Cut-elimination for the modal Grzegorczyk logic via non-well-founded proofs

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Abstract. We present a sequent calculus for the modal Grzegorczyk logic Grz allowing non-well-founded proofs and obtain the cut-elimination theorem for it by constructing a continuous cut-elimination mapping acting on these proofs.

Keywords: non-well-founded proofs, Grzegorczyk logic, cut elimination.

1 Introduction

The Grzegorczyk logic Grz is a well-known modal logic [3], which can be characterized by reflexive partially ordered Kripke frames without infinite ascending chains. This logic is complete w.r.t. the arithmetical semantics, where the modal connective $\Box$ corresponds to the strong provability operator *"... is true and provable"* in Peano arithmetic.

Recently a new proof-theoretic description for the Gödel-Löb provability logic GL in the form of a sequent calculus allowing so-called cyclic, or circular, proofs was given in [6]. A feature of cyclic proofs is that the graph underlying a proof is not a finite tree but is allowed to contain cycles. Since GL and Grz are closely connected, we wonder whether cyclic and, more generally, non-well-founded proofs can be fruitfully considered in the case of Grz.

In this paper, we present a sequent calculus for the modal Grzegorczyk logic allowing non-well-founded proofs and obtain the cut-elimination theorem for it by constructing a continuous cut-elimination mapping acting on these proofs.

In Section 2, we recall an ordinary sequent calculus for Grz. In Section 3 we introduce the infinitary proof system Grz_∞. In Section 4 we establish the cut elimination result for Grz_∞ syntactically. Then, in Section 5 we prove the equivalence of the two systems. In Section 6 we discuss possible applications of the new system.

2 Preliminaries

In this section we recall the modal Grzegorczyk logic Grz and define an ordinary sequent calculus for it.

Formulas of Grz, denoted by A, B, C, are built up as follows:

$$A ::= \bot \mid p \mid (A \to A) \mid \Box A \ ,$$

where p stands for atomic propositions. We treat other boolean connectives and the modal operator $\Diamond$ as abbreviations:

$$\neg A := A \to \bot, \qquad \top := \neg\bot, \qquad A \wedge B := \neg(A \to \neg B),$$
$$A \vee B := (\neg A \to B), \qquad \Diamond A := \neg\Box\neg A.$$

The Hilbert-style axiomatization of Grz is given by the following axioms and inference rules:

Axioms:

(i) Boolean tautologies;
(ii) $\Box(A \to B) \to (\Box A \to \Box B)$;
(iii) $\Box A \to \Box\Box A$;
(iv) $\Box A \to A$;
(v) $\Box(\Box(A \to \Box A) \to A) \to \Box A$.

Rules: modus ponens, $A/\Box A$.

Now we define an ordinary sequent calculus for Grz. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. For a multiset of formulas $\Gamma = A_1, \dots, A_n$, we set $\Box\Gamma := \Box A_1, \dots, \Box A_n$.

The system $\mathsf{Grz}_{\mathsf{Seq}}$, which is a variant of the sequent calculus from [2], is defined by the following initial sequents and inference rules:

$$\Gamma, A \Rightarrow A, \Delta\ , \qquad \Gamma, \bot \Rightarrow \Delta\ ,$$

$$\to_{\mathsf{L}} \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}\ , \qquad \to_{\mathsf{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}\ ,$$

$$\mathsf{refl}\ \frac{\Gamma, B, \Box B \Rightarrow \Delta}{\Gamma, \Box B \Rightarrow \Delta}\ , \qquad \Box_{\mathsf{Grz}} \frac{\Box\Pi, \Box(A \to \Box A) \Rightarrow A}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}\ .$$

Fig. 1. The system $\mathsf{Grz}_{\mathsf{Seq}}$

The cut rule has the form

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ ,$$

where A is called the *cut formula* of the given inference.

Lemma 21 $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathsf{Grz} \vdash \bigwedge \Gamma \to \bigvee \Delta$.

Proof. Standard transformations of proofs.

Theorem 22 *If* $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Gamma \Rightarrow \Delta$.

A syntactic cut-elimination for the logic Grz was obtained by M. Borga and P. Gentilini in [2]. In this paper, we will give another proof of this cut-elimination theorem in the next sections.

3 Non-well-founded proofs

Now we define a sequent calculus for the logic Grz allowing non-well-founded proofs. The cut-elimination theorem for it will be proved in the next section.

Inference rules and initial sequents of the sequent calculus Grz_∞ have the following form:

$$\Gamma, p \Rightarrow p, \Delta\ , \qquad \Gamma, \bot \Rightarrow \Delta\ ,$$

$$\to_{\mathsf{L}}\ \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}\ , \qquad \to_{\mathsf{R}}\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}\ ,$$

$$\mathsf{refl}\ \frac{\Gamma, A, \Box A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ , \qquad \Box\ \frac{\Gamma, \Box\Pi \Rightarrow A, \Delta \qquad \Box\Pi \Rightarrow A}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}\ .$$

Fig. 2. The system Grz_∞

The system $\mathsf{Grz}_\infty + \mathsf{cut}$ is defined by adding the rule (cut) to the system Grz_∞. An ∞*–proof* in Grz_∞ ($\mathsf{Grz}_\infty + \mathsf{cut}$) is a (possibly infinite) tree whose nodes are marked by sequents and whose leaves are marked by initial sequents and that is constructed according to the rules of the sequent calculus. In addition, every infinite branch in an ∞–proof must pass through a right premise of the rule $\Box$ infinitely many times. A sequent $\Gamma \Rightarrow \Delta$ is *provable* in Grz_∞ ($\mathsf{Grz}_\infty + \mathsf{cut}$) if there is an ∞–proof in Grz_∞ ($\mathsf{Grz}_\infty + \mathsf{cut}$) with the root marked by $\Gamma \Rightarrow \Delta$.

The *main fragment* of an ∞–proof is a finite tree obtained from the ∞–proof by cutting every infinite branch at the nearest to the root right premise of the rule $(\Box)$. The *local height* $|\pi|$ *of an* ∞*–proof* π is the length of the longest branch in its main fragment. An ∞–proof only consisting of an initial sequent has height 0.

