}^{3}$ with $\vec{x} \neq \vec{y}$ such that

$$
\begin{equation*}
u^{x_{u}} v^{x_{v}} w^{x_{w}} \equiv u^{y_{u}} v^{y_{v}} w^{y_{w}} . \tag{3}
\end{equation*}
$$

Consider the following system of linear equations:

$$
\left.\begin{array}{rl}
a_{u} x_{u}+a_{v} x_{v}+a_{w} x_{w} & =a_{u} y_{u}+a_{v} y_{v}+a_{w} y_{w}  \tag{4}\\
b_{u} x_{u}+b_{v} x_{v}+b_{w} x_{w} & =b_{u} y_{u}+b_{v} y_{v}+b_{w} y_{w}
\end{array}\right\}
$$

Using Gaussian elimination, we find a nontrivial rational solution. Hence the system (4) has an integer solution. Increasing all entries in the integer solution by some fixed number $n \in \mathbb{N}$ yields another solution. Hence we can choose $n$ large enough such that the resulting solution $\vec{x}=\left(x_{u}, x_{v}, x_{w}\right)$ and $\vec{y}=\left(y_{u}, y_{v}, y_{w}\right)$ satisfies

$$
\begin{aligned}
& -\vec{x}, \vec{y} \in \mathbb{N}^{3} \\
& -|p|+|q| \leq b_{w} \cdot x_{w} \cdot|q| \text { and }|p|+|q| \leq b_{w} \cdot y_{w} \cdot|q|, \text { and } \\
& -|p|+|q| \leq\left(a_{u} \cdot x_{u}+a_{v} \cdot x_{v}\right) \cdot|p| \text { and }|p|+|q| \leq\left(a_{u} \cdot y_{u}+a_{v} \cdot y_{v}\right) \cdot|p|
\end{aligned}
$$

Now we show that $\vec{x}$ and $\vec{y}$ is a solution to the Equation (3).
First, we have

$$
\begin{aligned}
\pi\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right) & =\left(p^{a_{u}}\right)^{x_{u}}\left(p^{a_{v}}\right)^{x_{v}}\left(p^{a_{w}}\right)^{x_{w}} \\
& =p^{a_{u} x_{u}+a_{v} x_{v}+a_{w} x_{w}} \\
& =p^{a_{u} y_{u}+a_{v} y_{v}+a_{w} y_{w}} \\
& =\pi\left(u^{y_{u}} v^{y_{v}} w^{y_{w}}\right)
\end{aligned}
$$

and similarly

$$
\bar{\pi}\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)=\bar{\pi}\left(u^{y_{u}} v^{y_{v}} w^{y_{w}}\right) .
$$

It remains to be shown that $\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)$ equals $\mu\left(u^{y_{u}} v^{y_{v}} w^{y_{w}}\right)$. Let $H$ denote the set of words that are both, a suffix of $q^{m}$ and a prefix of $p^{n}$ for some $m, n \in \mathbb{N}$. First note that $\mu(u)$ belongs to $H$ since it is a suffix of $\bar{\pi}(u)=q^{b_{u}}$ and a prefix of $\pi(u)=p^{a_{u}}$. By Lemma 2.6,

$$
\mu\left(u^{x_{u}}=\operatorname{ol}\left(\mu(u) q^{b_{u}\left(x_{u}-1\right)}, p^{a_{u}\left(x_{u}-1\right)} \mu(u)\right)\right.
$$

is a suffix of $\mu(u) q^{b_{u}\left(x_{u}-1\right)}$ which is a suffix of $q^{m}$ for some $m \in \mathbb{N}$ since $u \in H$. Symmetrically, $\mu\left(u^{x_{u}}\right)$ is a prefix of $p^{a_{u}\left(x_{u}-1\right)} \mu(u)$ and therefore a prefix of $p^{m}$ for some $m \in \mathbb{N}$ since $\mu(u) \in H$. Hence we get $\mu\left(u^{x_{u}}\right) \in H$. Using the analogous arguments, it follows that

$$
\mu\left(u^{x_{u}} v^{x_{v}}\right)=\operatorname{ol}\left(\mu\left(u^{x_{u}}\right) q^{b_{v} x_{v}}, p^{a_{u} x_{u}} \mu\left(v^{x_{v}}\right)\right) .
$$

belongs to $H$. Finally, also

$$
\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)=\operatorname{ol}\left(\mu\left(u^{x_{u}} v^{x_{v}}\right) q^{b_{w} x_{w}}, p^{a_{u} x_{u}+a_{v} x_{v}} \mu\left(w^{x_{w}}\right)\right)
$$

is an element of $H$ by analogous arguments. For our following argument, it is important to note that $\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)$ is a factor of $q^{m}$ and of $p^{m}$ for some $m \in \mathbb{N}$. Since $p \nsim q$, [8, Lemma7, p.282] implies $\left|\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)\right| \leq|p|+|q|$. Furthermore, we have $|p|+|q| \leq b_{w} \cdot x_{w} \cdot|q|=\left|q^{b_{w} x_{w}}\right|$ and $|p|+|q| \leq\left(a_{u} x_{u}+a_{v} x_{v}\right) \cdot|p|=\left|p^{a_{u} x_{u}+a_{v} x_{v}}\right|$ and therefore $\left|\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)\right| \leq\left|q^{b_{w} x_{w}}\right|$ and $\left|\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)\right| \leq\left|p^{a_{u} x_{u}+a_{v} x_{v}}\right|$. Consequently,

$$
\begin{aligned}
\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right) & =\operatorname{ol}\left(\mu\left(u^{x_{u}} v^{x_{v}}\right) q^{b_{w} x_{w}}, p^{a_{u} x_{u}+a_{v} x_{v}} \mu\left(w^{x_{w}}\right)\right) \\
& =\operatorname{ol}\left(q^{b_{w} x_{w}}, p^{a_{u} x_{u}+a_{v} x_{v}}\right) \\
& =\operatorname{ol}\left(q^{b_{u} x_{u}+b_{v} x_{v}+b_{w} x_{w}}, p^{a_{u} x_{u}+a_{v} x_{v}+a_{w} x_{w}}\right) \\
& =\operatorname{ol}\left(q^{b_{u} y_{u}+b_{v} y_{v}+b_{w} y_{w}}, p^{a_{u} y_{u}+a_{v} y_{v}+a_{w} y_{w}}\right) .
\end{aligned}
$$

By symmetric arguments, this last overlap equals $\mu\left(u^{y_{u}} v^{y_{v}} w^{y_{w}}\right)$. Thus, indeed,

$$
\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)=\mu\left(u^{y_{u}} v^{y_{v}} w^{y_{w}}\right) .
$$

Hence the two words $u^{x_{u}} v^{x_{v}} w^{x_{w}}$ and $u^{y_{u}} v^{y_{v}} w^{y_{w}}$ agree in their projections and their normal forms agree in their mixed part. Consequently, the normal forms of these two words coincide. Hence they are equivalent, i.e., as required, we found a non-trivial solution $\vec{x}, \vec{y}$ of Equation (3).

