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Cseh, A., Manlove, D. and Irving, R. W. (2016) The Stable Roommates Problem with Short Lists. In: 9th International Symposium on Algorithmic Game Theory (SAGT), Liverpool, UK, 19-21 Sept 2016, pp. 207-219. ISBN 9783662533536 (doi:10.1007/978-3-662-53354-3_17)

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Deposited on: 20 July 2016

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The Stable Roommates problem with short lists

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Abstract. We consider two variants of the classical Stable Roommates problem with Incomplete (but strictly ordered) preference lists (SRI) that are degree constrained, i.e., preference lists are of bounded length. The first variant, EGAL d-SRI, involves finding an egalitarian stable matching in solvable instances of SRI with preference lists of length at most d. We show that this problem is NP-hard even if $d = 3$. On the positive side we give a $\frac{2d+3}{7}$-approximation algorithm for $d \in \{3, 4, 5\}$ which improves on the known bound of 2 for the unbounded preference list case. In the second variant of SRI, called d-SRTI, preference lists can include ties and are of length at most d. We show that the problem of deciding whether an instance of d-SRTI admits a stable matching is NP-complete even if $d = 3$. We also consider the "most stable" version of this problem and prove a strong inapproximability bound for the $d = 3$ case. However for $d = 2$ we show that the latter problem can be solved in polynomial time.

1 Introduction

In the *Stable Roommates problem with Incomplete lists* (SRI), a graph $G = (A, E)$ and a set of preference lists $\mathcal{O}$ are given, where the vertices $A = \{a_1, \ldots, a_n\}$ correspond to *agents*, and $\mathcal{O} = \{\prec_1, \ldots, \prec_n\}$, where $\prec_i$ is a linear order on the vertices adjacent to a_i in G ($1 \leq i \leq n$). We refer to $\prec_i$ as a_i's *preference list*. The agents that are adjacent to a_i in G are said to be *acceptable* to a_i. If a_j and a_k are two acceptable agents for a_i where $a_j \prec_i a_k$ then we say that a_i *prefers* a_j to a_k.

Let M be a matching in G. If $a_ia_j \in M$ then we let $M(a_i)$ denote a_j. An edge $a_ia_j \notin M$ *blocks* M, or forms a *blocking edge* of M, if a_i is unmatched or prefers a_j to $M(a_i)$, and similarly a_j is unmatched or prefers a_i to $M(a_j)$. A matching is called *stable* if no edge blocks it. Denote by SR the special case of SRI in which $G = K_n$. Gale and Shapley [8] observed that an instance of SR need not admit a stable matching. Irving [13] gave a linear-time algorithm to find a stable matching or report that none exists, given an instance of SR. The

⋆ Supported by Icelandic Research Fund grant no. 152679-051, the Hungarian Academy of Sciences under its Momentum Programme (LP2016-3) and COST Action IC1205 on Computational Social Choice. Part of this work was carried out whilst visiting the University of Glasgow.

⋆⋆ Supported by EPSRC grant EP/K010042/1.

straightforward modification of this algorithm to the SRI case is described in [10]. We call an SRI instance *solvable* if it admits a stable matching.

In practice agents may find it difficult to rank a large number of alternatives in strict order of preference. One natural assumption, therefore, is that preference lists are short, which corresponds to the graph being of bounded degree. Given an integer $d \geq 1$, we define d-SRI to be the restriction of SRI in which G is of bounded degree d. This special case of SRI problem has potential applications in organising tournaments. As already pointed out in a paper of Kujansuu et al. [16], SRI can model a pairing process similar to the Swiss system, which is used in large-scale chess competitions. The assumption on short lists is reasonable, because according to the Swiss system, players can be matched only to other players with approximately the same score.

A second variant of SRI, which can be motivated in a similar fashion, arises if we allow ties in the preference lists, i.e., $\prec_i$ is now a strict weak ordering[3] $(1 \leq i \leq n)$. We refer to this problem as the *Stable Roommates problem with Ties and Incomplete lists* (SRTI) [15]. As in the SRI case, define d-SRTI to be the restriction of SRTI in which G is of bounded degree d. Denote by SRT the special case of SRTI in which $G = K_n$. In the context of the motivating application of chess tournament construction as mentioned in the previous paragraph, d-SRTI is naturally obtained if a chess player has several potential partners of the same score and match history in the tournament.