For instance, consider an ∞–proof of the sequent $\Box(\Box(p \to \Box p) \to p) \Rightarrow p$:

$$\mathsf{refl}\ \dfrac{\to_{\mathsf{L}}\ \dfrac{\overset{\mathsf{Ax}}{F, p \Rightarrow p} \qquad \Box\ \dfrac{\to_{\mathsf{R}}\ \dfrac{\overset{\mathsf{Ax}}{F, p \Rightarrow \Box p, p}}{F \Rightarrow p \to \Box p, p} \qquad \to_{R}\ \dfrac{\Box\ \dfrac{\overset{\mathsf{Ax}}{p, F \Rightarrow p} \qquad \overset{\vdots}{F \Rightarrow p}}{p, F \Rightarrow \Box p}}{F \Rightarrow p \to \Box p}}{F \Rightarrow \Box(p \to \Box p), p}}{\Box(p \to \Box p) \to p, F \Rightarrow p}}{F \Rightarrow p}\ ,$$

where $F = \Box(\Box(p \to \Box p) \to p)$. The local height of this ∞–proof equals to 4 and its main fragment has the form

$$\mathsf{refl}\ \dfrac{\to_{\mathsf{L}}\ \dfrac{\overset{\mathsf{Ax}}{F, p \Rightarrow p} \qquad \Box\ \dfrac{\to_{\mathsf{R}}\ \dfrac{\overset{\mathsf{Ax}}{F, p \Rightarrow \Box p, p}}{F \Rightarrow p \to \Box p, p}}{F \Rightarrow \Box(p \to \Box p), p}}{\Box(p \to \Box p) \to p, F \Rightarrow p}}{F \Rightarrow p}\ .$$

By $\mathcal{P}$ denote the set of all ∞-proofs in $\mathsf{Grz}_\infty + \mathsf{cut}$. For $n \in \mathbb{N}$, we define binary relations $\sim_n$ on $\mathcal{P}$ by simultaneous induction:

1. $\pi \sim_0 \tau$ for any π, τ;
2. if $|\pi| = 0$, then $\pi \sim_n \pi$;
3. if π and τ are obtained by the same instance of inference rules $(\to_\mathsf{L})$, (cut) from π', π'' and τ', τ'', where $\pi' \sim_n \tau'$ and $\pi'' \sim_n \tau''$, then $\pi \sim_n \tau$;
4. if π and τ are obtained by the same instance of inference rules $(\to_\mathsf{R})$, (refl) from π' and τ', where $\pi' \sim_n \tau'$, then $\pi \sim_n \tau$;
5. if π and τ are obtained by the same instance of an inference rule $(\Box)$ from π', π'' and τ', τ'', where π', τ' are ∞-proofs for the left premises of $(\Box)$, and $\pi' \sim_{n+1} \tau'$, $\pi'' \sim_n \tau''$, then $\pi \sim_{n+1} \tau$.

Notice that $\pi \sim_1 \tau$ if and only if π and τ have the same main fragment.

Lemma 31 *For any $n \in \mathbb{N}$, we have that*

1. *the relation $\sim_n$ is an equivalence relation;*
2. *the relation $\sim_{n+1}$ is finer than the relation $\sim_n$.*

In addition, the intersection of all relations $\sim_n$ is exactly the equality relation over $\mathcal{P}$.

Now we define a sequence $\mathcal{P}_n$ of subsets of $\mathcal{P}$ by simultaneous induction:

1. $\pi \in \mathcal{P}_0$ for any π;
2. if $|\pi| = 0$, then $\pi \in \mathcal{P}_n$;
3. if π is obtained by an instance of an inference rule $(\to_\mathsf{L})$ from π' and π'', where $\pi', \pi'' \in \mathcal{P}_n$, then $\pi \in \mathcal{P}_n$;
4. if π is obtained by an instance of inference rules $(\to_\mathsf{R})$, (refl) from π', where $\pi' \in \mathcal{P}_n$, then $\pi \in \mathcal{P}_n$;
5. if π is obtained by an instance of an inference rule $(\Box)$ from π' and π'', where π' is an ∞-proof for the left premise of $(\Box)$, and $\pi' \in \mathcal{P}_{n+1}$, $\pi'' \in \mathcal{P}_n$, then $\pi \in \mathcal{P}_{n+1}$.

Notice that $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{P}_1$ consists of the ∞-proofs that do not contain the cut rule in their main fragment.

Lemma 32 *We have that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for any $n \in \mathbb{N}$. In addition, the intersection of all sets $\mathcal{P}_n$ consists exactly of the ∞-proofs in Grz_∞.*

For $\pi, \tau \in \mathcal{P}$, we define $d(\pi, \tau) = 2^{-\sup\{n \in \mathbb{N} \mid \pi \sim_n \tau\}}$, where by convention $2^{-\infty} = 0$. We see that an equivalence $\pi \sim_n \tau$ holds if and only if $d(\pi, \tau) \leqslant 2^{-n}$.

Proposition 33 *$(\mathcal{P}, d)$ is a complete metric space.*

A mapping $\mathcal{U} \colon \mathcal{P}_m^k \to \mathcal{P}_m$ is *nonexpansive* if for any $n \in \mathbb{N}$

$$\pi_1 \sim_n \tau_1, \ldots, \pi_k \sim_n \tau_k \Rightarrow \mathcal{U}(\pi_1, \ldots, \pi_k) \sim_n \mathcal{U}(\tau_1, \ldots, \tau_k)\,,$$

which is equivalent to the standard condition

$$d(\mathcal{U}(\pi_1,\ldots,\pi_k),\mathcal{U}(\tau_1,\ldots,\tau_k)) \leqslant \max\{d(\pi_1,\tau_1),\ldots,d(\pi_k,\tau_k)\}\ .$$

Trivially, any nonexpansive mapping is continuous.

A nonexpansive mapping $\mathcal{U}\colon \mathcal{P}\to\mathcal{P}$ is called *adequate* if $\mathcal{U}(\mathcal{P}_1)\subset\mathcal{P}_1$ and $|\mathcal{U}(\pi)|\leqslant|\pi|$ for any $\pi\in\mathcal{P}$.

Recall that an inference rule is called *admissible* (in a given proof system) if, for any instance of the rule, the conclusion is provable whenever all premises are provable. In $\mathsf{Grz}_\infty+\mathsf{cut}$, we call a single-premise inference rule *strongly admissible* if there is an adequate mapping $\mathcal{U}\colon\mathcal{P}\to\mathcal{P}$ that maps any ∞-proof of the premise of the rule to an ∞-proof of the conclusion.

