Optimal Consistent Network Updates in Polynomial Time

(Extended Version)

Pavol Černý¹, Nate Foster², Nilesh Jagnik¹, and Jedidiah McClurg¹

¹University of Colorado Boulder ²Cornell University

Abstract. Software-defined networking (SDN) allows operators to control the behavior of a network by programatically managing the forwarding rules installed on switches. However, as is common in distributed systems, it can be difficult to ensure that certain consistency properties are preserved during periods of reconfiguration. The widely-accepted notion of *per-packet consistency* requires every packet to be forwarded using the new configuration or the old configuration, but not a mixture of the two. If switches can be updated in some (partial) order which guarantees that per-packet consistency is preserved, we call this order a *consistent order update*. In particular, switches that are incomparable in this order can be updated in parallel. We call a consistent order update *optimal* if it allows maximal parallelism. This paper presents a polynomial-time algorithm for finding an optimal consistent order update. This contrasts with other recent results in the literature, which show that for other classes of properties (e.g., loop-freedom and waypoint enforcement), the optimal update problem is NP-complete.

1 Introduction

Software-defined networking (SDN) replaces conventional network management interfaces with higher-level APIs. While SDN has been used to build a wide variety of useful applications, in practice, it can be difficult for operators to *correctly* and *efficiently* reconfigure the network, i.e., update the global set of forwarding rules installed on switches (known as a *configuration*). Even if the initial and final configurations are free of errors, naïvely updating individual switches (referred to in this paper as *switch-updates*) can lead to incorrect transient behaviors such as forwarding loops, blackholes, bypassing a firewall, etc. In certain cases, updating switches in parallel can lead to incorrect transient behavior, but in other cases we can correctly parallelize switch updates. Therefore, we need a partial order on switch-update.

Consistent order updates. This paper investigates the problem of computing a *consistent order update*. Given an initial and final network configuration, a consistent order update is a partial order on switch-updates, such that if the switches are updated according to this order, an important consistency property called *per-packet consistency* [15] is guaranteed throughout the update process. This property guarantees that each packet traversing the network will follow a single global configuration: either the initial one, or the final one, but not a mixture of the two. In particular, this means that if the initial and the final configurations are loop-free, blackhole-free, prevent bypassing a firewall, etc., then so do all intermediate configurations.

Optimal consistent order updates. In implementing a consistent order update, we would generally prefer to use one that is optimal. A consistent order update is *optimal* if it allows the most parallelism among all consistent order updates. Formally, recall that a consistent order update is a partial order on switch-updates—an optimal partial order is one where the length of the longest chain in the order is the smallest among all possible correct partial orders. Intuitively, this means the update can be performed in the smallest number of "rounds," where rounds are separated by waiting for in-flight packets to exit the network and by waiting for all the switch updates from the previous rounds to finish.

Single flow vs. multiple flows. A *flow* is a restriction of a network configuration to packets of a single type, corresponding to values in packet headers. A packet type might include the destination address, protocol number (TCP vs. UDP), etc. We show that if we consider flows to be *symbolic* (i.e. represented by predicates over packet headers, potentially matching multiple flows), then the problem is CO-NP-hard. In this paper, we focus on the problem of updating an *individual* flow—i.e., we are interested in the situation where the flows to be updated can be enumerated. Furthermore, as we are looking for efficient consistent order updates, we focus on the case where each switch can be updated at most once, from its initial to its final configuration.

Main result. Our main result is that for updating a single flow, there is a polynomialtime algorithm, with $O(n^2(n+m))$ complexity where n is the number of switches and m the number of links. The result is interesting both theoretically and practically. On the theoretical side, recent papers have presented complexity results for network updates. However, for many other consistency properties (loop-freedom, waypoint enforcement) and network models, the optimal network update problem is NP-hard [3, 5, 8, 9, 10, 11]. The same is true for results that study these problems with a model which is the same as ours (single flows, update every switch at most once). In contrast, we provide a positive result that there exists a polynomial-time algorithm for optimal order updates for a single flow, with respect to the per-packet consistency property. The consistency properties studied in these papers (loop-freedom and waypoint enforcement) are weaker than per-packet consistency, which offers a trade-off: enforcing only (for instance) loopfreedom allows more updates to be found, but it is an (exponentially) harder problem. In practice, network operators might wish to update only a small number of flows, and here our polynomial-time algorithm would be advantageous. A potential limitation is that if many flows are considered separately, it could lead to large forwarding tables.

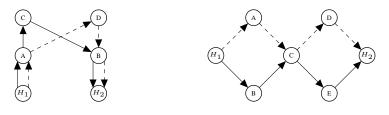


Fig. 1: Trivial update. Fig. 2: Double diamond: no consistent update order exists.

Algorithm. Our algorithm models a network configuration as a directed graph with unlabeled edges, and an update from an initial configuration to a final configuration as a sequence of individual switch-updates—i.e., updating the outgoing edges at each switch. In order to determine whether a switch n can be updated while properly respecting the per-packet consistency property, we define a set of conditions on the paths *upstream* and *downstream* from n. We show that these conditions can be checked in O(n(n + m))time. In this way, the algorithm produces a partial order on switches, representing the consistent order update (if such an order does not exist, our algorithm reports a failure). Additionally, we show that if the partial order is constructed greedily (i.e., all nodes that can be updated are immediately updated in parallel), it results in an *optimal* consistent order update. The challenging part of the proof is to show that this algorithm is complete (i.e., always finds a consistent order update if one exists) and optimal.

2 Overview

This section presents a number of simple examples to help develop intuition about the consistent order updates problem and the challenges that any solution must address.

Consistent order updates. Consider Figure 1. In the initial configuration C_i (denoted by solid edges), the forwarding-table rules (outgoing edges) on each switch are set up such that host H_1 is sending packets to H_2 along the path $H_1 \rightarrow A \rightarrow C \rightarrow B \rightarrow H_2$. Let us assume that switch C is scheduled for maintenance, meaning we must first transition to configuration C_f (denoted by the *dashed* edges). Note that the two configurations differ only for nodes A and D. If the node A is updated before node D, packets from H_1 will be dropped at D. On the other hand, updating D before A leads to a consistent order update. Note that since we model networks as graphs, we will use the terms *switch* and *node* interchangeably based on the context, and similarly for the terms *edge* and *forwarding rule. Path* will be used to describe a sequence of adjacent edges.

In Figure 2, regardless of the order in which we update nodes, there will always be inconsistency. Note that here the nodes A and D can be updated first, but a problem arises due to nodes H_1 and C. Specifically, if C is updated before H_1 , then the network is in a configuration containing a path $H_1 \rightarrow B \rightarrow C \rightarrow D \rightarrow H_2$, which is not in either C_i or C_f . In other words, H_1 cannot be updated unless the (downstream) path from C to H_2 is first updated. On the other hand, C cannot be updated unless the (upstream) path from H_1 to C is first updated. We refer to this case as a *double diamond*. If we consider the notion of dependency graphs [12], where there is an edge from a node x to node y if the update of y can only be executed after the update of x, then our double diamond example corresponds to a cyclic dependency graph between H_1 and C. Unfortunately, the presence of a double diamond (cyclic dependency) does not necessarily indicate that there cannot be a solution. Consider Figure 3, where there is a double diamond between D and J. Updating B removes the old traffic to D, and then after updating B, the nodes D, E, G, F, H, I, J have no incoming traffic. At this point, these nodes can be updated without violating per-packet consistency. Thus, the circular dependency has been eliminated, allowing a valid update order such as $[A, H_1, K, L, B, D, E, F, G, H, I, J, C, M]$. This shows that an approach (such as [6, 17]) based on a static dependency graph might miss some cases where a consistent order update exists—a limitation that our algorithm does not exhibit.

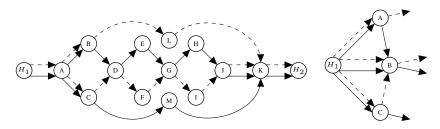


Fig. 3: Removable double diamond. Fig. 4: Wait example.

Waits. As mentioned, it may be impossible to parallelize certain updates—we may need to make sure that some node x is updated before another node y. We may need to *wait* during the sequence of switch-updates to ensure that such updates are executed one after the other. This requirement can arise because when updating a node, we may need to ensure that (1) all of the previous switch-updates have been completed, and (2) all of the packets that were in the network since before the previous update have exited the network. The former type we call a *switch-wait*, and the latter a *packet-wait*.