In the SRTI context, ties correspond to indifference in the preference lists. In particular, if $a_i a_j \in E$ and $a_i a_k \in E$ where $a_j \not\prec_i a_k$ and $a_k \not\prec_i a_j$ then a_i is said to be *indifferent between* a_j and a_k. Thus preference in the SRI context corresponds to strict preference in the case of SRTI. Relative to the strict weak orders in $\mathcal{O}$, we can define stability in SRTI instances in exactly the same way as for SRI. This means, for example, that if $a_i a_j \in M$ for some matching M, and a_i is indifferent between a_j and some agent a_k, then $a_i a_k$ cannot block M. The term *solvable* can be defined in the SRTI context in an analogous fashion to SRI. Using a highly technical reduction from a restriction of 3-SAT, Ronn [20] proved that the problem of deciding whether a given SRT instance is solvable is NP-complete. A simpler reduction was given by Irving and Manlove [15].

For solvable instances of SRI there can be many stable matchings. Often it is beneficial to work with a stable matching that is fair to all agents in a precise sense [9,14]. One such fairness concept can be defined as follows. Given two agents a_i, a_j in an instance $\mathcal{I}$ of SRI, where $a_i a_j \in E$, let $\mathrm{rank}(a_i, a_j)$ denote the rank of a_j in a_i's preference list (that is, 1 plus the number of agents that a_i prefers to a_j). Let A_M denote the set of agents who are matched in a given stable matching M. (Note that this set depends only on $\mathcal{I}$ and is independent of M by [10, Theorem 4.5.2].) Define $c(M) = \sum_{a_i \in A_M} \mathrm{rank}(a_i, M(a_i))$ to be the *cost* of M. An *egalitarian stable matching* is a stable matching M that minimises $c(M)$ over the set of stable matchings in $\mathcal{I}$. Finding an egalitarian stable matching in SR was shown to be NP-hard by Feder [6]. Feder [6,7] also gave a 2-approximation algorithm for this problem in the SRI setting. He also showed that an egalitarian

[3] That is, $\prec_i$ is a strict partial order in which incomparability is transitive.

stable matching in SR can be approximated within a factor of α of the optimum if and only if Minimum Vertex Cover can be approximated within the same factor α. Is was proved later that, assuming the Unique Games Conjecture, Minimum Vertex Cover cannot be approximated within $2-\varepsilon$ for any $\varepsilon > 0$ [17].

Given an unsolvable instance $\mathcal{I}$ of SRI or SRTI, a natural approximation to a stable matching is a *most-stable* matching [1]. Relative to a matching M in $\mathcal{I}$, define $bp(M)$ to be the set of blocking edges of M and let $bp(\mathcal{I})$ denote the minimum value of $|bp(M')|$, taken over all matchings M' in $\mathcal{I}$. Then M is a *most-stable* matching in $\mathcal{I}$ if $|bp(M)| = bp(\mathcal{I})$. The problem of finding a most-stable matching was shown to be NP-hard and not approximable within $n^{k-\varepsilon}$, for any $\varepsilon > 0$, unless P = NP, where $k = \frac{1}{2}$ if $\mathcal{I}$ is an instance of SR and $k = 1$ if $\mathcal{I}$ is an instance of SRT [1].

To the best of our knowledge, there has not been any previous work published on either the problem of finding an egalitarian stable matching in a solvable instance of SRI with bounded-length preference lists or the solvability of SRTI with bounded-length preference lists. This paper provides contributions in both of these directions, focusing on instances of d-SRI and d-SRTI for $d \geq 2$, with the aim of drawing the line between polynomial-time solvability and NP-hardness for the associated problems in terms of d.

Our contribution. In Section 2 we study the problem of finding an egalitarian stable matching in an instance of d-SRI. We show that this problem is NP-hard if $d = 3$, whilst there is a straightforward algorithm for the case that $d = 2$. We then consider the approximability of this problem for the case that $d \geq 3$. We give an approximation algorithm with a performance guarantee of $\frac{9}{7}$ for the case that $d = 3$, $\frac{11}{7}$ if $d = 4$ and $\frac{13}{7}$ if $d = 5$. These performance guarantees improve on Feder's 2-approximation algorithm for the general SRI case [6,7]. In Section 3 we turn to d-SRTI and prove that the problem of deciding whether an instance of 3-SRTI is solvable is NP-complete. We then show that the problem of finding a most-stable matching in an instance of d-SRTI is solvable in polynomial time if $d = 2$, whilst for $d = 3$ we show that this problem is NP-hard and not approximable within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless P = NP. Due to various complications, as explained in Appendix A of the full version of this paper [5], we do not attempt to define and study egalitarian stable matchings in instances of SRTI. A structured overview of previous results and our results (marked by *) for d-SRI and d-SRTI is contained in Table 1. All missing proofs are contained in Appendix B of the full version of this paper [5].