Lemma 34 *For any finite multisets of formulas Π and Σ, the inference rule*

$$\mathsf{wk}_{\Pi,\Sigma}\,\frac{\Gamma\Rightarrow\Delta}{\Pi,\Gamma\Rightarrow\Delta,\Sigma}$$

is strongly admissible in $\mathsf{Grz}_\infty+\mathsf{cut}$.

Lemma 35 *For any formulas A and B, the rules*

$$\mathsf{li}_{A\to B}\,\frac{\Gamma,A\to B\Rightarrow\Delta}{\Gamma,B\Rightarrow\Delta}\qquad \mathsf{ri}_{A\to B}\,\frac{\Gamma,A\to B\Rightarrow\Delta}{\Gamma\Rightarrow A,\Delta}$$

$$\mathsf{i}_{A\to B}\,\frac{\Gamma\Rightarrow A\to B,\Delta}{\Gamma,A\Rightarrow B,\Delta}\qquad \mathsf{i}_{\bot}\,\frac{\Gamma\Rightarrow\bot,\Delta}{\Gamma\Rightarrow\Delta}\qquad \mathsf{li}_{\Box A}\,\frac{\Gamma\Rightarrow\Box A,\Delta}{\Gamma\Rightarrow A,\Delta}$$

are strongly admissible in $\mathsf{Grz}_\infty+\mathsf{cut}$.

Lemma 36 *For any atomic proposition p, the rules*

$$\mathsf{acl}_p\,\frac{\Gamma,p,p\Rightarrow\Delta}{\Gamma,p\Rightarrow\Delta}\qquad \mathsf{acr}_p\,\frac{\Gamma\Rightarrow p,p,\Delta}{\Gamma\Rightarrow p,\Delta}$$

are strongly admissible in $\mathsf{Grz}_\infty+\mathsf{cut}$.

These lemmata can be obtained in a standard way, so we omit the proofs.

4 Cut elimination

In this section we construct a continuous cut elimination mapping from $\mathcal{P}$ to $\mathcal{P}$, which eliminates all applications of the cut rule from any ∞-proof in $\mathsf{Grz}_\infty+\mathsf{cut}$. In what follows, we use nonexpansive mappings $wk_{\Pi,\Sigma}$, $li_{A\to B}$, $ri_{A\to B}$, $i_{A\to B}$, $i_\bot$, $li_{\Box A}$, acl_p, acr_p from Lemma 34, Lemma 35 and Lemma 36.

For a modal formula A, a nonexpansive mapping $\mathcal{R}$ from $\mathcal{P}_1\times\mathcal{P}_1$ to $\mathcal{P}_1$ is called *A-reducing* if $\mathcal{R}(\pi',\pi'')$ is an ∞-proof of $\Gamma\Rightarrow\Delta$ whenever π' is an ∞-proof of $\Gamma\Rightarrow\Delta,A$ and π'' is an ∞-proof of $A,\Gamma\Rightarrow\Delta$.

Lemma 41 *For any atomic proposition p there is a p-reducing mapping $\mathcal{R}_p$.*

Lemma 42 *Given a B-reducing mapping $\mathcal{R}_B$, there is a $\Box B$-reducing mapping $\mathcal{R}_{\Box B}$.*

The proof of these two Lemmas can be found in the Appendix.

Lemma 43 *For any formula A, there is an A-reducing mapping $\mathcal{R}_A$.*

Proof. We define $\mathcal{R}_A$ by induction on the structure of the formula A.

Case 1: A has the form p. In this case, $\mathcal{R}_p$ is defined in Lemma 41.

Case 2: A has the form $\bot$. Then we put $\mathcal{R}_\bot(\pi', \pi'') := i_\bot(\pi')$, where $i_\bot$ is a nonexpansive mapping from Lemma 35.

Case 3: A has the form $B \to C$. Then we put

$$\mathcal{R}_{B\to C}(\pi', \pi'') := \mathcal{R}_C(\mathcal{R}_B(wk_{\emptyset,C}(ri_{B\to C}(\pi'')), i_{B\to C}(\pi')), li_{B\to C}(\pi'')) ,$$

where $ri_{B\to C}$, $i_{B\to C}$, $li_{B\to C}$ are nonexpansive mappings from Lemma 35 and $wk_{\emptyset,C}$ is a nonexpansive mapping from Lemma 34.

Case 4: A has the form $\Box B$. By the induction hypothesis, there is a B-reducing mapping $\mathcal{R}_B$. By Lemma 42 there is a $\Box B$-reducing mapping $\mathcal{R}_{\Box B}$.

A mapping $\mathcal{U} \colon \mathcal{P} \to \mathcal{P}$ is called *root-preserving* if it maps ∞-proofs to ∞-proofs of the same sequents. The set of all root-preserving nonexpansive mappings from $\mathcal{P}$ to $\mathcal{P}$ is denoted by $\mathcal{N}$. We consider $\mathcal{N}$ as a metric space with the uniform metric:

$$d(\mathcal{U}, \mathcal{V}) = \sup_{\pi\in\mathcal{P}} d(\mathcal{U}(\pi), \mathcal{V}(\pi)) .$$

Lemma 44 $(\mathcal{N}, d)$ *is a non-empty complete metric space.*

Proof. By Lemma 33, $\mathcal{P}$ is a complete metric space. Consequently the set $C(\mathcal{P}, \mathcal{P})$ of all continuous mappings from $\mathcal{P}$ to $\mathcal{P}$ with the uniform metric forms a complete metric space. The reader will easily prove that $\mathcal{N}$ is a closed subset of $C(\mathcal{P}, \mathcal{P})$. In addition, the set $\mathcal{N}$ is non-empty, because the identity mapping belongs to $\mathcal{N}$. Thus $(\mathcal{N}, d)$ is a non-empty complete metric space.

We define $\mathcal{N}_n := \{\mathcal{U} \in \mathcal{N} \mid \mathcal{U}(\mathcal{P}) \subset \mathcal{P}_n\}$.

Lemma 45 *There exists a mapping $\mathcal{E}^* \in \mathcal{N}_1$.*

Proof. Assume we have an ∞-proof π. We define $\mathcal{E}^*(\pi)$ by induction on $|\pi|$.

