# The Complexity of Routing with Few Collisions* 

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#### Abstract

We study the computational complexity of routing multiple objects through a network in such a way that only few collisions occur: Given a graph $G$ with two distinct terminal vertices and two positive integers $p$ and $k$, the question is whether one can connect the terminals by at least $p$ routes (e.g. paths) such that at most $k$ edges are time-wise shared among them. We study three types of routes: traverse each vertex at most once (paths), each edge at most once (trails), or no such restrictions (walks). We prove that for paths and trails the problem is NP-complete on undirected and directed graphs even if $k$ is constant or the maximum vertex degree in the input graph is constant. For walks, however, it is solvable in polynomial time on undirected graphs for arbitrary $k$ and on directed graphs if $k$ is constant. We additionally study for all route types a variant of the problem where the maximum length of a route is restricted by some given upper bound. We prove that this length-restricted variant has the same complexity classification with respect to paths and trails, but for walks it becomes NP-complete on undirected graphs.


## 1 Introduction

We study the computational complexity of determining bottlenecks in networks. Consider a network in which each link has a certain capacity. We want to send a set of objects from point $s$ to point $t$ in this network, each object moving at a constant rate of one link per time step. We want to determine whether it is possible to send our (predefined number of) objects without congestion and, if not, which links in the network we have to replace by larger-capacity links to make it possible.

Apart from determining bottlenecks, the above-described task arises when securely routing very important persons [15], or packages in a network [2], routing container transporting vehicles [18], and generally may give useful insights into the structure and robustness of a network. A further motivation is congestion avoidance in routing

[^0]fleets of vehicles, a problem treated by recent commercial software products (e.g. http://nunav.net/) and poised to become more important as passenger cars and freight cars become more and more connected. Assume that we have many requests on computing a route for a set of vehicles from a source location to a target location, as it happens in daily commuting traffic. Then the idea is to centrally compute these routes, taking into account the positions in space and time of all other vehicles. To avoid congestion, we try to avoid that on two of the routes the same street appears at the same time.

A first approximation to determine such bottlenecks would be to compute the set of minimum cuts between $s$ and $t$. However, by daisy chaining our objects, we may avoid such "bottlenecks" and, hence, save on costs for improving the capacity of our links. Apart from the (static) routes we have to take into account the traversals in time that our objects take.

Formally, we are given an undirected or directed graph with marked source and sink vertex. We ask whether we can construct routes between the source and the sink in such a way that these routes share as few edges as possible. By routes herein we mean either paths, trails, or walks, modeling different restrictions on the routes: A walk is a sequence of vertices such that for each consecutive pair of vertices in the sequence there is an edge in the graph. A trail is a walk where each edge of the graph appears at most once. A path is a trail that contains each vertex at most once. We say that an edge is shared by two routes, if the edge appears at the same position in the sequence of the two routes. The sequence of a route can be interpreted as the description of where the object taking this route is at which time. So we arrive at the following core problem:
Routing with Collision Avoidance (RCA)
Input: A graph $G=(V, E)$, two distinct vertices $s, t \in V$, and two integers $p \geq 1$ and $k \geq 0$.
Question: Are there $p$ s-t routes that share at most $k$ edges?
This definition is inspired by the Minimum Shared Edges (MSE) problem [6, $15,20]$, in which an edge is already shared if it occurs in two routes, regardless of the time of traversal. Finally, note that finding routes from $s$ to $t$ also models the general case of finding routes between a set of sources and a set of sinks.

Considering our introductory motivating scenarios, it is reasonable to restrict the maximal length of the routes. For instance, when routing vehicles in daily commuting traffic while avoiding congestion, the routes should be reasonably short. Motivated by this, we study the following variant of RCA.
Fast Routing with Collision Avoidance (FRCA)
Input: $\quad$ A graph $G=(V, E)$, two distinct vertices $s, t \in V$, and three integers $p, \alpha \geq 1$ and $k \geq 0$.
Question: Are there $p s-t$ routes each of length at most $\alpha$ that share at most $k$ edges?
In the problem variants Path-RCA, Trail-RCA, and Walk-RCA, the routes are restricted to be paths, trails, or walks, respectively (analogously for FRCA).

Table 1: Overview of our results: DAGs abbreviates directed acyclic graphs; NPc., W[2]-h., P abbreviate NP-complete, W[2]-hard, and containment in the class $P$, respectively; $\Delta$ denotes the maximum degree; $\Delta_{\mathrm{i} / \mathrm{o}}$ denotes the maximum over the inand outdegrees. ${ }^{a}(\mathrm{Thm} .1)^{b}(\mathrm{Thm.3)})^{c}$ (Cor. 2) ${ }^{d}$ (even on planar graphs)