Related work. Degree-bounded graphs, most-stable matchings and egalitarian stable matchings are widely studied concepts in the literature on matching under preferences [18]. As already mentioned, the problem of finding a most-stable matching has been studied previously in the context of SRI [1]. In addition to the results surveyed already, the authors of [1] gave an $O(m^{k+1})$ algorithm to find a matching M with $|bp(M)| \leq k$ or report that no such matching exists, where $m = |E|$ and $k \geq 1$ is any integer. Most-stable matchings have also been considered in the context of d-SRI [3]. The authors showed that, if $d = 3$, there is some constant $c > 1$ such that the problem of finding a most-stable matching is

	finding a stable matching	finding an egalitarian stable matching
d-SRI	in P [13,10]	in P for $d = 2$ (*) NP-hard even for $d = 3$ (*) $\frac{2d+3}{7}$-approximation for $d \in \{3, 4, 5\}$ (*) 2-approximation for $d \geq 6$ [6,7]
d-SRTI	in P for $d = 2$ (*) NP-hard even for $d = 3$ (*)	not well-defined (see [5, Appendix A])

Table 1. Summary of results for d-SRI and d-SRTI.

not approximable within c unless P = NP. On the other hand, they proved that the problem is solvable in polynomial time for $d \leq 2$. The authors also gave a $(2d-3)$-approximation algorithm for the problem for fixed $d \geq 3$. This bound was improved to $2d-4$ if the given instance satisfies an additional condition (namely the absence of a structure called an *elitist odd party*). Most-stable matchings have also been studied in the bipartite restriction of SRI called the *Stable Marriage problem with Incomplete lists* (SMI) [12,4]. Since every instance of SMI admits a stable matching M (and hence $bp(M) = \emptyset$), the focus in [12,4] was on finding maximum cardinality matchings with the minimum number of blocking edges.

Regarding the problem of finding an egalitarian stable matching in an instance of SRI, as already mentioned Feder [6,7] showed that this problem is NP-hard, though approximable within a factor of 2. A 2-approximation algorithm for this problem was also given independently by Gusfield and Pitt [11], and by Teo and Sethuraman [23]. These approximation algorithms can also be extended to the more general setting where we are given a weight function on the edges, and we seek a stable matching of minimum weight. Feder's 2-approximation algorithm requires monotone, non-negative and integral edge weights, whereas with the help of LP techniques [22,23], the integrality constraint can be dropped, while the monotonicity constraint can be partially relaxed.

2 The Egalitarian Stable Roommates problem

In this section we consider the complexity and approximability of the problem of computing an egalitarian stable matching in instances of d-SRI. We begin by defining the following problems.

Problem 1. EGAL d-SRI
Input: *A solvable instance $\mathcal{I} = \langle G, \mathcal{O} \rangle$ of d-SRI, where G is a graph and $\mathcal{O}$ is a set of preference lists, each of length at most d.*
Output: *An egalitarian stable matching M in $\mathcal{I}$.*

The decision version of EGAL d-SRI is defined as follows:

Problem 2. EGAL d-SRI DEC
Input: *$\mathcal{I} = \langle G, \mathcal{O}, K' \rangle$, where $\langle G, \mathcal{O} \rangle$ is a solvable instance $\mathcal{I}'$ of d-SRI and K' is an integer.*
Question: *Does $\mathcal{I}'$ admit a stable matching M with $c(M) \leq K'$?*

In [5, Appendix B] we give a reduction from the NP-complete decision version of Minimum Vertex Cover in cubic graphs to EGAL 3-SRI DEC, deriving the hardness of the latter problem.

Theorem 1. EGAL 3-SRI DEC *is* NP*-complete.*

Theorem 1 immediately implies the following result.