If $|\pi| = 0$, then we put $\mathcal{E}^*(\pi) = \pi$. Otherwise, consider the last application of an inference rule in π and define $\mathcal{E}^*$ as follows:

$$\to_{\mathsf{L}}\dfrac{\overset{\pi_1}{\Gamma, B \Rightarrow \Delta} \qquad \overset{\pi_2}{\Gamma \Rightarrow A, \Delta}}{\Gamma, A \to B \Rightarrow \Delta} \longmapsto \to_{\mathsf{L}}\dfrac{\overset{\mathcal{E}^*(\pi_1)}{\Delta, A} \qquad \overset{\mathcal{E}^*(\pi_2)}{\Gamma \Rightarrow A, \Delta}}{\Gamma, A \to B \Rightarrow \Delta} ,$$

$$\to_{\mathsf{R}}\ \dfrac{\begin{matrix}\pi_0\\ \Gamma, A \Rightarrow B, \Delta\end{matrix}}{\Gamma \Rightarrow A \to B, \Delta} \longmapsto \to_{\mathsf{R}}\ \dfrac{\begin{matrix}\mathcal{E}^*(\pi_0)\\ \Gamma, A \Rightarrow B, \Delta\end{matrix}}{\Gamma \Rightarrow A \to B, \Delta}\ ,$$

$$\mathsf{refl}\ \dfrac{\begin{matrix}\pi_0\\ \Gamma, A, \Box A \Rightarrow \Delta\end{matrix}}{\Gamma, \Box A \Rightarrow \Delta} \longmapsto \mathsf{refl}\ \dfrac{\begin{matrix}\mathcal{E}^*(\pi_0)\\ \Gamma, A, \Box A \Rightarrow \Delta\end{matrix}}{\Gamma, \Box A \Rightarrow \Delta}\ ,$$

$$\Box\ \dfrac{\begin{matrix}\pi_1\\ \Gamma, \Box\Pi \Rightarrow A, \Delta\end{matrix} \qquad \begin{matrix}\pi_2\\ \Box\Pi \Rightarrow A\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta} \longmapsto \Box\ \dfrac{\begin{matrix}\mathcal{E}^*(\pi_1)\\ \Gamma, \Box\Pi \Rightarrow A, \Delta\end{matrix} \qquad \begin{matrix}\pi_2\\ \Box\Pi \Rightarrow A\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}\ ,$$

$$\mathsf{cut}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad \begin{matrix}\pi_2\\ A, \Gamma \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta} \longmapsto \mathcal{R}_A(\mathcal{E}^*(\pi_1), \mathcal{E}^*(\pi_2))\ .$$

Clearly, the mapping $\mathcal{E}^*$ is root-preserving, and $\mathcal{E}^*(\mathcal{P}) \subset \mathcal{P}_1$. We also see that $\mathcal{E}^*$ is nonexpansive, i.e. for any $n \in \mathbb{N}$ and any $\pi, \tau \in \mathcal{P}$

$$\pi \sim_n \tau \Rightarrow \mathcal{E}^*(\pi) \sim_n \mathcal{E}^*(\tau)\ .$$

Now we define a contractive operator $\mathcal{F}\colon \mathcal{N} \to \mathcal{N}$. The required cut-elimination mapping will be obtained as the fixed-point of $\mathcal{F}$.

For a root-preserving nonexpansive mapping $\mathcal{U}$ and an ∞-proof π of a sequent $\Gamma \Rightarrow \Delta$, we define $\mathcal{F}(\mathcal{U})(\pi)$. In the case $\pi \in \mathcal{P}_1$, $\mathcal{F}(\mathcal{U})(\pi)$ is introduced by induction on $|\pi|$. If $|\pi| = 0$, then we put $\mathcal{F}(\mathcal{U})(\pi) = \pi$. Otherwise, consider the last application of an inference rule in π and define $\mathcal{F}(\mathcal{U})$ as follows:

$$\to_{\mathsf{L}}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma, B \Rightarrow \Delta\end{matrix} \qquad \begin{matrix}\pi_2\\ \Gamma \Rightarrow A, \Delta\end{matrix}}{\Gamma, A \to B \Rightarrow \Delta} \longmapsto \to_{\mathsf{L}}\ \dfrac{\begin{matrix}\mathcal{F}(\mathcal{U})(\pi_1)\\ \Delta, A\end{matrix} \qquad \begin{matrix}\mathcal{F}(\mathcal{U})(\pi_2)\\ \Gamma \Rightarrow A, \Delta\end{matrix}}{\Gamma, A \to B \Rightarrow \Delta}\ ,$$

$$\to_{\mathsf{R}}\ \dfrac{\begin{matrix}\pi_0\\ \Gamma, A \Rightarrow B, \Delta\end{matrix}}{\Gamma \Rightarrow A \to B, \Delta} \longmapsto \to_{\mathsf{R}}\ \dfrac{\begin{matrix}\mathcal{F}(\mathcal{U})(\pi_0)\\ \Gamma, A \Rightarrow B, \Delta\end{matrix}}{\Gamma \Rightarrow A \to B, \Delta}\ ,$$

$$\mathsf{refl}\ \dfrac{\begin{matrix}\pi_0\\ \Gamma, A, \Box A \Rightarrow \Delta\end{matrix}}{\Gamma, \Box A \Rightarrow \Delta} \longmapsto \mathsf{refl}\ \dfrac{\begin{matrix}\mathcal{F}(\mathcal{U})(\pi_0)\\ \Gamma, A, \Box A \Rightarrow \Delta\end{matrix}}{\Gamma, \Box A \Rightarrow \Delta}\ ,$$

$$\Box\ \dfrac{\begin{matrix}\pi_1\\ \Gamma, \Box\Pi \Rightarrow A, \Delta\end{matrix} \qquad \begin{matrix}\pi_2\\ \Box\Pi \Rightarrow A\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta} \longmapsto \Box\ \dfrac{\begin{matrix}\mathcal{F}(\mathcal{U})(\pi_1)\\ \Gamma, \Box\Pi \Rightarrow A, \Delta\end{matrix} \qquad \begin{matrix}\mathcal{U}(\pi_2)\\ \Box\Pi \Rightarrow A\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}\ .$$

The mapping $\mathcal{F}(\mathcal{U})$ is well defined on the set $\mathcal{P}_1$. If $\pi \notin \mathcal{P}_1$, then we put $\mathcal{F}(\mathcal{U})(\pi) := \mathcal{F}(\mathcal{U})(\mathcal{E}^*(\pi))$.

It can easily be checked that $\mathcal{F}(\mathcal{U})$ is a root-preserving nonexpansive mapping.