|  | Undirected, with $k$ | Directed, with $k$ | DAGs, with $k$ |  |
| :---: | :---: | :---: | :---: | :---: |
|  | constant arbitrary | constant arbitrary | const. | arbitrary |
| Path-(F)RCA | NP-c. ${ }^{d}$, Thm. 4 (Cor. 3) <br> each $k \geq 0 \quad$ each $\Delta \geq 4$ | NP-c. ${ }^{d}$, Thm. 4 (Cor. 3) <br> each $k \geq 0 \quad$ each $\Delta_{\mathrm{i} / \mathrm{o}} \geq 4$ | $\mathrm{P}^{a}$ | $\begin{aligned} & \text { NP-c., } \\ & \text { W[2]-h }{ }^{b / c} \end{aligned}$ |
| Trail-(F)RCA | NP-c. ${ }^{d}$, Thm. 5 (Cor. 4) <br> each $k \geq 0 \quad$ each $\Delta \geq 5$ | $\begin{aligned} & \text { NP-c. }{ }^{d} \text {, Thm. } 6 \text { (Cor. 5) } \\ & \text { each } k \geq 0 \quad \text { each } \Delta_{i / o} \geq 3 \end{aligned}$ | $\mathrm{P}^{a}$ | $\begin{aligned} & \text { NP-c., } \\ & \mathrm{W}[2]-\mathrm{h}^{b / c} \end{aligned}$ |
| WALK-RCA | P (Thm. 7) | $\begin{aligned} \mathrm{P}(\text { Thm. 9) } & \text { NP-c. }, \\ & \text { W[2]-h. }{ }^{b} \end{aligned}$ | $\mathrm{P}^{a}$ | $\begin{aligned} & \text { NP-c. } \\ & \text { W[2]-h. }{ }^{b} \end{aligned}$ |
| WALK-FRCA | open NP-c., W[2]-h. <br> (Thm. 8) | $\begin{aligned} & \mathrm{P}(\text { Thm. 9) } \mathrm{NP}-\mathrm{c} . \\ & \mathrm{W}[2]-\mathrm{h} . \\ & \end{aligned}$ | $\mathrm{P}^{a}$ | $\begin{aligned} & \text { NP-c. } \\ & \text { W[2]-h. }{ }^{c} \end{aligned}$ |

Our Contributions. We give a full computational complexity classification of RCA and FRCA (except WALK-FRCA) with respect to the three mentioned route types; with respect to undirected, directed, and directed acyclic input graphs; and distinguishing between constant and arbitrary budget. Table 1 summarizes our results.

To our surprise, there is no difference between paths and trails in our classification. Both Path-RCA (Section 4) and Trail-RCA (Section 5) are NP-complete in all of our cases except on directed acyclic graphs when $k \geq 0$ is constant (Section 3). We show that the problems remain NP-complete on undirected and directed graphs even if $k \geq 0$ is constant or the maximum degree is constant. We note that the Minimum Shared Edges problem is solvable in polynomial time when the number of shared edges is constant, highlighting the difference to its time-variant Path-RCA.

The computational complexity of the length-restricted variant FRCA for paths and trails equals the one of the variant without length restrictions. The variant concerning walks (Section 6) however differs from the other two variants as it is tractable in more cases, in particular on undirected graphs. (We note that almost all of our tractability results rely on flow computations in time-expanded networks (see, e.g., Skutella [19]).) Remarkably, the tractability does not transfer to the length-restricted variant WalkFRCA, as it becomes NP-complete on undirected graphs. This is the only case where RCA and FRCA differ with respect to their computational complexity.

Related Work. As mentioned, Minimum Shared Edges inspired the definition of RCA. MSE is NP-hard on directed [15] and undirected [5, 6] graphs. In contrast to RCA, if the number of shared edges equals zero, then MSE is solvable in polynomial time. Moreover, MSE is W[2]-hard with respect to the number of shared edges and fixed-parameter tractable with respect to the number of paths [6]. MSE is
polynomial-time solvable on graphs of bounded treewidth [20, 1].
There are various tractability and hardness results for problems related to RCA with $k=0$ in temporal graphs, in which edges are only available at predefined time steps $[3,10,14,13]$. The goal herein is to find a number of edge or vertex-disjoint time-respecting paths connecting two fixed terminal vertices. Time-respecting means that the time steps of the edges in the paths are nondecreasing. Apart from the fact that all graphs that we study are static, the crucial difference is in the type of routes: vehicles moving along time-respecting paths may wait an arbitrary number of time steps at each vertex, while we require them to move at least one edge per time step (unless they already arrived at the target vertex).

Our work is related to flows over time, a concept already introduced by Ford and Fulkerson [7] to measure the maximal throughput in a network over a fixed time period. This and similar problems were studied continually, see Skutella [19] and Köhler et al. [12] for surveys. In contrast, our throughput is fixed, our flow may not stand still or go in circles arbitrarily, and we want to augment the network to allow for our throughput.

## 2 Preliminaries

We define $[n]:=\{1, \ldots, n\}$ for every $n \in \mathbb{N}$. Let $G=(V, E)$ be an undirected (directed) graph. Let the sequence $P=\left(v_{1}, \ldots, v_{\ell}\right)$ of vertices in $G$ be a walk, trail, or path. We call $v_{1}$ and $v_{\ell}$ the start and end of $P$. For $i \in[\ell]$, we denote by $P[i]$ the vertex $v_{i}$ at position $i$ in $P$. Moreover, for $i, j \in[\ell], i<j$, we denote by $P[i, j]$ the subsequence $\left(v_{i}, \ldots, v_{j}\right)$ of $P$. By definition, $P$ has an alternative representation as sequence of edges (arcs) $P=\left(e_{1}, \ldots, e_{\ell-1}\right)$ with $e_{i}:=\left\{v_{i}, v_{i+1}\right\}\left(e_{i}:=\left(v_{i}, v_{i+1}\right)\right)$ for $i \in[\ell-1]$. Along this representation, we say that $P$ contains/uses edge (arc) e at time step $i$ if edge (arc) $e$ appears at the $i$ th position in $P$ represented as sequence of edges (arcs) (analogue for vertices). We call an edge/arc shared if two routes uses the edge/arc at the same time step. We say that a walk/trail/path $Q$ is an $s-t$ walk/trail/path, if $s$ is the start and $t$ is the end of $Q$. The length of a walk/trail/path is the number of edges (arcs) contained, where we also count multiple occurrences of an edge (arc) (we refer to a path of length $m$ as an $m$-chain). (We define the maximum over in- and outdegrees in $G$ by $\Delta_{\mathrm{i} / \mathrm{o}}(G):=\max _{v \in V(G)}\{\operatorname{outdeg}(v)$, indeg $(v)\}$.)