Corollary 2. EGAL 3-SRI *is* NP*-hard.*

We remark that EGAL 2-SRI is trivially solvable in polynomial time: the components of the graph are paths and cycles in this case, and the cost of a stable matching selected in one component is not affected by the matching edges chosen in another component. Therefore we can deal with each path and cycle separately, minimising the cost of a stable matching in each. Paths and odd cycles admit exactly one stable matching (recall that (i) the instance is assumed to be solvable, and (ii) the set of matched agents is the same in all stable matchings [10, Theorem 4.5.2]), whilst even cycles admit at most two stable matchings (to find them, just pick the two perfect matchings and test each for stability) – we can just pick the stable matching with lower cost in such a case. The following result is therefore immediate.

Proposition 3. EGAL 2-SRI *admits a linear-time algorithm.*

Corollary 2 naturally leads to the question of the approximabilty of EGAL d-SRI. As mentioned in the Introduction, Feder [6,7] provided a 2-approximation algorithm for the problem of finding an egalitarian stable matching in an instance of SRI. As Theorems 4, 6 and 7 show, this bound can be improved for instances with bounded-length preference lists.

Theorem 4. EGAL 3-SRI *is approximable within* $9/7$*.*

Proof. Let $\mathcal{I}$ be an instance of 3-SRI and let M_{egal} denote an egalitarian stable matching in $\mathcal{I}$. First we show that any stable matching in $\mathcal{I}$ is a $4/3$-approximation to M_{egal}. We then focus on the worst-case scenario when this ratio $4/3$ is in fact realised. Then we design a weight function on the edges of the graph and apply Teo and Sethuraman's 2-approximation algorithm [22,23] to find an approximate solution M' to a minimum weight stable matching M_{opt} for this weight function. This weight function helps M' to avoid the worst case for the $4/3$-approximation for a significant amount of the matching edges. We will ultimately show that M' is in fact a $9/7$-approximation to M_{egal}.

Claim 5. *In an instance of* EGAL 3-SRI*, any stable matching approximates* $c(M_{egal})$ *within a factor of* $4/3$*.*

Proof. Let M be an arbitrary stable matching in $\mathcal{I}$. Call an edge uv an (i,j)*-pair* $(i \leq j)$ if v is u's ith choice and u is v's jth choice. By Theorem 4.5.2 of [10], the set of agents matched in M_{egal} is identical to the set of agents matched in M. We will now study the worst approximation ratios in all cases of (i,j)-pairs, given that $1 \leq i \leq j \leq 3$ in 3-SRI.

- If $uv \in M_{\text{egal}}$ is a $(1,1)$-pair then u and v contribute 2 to $c(M_{\text{egal}})$ and also 2 to $c(M)$ since they must be also be matched in M (and in every stable matching).
- If $uv \in M_{\text{egal}}$ is a $(1,2)$-pair then u and v contribute 3 to $c(M_{\text{egal}})$ and at most 4 to $c(M)$. Since, if $uv \notin M$, then v must be matched to his 1st choice and u to his 2nd or 3rd, because one of u and v must be better off and the other must be worse off in M than in M_{egal}.
- If $uv \in M_{\text{egal}}$ is a $(1,3)$-pair then u and v contribute 4 to $c(M_{\text{egal}})$ and at most 5 to $c(M)$. Since, if $uv \notin M$, then v must be matched to his 1st or 2nd choice and u to his 2nd or 3rd.
- If $uv \in M_{\text{egal}}$ is a $(2,2)$-pair then u and v contribute 4 to $c(M_{\text{egal}})$ and at most 4 to $c(M)$. Since, if $uv \notin M$, then one must be matched to his 1st choice and the other to his 3rd.
- If $uv \in M_{\text{egal}}$ is a $(2,3)$-pair then u and v contribute 5 to $c(M_{\text{egal}})$ and at most 5 to $c(M)$. Since, if $uv \notin M$, then v must be matched to his 1st or 2nd choice and u to his 3rd.
- If $uv \in M_{\text{egal}}$ is a (3,3)-pair then u and v contribute 6 to $c(M_{\text{egal}})$ and also 6 to $c(M)$ since they must be also be matched in M (and in every stable matching – this follows by [10, Lemma 4.3.9]).