Finally we obtain

$$
\begin{aligned}
\eta\left(\left[(a b)^{x_{u}} b^{x_{v}}(b c)^{x_{w}}\right]_{I}\right) & =\left[u^{x_{u}} v^{x_{v}} w^{x_{w}}\right] \\
& =\left[u^{y_{u}} v^{y_{v}} w^{y_{w}}\right] \\
& =\eta\left(\left[(a b)^{y_{u}} b^{y_{v}}(b c)^{y_{w}}\right]_{I}\right) .
\end{aligned}
$$

Since $\eta$ is injective, and since $(a, b),(b, c) \in I$, this implies

$$
\begin{aligned}
a^{x_{u}} b^{x_{u}+x_{v}+x_{w}} c^{x_{w}} & \equiv_{I}(a b)^{x_{u}} b^{x_{v}}(b c)^{x_{w}} \\
& \equiv_{I}(a b)^{y_{u}} b^{y_{v}}(b c)^{y_{w}} \\
& \equiv_{I} a^{y_{u}} b^{y_{u}+y_{v}+y_{w}} c^{y_{w}} .
\end{aligned}
$$

Since the letters $a, b$, and $c$ are mutually distinct, we obtain

$$
\left(x_{u}, x_{u}+x_{v}+x_{w}, x_{w}\right)=\left(y_{u}, y_{u}+y_{v}+y_{w}, y_{w}\right)
$$

and therefore $\vec{x}=\vec{y}$. But this contradicts our choice of these two vectors as distinct. Thus there are no two distinct letters $a$ and $c$ with $(a, b),(b, c) \in I$.

Note that the above proof, essentially, proceeded as follows: we aimed at a nontrivial solution to Equation (3) in natural numbers. Length conditions on the positive and negative projections yielded the system of linear equations (4). Since this system consists of two equations in the unknown $x_{u}-y_{u}, x_{v}-y_{v}$ and $x_{w}-y_{w}$, it has an integer solution that can be increased by arbitrary natural numbers, i.e., there is a "sufficiently large" solution that makes the positive (and negative) projections of $u^{x_{u}} v^{x_{v}} w^{x_{w}}$ and $u^{y_{u}} v^{y_{v}} w^{y_{w}}$ equal. Using that this solution is "sufficiently large" and that $p$ and $q$ are not conjugated, we employed some combinatorics on words to prove that also the mixed parts of the normal forms of these two words were equal.

Conjugated roots We now want to prove a similar result in case $p$ and $q$ are conjugated. The proof, although technically more involved, will proceed similarly, i.e., we will determine a nontrivial solution of Equation (3). But presentationwise, we will proceed differently: First, Lemma 4.8 describes the mixed part of the normal form of $u^{x_{u}} v^{x_{v}} w^{x_{w}}$. Then, Lemma 4.9 determines a nontrival solution to (some rotation of) Equation (3), before, finally, Proposition 4.10 proves the analogous to Proposition 4.5 for conjugated roots.

We first prove a combinatorial lemma on words that are prefix of some power of $p$ and, at the same time, suffixes of some power of $q$ (where $p$ and $q$ are conjugated).

Lemma 4.6. Let $g \in A^{*}, h \in A^{+}$such that $p=g h$ and $q=h g$ are both primitive words. Let furthermore $y$ be some suffix of $q^{i}$ and some prefix of $p^{j}$ for some $i, j \geq 1$ such that $|y| \geq|q|$. Then $y=g q^{k}=p^{k} g$ where $k=\left\lfloor\frac{|y|}{|q|}\right\rfloor$.

Proof. Since $y$ is a suffix of $q^{i}$, there exist words $r \in A^{+}$and $s \in A^{*}$ with $y=s q^{k}$ and $q=r s$.
Since $p$ and $q$ are conjugate, their lengths are equal. Hence $k=\left\lfloor\frac{|y|}{|p|}\right\rfloor$. Since $y$ is a prefix of $p^{j}$, there exist words $s^{\prime} \in A^{*}$ and $t \in A^{+}$with $y=p^{k} s^{\prime}$ and $p=s^{\prime} t$.

Since $|p|=|q|, s q^{k}=y=p^{k} s^{\prime}$ implies $|s|=\left|s^{\prime}\right|$. Together with $s(r s)^{k}=s q^{k}=y=p^{k} s^{\prime}=$ $\left(s^{\prime} t\right)^{k} s^{\prime}=s^{\prime}\left(t s^{\prime}\right)^{k}$, this implies $s=s^{\prime}$. Since $k>0$, we also get $r=t$. Hence we obtained $q=r s$ and $p=s^{\prime} t=s r$. Since $p$ and $q$ are conjugate primitive words and $r \in A^{+}, ~ 9, ~ P r o p o s i t i o n ~ 1.3 .3, ~$ p. 8] implies $(g, h)=(s, r)$. This ensures in particular $g=s$ and therefore $y=g q^{k}=p^{k} g$.

Using this combinatorial lemma, we can often determine the overlap of two words via the following corollary:

Corollary 4.7. Let $g \in A^{*}, h \in A^{+}$such that $p=g h$ and $q=h g$ are both primitive words. Furthermore, let $p^{\prime}$ be a suffix of $p$ with $\left|p^{\prime}\right|<|p|$ and let $q^{\prime}$ be a prefix of $q$ with $\left|q^{\prime}\right|<|q|$.

Then for every $i, j \in \mathbb{N}$ we have $\operatorname{ol}\left(p^{\prime} g q^{i}, p^{j} g q^{\prime}\right)=g q^{\min (i, j)}$.
Proof. Let $y=\operatorname{ol}\left(p^{\prime} g q^{i}, p^{j} g q^{\prime}\right)$. Since $p^{\prime}$ is a suffix of $p=g h$, the word $p^{\prime} g q^{i}$ is a suffix of $g h g q^{i}=$ $g q^{i+1}$ and therefore of $q^{i+2}$. Hence also $y$ is a suffix of $q^{i+2}$. Similarly, $y$ is a prefix of $p^{j+2}$. By Lemma4.6, we obtain $y=g q^{k}=p^{k} g$ for some $k \in \mathbb{N}$ and it remains to be shown that $k=\min (i, j)$.