A parameterized problem $P$ is a set of tuples $(x, \ell) \in \Sigma^{*} \times \mathbb{N}$, where $\Sigma$ denotes a finite alphabet. A parameterized problem $P$ is fixed-parameter tractable if it admits an algorithm that decides every input $(x, \ell)$ in $f(\ell) \cdot|x|^{O(1)}$ time (FPT-time), where $f$ is a computable function. The class FPT is the class of fixed-parameter tractable problems. The classes $\mathrm{W}[q], q \geq 1$, contain parameterized problems that are presumably not fixed-parameter tractable. For two parameterized problems $P$ and $P^{\prime}$, a parameterized reduction from $P$ to $P^{\prime}$ is an algorithm that maps each input $(x, \ell)$ to $\left(x^{\prime}, \ell^{\prime}\right)$ in FPTtime such that $(x, \ell) \in P$ if and only if $\left(x^{\prime}, \ell^{\prime}\right) \in P^{\prime}$, and $\ell^{\prime} \leq g(\ell)$ for some function $g$. A parameterized problem $P$ is $\mathrm{W}[q]$-hard if for every problem contained in $\mathrm{W}[q]$ there is a parameterized reduction to $P$.

Preliminary Observations on RCA and FRCA. We state some preliminary observations on RCA and FRCA. If there is a shortest path between the terminals $s$ and $t$ of length at most $k$, then routing any number of paths along the shortest path introduces at most $k$ shared edges. Hence, we obtain the following.

Observation 1. Let $(G, s, t, p, k)$ be an instance of RCA with $\operatorname{dist}(s, t)<\infty$. If $k \geq \operatorname{dist}_{G}(s, t)$, then $(G, s, t, p, k)$ is a yes-instance.

If we consider walks, the length of an $s-t$ walk in a graph can be arbitrarily large. We prove, however, that for paths, trails, and walks, RCA and FRCA are contained in NP, that is, each variant allows for a certificate of size polynomial in the input size that can be verified in time polynomial in the input size.

Lemma 1. RCA and FRCA on undirected and on directed graphs are contained in NP.

Proof. Given an instance $(G, s, t, p, k)$ of Path-RCA and a set of $p s$ - $t$ paths, we can check in polynomial time whether they share at most $k$ edges. The same holds for Trail-RCA and Walk-RCA (the latter follows from Theorem 7). This is still true for all variants on directed graphs (for walks we refer to Lemma 12). Moreover, we can additionally check in linear time whether the length of each path/trail/walk is at most some given $\alpha \in \mathbb{N}$. Hence, the statement follows.

## 3 Everything is Equal on DAGs

Note that on directed acyclic graphs, every walk contains each edge and each vertex at most once. Hence, every walk is a path in DAGs, implying that all three types of routes are equivalent in DAGs.

We prove that RCA is solvable in polynomial time if the number $k$ of shared arcs is constant, but NP-complete if $k$ is part of the input. Moreover, we prove that the same holds for the length-restricted variant FRCA. We start the section with the case of constant $k \geq 0$.

### 3.1 Constant Number of Shared Arcs

Theorem 1. RCA and FRCA on n-vertex m-arc DAGs are solvable in $O\left(m^{k+1}\right.$. $\left.n^{3}\right)$ time and $O\left(m^{k+1} \cdot \alpha^{2} \cdot n\right)$ time, respectively.

We prove Theorem 1 as follows: We first show that RCA and FRCA on DAGs are solvable in polynomial time if $k=0$ (Theorem 2 below). We then show that an instance of RCA and FRCA on directed graphs is equivalent to deciding, for all $k$ sized subsets $K$ of arcs, the instance with $k=0$ and a modified input graph in which each arc in $K$ has been copied $p$ times:
Theorem 2. If $k=0, R C A$ on $n$-vertex $m$-arc $D A G$ s is solvable in $O\left(n^{3} \cdot m\right)$ time.
We need the notion of time-expanded graphs.

Definition 1. Given a directed graph $G$, we denote a directed graph $H$ the (directed) $\tau$-time-expanded graph of $G$ if $V(H)=\left\{v^{i} \mid v \in V(G), i=0, \ldots, \tau\right\}$ and $A(H)=$ $\left\{\left(v^{i-1}, w^{i}\right) \mid i \in[\tau],(v, w) \in A(G)\right\}$.

Note that for every directed $n$-vertex $m$-arc graph the $\tau$-time-expanded graph can be constructed in $O(\tau \cdot(n+m))$ time. We prove that we can decide RCA and FRCA by flow computation in the time-expanded graph of the input graph:

Lemma 2. Let $G=(V, A)$ be a directed graph with two distinct vertices $s, t \in V$. Let $p \in \mathbb{N}$ and $\tau:=|V|$. Let $H$ be the $\tau$-time-expanded graph of $G$ with $p$ additional $\operatorname{arcs}\left(t^{i-1}, t^{i}\right)$ between the copies of $t$ for each $i \in[\tau]$. Then, $G$ allows for at least $p$ $s$ - $t$ walks of length at most $\tau$ not sharing any arc if and only if $H$ allows for an $s^{0}-t^{\tau}$ flow of value at least $p$.