In Figure 3, we see that L must be updated before updating B. To ensure that edges outgoing from L are ready, we must wait after sending the update command to L, in order to ensure that its forwarding rules have been fully installed. In other words, we say that there is a *switch-wait* required between updates of L and B. After updating B, the switch D becomes disconnected, but there may still be some packets in transit on the $B \rightarrow D$ path. Before updating D, we must ensure that packets along these old removed paths have been flushed from the network. For this reason, we need a *packetwait* between updates of nodes D and B.

If we are interested only in finding a correct sequence of updates, we can wait (for an amount of time larger than the maximum switch-wait and packet-wait duration) after every node update. However, waits may not be necessary after every update if we update switches from separate parts of the network. For the Figure 3 example, the correct sequence with a *minimal* number of waits is $[A, H_1, K, L, \textcircled{s}, B, \textcircled{o}, D, E, F, G, H, I, J, \textcircled{s}, C, M]$, where o denotes a packet-wait and s denotes a switch-wait. In this example, nodes A, H_1, K, L can be updated in parallel. Similarly, nodes D, E, F, G, H, I can be updated in parallel, etc. There are three waits, meaning this consistent order update requires *four* switch-update *rounds*.

The example in Figure 4 highlights the relationship between switch-waits and packet-waits. Observing that the configurations are roughly symmetrical, let us examine the relationship between nodes A, B, C. The correct order of updates between these nodes is $H_1, A, \bigoplus, B, \bigotimes, C$. There must be a *switch-wait* between the updates of B and C, due to the presence of a C_f path $C \rightarrow B$. There must be a *packet-wait* between updates of switches A and B, due to the presence of a C_i path $A \rightarrow B$.

As is common elsewhere (e.g. [8]), in this paper, we do not distinguish between packet-waits and switch-waits, and only use the term *wait*—our goal is to maximize the parallelism of switch-updates, i.e. minimize the number of switch-update rounds.

3 Network Model

Network and Configurations. A topology of a network is a graph G = (N, E), where N is a set of nodes, and E is a set of directed edges. A configuration $C \in \mathcal{P}(E)$ is a subset of edges in E. A proper configuration is such that (a) it has one source H_1 and (b) it is acyclic. Here, a source is a designated node with no incoming edges, representing the point where packets enter the network. Note that cycles in a configuration are undesirable, as this would mean that traffic might loop forever in the network. We first consider the case with one source, and in Section 6, we describe a simple reduction for the case of multiple sources. Our goal is to transition from an initial configuration C_i to a final configuration C_f by updating individual nodes. Consider C_i and C_f to be fixed throughout the paper, and assume both are proper.

Updates. Let u be a node, and let C be a configuration. We define a function out(C, u) which returns the set of edges from C whose source is u. The function $upd_1(C, u)$ returns the configuration C' such that $C' = (C \setminus out(C_i, u)) \cup out(C_f, u)$. That is, C' has the node u updated to the final configuration. Let R be the set of all sequences that can be formed using nodes in N without repetition. We extend upd_1 to sequences of nodes by defining the function upd that, given a configuration C and a sequence of nodes S, returns a configuration C' = upd(C, S). The function upd is defined by $upd(C, \varepsilon) = C$ (where ε is the empty sequence), and $upd(C, uS) = upd(upd_1(C, u), S)$. We consider sequences of nodes without repetition, because our goal is to find update sequences that update every node at most once.

Paths. Given a configuration C, a C-path is a directed path (finite or infinite) whose edges are in C. For a path p, we write $p \in C$ if p is a C-path. A C_i -only path is one which is in C_i and not in C_f . Similarly, a C_f -only path is in C_f but not C_i . The function nodes takes a path q as an argument and returns a set Q of all nodes on a path. Let sand t be two nodes, and let C be a configuration. The function paths(s,t,C) returns the set of all paths between s and t in configuration C. A path p in a configuration Cis maximal if it is either (a) finite, and its last node has no outgoing edges in C, or (b) infinite. The function maxpaths(s, C) returns the set of all maximal paths starting at node s in configuration C.

Path and Configuration Consistency. We say that a path p is consistent if $p \in maxpaths(H_1, C_i) \lor p \in maxpaths(H_1, C_f)$, and a configuration C is consistent if and only if $\forall p \in maxpaths(H_1, C)$, we have that p is consistent. Intuitively, all maximal paths starting at H_1 are maximal paths in either the old configuration or the new

configuration—this corresponds to per-packet consistency [15]. If initial configuration C_i and final configuration C_f are proper, then so is every consistent configuration.

Waits. Let $U = u_1 u_2 \cdots u_k$ be a sequence of node updates. Let $C_j = upd(C_i, U_j)$ be the configuration reached after updating a sequence $U = u_1 u_2 \cdots u_j$ for $1 \le j \le k$, and let $C_0 = C_i$. For l, u such that $0 \le l \le u \le k$, let C_l^u be the configuration obtained as a union of configurations $C_l \cup \cdots \cup C_u$. We say that a *wait is needed* between u_j and u_k in U if and only if the configuration C_{j-1}^k is not consistent. To illustrate, let us return to the example in Figure 4 (note that we no longer distinguish between packet-waits and switch-waits). As mentioned, after updating H_1 and A, we need a wait before updating B. Let the configuration C_v be the union of all the intermediate configurations until after the update to B. Then C_v has the path $H_1 \rightarrow A \rightarrow B \rightarrow$, where we take the solid edge from A to B and a dashed outgoing edge from B, meaning a wait is needed. In this case, using the union of the configurations captures the reason for the wait.

Consistent update sequence. For any set of nodes S, let $\pi(S)$ be the set of sequences that can be formed by nodes in S, without repetition. Let $Z = S_1 S_2 \cdots S_k$ be a sequence such that each S_i is a subset of N. Let $\pi(Z)$ be the set of sequences defined by $\{r_1 r_2 \cdots r_k \mid r_1 \in \pi(S_1) \land r_2 \in \pi(S_2) \land \cdots \land r_k \in \pi(S_k)\}$.

The sequence $Z = S_1 S_2 \cdots S_k$ is a *consistent update sequence* if and only if

- 1. The sets S_1, S_2, \dots, S_k form a partition of the set of nodes N. Note that this ensures that $\forall U \in \pi(Z)$, we have $upd(C_i, U) = C_f$, i.e., after updating u, we are in C_f .
- 2. $\forall U \in \pi(Z)$, for every prefix U' of U, $C = upd(C_i, U')$ is a consistent configuration.
- ∀U ∈ π(Z), let U' = u₁u₂…u_j and U'' = u₁u₂…u_k be prefixes of u, s.t. k > j, then if a wait is needed between u_j, u_k in U, then u_j, u_k are in different sets S and S'.

Consistent Order Update Problem. Given an initial configuration C_i and the final configuration C_f , the consistent order update problem is to find a consistent update sequence if there exists one.

Optimal Consistent Order Update Problem. Given C_i and C_f , if a consistent update sequence exists, the optimal consistent update problem is to find a consistent update sequence of minimal length.

4 OrderUpdate Algorithm

This section presents an algorithm (Algorithm 1) that solves the consistent order update problem. It works by repeatedly finding and updating a node that can be updated without violating consistency. For clarity, we focus first on correctness. Section 5 presents an improved version that finds an optimal update.

Correct Sequence. A *correct* sequence of node updates $T = t_1 t_2 \cdots t_{|N|}$ refers to a consistent update sequence of singleton sets $Z = S_1 S_2 \cdots S_{|N|}$ s.t. $\forall j \in [1, |N|] : S_j = \{t_j\}$. Algorithm 1 uses a subroutine at Line 6 (in this section, the subroutine is Algorithm 2; in Section 5 we will replace it with Algorithm 3 to achieve optimality) to find a correct update sequence. It takes C_i, C_f as input and returns two sequences of nodes, R, R_w . Sequence R is the solution to the consistent order update problem (a sequence of singleton sets). Sequence R_w contains information about the placement of waits, which will be the same as R in this section, since we initially wait after every node update.