It follows that, for every pair $uv \in M_{\text{egal}}$,

$$\frac{\text{rank}(u, M(u)) + \text{rank}(v, M(v))}{\text{rank}(u, M_{\text{egal}}(u)) + \text{rank}(v, M_{\text{egal}}(v))} = \frac{\text{rank}(u, M(u)) + \text{rank}(v, M(v))}{\text{rank}(u, v) + \text{rank}(v, u)} \leq 4/3.$$

Hence $c(M)/c(M_{\text{egal}}) \leq 4/3$ and Claim 5 is proved. □

As shown in Claim 5, the only case when the approximation ratio 4/3 is reached is where M_{egal} consists of (1,2)-pairs exclusively, while the stable matching output by the approximation algorithm contains (1,3)-pairs only. We will now present an algorithm that either delivers a stable solution M' containing at least a significant amount of the (1,2)-pairs in M_{egal} or a certificate that M_{egal} contains only a few (1,2)-pairs and thus any stable solution is a good approximation.

To simplify our proof, we execute some basic pre-processing of the input graph. If there are any (1,1)-pairs in G, then these can be fixed, because they occur in every stable matching and thus can only lower the approximation ratio. Similarly, if an arbitrary stable matching contains a (3,3)-pair, then this edge appears in all stable matchings and thus we can fix it. Those (3,3)-pairs that do not belong to the set of stable edges can be deleted from the graph. From this point on, we assume that no edge is ranked first or last by both of its end vertices in G and prove the approximation ratio for such graphs.

Take the following weight function on all $uv \in E$:

$$w(uv) = \begin{cases} 0 & \text{if } uv \text{ is a (1,2)-pair,} \\ 1 & \text{otherwise.} \end{cases}$$

We designed $w(uv)$ to fit the necessary U-shaped condition of Teo and Sethuraman's 2-approximation algorithm [22,23]. This condition on the weight function is as follows. We are given a function f_p on the neighbouring edges of a vertex p. Function f_p is *U-shaped* if it is non-negative and there is a neighbour q of p so that f_p is monotone decreasing on neighbours in order of p's preference until q, and f_p is monotone increasing on neighbours in order of p's preference after q. The approximation guarantee of Teo and Sethuraman's algorithm holds for an edge weight function $w(uv)$ if for every edge $uv \in E$, $w(uv)$ can be written as $w(uv) = f_u(uv) + f_v(uv)$, where f_u and f_v are U-shaped functions.

Our $w(uv)$ function is clearly U-shaped, because at each vertex the sequence of edges in order of preference is either monotone increasing or it is $(1, 0, 1)$. Since w itself is U-shaped, it is easy to decompose it into a sum of U-shaped f_v functions, for example by setting $f_v(uv) = f_u(uv) = \frac{w(uv)}{2}$ for every edge uv.

Let M denote an arbitrary stable matching and $M^{(1,2)}$ be the set of (1,2)-pairs in a matching M and M_{opt} be a minimum weight stable matching with respect to the weight function $w(uv)$. Since M_{opt} is by definition the stable matching with the largest number of (1,2)-pairs, $|M^{(1,2)}_{\text{opt}}| \geq |M^{(1,2)}_{\text{egal}}|$. We also know that $w(M) = |M| - |M^{(1,2)}|$ for every stable matching M.

Due to Teo and Sethuraman's approximation algorithm [22,23], it is possible to find a stable matching M' whose weight approximates $w(M_{\text{opt}})$ within a factor of 2. Formally,

$$|M| - |M'^{(1,2)}| = w(M') \leq 2w(M_{\text{opt}}) = 2|M| - 2|M^{(1,2)}_{\text{opt}}|.$$

This gives us a lower bound on $|M'^{(1,2)}|$.

$$|M'^{(1,2)}| \geq 2|M^{(1,2)}_{\text{opt}}| - |M| \geq 2|M^{(1,2)}_{\text{egal}}| - |M| \tag{1}$$

We distinguish two cases from here on, depending on the sign of the term on the right. In both cases, we establish a lower bound on $c(M_{\text{egal}})$ and an upper bound on $c(M')$. These will give the desired upper bound of 9/7 on $\frac{c(M')}{c(M_{\text{egal}})}$.

1) $2|M^{(1,2)}_{\text{egal}}| - |M| \leq 0$

   The derived lower bound for $|M'^{(1,2)}|$ is negative or zero in this case. Yet we know that at most half of the edges in M_{egal} are (1,2)-pairs, and $c(e) \geq 4$ for the rest of the edges in M_{egal}. Let us denote $|M| - 2|M^{(1,2)}_{\text{egal}}| \geq 0$ by x. Thus, $|M^{(1,2)}_{\text{egal}}| = \frac{|M|-x}{2}$.

$$c(M_{\text{egal}}) \geq \frac{|M| - x}{2} \cdot 3 + \frac{|M| + x}{2} \cdot 4 = 3.5|M| + 0.5x \tag{2}$$