Note that

$$
k|q|+|g|=|y|
$$

$$
\leq\left|p^{\prime} g q^{i}\right| \quad \text { since } y \text { is a suffix of } p^{\prime} g q^{i}
$$

$$
<(i+1)|q|+|g| \quad \text { since }\left|p^{\prime}\right|<|p|=|q|
$$

This implies $k \leq i$ and, similarly, we can show $k \leq j$, i.e., $k \leq \min (i, j)$. On the other hand note that $g q^{\min (i, j)}=p^{\min (i, j)} g$ is a suffix of $p^{\prime} g q^{i}$ and a prefix of $p^{j} g q^{\prime} \operatorname{implying} k \geq \min (i, j)$ since $g q^{k}=\mathrm{ol}\left(p^{\prime} g q^{i}, p^{j} g q^{\prime}\right)$. Hence $k=\min (i, j)$.

Lemma 4.8. Let $g \in A^{*}, h \in A^{+}$such that $p=g h$ and $q=h g$ are primitive. Let $u, v, w \in Q$ such that the following holds for some $a_{u}, a_{v}, a_{w}, b_{u}, b_{v}, b_{w} \in \mathbb{N} \backslash\{0\}$, and $c_{u}, c_{v}, c_{w} \in \mathbb{Z}$ :

$$
\begin{array}{lll}
\pi(u)=p^{a_{u}} & \bar{\pi}(u)=q^{b_{u}} & c_{u}= \begin{cases}-1 & \text { if }|\mu(u)|<|g| \\
\left\lfloor\frac{|\mu(u)|}{|q|}\right\rfloor & \text { otherwise }\end{cases} \\
\pi(v)=p^{a_{v}} & \bar{\pi}(v)=q^{b_{v}} & c_{v}= \begin{cases}-1 & \text { if }|\mu(v)|<|g| \\
\left\lfloor\frac{|\mu(v)|}{|q|}\right\rfloor & \text { otherwise }\end{cases} \\
\pi(w)=p^{a_{w}} & \bar{\pi}(w)=q^{b_{w}} & c_{w}= \begin{cases}-1 & \text { if }|\mu(w)|<|g| \\
\left\lfloor\frac{\lfloor\mu(w) \mid}{|q|}\right\rfloor & \text { otherwise }\end{cases}
\end{array}
$$

Let $\vec{x}=\left(x_{u}, x_{v}, x_{w}\right) \in \mathbb{N}^{3}$ with $x_{u}, x_{v}, x_{w} \geq 2$. Then $\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)=g q^{X_{\vec{x}}}=p^{X_{\vec{x}}} g$ where

$$
X_{\vec{x}}=\min \left(\begin{array}{ccc}
\min \left(a_{u}, b_{u}\right) x_{u}+ & b_{v} x_{v}+ & b_{w} x_{w}+c_{u}-\min \left(a_{u}, b_{u}\right), \\
a_{u} x_{u}+\min \left(a_{v}, b_{v}\right) x_{v}+ & b_{w} x_{w}+c_{v}-\min \left(a_{v}, b_{v}\right) \\
a_{u} x_{u}+ & a_{v} x_{v}+\min \left(a_{w}, b_{w}\right) x_{w}+c_{w}-\min \left(a_{w}, b_{w}\right)
\end{array}\right) .
$$

Proof. From Lemma 2.6, we get

$$
\mu\left(u^{x_{u}}\right)=\operatorname{ol}\left(c(u) \bar{\pi}(u)^{x_{u}-1}, \pi(u)^{x_{u}-1} \mu(u)\right) .
$$

Depending on the length of $\mu(u)$, we distinguish three cases: First, let $|\mu(u)|<|g|$. Since $\mu(u)$ is a suffix of $\bar{\pi}(u) \in q^{*}=(h g)^{*}$, the word $\mu(u)$ is a suffix of $g$. Similarly, $\mu(u)$ is a prefix of $\pi(u) \in p^{*}=(g h)^{*}$ implying that $\mu(u)$ is a prefix of $g$. Then $a_{u}, b_{u}>0$ and $x_{u} \geq 2$ imply $b_{u}\left(x_{u}-1\right), a_{u}\left(x_{u}-1\right)>0$. Hence we can determine $\mu(u)$ as follows:

$$
\begin{aligned}
\mu\left(u^{x_{u}}\right) & =\operatorname{ol}\left(\mu(u) q^{b_{u}\left(x_{u}-1\right)}, p^{a_{u}\left(x_{u}-1\right)} \mu(u)\right) \\
& =\operatorname{ol}\left(\mu(u) h g q^{b_{u}\left(x_{u}-1\right)-1}, p^{a_{u}\left(x_{u}-1\right)-1} g h \mu(u)\right)
\end{aligned}
$$

$$
=g q^{\min \left(b_{u}\left(x_{u}-1\right)-1, a_{u}\left(x_{u}-1\right)-1\right)} \quad \text { by Corollary } 4.7
$$

$$
=g q^{\min \left(a_{u}, b_{u}\right) \cdot\left(x_{u}-1\right)+c_{u}} \quad \text { since } c_{u}=-1
$$

Next, consider the case $|g| \leq|\mu(u)|<|q|$. Then $\mu(u)$ is a prefix of $p=g h$ and a suffix of $q=h g$. Hence there are a prefix $h^{\prime}$ and a suffix $h^{\prime \prime}$ of $h$ with $\mu(u)=g h^{\prime}=h^{\prime \prime} g$. Now we can determine $\mu\left(u^{x_{u}}\right.$ as follows:

$$
\begin{aligned}
\mu\left(u^{x_{u}}\right) & =\operatorname{ol}\left(\mu(u) q^{b_{u}\left(x_{u}-1\right)}, p^{a_{u}\left(x_{u}-1\right)} \mu(u)\right) \\
& =\operatorname{ol}\left(h^{\prime} g q^{b_{u}\left(x_{u}-1\right)}, p^{a_{u}\left(x_{u}-1\right)} g h^{\prime \prime}\right)
\end{aligned}
$$

$$
=g q^{\min \left(b_{u}\left(x_{u}-1\right), a_{u}\left(x_{u}-1\right)\right)} \quad \text { by Corollary } 4.7
$$

$$
=g q^{\min \left(a_{u}, b_{u}\right) \cdot\left(x_{u}-1\right)+c_{u}} \quad \text { since } c_{u}=0
$$