	Upstream (Condition for $paths(H_1, s, C_c)$)	Downstream (Condition for $maxpaths(s, C_c)$)
А	$Y_a(s) = \nexists p \in paths(H_1, s, C_c)$	$Z_{a}^{\dagger}(s) = (out(s, C_{f}) = \emptyset) \lor \\ \forall p \in maxpaths(s, upd(C_{c}, s)) : \\ p \in maxpaths(s, C_{f})$
В	$Y_b(s) = \neg Y_a(s) \land \forall p \in paths(H_1, s, C_c) :$ $p \in paths(H_1, s, C_i)$ $\land p \in paths(H_1, s, C_f)$	$Z_b(s) = \forall p \in maxpaths(s, upd(C_c, s)):$ $p \in maxpaths(s, C_i)$ $\lor p \in maxpaths(s, C_f)$
С	$Y_{c}(s) = \neg Y_{a}(s) \land \neg Y_{b}(s)$ $\land \forall p \in paths(H_{1}, s, C_{c}) :$ $p \in paths(H_{1}, s, C_{f})$	$Z_{c}(s) = \forall p \in maxpaths(s, upd(C_{c}, s)):$ $p \in maxpaths(s, C_{f})$
D	$Y_d(s) = \neg Y_a(s) \land \neg Y_b(s)$ $\land \forall p \in paths(H_1, s, C_c) :$ $p \in paths(H_1, s, C_i)$	$Z_d(s) = \forall p \in maxpaths(s, upd(C_c, s)):$ $p \in maxpaths(s, C_i)$
E	$Y_e(s) = \neg Y_a(s) \land \neg Y_b(s)$ $\land \neg Y_c(s) \land \neg Y_d(s)$ $= (\exists p_f \in paths(H_1, s, C_c) :$ $p_f \in paths(H_1, s, C_f)$ $\land p_f \notin paths(H_1, s, C_i))$ $\land (\exists p_i \in paths(H_1, s, C_i) :$ $p_i \notin paths(H_1, s, C_i)$ $\land p_i \notin paths(H_1, s, C_f))$	$Z_{e}(s) = \forall p \in maxpaths(s, upd(C_{c}, s)):$ $p \in maxpaths(s, C_{i})$ $\land p \in maxpaths(s, C_{f})$

Fig. 5: Necessary conditions for updating a node s in current configuration C_c

4.1 Necessary Conditions for Updating a Node

To determine which node updates lead to consistent configurations, we assume that the network is in a consistent configuration C_c , and identify a set of necessary conditions which must hold in order for the update to preserve consistency. We classify nodes into five categories based on the types of paths that are incoming to them from H_1 . The classification is given in the left-hand side of Figure 5.

Upstream Paths and Candidate Nodes. Paths from source H_1 to a node *s* are called *upstream* paths to *s* (in some configuration). The condition on these paths is called the upstream condition. If a node satisfies the upstream condition for one of the five categories/types, it is known as a *candidate* of that type.

Downstream Paths and Valid Nodes. Downstream paths from a node s are maximal paths starting at s (in some configuration). For each of the upstream conditions, there is a downstream condition which must be satisfied, in order to ensure that all maximal paths starting from H_1 in $upd(C_c, s)$ through s are consistent. If a candidate node satisfies the corresponding downstream condition, it is called *valid*. A node which is not valid is called *invalid*. Note that upstream paths to s are the same in C_c and $upd(C_c, s)$.

Lemma 1. In a consistent configuration C_c , if a valid node s is updated, then $upd(C_c, s)$ is consistent.



Fig. 6: Type B Valid Node.

Fig. 7: Type E Valid Node.

Proof. Given a consistent configuration C_c , $\forall p \in maxpaths(H_1, upd(C_c, s)) : s \notin nodes(p) \to p \in maxpaths(H_1, C_c)$. Maximal paths that are not touched by s are retained from C_c in $upd(C_c, s)$. From consistency of C_c , these paths are consistent. For checking the consistency of $upd(C_c, s)$, it is enough to ensure that $\forall p \in maxpaths(H_1, upd(C_c, s)) : s \in nodes(p) \to p$ is consistent. We use this in the rest of the proof. Our necessary conditions for updating a node ensure that all maximal paths, starting from H_1 , in $upd(C_c, s)$ through s are consistent. Figure 5 identifies nodes as Types A-E based on upstream conditions. The upstream conditions are exhaustive and mutually exclusive, meaning each node is a candidate of exactly one of the types. For each type, we show that if the node is valid, then updating it preserves consistency.

- Type A: no upstream paths incoming to node s in C_c . Type A candidate nodes are also called a *disconnected* nodes. Updating s does not add downstream maximal paths starting from H_1 to C_c . So, $maxpaths(H_1, C_c) = maxpaths(H_1, upd(C_c, s))$, meaning updating s preserves consistency. However, to simplify the presentation, Algorithm 1 imposes a downstream condition. We will show that if a correct sequence exists, then there also exists some correct sequence that updates nodes with this optional downstream condition (Z_a in Figure 5).
- Type B: paths to s from H_1 in C_c , are in both $paths(H_1, s, C_i)$ and $paths(H_1, s, C_f)$. Downstream paths in $upd(C_c, s)$ from s must be in either $maxpaths(s, C_i)$ or $maxpaths(s, C_f)$. This s is a Type B valid node in Figure 6, where highlighted edges are in C_c .
- Type C: all paths to s from H_1 in C_c , are $paths(H_1, s, C_f)$. To ensure consistency of $upd(C_c, n)$, downstream maximal paths from s in $upd(C_c, s)$ must lie in $maxpaths(s, C_f)$.
- Type D: all paths to s from H_1 in C_c , are $paths(H_1, s, C_i)$. To ensure consistency of $upd(C_c, n)$, downstream maximal paths from s in $upd(C_c, s)$ must lie in $maxpaths(s, C_i)$.
- Type E: some non-empty set of upstream paths to s in C_c , are in $paths(H_1, s, C_f) \\ paths(H_1, s, C_i)$, and some non-empty set of upstream paths to s are in $paths(H_1, s, C_i) \\ paths(H_1, s, C_i) \\ paths(H_1, s, C_f)$. This s is a Type E valid node in Figure 7, where highlighted edges are in C_c . Downstream paths from s in $upd(C_c, s)$ must be in both $maxpaths(s, C_i)$ and $maxpaths(s, C_f)$.

Using Lemma 1, each node updated by OrderUpdate leads to a valid intermediate configuration. So, we change from C_i to C_f without going through an inconsistent state, and since we wait between all updates, we obtain a consistent sequence.

Theorem 1. Any sequence R of nodes produced by Algorithm 1 (using subroutine Algorithm 2) is correct.

Proof. Every node updated by OrderUpdate preserves consistency in the network. Let a sequence $S = s_1 \cdots s_{|N|}$ be generated by OrderUpdate. Then, using Lemma 1, $\forall r \in [1, |N|] : upd(C_i, s_1 \cdots s_{r-1})$ is consistent. Finally, since all nodes are updated in S, $upd(C_i, S) = C_f$. So, if a sequence of updates is generated by Algorithm 1 using subroutine SequentialPickAndWait, it is a correct sequence.

4.2 Careful Sequences

Previously, we said that Type A candidates (disconnected nodes) do not require a downstream condition to be updated. However, Algorithm 1 imposes a downstream condition on disconnected nodes for them to be valid and updated. We refer to sequences that respect this downstream condition (i.e., update only valid nodes) as *careful* sequences. Let s be a node and C be a configuration, and define $valid_1(C,s)$ to be *true* iff s in valid in configuration C. We extend $valid_1$ to a sequence of nodes by defining valid as $valid(\varepsilon, C) = true$ (where ε is the empty sequence) and $valid(C, uS) = valid(upd(C, u), S) \land valid_1(C, u)$.

Careful Sequence A careful sequence $T = t_1 t_2 \cdots t_{|N|}$ is a correct sequence of nodes s.t. $\forall l \in [1, |N|] : valid(upd(C_i, t_1 t_2 \cdots t_{l-1}), t_l).$

Type A candidates do not have to be valid to be updated, but we enforce the downstream condition for them to be valid. The downstream condition for a Type A valid node s in Figure 5 has two clauses:

- The first clause (final-connectivity condition) is true when s is connected in C_i , but disconnected in C_f . If there are no outgoing C_f edges from s after its update, then it is a node which will be disconnected in C_f . After s becomes disconnected, it remains disconnected, as it has no incoming/outgoing C_f edges, and can be updated.
- The second clause states that all maximal paths downstream, after update, are in $maxpaths(s, C_f)$. This simplifies the proof of claims about correct sequences.

We will now prove that if there exists a correct sequence of updates, then there is also a careful sequence of updates. Before proving this, we first observe the following properties of correct sequences:

Property 1. If we have two sequences A and a permutation A' of A s.t $valid(C, A) \land valid(C, A')$, then upd(C, A) = upd(C, A').