Proof. $(\Rightarrow)$ Let $G$ allow for $p s$ - $t$ walks $W_{1}, \ldots, W_{p}$ not sharing any arc. We construct an $s^{0}-t^{\tau}$ flow of value $p$ in $H$ as follows. Observe that $W_{i}=\left(v_{0}, \ldots, v_{\ell}\right)$ corresponds to a path $P=\left(v_{0}^{0}, v_{1}^{1}, \ldots, v_{\ell}^{\ell}\right)$ in $H$. If $\ell<\tau$, then extend this path to the path $P=$ $\left(v_{0}^{0}, \ldots, v_{\ell}^{\ell}, t^{\ell+1}, \ldots, t^{\tau}\right)$ (observe that $\left.v_{\ell}^{\ell}=t^{\ell}\right)$. Set the flow on the arcs of $P$ to one. From the fact that $W_{1}, \ldots, W_{p}$ are not sharing any arc in $G$, we extend the flow as described above for each walk by one, hence obtaining an $s^{0}-t^{\tau}$ flow of value $p$.
$(\Leftarrow)$ Let $H$ allow an $s^{0}-t^{\tau}$ flow of value $p$. It is well-known that any $s^{0}-t^{\tau}$ flow of value $p$ in $H$ can be turned into $p$ arc-disjoint $s^{0}-t^{\tau}$ paths in $H$ [11]. Let $P=\left(v_{0}^{0}, v_{1}^{1}, \ldots, v_{\ell}^{\ell}\right)$ be an $s^{0}-t^{\tau}$ path in $H$. Let $\ell^{\prime}$ be the smallest index such that $v_{\ell^{\prime}}^{\ell^{\prime}}=t^{\ell^{\prime}}$. Then $W=\left(v_{1}, v_{2}, \ldots, v_{\ell^{\prime}}\right)$ is an $s$ - $t$ walk in $G$. Let $\mathcal{P}$ be the set of $p s^{0}-t^{\tau}$ paths in $H$ obtained from an $s^{0}-t^{\tau}$ flow of value $p$, and let $\mathcal{W}$ be the set of $p s$ - $t$ walks in $G$ obtained from $\mathcal{P}$ as described above. As every pair of paths in $\mathcal{P}$ is arc-disjoint, no pair of walks in $\mathcal{W}$ share any arc in $G$.

Proof of Theorem 2. Let $(G=(V, A), s, t, p, 0)$ be an instance of Walk-RCA with $G$ being a directed acyclic graph. Let $n:=|V|$. We construct the directed $n$-timeexpanded graph $H$ of $G$ with $p$ additional $\operatorname{arcs}\left(t^{i-1}, t^{i}\right)$ for each $i \in[\tau]$. Note that any $s$ - $t$ path in $G$ is of length at most $n-1$ due to $G$ being directed and acyclic. The statement then follows from Lemma 2.

Lemma 2 is directly applicable to FRCA, by constructing an $\alpha$-expanded graph.
Corollary 1. If $k=0$, then FRCA on $n$-vertex $m$-arc $D A G$ s is solvable in $O\left(\alpha^{2} \cdot n\right.$. m) time.

Let $G=(V, A)$ be a directed graph and let $K \subseteq A$ and $x \in \mathbb{N}$. We denote by $G(K, x)$ the graph obtained from $G$ by replacing each $\operatorname{arc}(v, w) \in K$ in $G$ by $x$ copies $(v, w)_{1}, \ldots,(v, w)_{x}$.

Lemma 3. Let $(G=(V, A), s, t, p, k)$ be an instance of WALK-RCA with $G$ being a directed graph. Then, $(G, s, t, p, k)$ is a yes-instance of WaLk-RCA if and only if there exists a set $K \subseteq A$ with $|K| \leq k$ such that $(G(K, p), s, t, p, 0)$ is a yes-instance of Walk-RCA. The same statement holds true for Walk-FRCA.