   We use our arguments in the proof of Claim 5 to derive that an arbitrary stable matching approximates $c(M_{\text{egal}})$ on the $\frac{|M|-x}{2}$ (1,2)-edges within a ratio of $\frac{4}{3}$, while its cost on the remaining $\frac{|M|+x}{2}$ edges is at most 5. These imply the following inequalities for an arbitrary stable matching M.

$$c(M) \leq \frac{|M| - x}{2} \cdot 3 \cdot \frac{4}{3} + \frac{|M| + x}{2} \cdot 5 = 4.5|M| + 0.5x \tag{3}$$

We now combine (2) and (3). The last inequality holds for all $x \geq 0$.

$$\frac{c(M)}{c(M_{\text{egal}})} \leq \frac{4.5|M| + 0.5x}{3.5|M| + 0.5x} \leq \frac{9}{7}$$

2) $2|M_{\text{egal}}^{(1,2)}| - |M| > 0$

Let us denote $2|M_{\text{egal}}^{(1,2)}| - |M|$ by $\hat{x}$. Notice that $|M_{\text{egal}}^{(1,2)}| = \frac{\hat{x}+|M|}{2}$. We can now express now the number of edges with cost 3, and at least 4 in M_{egal}.

$$\begin{aligned} c(M_{\text{egal}}) &\geq 3 \cdot \frac{\hat{x} + |M|}{2} + 4 \cdot \left(|M| - \frac{\hat{x} + |M|}{2} \right) \\ &= 3.5|M| - 0.5\hat{x} \end{aligned} \tag{4}$$

Let $|M'^{(1,2)}| = z_1$. Then exactly z_1 edges in M' have cost 3. It follows from (1) that $z_1 \geq \hat{x}$. Suppose that $z_2 \leq z_1$ edges in $M'^{(1,2)}$ correspond to edges in $M_{\text{egal}}^{(1,2)}$. Recall that $|M_{\text{egal}}^{(1,2)}| = \frac{\hat{x}+|M|}{2}$. The remaining $\frac{|M|+\hat{x}}{2} - z_2$ edges in $M_{\text{egal}}^{(1,2)}$ have cost at most 4 in M'. This leaves $|M| - |M_{\text{egal}}^{(1,2)}| - (z_1 - z_2) = \frac{|M|-\hat{x}}{2} - z_1 + z_2$ edges in M_{egal} that are as yet unaccounted for; these have cost at most 5 in both M_{egal} and M'. We thus obtain:

$$\begin{aligned} c(M') &\leq 3z_1 + 4\left(\frac{|M| + \hat{x}}{2} - z_2 \right) + 5\left(\frac{|M| - \hat{x}}{2} - z_1 + z_2 \right) \\ &= 4.5|M| - 0.5\hat{x} - 2z_1 + z_2 \\ &\leq 4.5|M| - 1.5\hat{x} \end{aligned} \tag{5}$$

Combining (4) and (5) delivers the following bound.

$$\frac{c(M')}{c(M_{\text{egal}})} \leq \frac{4.5|M| - 1.5\hat{x}}{3.5|M| - 0.5\hat{x}} < \frac{9}{7}$$

The last inequality holds for every $\hat{x} > 0$.

We derived that M', the 2-approximate solution with respect to the weight function $w(uv)$ delivers a $\frac{9}{7}$-approximation in both cases. □

Using analogous techniques we can establish similar approximation bounds for EGAL 4-SRI and EGAL 5-SRI, as follows.

Theorem 6. EGAL 4-SRI *is approximable within* 11/7.

Theorem 7. EGAL 5-SRI *is approximable within* 13/7.

Using a similar reasoning for each $d \geq 6$, our approach gives a c_d-approximation algorithm for EGAL d-SRI where $c_d > 2$. In these cases the 2-approximation algorithm of Feder [6,7] should be used instead.

3 Solvability and most-stable matchings in d-SRTI

In this section we study the complexity and approximability of the problem of deciding whether an instance of d-SRTI admits a stable matching, and the problem of finding a most-stable matching given an instance of d-SRTI.

We begin by defining two problems that we will be studying in this section from the point of view of complexity and approximability.