Proof. This is because A and A' both update the same nodes in the graph. Additionally, the final configuration after both updates has the same edges regardless of the update order between A and A'. \Box

Lemma 2. Let T = UnV be a correct sequence where n is an invalid Type A candidate, then $\exists T' = Un'V'$, a correct sequence in which n' is a valid node, and V' is a sequence s.t. n'V' is a permutation of nV.

Proof. If n is an invalid disconnected node, it was not disconnected in C_f (finalconnectivity condition). Let v_p be the first node in sequence $V = v_1 v_2 \cdots v_k$ s.t. there is a path from H_1 to n in $upd(C_i, Unv_1v_2\cdots v_p)$. Let us consider a sequence

Algorithm 1: OrderUpdate

Input: Set of all nodes N, Initial configuration C_i , Final configuration C_f **Result**: An consistent order of node updates R, Updates before which there are waits R_w 1 $R = R_w = P_0 \leftarrow \emptyset; k \leftarrow 1$ // initialize R, R_w , P_0 and k2 $C_c \leftarrow C_i$ // C_c starts with the initial value of C_i // stop when C_c and C_f are equal 3 while $C_c \neq C_f$ do $U \leftarrow \{s \mid s \in N \land ((Y_a(s) \land Z_a(s)) \lor (Y_b(s) \land Z_b(s)) \lor$ 4 $(Y_c(s) \land Z_c(s)) \lor (Y_d(s) \land Z_d(s)) \lor (Y_e(s) \land Z_e(s))) \} // \text{ valid nodes}$ if $U = \emptyset$ then EXIT; // no consistent order of updates exists 5 s = PickAndWait()// by default, use Algorithm 2 $C_c \leftarrow (C_c \smallsetminus out(s, C_i)) \cup out(s, C_f)$ // update C_c $N \leftarrow N - \{s\}$ // remove updated nodes from node list 9 return (R, R_w)

Algorithm 2: SequentialPickAndWait

1 $s = Pick(U)$	<pre>// pick any valid node</pre>
2 $R_w \leftarrow R_w.s$	<pre>// by default, there is a wait after every update</pre>
3 $R \leftarrow R.s$	// append s to the end of result R

 $V'' = Uv_1v_2\cdots v_{p-1}nv_p\cdots v_k$. Let us define $\forall r \in [1,p) : C_r = upd(C_i, Unv_1\cdots v_r)$ and $C'_r = upd(C_i, Uv_1 \cdots v_r))$. $\forall r \in [1, p) : maxpaths(H_1, C_r) = maxpaths(H_1, C'_r)$ because there is no path from H_1 to n in all configurations C_r and C'_r . So in V'', updates of nodes v_1, v_2, \dots, v_{p-1} lead to consistent configurations. In V", n was disconnected before v_p was updated, so updating n after v_{p-1} leads to a consistent configuration. Finally, from Property 1, $\forall r \in [p,k] : upd(C_i, Unv_1 \cdots v_r) = upd(C_i, Uv_1 \cdots nv_p \cdots v_r)$, so every node after v_p can be updated in V", since it could be updated in T. Let $C_1 = upd(C_i, Unv_1v_2\cdots v_{p-1})$ be the configuration before updating v_p in T. To connect n to H_1 , the update of v_p when the network is in configuration C_1 will add a C_f -only edge upstream to n and create a C_f path between v_p and n. For consistency with this C_f -only edge, in C_1 , all downstream maximal paths from n are in maxpaths (n, C_f) . In C_1 , n satisfies the Type A downstream condition. $C_1 = upd(C_i, Unv_1v_2v_3\cdots v_{p-1}) =$ $upd(C_i, Uv_1v_2\cdots v_{p-1}n)$, so in V", n satisfies the downstream condition and is a Type A valid node. If V'' starts with a disconnected invalid node, we repeat this process until we find V''' = n'V' where n' is a valid node. We are guaranteed to find V''', because we continue changing invalid disconnected nodes to valid nodes, and there can be only a finite number of invalid disconnected nodes in T.

Theorem 2. If a correct sequence of updates exists, then a careful sequence also exists.

Proof. Let $Q = s_1 s_2 \cdots s_n$ be a correct update sequence. Let r be the first index s.t. $\forall i < r : s_i$ is valid and s_r is invalid. Then using Lemma 2, there is a sequence $Q' = s'_1 s'_2 \cdots s'_n$ s.t. $\forall i \le r : s'_i$ is valid. Using this argument for every index up to n, we can find a Q'' s.t. Q'' is a careful sequence.

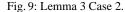
Algorithm 3: OptimalPickAndWait

4.3 Completeness of the OrderUpdate Algorithm

The OrderUpdate Algorithm (with the SequentialPickAndWait subroutine) is complete, i.e., if there exists any correct sequence, we find one. We can observe that if two nodes a and b are both valid in configuration C_c , then $upd(C_c, ab)$ and $upd(C_c, ba)$ are both consistent configurations. This property holds for any number of nodes and for all *careful* sequences, but not for all *correct* sequences. We prove this behavior in the following lemma, which is the key to observe completeness of OrderUpdate Algorithm. Lemma 3. If T = UVnY is a careful sequence, and $valid(upd(C_i, U), n)$, then T' = UnVY is also careful.



Fig. 8: Lemma 3 Case 1.



Proof. Let $V = v_1 \cdots v_k$, then $\forall r \in [1,k] : C_r = upd(C_i, Uv_1 \cdots v_r)$ and $C'_r = upd(C_i, Unv_1 \cdots v_r)$ are the configurations after updating v_r in T and T' respectively. We will argue for each node v_r in V, that C'_r is consistent. It is trivial to see that $\forall p \in maxpaths(H_1, C'_r) \cap maxpaths(H_1, C_r) : p$ is consistent. So, we only need to prove that $\forall p \in maxpaths(H_1, C'_r) \setminus maxpaths(H_1, C_r) : p$ is consistent. Each v_r can be classified into one of several types based on maximal paths in $maxpaths(H_1, C'_r) \setminus maxpaths(H_1, C_r)$.

- Case 1: $\exists p \in paths(H_1, v_r, C'_r) : p \notin paths(H_1, v_r, C_r) \land \neg(\exists p \in maxpaths(v_r, C'_r) : p \notin maxpaths(v_r, C_r))$. See Figure 8. There are upstream paths to v_r in C'_r not present in C_r . No downstream maximal paths from v_r were added in C'_r . Consider sets of paths in C'_r touching v_r :
 - 1. $up = paths(H_1, v_r, C_r)$ set of upstream paths to v_r in C_r .

- up' = {p | p ∈ paths(H₁, v_r, C'_r) : p ∉ C_r} set of upstream paths to v_r in C'_r which are not in C_r. Updating a node adds C_f-only edge(s) to the network, so for any path p containing any of these edges p ∈ C_f ∧ p ∉ C_i. Hence, ∀p ∈ up' : p ∈ C_f ∧ p ∉ C_i.
- 3. down = maxpaths $(v_r, C'_r) \subseteq maxpaths (v_r, C'_r)$ set of downstream paths from v_r in C'_r .

Let us define the \cdot operator on two sets of paths S and S'. We use $S \cdot S'$ to mean the set of all paths formed by the concatenation of any two paths $p \in S$ and $p' \in S'$ s.t. p' starts at the same node where p ends. All paths in $maxpaths(H_1, C_r) \supseteq up \cdot down$ are consistent.

 $\forall p \in (up \cdot down) : p \in maxpaths(H_1, C_i) \lor p \in maxpaths(H_1, C_f)$ (1) Let us partition down into down_1 and down_2. The set down_1 contains downstream maximal paths from v_r that existed in C'_{r-1} and $down_2 = down \lor down_1$. We inductively assume $maxpaths(H_1, C'_{r-1}) \supseteq (up \cup up') \cdot down_1$ is consistent.