Finally, let $|q| \leq|\mu(u)|$. Then $c_{2}=\left\lfloor\frac{\lfloor\mu(u) \mid}{|q|}\right\rfloor$. Furthermore, $\mu(u)$ is a prefix of $\pi(u) \in p^{*}$ and a suffix of $\bar{\pi}(u) \in q^{*}$. Hence, by Lemma 4.6, $\mu(u)=g q^{c_{u}}=p^{c_{u}} g$. Hence we can determine $\mu\left(u^{x_{u}}\right)$ as follows:

$$
\begin{array}{rlr}
\mu\left(u^{x_{u}}\right) & =\mathrm{ol}\left(\mu(u) q^{b_{u}\left(x_{u}-1\right)}, p^{a_{u}\left(x_{u}-1\right)} \mu(u)\right) & \\
& =\operatorname{ol}\left(g q^{c_{u}+b_{u}\left(x_{u}-1\right)}, p^{a_{u}\left(x_{u}-1\right)+c_{u}} g\right) & \\
& =g q^{\min \left(c_{u}+b_{u}\left(x_{u}-1\right), a_{u}\left(x_{u}-1\right)+c_{u}\right)} & \text { by Corollary } 4.7 \\
& =g q^{\min \left(a_{u}, b_{u}\right) \cdot\left(x_{u}-1\right)+c_{u}} &
\end{array}
$$

In other words, we proved

$$
\mu\left(u^{x_{u}}\right)=g q^{e_{u}}=p^{e_{u}} g
$$

with

$$
e_{u}=\min \left(a_{u}, b_{u}\right) \cdot\left(x_{u}-1\right)+c_{u} .
$$

Clearly, similar statements hold for $\mu\left(v^{x_{v}}\right)$ and $\mu\left(w^{x_{w}}\right)$.
In a second step, we determine $\mu\left(u^{x_{u}} v^{x_{v}}\right)$. We get

$$
\begin{aligned}
\mu\left(u^{x_{u}} v^{x_{v}}\right) & =\operatorname{ol}\left(\mu\left(u^{x_{u}}\right) \bar{\pi}\left(v^{x_{v}}\right), \pi\left(u^{x_{u}}\right) \mu\left(v^{x_{v}}\right)\right) \\
& =\operatorname{ol}\left(g q^{e_{u}} q^{b_{v} x_{v}}, p^{a_{u} x_{u}} p^{e_{v}} g\right) \\
& \left.=g q^{\min \left(e_{u}+b_{v} x_{v}, a_{u} x_{u}+e_{v}\right.}\right)
\end{aligned}
$$

In other words,

$$
\mu\left(u^{x_{u}} v^{x_{v}}\right)=g q^{e_{u v}}=p^{e_{u v}} g
$$

with

$$
e_{u v}=\min \left(e_{u}+b_{v} x_{v}, a_{u} x_{u}+e_{v}\right)
$$

In a third and last step, we determine $\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)$. Note that $\mu\left(w^{x_{w}}\right)=p^{e_{w}} g$. Then we get

$$
\begin{aligned}
\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right) & =\operatorname{ol}\left(\mu\left(u^{x_{u}} v^{x_{v}}\right) \bar{\pi}\left(w^{x_{w}}\right), \pi\left(u^{x_{u}} v^{x_{v}}\right) \mu\left(w^{x_{w}}\right)\right) \\
& =\operatorname{ol}\left(g q^{e_{u v}} q^{b_{w} x_{w}}, p^{a_{u} x_{u}+a_{v} x_{v}} q^{e_{w}} g\right) \\
& =g q^{\min \left(e_{u v}+b_{w} x_{w}, a_{u} x_{u}+a_{v} x_{v}+e_{w}\right)}
\end{aligned}
$$

Unraveling the definitions of $e_{u}, e_{v}, e_{w}$, and $e_{u v}$ yields

$$
\begin{aligned}
\min \left(e_{u v}+b_{w} x_{w}, a_{u} x_{u}+a_{v} x_{v}+e_{w}\right) & =\min \binom{\min \left(e_{u}+b_{v} x_{v}, a_{u} x_{u}+e_{v}\right)+b_{w} x_{w},}{a_{u} x_{u}+a_{v} x_{v}+e_{w}} \\
& =\min \left(\begin{array}{l}
e_{u}+b_{v} x_{v}+b_{w} x_{w} \\
a_{u} x_{u}+e_{v}+b_{w} x_{w} \\
a_{u} x_{u}+a_{v} x_{v}+e_{w}
\end{array}\right) \\
& =X_{\vec{x}} .
\end{aligned}
$$

Hence, we have indeed $\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)=g q^{X_{\vec{x}}}$.
Lemma 4.9. Let $g \in A^{*}, h \in A^{+}$such that $p=g h$ and $q=h g$ are primitive. Let $u^{\prime}, v^{\prime}, w^{\prime} \in \Sigma^{+}$ with $\pi\left(u^{\prime}\right), \pi\left(v^{\prime}\right), \pi\left(w^{\prime}\right) \in p^{+}$and $\bar{\pi}\left(u^{\prime}\right), \bar{\pi}\left(v^{\prime}\right), \bar{\pi}\left(w^{\prime}\right) \in q^{+}$.

Then there exist a rotation $(u, v, w)$ of $\left(u^{\prime}, v^{\prime}, w^{\prime}\right)$ and vectors $\vec{x}=\left(x_{u}, x_{v}, x_{w}\right) \in \mathbb{N}^{3}$ and $\vec{y}=\left(y_{u}, y_{v}, y_{w}\right) \in \mathbb{N}^{3}$ with $\vec{x} \neq \vec{y}$ such that

$$
\begin{equation*}
u^{x_{u}} v^{x_{v}} w^{x_{w}} \equiv u^{y_{u}} v^{y_{v}} w^{y_{w}} \tag{5}
\end{equation*}
$$

Proof. We choose the rotation $(u, v, w)$ such that one of the following three conditions hold:

1. $|\pi(u)|=|\bar{\pi}(u)|,|\pi(v)|=|\bar{\pi}(v)|$, and $|\pi(w)|=|\bar{\pi}(w)|$ or
2. $|\pi(u)|>|\bar{\pi}(u)|$ or
3. $|\pi(w)|<|\bar{\pi}(w)|$.