Proof. $(\Rightarrow)$ Let $G$ allow for a set of $p$ s- $t$ walks $\mathcal{W}=\left\{W_{1}, \ldots, W_{p}\right\}$ sharing at most $k$ edges. Let $K \subseteq A$ denote the set of at most $k$ arcs shared by the walks in $\mathcal{W}$. We construct a set of $p s$ - $t$ walks $\mathcal{W}^{\prime}=\left\{W_{1}^{\prime}, \ldots, W_{p}^{\prime}\right\}$ in $G(K, p)$ from $\mathcal{W}$ as follows. For each $i \in[p]$, let $W_{i}^{\prime}=W_{i}$. Whenever an $\operatorname{arc}(v, w) \in K$ appears in $W_{i}^{\prime}$, we replace the arc by its copy $(v, w)_{i}$. Observe that (i) $W_{i}^{\prime}$ forms an $s-t$ walk in $G(K, p)$, (ii) the positions of the arcs in $A \backslash K$ in the walks remain unchanged, and (iii) for each arc $(v, w) \in K$, no walk contains the same copy of the arc. As the arcs in $K$ are the only shared arcs of the walks in $\mathcal{W}$, the walks in $\mathcal{W}^{\prime}$ do not share any arc in $G(K, p)$.
$(\Leftarrow)$ Let $K \subseteq A$ be a subset of arcs in $G$ with $|K| \leq k$ such that $G(K, p)$ allows for a set of $p s$ - $t$ walks $\mathcal{W}^{\prime}=\left\{W_{1}^{\prime}, \ldots, W_{p}^{\prime}\right\}$ with no shared arc. We construct a set of $p$ $s$ - $t$ walks $\mathcal{W}=\left\{W_{1}, \ldots, W_{p}\right\}$ in $G$ from $\mathcal{W}^{\prime}$ as follows. For each $i \in[p]$, let $W_{i}=W_{i}^{\prime}$. Whenever an $\operatorname{arc}(v, w)_{x}$, for some $x \in[p]$, with $(v, w) \in K$ appears in $W_{i}^{\prime}$, we replace the arc by its original $(v, w)$. Observe that (i) $W_{i}$ forms an $s$ - $t$ walk in $G$, (ii) the positions of the arcs in $A \backslash K$ in the walks remain unchanged. As the arcs in the set $K$ of at most $|K| \leq k$ arcs can appear at the same positions in any pair of two walks in $\mathcal{W}$, the $s$ - $t$ walks in $\mathcal{P}$ share at most $k$ arcs in $G$.
Note that as the length of the walks do not change in the proof, the statement of the lemma also holds for Walk-FRCA.

Proof of Theorem 1. Let $(G=(V, A), s, t, p, k)$ be an instance of WALK-RCA with $G$ being a directed acyclic graph. For each $k$-sized subset $K \subseteq A$ of $\operatorname{arcs}$ in $G$, we decide the instance $(G(K, p), s, t, p, 0)$. The statement for RCA then follows from Lemma 3 and Theorem 2. We remark that the value of a maximum flow between two terminals in an $n$-vertex $m$-arc graph can be computed in $O(n \cdot m)$ time [16]. The running time of the algorithm is in $O\left(|A|^{k} \cdot\left(|V|^{3} \cdot|A|\right)\right)$. The statement for FRCA follows analogously with Lemma 3 and Corollary 1.

### 3.2 Arbitrary Number of Shared Arcs

If the number $k$ of shared arcs is part of the input, then both RCA and FRCA are NP-complete and W[2]-hard with respect to $k$.

Theorem 3. RCA on DAGs is NP-complete and $\mathrm{W}[2]$-hard with respect to $k$.
The construction in the reduction for Theorem 3 is similar to the one used by Omran et al. [15, Theorem 2]. Herein, we give a (parameterized) many-one reduction from the NP-complete [9] Set Cover problem: given a set $U=\left\{u_{1}, \ldots, u_{n}\right\}$, a set of subsets $\mathcal{F}=\left\{F_{1}, \ldots, F_{m}\right\}$ with $F_{i} \subseteq U$ for all $i \in[m]$, and an integer $\ell \leq m$, is there a subset $\mathcal{F}^{\prime} \subseteq \mathcal{F}$ with $\left|\mathcal{F}^{\prime}\right| \leq \ell$ such that $\bigcup_{F \in \mathcal{F}^{\prime}} F=U$. We say that $\mathcal{F}^{\prime}$ is a set cover and we say that the elements in $F \in \mathcal{F}$ are covered by $F$. Note that Set Cover is $\mathrm{W}[2]$-complete with respect to the solution size $\ell$ in question [4]. In the following Construction 1, given a SET Cover instance, we construct the DAG in an equivalent RCA or FRCA instance.

Construction 1. Let a set $U=\left\{u_{1}, \ldots, u_{n}\right\}$, a set of subsets $\mathcal{F}=\left\{F_{1}, \ldots, F_{m}\right\}$ with $F_{i} \subseteq U$ for all $i \in[m]$, and an integer $\ell \leq m$ be given. Construct a directed acyclic graph $G=(V, A)$ as follows. Initially, let $G$ be the empty graph. Add the vertex sets
$V_{U}=\left\{v_{1}, \ldots, v_{n}\right\}$ and $V_{\mathcal{F}}=\left\{w_{1}, \ldots, w_{m}\right\}$, corresponding to $U$ and $\mathcal{F}$, respectively. Add the edge $\left(v_{i}, w_{j}\right)$ to $G$ if and only if $u_{i} \in F_{j}$. Next, add the vertex $s$ to $G$. For each $w \in V_{\mathcal{F}}$, add an $(\ell+2)$-chain to $G$ connecting $s$ with $w$, and direct all edges in the chain from $s$ towards $w$. For each $v \in V_{U}$, add an $(\ell+1)$-chain to $G$ connecting $s$ with $v$, and direct all edges in the chain from $s$ towards $v$. Finally, add the vertex $t$ to $G$ and add the $\operatorname{arcs}(w, t)$ for all $w \in V_{\mathcal{F}}$.

Lemma 4. Let $U, \mathcal{F}, \ell$, and $G$ as in Construction 1. Then there are at most $\ell$ sets in $\mathcal{F}$ such that their union is $U$ if and only if $G$ admits $n+m$ s-t walks sharing at most $\ell$ arcs in $G$.