 $\forall p \in (up' \cdot down_1) : p \in maxpaths(H_1, C_i) \lor p \in maxpaths(H_1, C_f)$ (2) We know $down_2 \in maxpaths(v_r, C_f)$ since they were added by some update. Paths in up' are C_f paths.

$$\forall p \in (up' \cdot down_2) : p \in maxpaths(H_1, C_f)$$
(3)

From Equations 1, 2, and 3, we conclude that:

 $\forall p \in ((up' \cup up) \cdot down) : p \in maxpaths(H_1, C_i) \lor p \in maxpaths(H_1, C_f)$

- Thus, C'_r is consistent, since all maximal paths from H_1 that touch v_r are consistent. - Case 2: $\neg(\exists p \in paths(H_1, v_r, C'_r) : p \notin paths(H_1, v_r, C_r)) \land (\exists p \in maxpaths(v_r, C'_r) : p \notin maxpaths(v_r, C_r))$. See Figure 9. There are downstream maximal paths from v_r in C'_r which were not present in C_r . No upstream paths to v_r were added. Similar to the previous case, let us define three sets of paths in C'_r that touch v_r :
 - 1. $down = maxpaths(v_r, C_r)$ set of downstream paths in C_r .
 - 2. $down' = \{p \mid p \in maxpaths(v_r, C'_r) : p \notin C_r\}$ set of downstream maximal paths from v_r not present in C_r but are present in C'_r . Similar to up' in *Case 1*, $\forall p \in down' : p \in C_f \land p \notin C_i$.
 - 3. $up = paths(H_1, v_r, C'_r) \subseteq paths(H_1, v_r, C_r)$ set of upstream paths to v_r in C_r .

We know that $maxpaths(H_1, C_r) \supseteq up \cdot down$ is a consistent configuration, so Equation 1 holds. Since updating n made changes to the downstream paths from v_r , node n lies on a downstream maximal path from v_r . Also, $\forall p \in paths(v_r, n, C'_r) :$ $p \in C_f$, because if v_r and n are connected by a path only in C_i , then updating n before v_r in T' would not be able to add C_f paths to C'_r (due to consistency reasons). This leads to one of two cases:

- ∀p ∈ paths(v_r, n, C'_r) : p ∈ C_f ∧ p ∈ C_i, i.e. v_r and n were connected from the start. Since all paths in down' touch n (C_r and C'_r were different because n was updated in C'_r), the update of v_r in C_{r-1} does not add any paths to down'.
 ∀p ∈ down' : p ∈ maxpaths(v_r, C_{r-1}). Configuration C_{r-1} is consistent and maxpaths(H₁, C_{r-1}) ⊇ up · down', ∀p ∈ (up · down') : p was consistent.
- $\exists p \in paths(v_r, n, C'_r) : p \in C_f \land p \notin C_i$, i.e. v_r and n are connected by a C_f -only path. This path existed in C_r , so paths in up can exist in a consistent

configuration with downstream maximal C_f -only paths. Paths in up can exists with paths in down' in a consistent configuration.

 $\forall p \in (up \cup down') : p \in maxpaths(H_1, C_i) \lor p \in maxpaths(H_1, C_f)$ (4) From Equation 1 and Equation 4: $\forall p \in maxpaths(H_1, C'_r = up \cup down \cup down') : p \in maxpaths(H_1, C_i) \lor p \in maxpaths(H_1, C_f)$, meaning C'_r is a consistent state and v_r can be updated.

- Case 3: $\exists p \in maxpaths(v_r, C'_r) : p \notin maxpaths(v_r, C_r) \land \exists p \in paths(H_1, v_r, C'_r) : p \notin C_r$, i.e. updating *n* added some upstream paths to v_r and some downstream maximal paths from v_r . So, *n* was both upstream to v_r and downstream from v_r . This case is not possible because updating *n* does not add any cycles to the network.
- Case 4: $\nexists p \in maxpaths(v_r, C'_r) : p \notin maxpaths(v_r, C_r) \land \nexists p \in paths(H_1, v_r, C'_r) : p \notin C_r$, i.e. there has been no change in upstream and down-stream paths. So, C'_r is a consistent state.

We have seen that every v_r in the sequence V can be updated in T'. Also, using Property 1, $upd(C_i, UnV) = upd(C_i, UVn)$, nodes in Y can be updated in sequence. Hence we showed that if T = UVnY is a correct careful sequence, T' = UnVY is a correct careful sequence.

Lemma 3 shows that if there are multiple valid nodes in some configuration C, then these nodes can be updated in any order. This is because once a node becomes valid, it does not become invalid. This is why we introduced careful sequences because this lemma is not true for arbitrary correct sequences. Using this lemma, we can prove the completeness of Algorithm 1 (with the Algorithm 2 subroutine).

Theorem 3. Algorithm 1, using subroutine Algorithm 2, generates a correct order of updates R if there exists one, or fails (in Line 5) if such an order does not exist.

Proof. We proved the correctness of Algorithm 1, using subroutine SequentialPickAnd-Wait, in Theorem 1. So we know that if it generates an order of updates, it is correct.

Let us consider the case where a correct sequence of updates exists but Algorithm 1 fails. Let $Q_{careless}$ be the correct sequence of updates, and $Q_{alg} = a_1 a_2 \cdots a_k$ be the sequence of nodes updated by Algorithm 1 before it fails. Using Theorem 2, let $Q_{careful} = s_1 s_2 \cdots s_n$ be a careful sequence. Let r be the first index s.t. $\forall i < r : s_i = a_i \land s_r \neq a_i$. If r < k, then using Lemma 3, there is another careful sequence $Q'_{careful} = s'_1 s'_2 \cdots s'_n$ s.t. $\forall i \leq r : s'_i = a_i$. Using this argument for every index up to k, we can find a correct careful sequence $Q''_{careful}$ s.t. Q_{alg} is a prefix sequence of $Q''_{careful}$. So, there is a correct node after nodes in Q_{alg} were updated and Algorithm 1 could not have failed. Therefore, if Algorithm 1 fails, then no correct sequence of updates exists.

Running Time. Let |V| be the number of nodes and |E| be the number of edges in G. In each iteration of its outer loop, Algorithm 1 using SequentialPickAndWait (Algorithm 2) as a subroutine, makes a list of valid nodes and picks one to update. The set of valid nodes U in Line 4 can be found using a graph search on C_c for each node, which takes O(|V|(|V| + |E|)) steps. The loop runs |V| times and updates each node, so the overall runtime is $O(|V|^2(|V| + |E|))$. This analysis relies on the fact that the graph search is implemented in a way that goes through each edge and node a constant number of times. Once a node has been visited, it is marked F, I, or B, based on whether the maximal paths downstream from it are maximal paths starting from it in C_i , C_f , or both. This would avoid visiting the node (and its outgoing edges) again.

5 Optimal OrderUpdate Algorithm

Thus far, we solved the consistent order update problem by generating a consistent sequence with only singleton sets. This corresponds to requiring a wait at every step of the update sequence, which does not allow any parallelism. However, we have seen in Section 2 that some nodes can be updated in parallel. In Section 3, we defined when a wait is needed in the sequence of updates. In this section, we provide a sequence of updates where there is a wait if and only if it is needed, solving the optimal version of the problem. We use Algorithm 1, but replace the subroutine *SequentialPickAndWait* (Algorithm 2) with *OptimalPickAndWait* (Algorithm 3). The algorithm returns a solution for the optimal consistent update problem in the following format.

Correct Waited Sequence. A correct waited sequence of updates is a tuple (T, W) of node sequences without repetition, where W is a subsequence of T and $(T, W) = (t_1t_2\cdots t_{|N|}, w_1w_2\cdots w_{k-1})$, such that a consistent update sequence $S_1S_2\cdots S_k$ can be formed by taking $S_1 = \{t_1, \cdots, t_m\}$ where $t_{m_1} = w_1, \forall i \in (1, k) : S_i = \{t_{l_i}, \cdots, t_{m_i}\}$ where $t_{l_i} = w_{i-1}$ and $t_{m_i} = w_i$, and $S_k = \{t_{l_k}, \cdots, t_{|N|}\}$ where $t_{l_k} = w_{k-1}$.

Intuitively, T specifies a correct sequence of updates, with some waits, while W specifies the nodes, immediately before which a wait is placed. If we simply group the nodes between *i*-th and (i + 1)-st waits into a set S_{i+1} we obtain the consistent update sequence of Section 3. Considering solutions to the problem in the form of a sequence of nodes and waits simplifies the arguments we use to prove correctness and optimality.

Minimal Correct Waited Sequence. A minimal correct waited sequence is a correct waited sequence (T, W) such that |W| is minimal.

Since we always pick valid nodes, we need to prove that if there exists a minimal correct waited sequence, then there exists a minimal correct waited sequence that updates only valid nodes.