Given this rotation, we define the natural numbers $a_{u}, a_{v}, a_{w}, b_{u}, b_{v}, b_{w}, c_{u}, c_{v}, c_{w}$ as in Lemma 4.8.
Consider the following system of linear equations:

$$
\left.\begin{array}{r}
a_{u} x_{u}+a_{v} x_{v}+a_{w} x_{w}=a_{u} y_{u}+a_{v} y_{v}+a_{w} y_{w}  \tag{6}\\
b_{u} x_{u}+b_{v} x_{v}+b_{w} x_{w}=b_{u} y_{u}+b_{v} y_{v}+b_{w} y_{w}
\end{array}\right\}
$$

Using Gaussian elimination, we find a nontrivial rational solution. Hence the system (6) has an integer solution. Increasing all entries in this solution by the minimal entry plus 2 yields a nontrivial solution $\overrightarrow{x^{\prime}}=\left(x_{u}^{\prime}, x_{v}^{\prime}, x_{w}^{\prime}\right)$ and $\overrightarrow{y^{\prime}}=\left(y_{u}^{\prime}, y_{v}^{\prime}, y_{w}^{\prime}\right)$ with $\overrightarrow{x^{\prime}}, \overrightarrow{y^{\prime}} \in \mathbb{N}^{3}$ and $x_{u}^{\prime}, x_{v}^{\prime}, x_{w}^{\prime}, y_{u}^{\prime}, y_{v}^{\prime}, y_{w}^{\prime} \geq 2$.

From this solution of the system (6) of linear equations, we now construct a nontrivial solution $\vec{x}, \vec{y}$ that, in addition, satisfies $X_{\vec{x}}=X_{\vec{y}}$. This is done by considering the three possible cases for the rotation $(u, v, w)$ separately.

First, let $|\pi(u)|=|\bar{\pi}(u)|,|\pi(v)|=|\bar{\pi}(v)|$, and $|\pi(w)|=|\bar{\pi}(w)|$, i.e., $a_{u}=b_{u}, a_{v}=b_{v}$, and $a_{w}=b_{w}$. We obtain for the values $X_{\overrightarrow{x^{\prime}}}$ and $X_{\vec{y}^{\prime}}$ from Lemma 4.8

$$
\begin{aligned}
X_{\overrightarrow{x^{\prime}}} & =\min \left(\begin{array}{l}
a_{u} x_{u}^{\prime}+a_{v} x_{v}^{\prime}+a_{w} x_{w}^{\prime}+c_{u}-a_{u} \\
a_{u} x_{u}^{\prime}+a_{v} x_{v}^{\prime}+a_{w} x_{w}^{\prime}+c_{v}-a_{v} \\
a_{u} x_{u}^{\prime}+a_{v} x_{v}^{\prime}+a_{w} x_{w}^{\prime}+c_{w}-a_{w}
\end{array}\right) \\
& =\min \left(\begin{array}{l}
a_{u} y_{u}^{\prime}+a_{v} y_{v}^{\prime}+a_{w} y_{w}^{\prime}+c_{u}-a_{u} \\
a_{u} y_{u}^{\prime}+a_{v} y_{v}^{\prime}+a_{w} y_{w}^{\prime}+c_{v}-a_{v} \\
a_{u} y_{u}^{\prime}+a_{v} y_{v}^{\prime}+a_{w} y_{w}^{\prime}+c_{w}-a_{w}
\end{array}\right) \\
& =X_{\overrightarrow{y^{\prime}}} .
\end{aligned}
$$

This solves the first case.
Now, suppose $|\pi(u)|>|\bar{\pi}(u)|$ and therefore $a_{u}>b_{u}$. Then we find $k \geq 0$ such that the following hold:

$$
\begin{aligned}
b_{u}\left(x_{u}^{\prime}+k\right)+b_{v} x_{v}^{\prime}+b_{w} x_{w}^{\prime}+c_{u}-\min \left(a_{u}, b_{u}\right) & \leq a_{u}\left(x_{u}^{\prime}+k\right)+\min \left(a_{v}, b_{v}\right) x_{v}^{\prime}+b_{w} x_{w}^{\prime}+c_{v}-\min \left(a_{v}, b_{v}\right) \\
b_{u}\left(x_{u}^{\prime}+k\right)+b_{v} x_{v}^{\prime}+b_{w} x_{w}^{\prime}+c_{u}-\min \left(a_{u}, b_{u}\right) & \leq a_{u}\left(x_{u}^{\prime}+k\right)+a_{v} x_{v}^{\prime}+\min \left(a_{w}, b_{w}\right) x_{w}^{\prime}+c_{w}-\min \left(a_{w}, b_{w}\right) \\
b_{u}\left(y_{u}^{\prime}+k\right)+b_{v} y_{v}^{\prime}+b_{w} y_{w}^{\prime}+c_{u}-\min \left(a_{u}, b_{u}\right) & \leq a_{u}\left(y_{u}^{\prime}+k\right)+\min \left(a_{v}, b_{v}\right) y_{v}^{\prime}+b_{w} y_{w}^{\prime}+c_{v}-\min \left(a_{v}, b_{v}\right) \\
b_{u}\left(y_{u}^{\prime}+k\right)+b_{v} y_{v}^{\prime}+b_{w} y_{w}^{\prime}+c_{u}-\min \left(a_{u}, b_{u}\right) & \leq a_{u}\left(y_{u}^{\prime}+k\right)+a_{v} y_{v}^{\prime}+\min \left(a_{w}, b_{w}\right) y_{w}^{\prime}+c_{w}-\min \left(a_{w}, b_{w}\right)
\end{aligned}
$$

The reason is that in all cases, when increasing $k$, the right-hand side grows faster than the left-hand side. Set

$$
\vec{x}=\left(x_{u}^{\prime}+k, x_{v}^{\prime}, x_{w}^{\prime}\right) \text { and } \vec{y}=\left(y_{u}^{\prime}+k, y_{v}^{\prime}, y_{w}^{\prime}\right)
$$

Then this pair of vectors forms a non-trivial solution of the system (6). Since $b_{u}=\min \left(a_{u}, b_{u}\right)$, as a consequence we get in addition

$$
\begin{aligned}
X_{\vec{x}} & =b_{u} x_{u}+b_{v} x_{v}+b_{w} x_{w}+c_{u}-\min \left(a_{u}, b_{u}\right) \\
& =b_{u} y_{u}+b_{v} y_{v}+b_{w} y_{w}+c_{u}-\min \left(a_{u}, b_{u}\right) \\
& =X_{\vec{y}} .
\end{aligned}
$$