Proof. $(\Rightarrow)$ Suppose there are $\ell$ sets $F_{1}^{\prime}, \ldots, F_{\ell}^{\prime} \in \mathcal{F}$ such that $\bigcup_{i \in[\ell]} F_{i}^{\prime}=U$. Let $w_{1}^{\prime}, \ldots, w_{\ell}^{\prime}$ be the vertices in $V_{\mathcal{F}}$ corresponding to $F_{1}^{\prime}, \ldots, F_{k}^{\prime}$. We construct $n+m$ $s$ - $t$ walks as follows. Each outgoing chain on $s$ corresponds to exactly one $s$ - $t$ walk. Those walks that start with the chains connecting $s$ with a $w \in V_{\mathcal{F}}$ are extended directly to $t$ (there is no other choice). For all the other walks, as $F_{1}^{\prime}, \ldots, F_{k}^{\prime}$ cover $U$, for each $v_{i} \in V_{U}$ there is at least one arc towards $\left\{w_{1}^{\prime}, \ldots, w_{k}^{\prime}\right\}$. Route the walks arbitrarily towards one out of $\left\{w_{1}^{\prime}, \ldots, w_{k}^{\prime}\right\}$ and then forward to $t$. Observe that all walks contain exactly one vertex in $V_{\mathcal{F}}$ at time step $\ell+3$. Moreover, only the arcs $\left(w_{i}^{\prime}, t\right)$ for $i \in[\ell]$ are contained in more than one walk. As they are at most $\ell$, the claim follows.
$(\Leftarrow)$ Suppose $G$ admits a set $\mathcal{W}$ of $n+m s-t$ walks sharing at most $\ell$ arcs in $G$. Observe first that the arcs of the form $(w, t), w \in V_{\mathcal{F}}$, are the only arcs that can be shared whenever at most $\ell$ arcs are shared, due to the fact that each outgoing chain on $s$ is of length longer than $k$. Moreover, each $\operatorname{arc}(w, t), w \in V_{\mathcal{F}}$ is contained in at least one $s$ - $t$ walk in $\mathcal{W}$, because no two walks in $\mathcal{W}$ can share a chain outgoing from $s$, and for each $w \in V_{\mathcal{F}}$ the only outgoing arc on $w$ has endpoint $t$. Denote by $W \subseteq V_{\mathcal{F}}$ the set of vertices such that the set $\{(w, t) \mid w \in W\}$ is exactly the set of shared arcs by the $n+m s$ - $t$ walks in $\mathcal{W}$. Observe that $|W| \leq \ell$. We claim that the set of sets $\mathcal{F}^{\prime}$, containing the sets that by construction correspond to the vertices in $W$, forms a set cover for $U$. We show that for each element $u \in U$ there is a $w \in W$ such that the set corresponding to $w$ is containing $u$.

Let $u \in U$ be an arbitrary element of $U$. Consider the walk $P \in \mathcal{W}$ containing the vertex $v \in V_{U}$ corresponding to element $u$. As $P$ forms an $s$ - $t$ walk in $G$, walk $P$ contains a vertex $w^{\prime} \in V_{\mathcal{F}}$. As discussed before, there is a walk $P^{\prime}$ containing the chain from $s$ to $w^{\prime}$ not containing any vertex in $V_{U}$. By construction, $w^{\prime}$ is at time step $\ell+3$ in both $P$ and $P^{\prime}$. As the only outgoing arc on $w^{\prime}$ is $\left(w^{\prime}, t\right)$, both $P$ and $P^{\prime}$ use the arc $\left(w^{\prime}, t\right)$ at time step $\ell+3$, and hence $(w, t)$ is shared by $P$ and $P^{\prime}$. It follows that $w^{\prime} \in W$, and hence $u$ is covered by the set in $\mathcal{F}^{\prime}$ corresponding to $w^{\prime}$.

Proof of Theorem 3. We give a (parameterized) many-one reduction from SET Cover to RCA. Let $(U, \mathcal{F}, \ell)$ be an instance of Set Cover. We construct the instance $(G, s, t, p, k)$, where $G$ is obtained by applying Construction $1, p=|U|+|\mathcal{F}|$, and $k=\ell$. The correctness of the reduction then follows from Lemma 4. Finally, note that as $k=\ell$ and Set Cover is W[2]-hard with respect to the size $\ell$ of the set cover, it follows that RCA is $\mathrm{W}[2]$-hard with respect to the number $k$ of shared arcs.

Observe that each $s-t$ walk in the graph obtained from Construction 1 is of length at most $\ell+3$. Hence, in the proof of Theorem 3, we can instead reduce to an instance $(G, s, t, p, k, \alpha)$ of FRCA, where $G$ is obtained by applying Construction 1, $p=|U|+|\mathcal{F}|$, and $k=\ell$, and $\alpha=\ell+3$. Therewith, we obtain the following.
Corollary 2. FRCA on DAGs is NP -complete and $\mathrm{W}[2]$-hard with respect to $k+\alpha$.

## 4 Path-RCA

In this section, we prove the following theorem.
Theorem 4. Path-RCA both on undirected planar and directed planar graphs is NP-complete, even if $k \geq 0$ is constant or $\Delta \geq 4$ is constant.