Careful Waited Sequence. A careful waited sequence of updates $(T, W) = (t_1t_2\cdots t_{|N|}, w_1w_2\cdots w_{k-1})$ is a correct waited sequence s.t. $\forall j \in [1, |N|] :$ valid $(upd(C_i, t_1\cdots t_{j-1}), t_j)$ A minimal careful waited sequence is a careful waited sequence (T, W) s.t. |W| is minimal. We prove the following for such sequences.

Lemma 4. Let $Z = (UnV, W = w_1 \cdots w_k)$ be a correct waited sequence where *n* is an invalid disconnected node, then $\exists Z' = (Un'V', W')$, a correct waited sequence in which n' is a valid node, and V' is a sequence s.t. $n'V' = \pi(nV)$ and |W| = |W'|.

Proof. To prove Lemma 4, we use the same transformation as Lemma 2 and update n immediately before v_p , the node that connects it to the network, in a waited sequence Z' = (V'', W'), where $V'' = Uv_1v_2\cdots v_{p-1}nv_p\cdots v_k$, and prove that |W| = |W'|.

Let us consider the case where there was no wait before n in Z, i.e. n was not in sequence W. For each node $s \neq n$, let C_s and C'_s be configurations after updating s in Z and Z' respectively. For any node $s \neq n$, let r be the latest node updated before s in Z which had a wait before it (r is the last node in W). Let us form two unions

 $S = C_r \cup \cdots \cup C_s$ and $S' = C'_r \cup \cdots \cup C'_s$, consisting of unions of all intermediate configurations between r and s in Z and Z'.

- Node s was updated before n in Z. In this case S = S' as there was no change in updates before n in Z'. Since S = S', no wait is required before s in Z' if no wait was required in Z.
- Node s was updated between n and v_p in Z. In Z', n was not updated. There are two subcases:
 - Node r was updated after n in Z. For this subcase S' \ S = out(n,C_i) \ out(n,C_f). However, since n was disconnected in all configurations between C_r and C_s, consistency of S' is not affected by these edges, as there are no maximal paths from H₁ that go through n. Hence S' is consistent if S is consistent.
 - Node r was updated before n in Z. For this subcase, S' had only edges from out(s, C_i). Additionally, S had edges from both out(s, C_i) and out(s, C_f). So, S' \ S = Ø. S' is consistent if S is consistent.

In both subcases, no additional waits are required before s in Z'.

- We have $s = v_p$, or s was updated after v_p . There are again two subcases here:
 - Node r was updated before v_p in Z. In this subcase, S' \ S = out(s, C_i) \ out(s, C_f). Let us consider C₁ = upd(C_i, Uv₁...v_{p-1}n) and C₂ = upd(C_i, Uv₁...v_{p-1}nv_p). Configuration C₂ adds a C_i path p from v_p to n which was not present in C₁. Since there was no wait between n and v_p, C₁∪C₂ in consistent. So, because there was C_f upstream path from H₁ to n in C₂, C₁ had downstream maximal paths from n which were all in C_f. However, C₁ had paths in out(n, C_i). This is only possible if out(n, C_i) ⊆ out(n, C_f). So, out(s, C_i) \ out(s, C_f) = Ø and S = S'. S' is consistent if S is consistent.
 - We have r = v_p, or r was updated after v_p in Z. In this case, S = S' because ∀j > p: C_j = C'_j. So, S' is consistent if S is consistent.

We argued for all $s \neq n$ that the waits do not move. Now, let us argue for n. Let m be the latest node before n s.t. for some $j, w_j = m$. Then two cases are possible:

- In Z, no node in the sequence $v_1 \cdots v_{p-1}$ is in W. Let C_m be the configuration before updating m in Z. Since there was no wait before n in Z, we know that $S = C_m \cup \cdots \cup upd(C_i, U) \cup upd(C_i, Un)$ is consistent. We proved that waits in Z' for nodes $s \neq n$ are required at the same location as Z. So, $S' = C_m \cup$ $\cdots \cup upd(C_i, U) \cup upd(C_i, Uv_1) \cup \cdots \cup upd(C_i, Uv_1 \cdots v_{p-1})$ is consistent. Let us consider $S'' = S' \cup upd(C_i, Uv_1 \cdots v_{p-1}n)$. If there were any inconsistent paths in S'', they were also a part of S (since n is not connected to H_1 in any configuration $upd(C_i, Uv_1 \cdots v_l)$ where l < p). So, there is no wait needed before n.
- In Z, $\exists r \in [1, p)$ s.t. v_r is in W. Let q be the greatest index for which v_q satisfies this condition. Consider $S = upd(C_i, Unv_1 \cdots v_q) \cup \cdots \cup upd(C_i, Unv_1 \cdots v_{p-1})$ and $S' = upd(C_i, Uv_1 \cdots v_q) \cup \cdots \cup upd(C_i, Uv_1 \cdots v_{p-1}) \cup upd(C_i, Uv_1 \cdots v_{p-1}n)$. We proved that waits in Z' for nodes $s \neq n$ are required at the same location as Z. So, in Z', v_q was the latest node in V before which there was a wait. Then, maximal paths from H_1 in both S and S' are the same, since n was not connected to H_1 before v_p is updated. So there is no wait needed before n.

In case there was a wait before n in Z, we consider a sequence $Z'' = (Uv_1nv_2\cdots v_kY, W'')$. In Z'' there is a wait before v_1 but not before n. This is because n adds edges that are disconnected from the network. So, there is no requirement for a wait between v_1 and n. For Z'', this becomes the case with no wait before n.

Theorem 4. If a minimal correct waited sequence exists, then a minimal careful sequence exists as well.

Proof. The proof uses Lemma 4 and is similar to the proof of Theorem 2. \Box

5.1 Condition for Waits

Partial Careful Waited Sequence. Given careful waited sequence $Z = (T = t_1 \cdots t_{|N|}, W = w_1 \cdots w_{k-1})$, a partial careful waited sequence is $Z' = (T' = t_1 \cdots t_r, W' = w_1 \cdots w_s)$ such that T' is a prefix of T and W' is a prefix of W. The update mechanism starts with a partial careful waited sequence with no nodes and at every step, it adds a node in a way that ensures that the obtained sequence is a partial careful waited sequence, i.e., it can be extended to a careful waited sequence.

Wait Condition. Let us define a function *wait* that takes a partial careful waited sequence $S = (t_1t_2\cdots t_r, w_1w_2\cdots w_s)$ and node n s.t. $valid(C_i, Ut_1\cdots t_r)$ as an argument and returns true if there needs to be a wait before its update. It is defined as follows: wait(n, S) = true iff node $\exists x \in [1, r] : \neg valid(upd(C_i, t_1\cdots t_x), n) \land \neg (\exists y \in [1, s], \exists z \in (x, r] : w_y = t_z)$. In other words, in the partial careful waited sequence, there must be a wait before updating a valid node n if and only if it was not valid until its dependencies were updated, and there was no wait after their update. If this is true, then n must be updated in a new round, after a wait.

The following shows *completeness* of the wait condition, i.e., if a wait is needed (as defined in Section 3) after updating S and before updating n, then wait(n, S) is true.

Lemma 5. If (1) *n* is the node picked for update, and (2) the partial careful waited sequence built before updating *n* is $S = (t_1t_2\cdots t_r, w_1w_2\cdots w_s)$, and (3) $w_s = t_y$ for some $y \in [1, r]$, and (4) we define $\forall x \in [1, r] : C_{t_x} = upd(C_i, t_1\cdots t_x)$, and then $wait(n, S) \leftrightarrow C_{t_y} \cup \cdots \cup C_{t_r} \cup upd(C_{t_r}, n)$ is inconsistent.

Proof. Let us first prove that $wait(n, S) \to C_{t_y} \cup \cdots \cup C_{t_r} \cup upd(C_{t_r}, n)$ is inconsistent. For some a > y, let C_{t_a} be the configuration of the network in which n was invalid. We know t_a was updated after t_y , so there was no wait between the update of t_a and t_r . Updating n in C_{t_a} would lead to a inconsistent configuration $C'_{t_a} = upd(C_{t_a}, n) = (C_{t_a} \setminus out(n, C_i)) \cup out(n, C_f)$. Now, $C_{t_a} \setminus out(n, C_i) \subset C_{t_a}$ and $out(n, C_f) \subset upd(C_{t_r}, n)$. Therefore, $(C_{t_a} \setminus out(b, C_i)) \cup out(b, C_f) = C'_{t_a} \subseteq C_{t_a} \cup upd(C_{t_r}, n)$. Therefore, if wait(n, S) = false, then $C_{t_a} \cup upd(C_{t_r}, n)$ cannot be consistent.