Problem 3. SOLVABLE d-SRTI
Input: *$\mathcal{I} = \langle G, \mathcal{O} \rangle$, where G is a graph and $\mathcal{O}$ is a set of preference lists, each of length at most d, possibly involving ties.*
Question: *Is $\mathcal{I}$ solvable?*

Problem 4. MIN BP d-SRTI
Input: *An instance $\mathcal{I}$ of d-SRTI.*
Output: *A matching M in $\mathcal{I}$ such that $|bp(M)| = bp(\mathcal{I})$.*

We will show that SOLVABLE 3-SRTI is NP-complete and MIN BP 3-SRTI is hard to approximate. In both cases we will use a reduction from the following satisfiability problem:

Problem 5. (2,2)-E3-SAT
Input: *$\mathcal{I} = B$, where B is a Boolean formula in CNF, in which each clause comprises exactly 3 literals and each variable appears exactly twice in unnegated and exactly twice in negated form.*
Question: *Is there a truth assignment satisfying B?*

(2,2)-E3-SAT is NP-complete, as shown by Berman et al. [2]. We begin with the hardness of SOLVABLE 3-SRTI.

Theorem 8. SOLVABLE 3-SRTI *is* NP*-complete.*

Proof. Clearly SOLVABLE 3-SRTI belongs to NP. To show NP-hardness, we reduce from (2,2)-E3-SAT as defined in Problem 5. Let B be a given instance of (2,2)-E3-SAT, where $X = \{x_1, x_2, \ldots, x_n\}$ is the set of variables and $C = \{c_1, c_2, \ldots, c_m\}$ is the set of clauses. We form an instance $\mathcal{I} = (G, \mathcal{O})$ of 3-SRTI as follows. Graph G consists of a *variable gadget* for each x_i ($1 \le i \le n$), a *clause gadget* for each c_j ($1 \le j \le m$) and a set of *interconnecting edges* between them; these different parts of the construction, together with the preference orderings that constitute $\mathcal{O}$, are shown in Figure 1 and will be described in more detail below.

When constructing G, we will keep track of the order of the three literals in each clause of B and the order of the two unnegated and two negated occurrences of each variable in B. Each of these four occurrences of each variable is represented by an interconnecting edge.

A variable gadget for a variable x_i ($1 \le i \le n$) of B comprises the 4-cycle $\langle v_i^1, v_i^2, v_i^3, v_i^4 \rangle$ with cyclic preferences. Each of these four vertices is incident to an interconnecting edge. These edges end at specific vertices of clause gadgets.

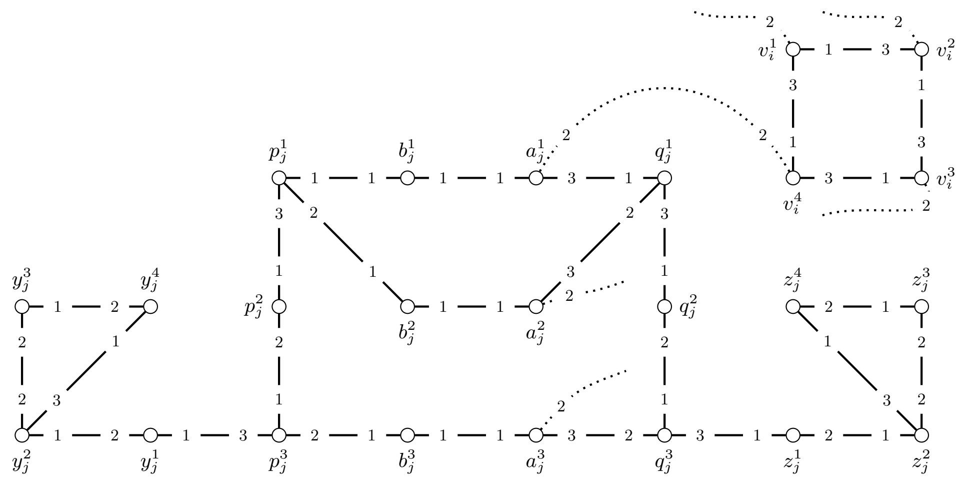

Fig. 1. Clause and variable gadgets for 3-SRTI. The dotted edges are the interconnecting edges. The notation used for edge $a_j^1 v_i^4$ implies that the first literal of the corresponding clause c_j is the second occurrence of the corresponding variable x_i in negated form.

The clause gadget for a clause c_j $(1 \leq j \leq m)$ contains 20 vertices, three of which correspond to the literals in c_j; these vertices are also incident to an interconnecting edge.