In the proof of Theorem 4, we reduce from the following NP-complete [8] problem (a cubic graph is a graph where every vertex has degree exactly three):

Planar Cubic Hamiltonian Cycle (PCHC)
Input: An undirected, planar, cubic graph $G$.
Question: Is there a cycle in $G$ that visits each vertex exactly once?
Roughly, the instance of Path-RCA obtained in the reduction consists of the original graph $G$ connected to the terminals $s, t$ via a bridge (see Figure 1). We ask for constructing roughly $n$ paths connecting the terminals, where $n$ is the number of vertices in the input graph of PCHC. All but one of these paths will use the bridge to $t$ in the constructed graph for $n$ time steps in total, each in a different time step. Thus, this bridge is occupied for roughly $n$ time steps, and the final path is forced to stay in the input graph of PCHC for $n$ time steps. For a path, this is only possible by visiting each of the $n$ vertices in the graph exactly once, and hence it corresponds to a Hamiltonian cycle.

The reduction to prove Theorem 4 uses the following Construction 2.
Construction 2. Let $G=(V, E)$ be an undirected, planar, cubic graph with $n=|V|$. Construct in time polynomial in the size of $G$ an undirected planar graph $G^{\prime}$ as follows (refer to Figure 1 for an illustration of the constructed graph). Let initially $G^{\prime}$ be the empty graph. Add a copy of $G$ to $G^{\prime}$. Denote the copy of $G$ in $G^{\prime}$ by $H$. Next, add the new vertices $s, t, v, w$ to $G$. Connect $s$ with $v$, and $w$ with $t$ by an edge. For each $m \in\{4,5, \ldots, n+1\}$, add an $m$-chain connecting $s$ with $w$. Next, consider a fixed plane embedding $\phi(G)$ of $G$. Let $x_{1}$ denote a vertex incident to the outer face in $\phi(G)$. Then, there are two neighbors $x_{2}$ and $x_{3}$ of $x_{1}$ also incident to the outer face in $\phi(G)$. Add the edges $\left\{v, x_{1}\right\},\left\{x_{2}, w\right\}$ and $\left\{x_{3}, w\right\}$ to $G^{\prime}$ completing the construction of $G^{\prime}$. We remark that $G^{\prime}$ is planar as it allows a plane embedding (see Figure 1) using $\phi$ as an embedding of $H$.

Lemma 5. Let $G$ and $G^{\prime}$ be as in Construction 2. Then $G$ admits a Hamiltonian cycle if and only if $G^{\prime}$ allows for at least $n-1 s$-t paths with no shared edge.


Figure 1: Graph $G^{\prime}$ obtained in Construction 2. The gray part represents the graph $H$. Dashed lines represent chains.

Proof. $(\Leftarrow)$ Let $\mathcal{P}$ denote a set of $n-1 s$ - $t$ paths in $G^{\prime}$ with no shared edge. Note that the degree of $s$ is equal to $n-1$. As no two paths in $\mathcal{P}$ share any edge in $G^{\prime}$, each path in $\mathcal{P}$ uses a different edge incident to $s$. This implies that $n-2$ paths in $\mathcal{P}$ uniquely contain each of the chains connecting $s$ with $w$, and one path $P \in \mathcal{P}$ contains the edge $\{s, v\}$. Note that each of the $n-2$ paths contain the vertex $w$ at most once, and since they contain the chains connecting $s$ with $w$, the edge $\{w, t\}$ appears at the time steps $\{5,6, \ldots, n+2\}$ in these $n-2$ paths $\mathcal{P}$. Hence, the path $P$ has to contain the edge $\{w, t\}$ at a time step smaller than five or larger than $n+2$. Observe that, by construction, the shortest path between $s$ and $w$ is of length 4 and, thus, $P$ cannot contain the edge $\{w, t\}$ on any time step smaller than five. Hence, $P$ has to contain the edge at time step at least $n+3$. Since the distance between $s$ and $x_{1}$ is two, and the distance from $x_{2}, x_{3}$ to $w$ is one, $P$ has to visit each vertex in $H$ exactly once, starting at $x_{1}$, and ending at one of the two neighbors $x_{2}$ or $x_{3}$ of $x_{1}$. Hence, $P$ restricted to $H$ describes a Hamiltonian path in $H$, which can be extended to an Hamiltonian cycle by adding the edge $\left\{x_{1}, x_{2}\right\}$ in the first or $\left\{x_{1}, x_{3}\right\}$ in the second case.
$(\Rightarrow)$ Let $G$ admit a Hamiltonian cycle $C$. Since $C$ contains every vertex in $G$ exactly once, it contains $x_{1}$ and its neighbors $x_{2}$ and $x_{3}$. Since $C$ forms a cycle in $G$ and $G$ is cubic, at least one of the edges $\left\{x_{1}, x_{2}\right\}$ or $\left\{x_{1}, x_{3}\right\}$ appears in $C$. Let $C^{\prime}$ denote an ordering of the vertices in $C$ such that $x_{1}$ appears first and the neighbor $x \in\left\{x_{2}, x_{3}\right\}$ of $x_{1}$ with $\left\{x_{1}, x\right\}$ contained in $C$ appears last. We construct $n-1 s$ - $t$ paths without sharing an edge. First, we construct $n-2 s$ - $t$ paths, each containing a different chain connecting $s$ with $w$ and the edge $\{w, t\}$. Observe that since the lengths of each chain is unique, no edge (in particular, not $\{w, t\}$ ) is shared. Finally, we construct the one remaining $s$ - $t$ path $P$ as follows. We lead $P$ from $s$ to $x_{1}$ via $v$, then following $C^{\prime}$ in $H$ to $x$, and then from $x$ to $t$ via $w$. Observe that $P$ has length $n+3$ and contains the edge $\{w, t\}$ at time step $n+3$. Hence, no edge is shared as the path containing the $(n+1)$-chain contains the edge $\{w, t\}$ at time step $(n+2)$. We constructed $n-1$ $s$ - $t$ paths in $G^{\prime}$ with no shared edge.