Now let us prove that $\neg wait(n, S) \rightarrow C_{t_y} \cup \cdots \cup C_{t_r} \cup upd(C_{t_r}, n)$ is consistent. Since wait(n, S) = false, there are no waits between t_y and t_r , n was valid in every configuration reached between the updates of t_y and t_r . This means $\forall z \in [y, r] : upd(C_{t_z}, n)$ is consistent. Also $W = C_{t_y} \cup \cdots \cup C_{t_r}$ is consistent. Let us assume that $W' = C_{t_y} \cup \cdots \cup C_{t_r} \cup upd(C_{t_r}, n)$ is inconsistent. Then there is an inconsistent path in W'. However, since W was consistent, this path was not from the union of configurations in W. So, this path had edges from set $W' \setminus W = out(n, C_f) \setminus out(n, C_i)$. Let us form the set $add(t_l) = out(t_l, C_i) \setminus out(t_l, C_f)$ which is the set of edges that are added to C_{t_l} after its update. Consider these cases for each inconsistent path p in W'.

- p has no edges from $add(t_l)$ for any t_l , so $upd(C_{t_u}, n)$ is inconsistent (impossible).
- p has edges from sets $add(t_{l_1}) \cup \cdots \cup add(t_{l_z})$ for some nodes $t_{l_1} \cdots t_{l_z}$ between t_y and t_r (inclusive), then let t_{l_g} be the node in set $\{t_{l_1}, \cdots, t_{l_z}\}$ that occurs latest in sequence $t_1 \cdots t_r$. So, p existed in $upd(C_{t_g}, n)$. However, since we know that $upd(C_{t_g}, n)$ is consistent, this condition is also impossible.

Using this argument for every inconsistent path in W', we prove W is consistent. So, we have proved that the wait condition defined by function *wait* is complete.

5.2 Algorithm for Optimal Consistent Order Updates

We now present the *OptimalPickAndWait* (Algorithm 3) subroutine, that minimizes the number of waits, solving the optimal consistent update problem. Our strategy for minimizing waits is to assign one of two priorities to nodes: P_0 (higher priority) and P_1 (lower priority). Let S be a partial sequence. A node is in P_0 iff $\neg wait(n, S)$, i.e. P_0 nodes do not require waiting before update. A node is in P_1 iff wait(n, S), i.e. we must wait before updating a P_1 node. We greedily update P_0 nodes first.

Correctness and optimality follow from the correctness argument in the previous section, and from Lemma 5. Intuitively, updating a node in P_0 which does not need a wait allows the P_1 list to build up. This means we need to place a single wait for as many P_1 nodes as possible. When we place a wait in the partial careful waited sequence, every valid node that was in P_1 moves to P_0 . The last key property needed for the following theorems is that once a node acquires priority P_0 , it remains in P_0 .

Lemma 6. If a node n is valid in configuration C, then it is valid in configuration upd(C, n') for some valid node $n' \neq n$.

Proof. For validity, we do not consider the waits. We can directly apply Lemma 3. If a node n is valid in a correct sequence T = Unn'V, then if $valid(upd(C_i, U), n')$, T' = Un'nV is a correct sequence, meaning $valid(upd(C_i, Un'), n)$. So, the update of any other node does not affect the validity of n.

Lemma 7. If during the update, a node has priority P_0 , it retains priority P_0 until it is updated.

Proof. Node *n* is a priority P_0 node when the partial careful waited sequence $Z = (t_1t_2\cdots t_r, w_1w_2\cdots w_s)$ has been built. If *n* is updated after t_r , wait(n, Z) = false. However, from Lemma 6, since *n* stays valid in every configuration after the update of t_r , wait(n, Z) = false no matter where *n* is updated.

Theorem 5. Algorithm 1 with Algorithm 3 as its subroutine on Line 6 produces a correct waited sequence.

Proof. Using Lemma 5, every node that is not valid at the start is a priority P_1 node when it becomes valid. We pick P_0 nodes with higher priority, and do not wait before them. When $P_0 = \emptyset$, we wait before we pick any node in P_1 . By definition, adding a wait changes the priority of all nodes in P_1 to P_0 . From Lemma 7, these nodes retain priority P_0 until they are updated, showing that waits are correctly placed.

We now prove that our greedy scheme is optimal. For this purpose, let us prove the following two lemmas:

Lemma 8. If $Z = (T, W) = (UVnY, w_1 \cdots w_k)$ is a careful waited sequence, and in Z, after updating nodes in $U, n \in P_0$, then $Z' = (T', W') = (UnVY, w'_1 \cdots w'_k)$ is a careful waited sequence.

Proof. From Lemma 3, we know that T' is a correct sequence. Here, in addition to n being a valid node, n is a Priority P_0 node. Since $n \in P_0$ after updating U, from Lemma 7, $s \in P_0$ in both Z and Z'. So, n does not get added in W'. The partial careful waited sequence consisting only of nodes in U is the same for both Z and Z'. Let us complete this sequence by arguing for each node s in VY.

- Case 1: In $Z, s \in P_1$ (s was in W). In Z', we keep s in W'. We do not add any nodes in W' as compared with W.
- Case 2: In $Z, s \in P_0$ (s was not in W). In $Z', s \in P_0$. Since we have kept the waits at the same position as Z, if a wait was needed between any two nodes (excluding n) in Z, there is a wait in Z'. In Z, if s became valid in some configuration C, then s is also valid in upd(C, n) (Lemma 6). A wait is needed before updating s in Z', if it was needed in Z.

Hence we proved that Z' is a careful waited sequence with |W| = |W'|.

Lemma 9. If $Z = (T, W) = (UVnY, w_1 \cdots w_k)$ is a careful waited sequence, and in Z, after updating nodes in $U, P_0 = \emptyset \land n \in P_1$, then $Z' = (T', W') = (UnVY, w'_1 \cdots w'_k)$ is a careful waited sequence.

Proof. Similar to Lemma 3, the partial careful waited sequence consisting only of nodes in U is the same for both Z and Z'. Let $V = v_1 \cdots v_g$. Then since $P_0 = \emptyset$ after updating U, v_1 is in W. To construct Z', let us swap n for v_1 in W'. After this wait, $v_1 \in P_0$, so we do not need to add v_1 to W'. Then for all nodes s in $v_2 \cdots v_g Y$, we argue in the same way as in Lemma 3, and prove that Z' is a careful waited sequence with |W| = |W'|. \Box

Theorem 6. Algorithm 1 with Algorithm 3 as its subroutine on Line 6 produces a correct and optimal waited sequence of updates, if there exists a correct waited sequence of updates.

Proof. We have seen the correctness and completeness of Algorithm 1. We also proved the correctness of our approach for minimizing waits (Theorem 5). We will now prove the optimality of Algorithm 1 with the Algorithm 3 modification. Let $Q_{careless} = (T_{careless}, W_{careless})$ be an minimal correct waited sequence, and $Q_{alg} = (a_1a_2\cdots a_n, b_1\cdots b_n)$ be the sequence generated by Algorithm 1 with Algorithm 3 as its subroutine. Using Lemma 4, we know there is a minimal careful waited sequence $Q_{careful} = (s_1s_2\cdots s_n, w_1\cdots w_k)$. Let r be the first index s.t. $\forall i < r : s_i = a_i \land s_r \neq a_r$. In Q_{alg} , if $a_r \in P_0$, then by Lemma 8, we can generate a careful sequence $Q' = (s'_1s'_2\cdots s'_n, w'_1\cdots w'_k)$ s.t. $\forall i \leq r : s'_i = a_i$. In Q_{alg} , if $a_r \in P_1$, then from Algorithm 3 we know that a_r was picked because $P_0 = \emptyset$ after updating nodes $s_1s_2\cdots s_{r-1} = a_1a_2\cdots a_{r-1}$. By Lemma 9, we can again generate a minimal careful waited sequence $Q' = (s'_1s'_2\cdots s'_n, w'_1\cdots w'_k)$ s.t. $\forall i \leq r : s'_i = a_i$. Using this argument for every index from *i* to *n*, we can find a minimal careful waited sequence

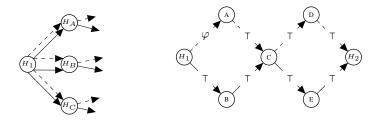


Fig. 10: Multiple sources.