This solves the second case.
Finally, suppose $|\pi(w)|<|\bar{\pi}(w)|$ and therefore $a_{w}<b_{w}$. The argument now is dual to the previous case: We find $k \geq 0$ such that the following hold:

$$
\begin{aligned}
& a_{u} x_{u}^{\prime}+a_{v} x_{v}^{\prime}+a_{w}\left(x_{w}^{\prime}+k\right)+c_{u}-\min \left(a_{w}, b_{w}\right) \leq \min \left(a_{u}, b_{u}\right) x_{u}^{\prime}+b_{v} x_{v}^{\prime}+b_{w}\left(x_{w}^{\prime}+k\right)+c_{u}-\min \left(a_{u}, b_{u}\right) \\
& a_{u} x_{u}^{\prime}+a_{v} x_{v}^{\prime}+a_{w}\left(x_{w}^{\prime}+k\right)+c_{u}-\min \left(a_{w}, b_{w}\right) \leq a_{u} x_{u}^{\prime}+\min \left(a_{v}, b_{v}\right) x_{v}^{\prime}+b_{w}\left(x_{w}^{\prime}+k\right)+c_{v}-\min \left(a_{v}, b_{v}\right) \\
& a_{u} y_{u}^{\prime}+a_{v} y_{v}^{\prime}+a_{w}\left(y_{w}^{\prime}+k\right)+c_{u}-\min \left(a_{w}, b_{w}\right) \leq \min \left(a_{u}, b_{u}\right) y_{u}^{\prime}+b_{v} y_{v}^{\prime}+b_{w}\left(y_{w}^{\prime}+k\right)+c_{u}-\min \left(a_{u}, b_{u}\right) \\
& a_{u} y_{u}^{\prime}+a_{v} y_{v}^{\prime}+a_{w}\left(y_{w}^{\prime}+k\right)+c_{u}-\min \left(a_{w}, b_{w}\right) \leq a_{u} y_{u}^{\prime}+\min \left(a_{v}, b_{v}\right) y_{v}^{\prime}+b_{w}\left(y_{w}^{\prime}+k\right)+c_{v}-\min \left(a_{v}, b_{v}\right)
\end{aligned}
$$

The reason is that in all cases, when increasing $k$, the right-hand side grows faster than the left-hand side. This time, set

$$
\vec{x}=\left(x_{u}^{\prime}, x_{v}^{\prime}, x_{w}^{\prime}+k\right) \text { and } \vec{y}=\left(y_{u}^{\prime}, y_{v}^{\prime}, y_{w}^{\prime}+k\right)
$$

Then this pair of vectors forms a non-trivial solution of the system (6). Since $a_{w}=\min \left(a_{w}, b_{w}\right)$, as a consequence we get in addition

$$
\begin{aligned}
X_{\vec{x}} & =a_{u} x_{u}+a_{v} x_{v}+a_{w} x_{w}+c_{w}-\min \left(a_{w}, b_{w}\right) \\
& =a_{u} y_{u}+a_{v} y_{v}+a_{w} y_{w}+c_{w}-\min \left(a_{w}, b_{w}\right) \\
& =X_{\vec{y}}
\end{aligned}
$$

This solves the third and last case.
So far, we constructed a nontrivial solution $\vec{x}, \vec{y}$ with natural coefficients of the system (6) that, in addition, satisfies $X_{\vec{x}}=X_{\vec{y}}$. Furthermore, all entries in these two vectors are at least 2 . We finally show that this is a solution to the Equation (5):

First, we have

$$
\begin{aligned}
\pi\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right) & =\left(p^{a_{u}}\right)^{x_{u}}\left(p^{a_{v}}\right)^{x_{v}}\left(p^{a_{w}}\right)^{x_{w}} \\
& =p^{a_{u} x_{u}+a_{v} x_{v}+a_{w} x_{w}} \\
& =p^{a_{u} y_{u}+a_{v} y_{v}+a_{w} y_{w}} \\
& =\pi\left(u^{y_{u}} v^{y_{v}} w^{y_{w}}\right)
\end{aligned}
$$

and similarly

$$
\bar{\pi}\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right)=\bar{\pi}\left(u^{y_{u}} v^{y_{v}} w^{y_{w}}\right) .
$$

By Lemma 4.8, we get

$$
\begin{aligned}
\mu\left(u^{x_{u}} v^{x_{v}} w^{x_{w}}\right) & =g q^{X_{\vec{x}}} \\
& =g q_{\vec{y}}^{X_{\vec{y}}} \\
& =\mu\left(u^{y_{u}} v^{y_{v}} w^{y_{w}}\right) .
\end{aligned}
$$

Hence the two words $u^{x_{u}} v^{x_{v}} w^{x_{w}}$ and $u^{y_{u}} v^{y_{v}} w^{y_{w}}$ agree in their projections and their normal forms agree in their mixed part. Consequently, the normal forms of these two words coincide. Hence they are equivalent, i.e., as required, we found a non-trivial solution $\vec{x}, \vec{y}$ of equation Equation (5).

Proposition 4.10. Let $(\Gamma, I)$ be an independence alphabet and let $\eta: \mathbb{M}(\Gamma, I) \hookrightarrow Q$ be an embedding. Let furthermore $b \in \Gamma$ and $p, q \in A^{+}$be primitive with $p \sim q$ such that

$$
\pi(\eta(b)) \in p^{+} \text {and } \bar{\pi}(\eta(b)) \in q^{+}
$$

Then there is at most one letter $a \in \Gamma$ with $(a, b) \in I$.
Proof. Towards a contradiction, suppose there are distinct letters $a$ and $c$ in $\Gamma$ with $(a, b),(b, c) \in I$. Let

$$
u^{\prime}=\operatorname{nf}\left(\eta\left([a b]_{I}\right)\right), v^{\prime}=\operatorname{nf}(\eta(b)), \text { and } w^{\prime}=\operatorname{nf}\left(\eta\left([b c]_{I}\right)\right)
$$

Since $(a, b) \in I$, we have $a b \equiv_{I} b a$ and therefore $\eta\left([a b]_{I}\right)=\eta\left([b a]_{I}\right)$. This implies $\pi(\eta(a)) \pi(\eta(b))=$ $\pi(\eta(b)) \pi(\eta(a))$, i.e., the two words $\pi(\eta(a))$ and $\pi(\eta(b))$ commute in the free monoid. Since $\pi(\beta(b)) \in p^{+}$and $p$ is primitive, this implies $\pi(\eta(a)) \in p^{*}$ and therefore $\pi\left(u^{\prime}\right)=\pi(\eta(a)) \pi(\eta(b)) \in$ $p^{+}$. Similarly, $\bar{\pi}\left(u^{\prime}\right) \in q^{+}$as well as $\pi\left(w^{\prime}\right) \in p^{+}$and $\pi\left(w^{\prime}\right) \in q^{+}$.

Hence, by Lemma 4.9, there exists a rotation $(u, v, w)$ of $\left(u^{\prime}, v^{\prime}, w^{\prime}\right)$ and distinct vectors $\vec{x}$, $\vec{y} \in \mathbb{N}^{3}$ satisfying Equation (5). We consider the three possible rotations separately.