Lemma 46 *We have that $d(\mathcal{F}(\mathcal{U}), \mathcal{F}(\mathcal{V})) \leqslant \frac{1}{2} \cdot d(\mathcal{U}, \mathcal{V})$ for any mappings $\mathcal{U}, \mathcal{V} \in \mathcal{N}$.*

Proof. Let us write $\mathcal{U} \sim_n \mathcal{V}$ if $\mathcal{U}(\pi) \sim_n \mathcal{V}(\pi)$ for any $\pi \in \mathcal{P}$. We claim that for any $n \in \mathbb{N}$

$$\mathcal{U} \sim_n \mathcal{V} \Rightarrow \mathcal{F}(\mathcal{U}) \sim_{n+1} \mathcal{F}(\mathcal{V}) \,.$$

Assume we have an ∞-proof π and $\mathcal{U} \sim_n \mathcal{V}$. Now it can be easily proved by induction on $|\pi|$ that $\mathcal{F}(\mathcal{U})(\pi) \sim_{n+1} \mathcal{F}(\mathcal{V})$.

Further, we see that $\mathcal{U} \sim_n \mathcal{V}$ if and only if $d(\mathcal{U}, \mathcal{V}) \leqslant 2^{-n}$. Thus, the condition

$$\forall n \, (\mathcal{U} \sim_n \mathcal{V} \Rightarrow \mathcal{F}(\mathcal{U}) \sim_{n+1} \mathcal{F}(\mathcal{V}))$$

is equivalent to $d(\mathcal{F}(\mathcal{U}), \mathcal{F}(\mathcal{V})) \leqslant \frac{1}{2} \cdot d(\mathcal{U}, \mathcal{V})$.

Lemma 47 *If $\mathcal{U} \in \mathcal{N}_n$, then $\mathcal{F}(\mathcal{U}) \in \mathcal{N}_{n+1}$.*

Proof. Assume we have an ∞-proof π and $\mathcal{U} \in \mathcal{N}_n$. We claim $\mathcal{F}(\mathcal{U})(\pi) \in \mathcal{P}_n$.

If $\pi \in \mathcal{P}_1$, then it is not hard to prove by induction on $|\pi|$ that $\mathcal{F}(\mathcal{U})(\pi) \in \mathcal{P}_n$. If $\pi \notin \mathcal{P}_1$, then $\mathcal{E}^*(\pi) \in \mathcal{P}_1$ by Lemma 45. Thus $\mathcal{F}(\mathcal{U})(\pi) = \mathcal{F}(\mathcal{U})(\mathcal{E}^*(\pi)) \in \mathcal{P}_n$ by the previous case.

Lemma 48 *There exists a mapping $\mathcal{E}$ such that $\mathcal{E} \in \mathcal{N}_n$ for any $n \in \mathbb{N}$.*

Proof. We have that $\mathcal{F} \colon \mathcal{N} \to \mathcal{N}$ is a contractive operator. By the Banach fixed-point theorem, there exists a root-preserving nonexpansive mapping $\mathcal{E}$ such that $\mathcal{F}(\mathcal{E}) = \mathcal{E}$. Trivially, $\mathcal{E} \in \mathcal{N}_0 = \mathcal{N}$. Hence $\mathcal{E}$ belongs to the intersection of all $\mathcal{N}_n$ for $n \in \mathbb{N}$ by Lemma 47.

Theorem 49 (cut-elimination) *If $\mathsf{Grz}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{Grz}_\infty \vdash \Gamma \Rightarrow \Delta$.*

Proof. Take an ∞-proof of the sequent $\Gamma \Rightarrow \Delta$ in the system $\mathsf{Grz}_\infty + \mathsf{cut}$ and apply the mapping $\mathcal{E}$ to it. You will get an ∞-proof of the same sequent in the system Grz_∞.

5 Ordinary and non-well-founded proofs

In this section we define two translations that connect ordinary and non-well-founded sequent calculi for Grz.

Lemma 51 *We have $\mathsf{Grz}_\infty \vdash \Gamma, A \Rightarrow A, \Delta$ for any sequent $\Gamma \Rightarrow \Delta$ and any formula A.*

Proof. Standard induction on the structure of A.

Lemma 52 *We have* $\mathsf{Grz}_\infty \vdash \Box(\Box(A \to \Box A) \to A) \Rightarrow A$ *for any formula* A.

Proof. Consider an example of ∞–proof for the sequent $\Box(\Box(p \to \Box p) \to p) \Rightarrow p$ from Section 3. We transform this example into an ∞–proof for $\Box(\Box(A \to \Box A) \to A) \Rightarrow A$ by replacing p with A and adding required ∞–proofs instead of initial sequents using Lemma 51.

Theorem 53 *If* $\mathsf{Grz_{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{Grz}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$.

Proof. Assume π is a proof of $\Gamma \Rightarrow \Delta$ in $\mathsf{Grz_{Seq}} + \mathsf{cut}$. By induction on the size of π we prove $\mathsf{Grz}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$.

If $\Gamma \Rightarrow \Delta$ is an initial sequent of $\mathsf{Grz_{Seq}} + \mathsf{cut}$, then it is provable in $\mathsf{Grz}_\infty + \mathsf{cut}$ by Lemma 51. Otherwise, consider the last application of an inference rule in π.

The only non-trivial case is when the proof π has the form

$$\Box_{\mathsf{Grz}}\,\dfrac{\begin{array}{c}\pi' \\ \Box\Pi, \Box(A \to \Box A) \Rightarrow A\end{array}}{\Sigma, \Box\Pi \Rightarrow \Box A, \Lambda}\ ,$$

where $\Sigma, \Box\Pi = \Gamma$ and $\Box A, \Lambda = \Delta$. By the induction hypothesis there is an ∞–proof ξ of $\Box\Pi, \Box(A \to \Box A) \Rightarrow A$ in $\mathsf{Grz}_\infty + \mathsf{cut}$.

We have the following ∞–proof λ of $\Box\Pi \Rightarrow A$ in $\mathsf{Grz}_\infty + \mathsf{cut}$:

$$\mathsf{cut}\,\dfrac{\Box\,\dfrac{\to_{\mathsf{R}}\,\dfrac{\begin{array}{c}\xi' \\ \Box\Pi, \Box(A \to \Box A) \Rightarrow A, A\end{array}}{\Box\Pi \Rightarrow G, A} \qquad \to_{\mathsf{R}}\,\dfrac{\begin{array}{c}\xi \\ \Box\Pi, \Box(A \to \Box A) \Rightarrow A\end{array}}{\Box\Pi \Rightarrow G}}{\Box\Pi \Rightarrow \Box G, A} \qquad \begin{array}{c}\theta \\ \Box\Pi, \Box G \Rightarrow A\end{array}}{\Box\Pi \Rightarrow A}\ ,$$

where $G = \Box(A \to \Box A) \to A$, ξ' is an ∞–proof of $\Box\Pi, \Box(A \to \Box A) \Rightarrow A, A$ obtained from ξ by Lemma 34 and θ is an ∞–proof of $\Box\Pi, \Box G \Rightarrow A$, which exists by Lemma 52 and Lemma 34.