Fig. 11: Double diamond case with symbolic forwarding rules.

 $Q'' = (s''_1 s''_2 \cdots s''_n, w''_1 \cdots w''_k)$ s.t $\forall i : s''_i = a_i$. Now since $\forall i : s''_i = a_i$, and our wait condition is complete (Lemma 5), so n' = k.

Running Time. The OrderUpdate Algorithm with the *OptimalPickAndWait* subroutine has the same time complexity that it had with the *SequentialPickAndWait* subroutine. The *OptimalPickAndWait* subroutine introduces a priority-based node selection mechanism—after every wait, it simply moves nodes from the valid set U to the higher priority list P_0 , which requires only O(|N|) additional steps in each iteration.

6 Discussion

Multiple hosts and sinks. We can extend our single-source approach to a network with multiple sources H_A, H_B, H_C, \cdots . To do this, we assume that there is a master source H_1 , and every actual source is connected to H_1 , as shown in Figure 10. This approach works because we update every node only once, meaning we cannot artificially disable and then re-enable some sources and keep others.

Multiple packet types. Our approach can be applied in contexts where there are multiple (discrete) packet types, as long as each forwarding rule matches on a *single* packet type—in this case, we simply compute an update for each packet type, and perform these (rule-granularity) updates independently. In the more realistic case with *symbolic* forwarding rules (i.e., matching based on *first-order formulae over packet header fields*), deciding whether a consistent update exists is CO-NP-hard. Specifically, there is a reduction from SAT to this problem. In this case, we can consider each edge in a configuration as being labeled by a formula, and only packets whose header fields satisfy this formula can be forwarded along that edge. To show the reduction, we consider a double diamond (Figure 11) with one edge labelled by such a formula φ , and all other edges labelled with *true* (\top). We have already seen that a consistent update for this double diamond example is not possible in the situation where packets (of any type) can flow along all of the edges, so we can see that *there exists a consistent update if and only if* φ *is unsatisfiable*. This completes the reduction.

7 Related Work

Consistency. Our core problem is motivated by earlier work by Reitblatt et al. [15] that proposed *per-packet consistency* and provided basic update mechanisms.

Exponential Search-Based Network Update Algorithms. There are various approaches for producing a sequence of switch updates guaranteed to respect certain path-based

consistency properties (e.g., properties representable using temporal logic, etc.). For example, McClurg et al. [14] use counter-example guided search and incremental LTL model checking, FLIP [16] uses integer linear programming, and CCG [18] uses custom reachability-based graph algorithms. Other works such as Dionysus [6], zUpdate [7], and Luo et al. [11], seek to perform updates with respect to quantitative properties.

Complexity results. Mahajan and Wattenhofer [12] introduce dependency-graphs for network updates, and propose properties which could be addressed via this general approach. They show how to handle one of the properties (*loop-freedom*) in a minimal way. Yuan et al. [17] detail general algorithms for building dependency graphs and using these graphs to perform a consistent update. Förster et al. [5] extend [12], and show that for *blackhole-freedom*, computing an update with a minimal number of rounds is NP-hard (when memory limits are assumed on switches). They also show NP-hardness results for rule-granular loop-free updates with maximal parallelism. Per-packet consistency in our problem is stronger than loop freedom and blackhole freedom, but we only consider solutions where each switch is updated *once*, and where a switch update swaps the entire old forwarding table with the new one simultaneously.

Förster and Wattenhofer [4] examine loop-freedom, showing that maximizing the number for forwarding rules updated simultaneously is NP-hard. Ludwig et al. [9] show how to minimize number of update rounds with respect to loop-freedom. They show that deciding whether a k-round schedule exists is NP-complete, and they present a polynomial algorithm for computing a weaker variant of loop-freedom. Amiri et al. [1] present an NP-hardness result for greedily updating a maximal number of forwarding rules in this context. Additionally, Ludwig et al. [8] investigate optimal updates with respect to a stronger property, namely *waypoint enforcement* in addition to loop freedom. They produce an update sequence with a minimal number of waits, using mixed-integer programming. Ludwig et al. [10] show that the decision problem is NP-hard.

Mattos et al. [13] propose a relaxed variant of per-packet consistency, where a packet may be processed by several subsequent configurations (rather than a *single* configuration), and they present a corresponding polynomial graph-based algorithm for computing updates. Dudycz et al. [3] show that simultaneously computing *two* network updates while requiring a minimal number of switch updates ("touches") is NP-hard. Brandt et al. [2] give a polynomial algorithm to decide if congestion-free update is possible when flows are "splittable" and/or not restricted to be integer.

8 Conclusion

We presented a polynomial-time algorithm to find a consistent update order on a single packet type. We then presented a modification to the algorithm, which finds a consistent update order with a minimal number of waits. Finally, we proved that this modification is correct, complete, and optimal.

References

- [1] Saeed Akhoondian Amiri, Arne Ludwig, Jan Marcinkowski, and Stefan Schmid. Transiently Consistent SDN Updates: Being Greedy is Hard. *SIROCCO*, 2016.
- [2] Sebastian Brandt, Klaus-Tycho Förster, and Roger Wattenhofer. On Consistent Migration of Flows in SDNs. *INFOCOM*, 2016.
- [3] Szymon Dudycz, Arne Ludwig, and Stefan Schmid. Can't Touch This: Consistent Network Updates for Multiple Policies. *DSN*, 2016.
- [4] Klaus-Tycho Förster and Roger Wattenhofer. The Power of Two in Consistent Network Updates: Hard Loop Freedom, Easy Flow Migration. *ICCCN*, 2016.
- [5] Klaus-Tycho Förster, Ratul Mahajan, and Roger Wattenhofer. Consistent Updates in Software Defined Networks: On Dependencies, Loop Freedom, and Blackholes. *IFIP*, 2016.
- [6] Xin Jin, Hongqiang Harry Liu, Rohan Gandhi, Srikanth Kandula, Ratul Mahajan, Ming Zhang, Jennifer Rexford, and Roger Wattenhofer. Dynamic Scheduling of Network Updates. *SIGCOMM*, 2014.
- [7] Hongqiang Harry Liu, Xin Wu, Ming Zhang, Lihua Yuan, Roger Wattenhofer, and David Maltz. zUpdate: Updating Data Center Networks with Zero Loss. SIG-COMM, 2013.
- [8] Arne Ludwig, Matthias Rost, Damien Foucard, and Stefan Schmid. Good Network Updates for Bad Packets: Waypoint Enforcement Beyond Destination-Based Routing Policies. *HotNets*, 2014.
- [9] Arne Ludwig, Jan Marcinkowski, and Stefan Schmid. Scheduling Loop-free Network Updates: It's Good to Relax! *PODC*, 2015.
- [10] Arne Ludwig, Szymon Dudycz, Matthias Rost, and Stefan Schmid. Transiently Secure Network Updates. SIGMETRICS, 2016.
- [11] Shouxi Luo, Hongfang Yu, Long Luo, and Le Min Li. Arrange Your Network Updates as You Wish. *IFIP*, 2016.
- [12] Ratul Mahajan and Roger Wattenhofer. On Consistent Updates in Software Defined Networks. *HotNets*, 2013.
- [13] Diogo Menezes Ferrazani Mattos, Otto Carlos Muniz Bandeira Duarte, and Guy Pujolle. Reverse Update: A Consistent Policy Update Scheme for Software Defined Networking. *IEEE Communications Letters*, 2016.
- [14] Jedidiah McClurg, Hossein Hojjat, Pavol Černý, and Nate Foster. Efficient Synthesis of Network Updates. PLDI, 2015.
- [15] Mark Reitblatt, Nate Foster, Jennifer Rexford, Cole Schlesinger, and David Walker. Abstractions for Network Update. *SIGCOMM*, 2012.
- [16] Stefano Vissicchio and Luca Cittadini. FLIP the (Flow) Table: Fast LIghtweight Policy-preserving SDN Updates. *INFOCOM*, 2016.
- [17] Yifei Yuan, Franjo Ivančić, Cristian Lumezanu, Shuyuan Zhang, and Aarti Gupta. Generating Consistent Updates for Software-Defined Network Configurations. *HotSDN*, 2014.
- [18] Wenxuan Zhou, Dong Jin, Jason Croft, Matthew Caesar, and P. Brighten Godfrey. Enforcing Customizable Consistency Properties in Software-Defined Networks. *NSDI*, May 2015.