First suppose the rotation is trivial, i.e., $(u, v, w)=\left(u^{\prime}, v^{\prime}, w^{\prime}\right)$. Then we obtain

$$
\begin{aligned}
\eta\left(\left[(a b)^{x_{u}} b^{x_{v}}(b c)^{x_{w}}\right]_{I}\right) & =\left[u^{x_{u}} v^{x_{v}} w^{x_{w}}\right] \\
& =\left[u^{y_{u}} v^{y_{v}} w^{y_{w}}\right] \\
& =\eta\left(\left[(a b)^{y_{u}} b^{y_{v}}(b c)^{y_{w}}\right]_{I}\right) .
\end{aligned}
$$

Since $\eta$ is injective, and since $(a, b),(b, c) \in I$, this implies

$$
\begin{aligned}
a^{x_{u}} b^{x_{u}+x_{v}+x_{w}} c^{x_{w}} & \equiv_{I}(a b)^{x_{u}} b^{x_{v}}(b c)^{x_{w}} \\
& \equiv_{I}(a b)^{y_{u}} b^{y_{v}}(b c)^{y_{w}} \\
& \equiv_{I} a^{y_{u}} b^{y_{u}+y_{v}+y_{w}} c^{y_{w}} .
\end{aligned}
$$

Since the letters $a, b$, and $c$ are mutually distinct, we obtain

$$
\left(x_{u}, x_{u}+x_{v}+x_{w}, x_{w}\right)=\left(y_{u}, y_{u}+y_{v}+y_{w}, y_{w}\right)
$$

and therefore $\vec{x}=\vec{y}$. But this contradicts our choice of these two vectors as distinct.
Secondly, suppose $(u, v, w)=\left(v^{\prime}, w^{\prime}, u^{\prime}\right)$. Then we obtain

$$
\begin{aligned}
\eta\left(\left[b^{x_{u}}(b c)^{x_{v}}(a b)^{x_{w}}\right]_{I}\right) & =\left[u^{x_{u}} v^{x_{v}} w^{x_{w}}\right] \\
& =\left[u^{y_{u}} v^{y_{v}} w^{y_{w}}\right] \\
& =\eta\left(\left[b^{y_{u}}(b c)^{y_{v}}(a b)^{y_{w}}\right]_{I}\right) .
\end{aligned}
$$

As in the previous case, injectivity of $\eta$ and commutation of $b$ with $a$ and with $c$ yields

$$
c^{x_{v}} b^{x_{u}+x_{v}+x_{w}} a^{x_{w}} \equiv_{I} c^{y_{v}} b^{y_{u}+y_{v}+y_{w}} a^{y_{w}} .
$$

From the distinctness of $a, b$ and $c$, we again get $\vec{x}=\vec{y}$ which contradicts our choice of these two vectors as distinct.

Finally, suppose $(u, v, w)=\left(w^{\prime}, u^{\prime}, v^{\prime}\right)$. Then we obtain

$$
\begin{aligned}
\eta\left(\left[(b c)^{x_{u}}(a b)^{x_{v}} b^{x_{w}}\right]_{I}\right) & =\left[u^{x_{u}} v^{x_{v}} w^{x_{w}}\right] \\
& =\left[u^{y_{u}} v^{y_{v}} w^{y_{w}}\right] \\
& =\eta\left(\left[(b c)^{y_{u}}(a b)^{y_{v}} b^{y_{w}}\right]_{I}\right) .
\end{aligned}
$$

As in the previous cases, this yields a contradiction to our choice of the two vectors $\vec{x}$ and $\vec{y}$ as distinct.

Thus, indeed, there are no two distinct letters $a$ and $c$ with $(a, b),(b, c) \in I$.
The following corollary is the main result of this section. Its proof is an immediate consequence of Propositions 4.5 and 4.10 (depending on whether the roots of the two projections of $\eta(a)$ are conjugated or not).
Corollary 4.11. Let $(\Gamma, I)$ be an independence alphabet, let $\eta: \mathbb{M}(\Gamma, I) \hookrightarrow Q$ be an embedding, and let $a \in \Gamma$. If $\pi(\eta(a)) \neq \varepsilon$ and $\bar{\pi}(\eta(b)) \neq \varepsilon$, then the degree of $a$ is $\leq 1$.

## $4.4(\Gamma, I)$ is $P_{4}$-free

Lemma 4.12. Let $t, u, v, w \in \Sigma^{+}$such that $\bar{\pi}(u)=\varepsilon, \pi(v)=\varepsilon$, $v w \equiv w v$, and tu $\equiv$ ut. Then there exists a tuple $\vec{x}=\left(x_{t}, x_{u_{1}}, x_{u_{2}}, x_{v}, x_{w}\right)$ of natural numbers with $x_{t}, x_{w} \neq 0$ and

$$
\begin{equation*}
u^{x_{u_{1}}} v^{x_{v}} w t^{x_{t}} w^{x_{w}} u^{x_{u_{2}}} \equiv u^{x_{u_{1}}} w u^{x_{u_{2}}} w^{x_{w}} t^{x_{t}} v^{x_{v}} \tag{7}
\end{equation*}
$$

Proof. Since $\bar{\pi}(u)=\varepsilon$ and $\pi(v)=\varepsilon$, there are primitive words $p$ and $q$ and natural numbers $a_{u}, b_{v}>0$ with

$$
u=\pi(u)=p^{a_{u}} \text { and } v=\bar{\pi}(v)=q^{b_{v}} .
$$

Since $t u \equiv u t$ and $v w \equiv w v$, there are $a_{t}, b_{w} \in \mathbb{N}$ with

$$
\pi(t)=p^{a_{t}} \text { and } \bar{\pi}(w)=q^{b_{w}}
$$

Then we have

$$
\begin{aligned}
\pi\left(v^{b_{w}} w t^{a_{u}} w^{b_{v}} u^{a_{t}}\right) & =\varepsilon^{b_{w}} \pi(w) p^{a_{t} a_{u}} \pi\left(w^{b_{v}}\right) p^{a_{u} a_{t}} \\
& =\pi(w) p^{a_{u} a_{t}} \pi\left(w^{b_{v}}\right) p^{a_{t} a_{u}} \varepsilon \\
& =\pi\left(w u^{a_{t}} w^{b_{v}} t^{a_{u}} v^{b_{w}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\pi}\left(v^{b_{w}} w t^{a_{u}} w^{b_{v}} u^{a_{t}}\right) & =q^{b_{v} b_{w}} q^{b_{w}} \bar{\pi}\left(t^{a_{u}}\right) q^{b_{w} b_{v}} \varepsilon \\
& =q^{b_{w}} \varepsilon q^{b_{w} b_{v}} \bar{\pi}\left(t^{a_{u}}\right) q^{b_{v} b_{w}} \\
& =\bar{\pi}\left(w u^{a_{t}} w^{b_{v}} t^{a_{u}} v^{b_{w}}\right)
\end{aligned}
$$