Due to the properties of (2,2)-E3-SAT, x_i occurs twice in unnegated form, say in clauses c_j and c_k of B. Its first appearance, as the rth literal of c_j $(1 \leq r \leq 3)$, is represented by the interconnecting edge between vertex v_i^1 in the variable gadget corresponding to x_i and vertex a_j^r in the clause gadget corresponding to c_j. Similarly the second occurrence of x_i, say as the sth literal of c_k $(1 \leq s \leq 3)$ is represented by the interconnecting edge between v_i^3 and a_k^s. The same variable x_i also appears twice in negated form. Appropriate a-vertices in the gadgets representing those clauses are connected to v_i^2 and v_i^4. We remark that this construction involves a gadget similar to one presented by Biró et al. [3] in their proof of the NP-hardness of MIN BP 3-SRI.

In [5, Appendix B] we prove that there is a truth assignment satisfying B if and only if there is a stable matching M in $\mathcal{I}$. $\square$

Our construction shows that the complexity result holds even if the preference lists are either strictly ordered or consist of a single tie of length two. Moreover, Theorem 8 also immediately implies the following result.

Corollary 9. MIN BP 3-SRTI *is* NP*-hard.*

The following result strengthens Corollary 9.

Theorem 10. MIN BP 3-SRTI *is not approximable within* $n^{1-\varepsilon}$*, for any* $\varepsilon > 0$*, unless* $\mathsf{P} = \mathsf{NP}$*, where* n *is the number of agents.*

Proof (sketch). The core idea of our proof is to gather several copies of the 3-SRTI instance created in the proof of Theorem 8, together with a small unsolvable 3-SRTI instance. By doing so, we create a MIN BP 3-SRTI instance $\mathcal{I}$ in which $bp(\mathcal{I})$ is large if the Boolean formula B (originally given as an instance of (2,2)-E3-SAT) is not satisfiable, and $bp(\mathcal{I}) = 1$ otherwise. Therefore, finding a good approximation for $\mathcal{I}$ will imply a polynomial-time algorithm to decide the satisfiability of B. $\square$

To complete the study of cases of MIN BP d-SRTI, we establish a positive result for instances with degree at most 2.

Theorem 11. MIN BP 2-SRTI *is solvable in* $\mathcal{O}(|V|)$ *time.*

Proof. For an instance $\mathcal{I}$ of MIN BP 2-SRTI, clearly every component of the underlying graph G is a path or cycle. We claim that $bp(\mathcal{I})$ equals the number of *odd parties* in G, where an *odd party* is a cycle $C = \langle v_1, v_2, ..., v_k \rangle$ of odd length, such that v_i strictly prefers v_{i+1} to v_{i-1} (addition and subtraction are taken modulo k).

Since an odd party never admits a stable matching, $bp(\mathcal{I})$ is bounded below by the number of odd parties [21]. This bound is tight: by taking an arbitrary maximum matching in an odd party component, a most-stable matching is already reached. Now we show that a stable matching M can be constructed in all other components.

Each component that is not an odd cycle is therefore a bipartite subgraph (indeed either a path or an even cycle). Such a subgraph therefore gives rise to the restriction of SRTI called the *Stable Marriage problem with Ties and Incomplete lists* (SMTI). An instance of SMTI always admits a stable solution and it can be found in linear time [19]. Thus these components contribute no blocking edge.

Regarding odd-length cycles that are not odd parties, we will show that there is at least one vertex not strictly preferred by either of its adjacent vertices. Leaving this vertex uncovered and adding a perfect matching in the rest of the cycle results in a stable matching.

Assume that every vertex along a cycle C_k (where k is an odd number) is strictly preferred by at least one of its neighbours. Since each of the k vertices is strictly preferred by at least one vertex, and a vertex v can prefer at most one other vertex strictly, every vertex along C_k has a strictly ordered preference list. Now every vertex can point at its unique first-choice neighbour. To avoid an odd cycle, there must be a vertex pointed at by both of its neighbours. This implies that there is also a vertex v pointed at by no neighbour, and v is hence ranked second by both of its neighbours. $\square$

Open questions. Theorems 4, 6 and 7 improve on the best known approximation factor for EGAL d-SRI for small d. It remains open to come up with an even better approximation or to establish an inapproximability bound matching our algorithm's guarantee. A more general direction is to investigate whether the problem of finding a minimum weight stable matching can be approximated within a factor less than 2 for instances of d-SRI for small d.

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