The required ∞–proof for $\Sigma, \Box\Pi \Rightarrow \Box A, \Delta$ has the form

$$\Box\,\dfrac{\begin{array}{c}\lambda' \\ \Sigma, \Box\Pi \Rightarrow A, \Lambda\end{array} \qquad \begin{array}{c}\lambda \\ \Box\Pi \Rightarrow A\end{array}}{\Sigma, \Box\Pi \Rightarrow \Box A, \Lambda}\ ,$$

where λ' is an ∞–proof for the sequent $\Gamma, \Box\Pi \Rightarrow A, \Delta$ obtained from λ by Lemma 54.

The cases of other inference rules being last in π are straightforward, so we omit them.

Lemma 54 *The rule*

$$\mathsf{weak}\,\dfrac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}$$

is admissible in $\mathsf{Grz_{Seq}}$.

Proof. Standard induction on the structure of a proof of $\Gamma \Rightarrow \Delta$.

For a sequent $\Gamma \Rightarrow \Delta$, let $Sub(\Gamma \Rightarrow \Delta)$ be the set of all subformulas of the formulas from $\Gamma \cup \Delta$. For a finite set of formulas Λ, set $\Lambda^* := \{\Box(A \to \Box A) \mid A \in \Lambda\}$.

Lemma 55 *If* $\mathsf{Grz}_\infty \vdash \Gamma \Rightarrow \Delta$*, then* $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Lambda^*, \Gamma \Rightarrow \Delta$ *for any finite set of formulas* Λ*.*

Proof. Assume π is an ∞–proof of the sequent $\Gamma \Rightarrow \Delta$ in Grz_∞ and Λ is a finite set of formulas. By induction on the number of elements in the finite set $Sub(\Gamma \Rightarrow \Delta) \setminus \Lambda$ with a subinduction on $|\pi|$, we prove $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Lambda^*, \Gamma \Rightarrow \Delta$.

If $|\pi| = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent. We see that the sequent $\Lambda^*, \Gamma \Rightarrow \Delta$ is an initial sequent and it is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Otherwise, consider the last application of an inference rule in π.

Case 1. Suppose that π has the form

$$\to_{\mathsf{R}} \frac{\overset{\pi'}{\Gamma, A \Rightarrow B, \Sigma}}{\Gamma \Rightarrow A \to B, \Sigma} \;,$$

where $A \to B, \Sigma = \Delta$. Notice that $|\pi'| < |\pi|$. By the induction hypothesis for π' and Λ, the sequent $\Lambda^*, \Gamma, A \Rightarrow B, \Sigma$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Applying the rule ($\to_{\mathsf{R}}$) to it, we obtain that the sequent $\Lambda^*, \Gamma \Rightarrow \Delta$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$.

Case 2. Suppose that π has the form

$$\to_{\mathsf{L}} \frac{\overset{\pi'}{\Sigma, B \Rightarrow \Delta} \qquad \overset{\pi''}{\Sigma \Rightarrow A, \Delta}}{\Sigma, A \to B \Rightarrow \Delta} \;,$$

where $\Sigma, A \to B = \Gamma$. We see that $|\pi'| < |\pi|$. By the induction hypothesis for π' and Λ, the sequent $\Lambda^*, \Sigma, B \Rightarrow \Delta$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Analogously, we have $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Lambda^*, \Sigma \Rightarrow A, \Delta$. Applying the rule ($\to_{\mathsf{L}}$), we obtain that the sequent $\Lambda^*, \Sigma, A \to B \Rightarrow \Delta$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$.

Case 3. Suppose that π has the form

$$\mathsf{refl} \frac{\overset{\pi'}{\Sigma, A, \Box A \Rightarrow \Delta}}{\Sigma, \Box A \Rightarrow \Delta} \;,$$

where $\Sigma, \Box A = \Gamma$. We see that $|\pi'| < |\pi|$. By the induction hypothesis for π' and Λ, the sequent $\Lambda^*, \Sigma, A, \Box A \Rightarrow \Delta$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Applying the rule ($\mathsf{refl}$), we obtain $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Lambda^*, \Sigma, \Box A \Rightarrow \Delta$.

Case 4. Suppose that π has the form

$$\Box \frac{\overset{\pi'}{\Phi, \Box \Pi \Rightarrow A, \Sigma} \qquad \overset{\pi''}{\Box \Pi \Rightarrow A}}{\Phi, \Box \Pi \Rightarrow \Box A, \Sigma} \;,$$

where $\Phi, \Box\Pi = \Gamma$ and $\Box A, \Sigma = \Delta$.

Subcase 4.1: the formula A belongs to Λ. We see that $|\pi'| < |\pi|$. By the induction hypothesis for π' and Λ, the sequent $\Lambda^*, \Phi, \Box\Pi \Rightarrow A, \Sigma$ is provable in $\mathsf{Grz_{Seq}}$. Then we see

$$\mathsf{refl}\,\dfrac{\to_{\mathsf{L}}\,\dfrac{\dfrac{\mathsf{Ax}}{\Lambda^*, \Box A, \Phi, \Box\Pi \Rightarrow \Box A, \Sigma} \qquad \mathsf{weak}\,\dfrac{\Lambda^*, \Phi, \Box\Pi \Rightarrow A, \Sigma}{\Lambda^*, \Phi, \Box\Pi \Rightarrow A, \Box A, \Sigma}}{(\Lambda\backslash\{A\})^*, A \to \Box A, \Box(A \to \Box A), \Phi, \Box\Pi \Rightarrow \Box A, \Sigma}}{(\Lambda\backslash\{A\})^*, \Box(A \to \Box A), \Phi, \Box\Pi \Rightarrow \Box A, \Sigma}\ ,$$

where the rule (weak) is admissible by Lemma 54.