Let $y \in \mathbb{N}$ such that $\left|\bar{\pi}\left(v^{b_{w}} w t^{a_{u}} w^{b_{v}} u^{a_{t}}\right)\right|=\left|\bar{\pi}\left(w u^{a_{t}} w^{b_{v}} t^{a_{u}} v^{b_{w}}\right)\right| \leq\left|u^{y}\right|$. We obtain

$$
\begin{array}{rlr}
u^{y} v^{b_{w}} w t^{a_{u}} w^{b_{v}} u^{a_{t}} & \equiv u^{y} \overline{\bar{\pi}\left(v^{b_{w}} w t^{a_{u}} w^{b_{v}} u^{a_{t}}\right)} \pi\left(v^{b_{w}} w t^{a_{u}} w^{b_{v}} u^{a_{t}}\right) & \\
& =u^{y} \overline{\bar{\pi}\left(w u^{a_{t}} w^{b_{v}} t^{a_{u}} v^{b_{w}}\right)} \pi\left(w u^{a_{t}} w^{b_{v}} t^{a_{u}} v^{b_{w}}\right) & \\
& \equiv u^{y} w u^{a_{t}} w^{b_{v}} t^{a_{u}} v^{b_{w}} & \\
\end{array}
$$

Hence the tuple $\left(x_{t}, x_{u_{1}}, x_{u_{2}}, x_{v}, x_{w}\right)=\left(a_{u}, y, a_{t}, b_{w}, b_{v}\right)$ has the desired properties.
Proposition 4.13. Let $(\Gamma, I)$ be an independence alphabet and let $\eta: \mathbb{M}(\Gamma, I) \hookrightarrow Q$ be an embedding. Then $(\Gamma, I)$ is $P_{4}$-free.
Proof. Suppose there are mutually distinct nodes $a, b, c, d \in \Gamma$ with $(a, b),(b, c),(c, d) \in I$. Then $b$ and $c$ both have degree $\geq 2$ in $(\Gamma, I)$, i.e., they belong to $\Gamma_{+} \cup \Gamma_{-}$by Corollary 4.11. Since $\left(\Gamma_{+}, I\right)$ and $\left(\Gamma_{-}, I\right)$ are both discrete by Proposition 4.2, we can assume w.l.o.g. that $b \in \Gamma_{+}$and $c \in \Gamma_{-}$.

There are words $t, u, v, w \in \Sigma^{+}$with $\eta(a)=[t], \eta(b)=[u], \eta(c)=[v]$, and $\eta(d)=[w]$.
Since $(a, b) \in I$, we get $[t u]=\eta(a b)=\eta(b a)=[u t]$ and therefore $t u \equiv u t$. Since $(c, d) \in I$, we get $[v w]=\eta(c d)=\eta(d c)=[w v]$ and therefore $v w \equiv w v$.

Since $b \in \Gamma_{+}$, we get $\bar{\pi}(u)=\bar{\pi}(\eta(b))=\varepsilon$. Similarly, from $c \in \Gamma_{-}$, we obtain $\pi(v)=\pi(\eta(c))=\varepsilon$.
From Lemma 4.12, we find natural numbers $x_{t}, x_{u_{1}}, x_{u_{2}}, x_{v}, x_{w}$ such that $x_{t}, x_{w} \neq 0$ and

$$
u^{x_{u_{1}}} v^{x_{v}} w t^{x_{t}} w^{x_{w}} u^{x_{u_{2}}} \equiv u^{x_{u_{1}}} w u^{x_{u_{2}}} w^{x_{w}} t^{x_{t}} v^{x_{v}} .
$$

Consequently,

$$
\begin{aligned}
\eta\left(b^{x_{u_{1}}} c^{x_{v}} d a^{x_{t}} d^{x_{w}} b^{x_{u_{2}}}\right) & =\left[u^{x_{u_{1}}} v^{x_{v}} w t^{x_{t}} w^{x_{w}} u^{x_{u_{2}}}\right] \\
& =\left[u^{x_{u_{1}}} w u^{x_{u_{2}}} w^{x_{w}} t^{x_{t}} v^{x_{v}}\right] \\
& =\eta\left(b^{x_{u_{1}}} d b^{x_{u_{2}}} d^{x_{w}} a^{x_{t}} c^{x_{v}}\right) .
\end{aligned}
$$

Since $\eta$ is injective, this implies

$$
b^{x_{u_{1}}} c^{x_{v}} d a^{x_{t}} d^{x_{w}} b^{x_{u_{2}}} \equiv_{I} b^{x_{u_{1}}} d b^{x_{u_{2}}} d^{x_{w}} a^{x_{t}} c^{x_{v}} .
$$

Since $x_{t}, x_{w} \neq 0$ and $a \neq d$, we obtain $(a, d) \in I$. Hence the mutually disjoint nodes $a, b, c, d$ do not induce $P_{4}$ in $(\Gamma, I)$.

### 4.5 Proof of the implication $(1) \Rightarrow(3)$ in Theorem 2.7

Theorem 4.14. Let $(\Gamma, I)$ be an independence alphabet and $\eta: \mathbb{M}(\Gamma, I) \rightarrow Q$ be an embedding. Then one of the following conditions holds:

1. all nodes in $(\Gamma, I)$ have degree $\leq 1$ or
2. $(\Gamma, I)$ has only one non-trivial connected component and this component is complete bipartite.

Proof. Suppose $(\Gamma, I)$ contains a node $a$ of degree $\geq 2$. Then, by Corollary 4.11 $a \in \Gamma_{+} \cup \Gamma_{-}$. From Proposition 4.4, we obtain that $a$ is connected to any edge, i.e., it belongs to the only nontrivial connected component $C$ of $(\Gamma, I)$. Note that $|C| \geq 3$ since it contains $a$ and its $\geq 2$ neighbors. Hence the induced subgraph $(C, I)$ contains at least one edge. Therefore Proposition 4.4 implies $\Gamma_{+} \cup \Gamma_{-} \subseteq C$. Note that all nodes in $C \backslash\left(\Gamma_{+} \cup \Gamma_{-}\right)$have degree 1 by Corollary 4.11 Hence, by Proposition 4.2, the connected graph $(C, I)$ is a complete bipartite graph together with some additional nodes of degree 1. It follows that $(C, I)$ is bipartite. By Proposition4.13, it is a connected and $P_{4}$-free graph. Hence its complementary graph $(C, D)$ is not connected [11. But this implies that $(C, I)$ is complete bipartite.

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