Subcase 4.2: the formula A doesn't belong to Λ. We have that the number of elements in $Sub(\Box\Pi \Rightarrow A) \setminus (\Lambda \cup \{A\})$ is strictly less than the number of elements in $Sub(\Phi, \Box\Pi \Rightarrow \Box A, \Sigma) \setminus \Lambda$. Therefore, by the induction hypothesis for π'' and $\Lambda \cup \{A\}$, the sequent $\Lambda^*, \Box(A \to \Box A), \Box\Pi \Rightarrow A$ is provable in $\mathsf{Grz_{Seq}}$. Then we have

$$\Box_{\mathsf{Grz}}\,\dfrac{\Lambda^*, \Box(A \to \Box A), \Box\Pi \Rightarrow A}{\Lambda^*, \Phi, \Box\Pi \Rightarrow \Box A, \Sigma}\ .$$

From Lemma 55 we immediately obtain the following theorem.

Theorem 56 *If* $\mathsf{Grz_\infty} \vdash \Gamma \Rightarrow \Delta$*, then* $\mathsf{Grz_{Seq}} \vdash \Gamma \Rightarrow \Delta$.

Theorem 22 is now established as a direct consequence of Theorem 53, Theorem 49, and Theorem 56.

6 Conclusion and Future Work

Recall that the Craig interpolation property for a logic L says that if A implies B, then there is an interpolant, that is, a formula I containing only common variables of A and B such that A implies I and I implies B. The Lyndon interpolation property is a strengthening of the Craig one that also takes into consideration negative and positive occurrences of the shared propositional variables; that is, the variables occurring in I positively (negatively) must also occur both in A and B positively (negatively).

Though the Grzegorczyk logic has the Lyndon interpolation property [4], there were seemingly no syntactic proofs of this result. It is unclear how Lyndon interpolation can be obtained from previously introduced sequent systems for Grz [1,2,5] by direct proof-theoretic arguments because these systems contain inference rules in which a polarity change occurs under the passage from the principal formula in the conclusion to its immediate ancestors in the premise. Using our system $\mathsf{Grz_\infty}$ we believe that we can obtain a syntactic proof of Lyndon interpolation for the modal Grzegorczyk logic as an application of our cut-elimination theorem.

We also believe that every provable $\mathsf{Grz_\infty}$ sequent has a proof that is a regular tree (has only finite amout of distinct subtrees). This gives a possibility of proof system for the logic Grz with cyclical proofs, like the system introduced in [6].

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Appendix.

Proof of Lemma 41

Assume we have two ∞-proofs π' and π'' from $\mathcal{P}_1$. If there is no application of the cut rule to these ∞-proofs with the cut formula p, then we put $\mathcal{R}_p(\pi', \pi'') := \pi'$. In the converse case, there is a sequent $\Gamma \Rightarrow \Delta$ such that π' is an ∞-proof of $\Gamma \Rightarrow \Delta, p$ and π'' is an ∞-proof of $p, \Gamma \Rightarrow \Delta$. We define $\mathcal{R}_p(\pi', \pi'')$ by induction on $|\pi'|$.

If $|\pi'| = 0$, then $\Gamma \Rightarrow \Delta, p$ is an initial sequent. Suppose that $\Gamma \Rightarrow \Delta$ is also an initial sequent. Then $\mathcal{R}_p(\pi', \pi'')$ is defined as the ∞-proof consisting only of this initial sequent. Otherwise, Γ has the form p, Φ, and π'' is an ∞-proof of $p, p, \Phi \Rightarrow \Delta$. Applying the nonexpansive mapping acl_p from Lemma 36, we put $\mathcal{R}_p(\pi', \pi'') := acl_p(\pi'')$.

Now suppose that $|\pi'| > 0$. We consider the last application of an inference rule in π'.

Case 1. The ∞-proof π' has the form

$$\dfrac{\begin{matrix}\pi'_0\\ \Gamma, A \Rightarrow B, \Sigma, p\end{matrix}}{\Gamma \Rightarrow A \to B, \Sigma, p}\,\to_{\mathsf{R}}\ ,$$

where $A \to B, \Sigma = \Delta$. Notice that $|\pi'_0| < |\pi'|$. In addition, π'' is an ∞-proof of $p, \Gamma \Rightarrow A \to B, \Sigma$. We define $\mathcal{R}_p(\pi', \pi'')$ as

$$\dfrac{\begin{matrix}\mathcal{R}_p(\pi'_0, i_{A\to B}(\pi''))\\ \Gamma, A \Rightarrow B, \Sigma\end{matrix}}{\Gamma \Rightarrow A \to B, \Sigma}\,\to_{\mathsf{R}}\ ,$$

where $i_{A\to B}$ is a nonexpansive mapping from Lemma 35.

Case 2. The ∞-proof π' has the form

$$\dfrac{\begin{matrix}\pi'_0\\ \Sigma, B \Rightarrow \Delta, p\end{matrix} \qquad \begin{matrix}\pi'_1\\ \Sigma \Rightarrow A, \Delta, p\end{matrix}}{\Sigma, A \to B \Rightarrow \Delta, p}\,\to_{\mathsf{L}}\ ,$$

where $\Sigma, A \to B = \Gamma$. We see that $|\pi'_0| < |\pi'|$ and $|\pi'_1| < |\pi'|$. Also, π'' is an ∞-proof of $p, \Sigma, A \to B \Rightarrow \Delta$. We define $\mathcal{R}_p(\pi', \pi'')$ as

$$\dfrac{\begin{matrix}\mathcal{R}_p(\pi'_0, li_{A\to B}(\pi''))\\ \Sigma, B \Rightarrow \Delta, p\end{matrix} \qquad \begin{matrix}\mathcal{R}_p(\pi'_1, ri_{A\to B}(\pi''))\\ \Sigma \Rightarrow A, \Delta, p\end{matrix}}{\Sigma, A \to B \Rightarrow \Delta, p}\,\to_{\mathsf{L}}\ ,$$

where $li_{A\to B}$ and $ri_{A\to B}$ are nonexpansive mappings from Lemma 35.

Case 3. The ∞-proof π' has the form

$$\dfrac{\begin{matrix}\pi'_0\\ \Sigma, A, \Box A \Rightarrow \Delta, p\end{matrix}}{\Sigma, \Box A \Rightarrow \Delta, p}\,\mathsf{refl}\ ,$$

where $\Sigma, \Box A = \Gamma$. We have that $|\pi'| < |\pi|$. Define $\mathcal{R}_p(\pi', \pi'')$ as

$$\mathsf{refl}\ \cfrac{\begin{array}{c}\mathcal{R}_p(\pi'_0, wk_{A,\emptyset}(\pi''))\\ \Sigma, A, \Box A \Rightarrow \Delta\end{array}}{\Sigma, \Box A \Rightarrow \Delta}\ ,$$

where $wk_{A,\emptyset}$ is the nonexpansive mapping from Lemma 34.

Case 4. Now consider the final case when π' has the form

$$\Box\ \cfrac{\begin{array}{c}\pi'_0\\ \Phi, \Box\Pi \Rightarrow A, \Sigma, p\end{array} \qquad \begin{array}{c}\pi'_1\\ \Box\Pi \Rightarrow A\end{array}}{\Phi, \Box\Pi \Rightarrow \Box A, \Sigma, p}\ ,$$

where $\Phi, \Box\Pi = \Gamma$ and $\Box A, \Sigma = \Delta$. Notice that $|\pi'_0| < |\pi'|$. In addition, π'' is an ∞-proof of $p, \Phi, \Box\Pi \Rightarrow \Box A, \Sigma$. We define $\mathcal{R}_p(\pi', \pi'')$ as

$$\Box\ \cfrac{\begin{array}{c}\mathcal{R}_p(\pi'_0, li_{\,\Box A}(\pi''))\\ \Phi, \Box\Pi \Rightarrow A, \Sigma\end{array} \qquad \begin{array}{c}\pi'_1\\ \Box\Pi \Rightarrow A\end{array}}{\Phi, \Box\Pi \Rightarrow \Box A, \Sigma}\ ,$$

where $li_{\,\Box A}$ is a nonexpansive mapping from Lemma 35.

The mapping $\mathcal{R}_p$ is well defined. It remains to check that $\mathcal{R}_p$ is nonexpansive, i.e. for any $n \in \mathbb{N}$ and any π', π'', τ', τ'' from $\mathcal{P}_0$

$$(\pi' \sim_n \tau' \wedge \pi'' \sim_n \tau'') \Rightarrow \mathcal{R}_p(\pi', \pi'') \sim_n \mathcal{R}_p(\tau', \tau'')\ .$$

This condition is checked by structural induction on the inductively defined relation $\pi' \sim_n \tau'$ in a straightforward way. So we omit further details.

Proof of Lemma 42

Assume we have two ∞-proofs π' and π'' from $\mathcal{P}_1$. If there is no application of the cut rule to these ∞-proofs with the cut formula $\Box B$, then we put $\mathcal{R}_{\Box B}(\pi', \pi'') := \pi'$. In the converse case, we define $\mathcal{R}_{\Box B}(\pi', \pi'')$ by induction on $|\pi'| + |\pi''|$.

If $|\pi'| = 0$ or $|\pi''| = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent. Then $\mathcal{R}_{\Box B}(\pi', \pi'')$ is defined as the ∞-proof consisting only of this initial sequent.

Now suppose that $|\pi'| > 0$. We consider the last application of an inference rule in π'. If the principal formula of this inference is not $\Box B$, then $\mathcal{R}_{\Box B}(\pi', \pi'')$ is defined similarly to the four cases of Lemma 41.

We can now assume that π' has the form

$$\Box\ \cfrac{\begin{array}{c}\pi'_0\\ \Phi, \Box\Pi \Rightarrow B, \Sigma\end{array} \qquad \begin{array}{c}\pi'_1\\ \Box\Pi \Rightarrow B\end{array}}{\Phi, \Box\Pi \Rightarrow \Box B, \Sigma}\ ,$$

Consider the last application of an inference rule in π''. If the rule used was $\to_{\mathsf{L}}$, $\to_{\mathsf{R}}$, refl with the principal formula being not $\Box B$, or the rule $\Box$ without the formula $\Box B$ in the right premise, then $\mathcal{R}_{\Box B}(\pi', \pi'')$ can also be defined similarly to the previous case.

Otherwise, we have the following cases.

Case A. The ∞-proof π'' has the form

$$\dfrac{\begin{array}{c}\pi_0''\\ \Gamma, B, \Box B \Rightarrow \Delta\end{array}}{\Gamma, \Box B \Rightarrow \Delta}\,\mathsf{refl}\ .$$

Since that $|\pi_0''| < |\pi''|$, we can define $\mathcal{R}_{\Box B}(\pi', \pi'')$ as

$$\mathcal{R}_B(\pi_0', \mathcal{R}_{\Box B}(\pi', \pi_0'')).$$

Case B. The ∞-proof π'' has the form

$$\Box\,\dfrac{\begin{array}{c}\pi_0''\\ \Phi', \Box B, \Box \Pi' \Rightarrow C, \Sigma'\end{array} \qquad \begin{array}{c}\pi_1''\\ \Box B, \Box \Pi' \Rightarrow C\end{array}}{\Phi', \Box B, \Box \Pi' \Rightarrow \Box C, \Sigma'}\ ,$$

Since $|\pi_0''| < |\pi''|$ and the sequents $\Phi', \Box \Pi' \Rightarrow \Box C, \Sigma'$ and $\Gamma \Rightarrow \Delta$ are equal, we can define $\mathcal{R}_{\Box B}(\pi', \pi'')$ as

$$\Box\,\dfrac{\begin{array}{c}\mathcal{R}_{\Box B}(\pi', \pi_0'')\\ \Phi', \Box \Pi' \Rightarrow C, \Sigma'\end{array} \qquad \mathsf{cut}\,\dfrac{\Box\,\dfrac{\begin{array}{c}wk_{\Box \Pi' \setminus \Box \Pi, C}(\pi_1')\\ \Box \Pi \cup \Box \Pi' \Rightarrow B, C\end{array} \qquad \begin{array}{c}wk_{\Box \Pi' \setminus \Box \Pi, \varnothing}(\pi_1')\\ \Box \Pi \cup \Box \Pi' \Rightarrow B\end{array}}{\Box \Pi \cup \Box \Pi' \Rightarrow \Box B, C} \qquad \begin{array}{c}wk_{\Box \Pi \setminus \Box \Pi', \varnothing}(\pi_1'')\\ \Box \Pi \cup \Box \Pi', \Box B \Rightarrow C\end{array}}{\Box \Pi \cup \Box \Pi' \Rightarrow C}}{\Phi', \Box \Pi' \Rightarrow \Box C, \Sigma'}$$

where $wk_{-,-}$ is a nonexpansive mapping from Lemma 34. Since the instance of the rule cut is not in the main fragment, this proof is in $\mathcal{P}_1$.