Note that the maximum degree of the graph obtained in the Construction 2 depends on the number of vertices in the input graph. In what follows, we give a second construction where the obtained graph has constant maximum degree $\Delta=4$.


Figure 2: $T_{s}$ and $T_{w}$ refer to complete binary trees with $\eta$ leaves.

Construction 3. Let $G=(V, E)$ be an undirected, planar, cubic graph with $n=|V|$. Construct in time polynomial in the size of $G$ an undirected planar graph $G^{\prime}$ as follows (refer to Figure 2 for an illustration of the constructed graph). Let initially $G^{\prime}$ be the graph obtained from Construction 2. Remove $s, w$, and the chains connecting $s$ with $w$ from $G^{\prime}$. Add a vertex $u$ to $G^{\prime}$ and add the edges $\left\{x_{2}, u\right\}$ and $\left\{x_{3}, u\right\}$ to $G^{\prime}$. Let $\eta$ be the smallest power of two larger than $n$ (note that $n \leq \eta \leq 2 n-2$ ). Add a complete binary tree $T_{s}$ with $\eta$ leaves to $G^{\prime}$, and denote its root by $s$. Denote the leaves by $a_{1}, \ldots, a_{\eta}$, ordered by a post-order traversal on $T_{s}$. Next add a copy of $T_{s}$ to $G^{\prime}$, and denote the copy by $T_{w}$ and its root by $w$. If $a_{i}$ is a leaf of $T_{s}$, denote by $a_{i}^{\prime}$ its copy in $T_{w}$. Next, for each $i \in[\eta]$, connect $a_{i}$ and $a_{i}^{\prime}$ via a $(\eta+i)$-chain. Finally, connect $s$ with $v$ via an $(\eta+\log (\eta)+(\eta-n))$-chain, connect $u$ with $w$ via a $\log (\eta)$-chain, and add the edge $\{w, t\}$ to $G^{\prime}$, which completes the construction of $G^{\prime}$. Note that $T_{s}$ and $T_{w}$ allow plane drawings, and the chains connecting the leaves can be aligned as illustrated in Figure 2. It follows that $G^{\prime}$ allows for a plane embedding.

Lemma 6. Let $G$ and $G^{\prime}$ be as in Construction 3. Then $G$ admits a Hamiltonian cycle if and only if $G^{\prime}$ allows for at least $\eta+1$ s-t paths with at most $\eta-2$ shared edges.

Proof. $(\Rightarrow)$ Let $G$ admit a Hamiltonian cycle $C$. As discussed in the proof of Lemma 5, there is an ordering $C^{\prime}$ of the vertices in $C$ such that $x_{1}$ appears first and $x \in\left\{x_{2}, x_{3}\right\}$ appears last in $C^{\prime}$. We construct $\eta s$ - $t$ paths as follows. We route them from $s$ in $T_{s}$ to the leaves of $T_{s}$ in such a way that each path contains a different leaf of $T_{s}$. Herein, $\eta-2$ edges are shared. Next, route each of them via the chain connecting the leaf to the corresponding leaf in $T_{w}$, then to $w$, and finally to $t$. In this part, no edge is shared, as the lengths of the chains connecting the leaves of $T_{s}$ and $T_{w}$ are pairwise different. Hence, the $\eta s$ - $t$ paths contain the edge $\{w, t\}$ at the time steps $2 \log (\eta)+\eta+i+1$
for each $i \in[\eta]$. We construct the one remaining $s-t$ path $P$ as follows. The path $P$ contains the chain connecting $s$ with $v$, the edge $\left\{v, x_{1}\right\}$. Then $P$ follows $C^{\prime}$ in $H$ to $x \in\left\{x_{2}, x_{3}\right\}$, via the edge $\{x, u\}$ to $u$, via the chain connecting $u$ with $w$ to $w$, and finally to $t$ via the edge $\{w, t\}$. Observe that $P$ contains the edge $\{w, t\}$ at the time step $2 \eta+2 \log (\eta)+2$, and hence no sharing any further edge in $G^{\prime}$.
$(\Leftarrow)$ Let $\mathcal{P}$ be a set of $\eta+1 s$ - $t$ paths in $G^{\prime}$ sharing at most $k:=\eta-2$ edges. First observe that no two paths contain the chain connecting $s$ with $v$, as otherwise more that $k$ edges are shared. Hence, at most one $s$ - $t$ path leaves $s$ via the chain to $v$. It follows that at least $\eta$ paths leave $s$ via the edges in $T_{s}$. By the definition of paths, observe that each of $s-t$ paths arrive at a leaf of $T_{s}$ at same time step. Suppose at least two $s$ - $t$ paths contain the same leaf of $T_{s}$. As each leaf is of degree two, the $s$ - $t$ paths follow the chain towards a leaf of $T_{w}$ simultaneously. This introduces at least $\eta+1>k$ shared edges, contradicting the choice of $\mathcal{P}$. It follows that exactly $\eta$ $s$ - $t$ paths leave $s$ via $T_{s}$ (denote the set by $\mathcal{P}^{\prime}$ ), and they arrive each at a different leaf of $T_{s}$ at time step $\log (\eta)$. Moreover, by construction, each path in $\mathcal{P}^{\prime}$ arrives at a different leaf of $T_{w}$ at the time steps $\log (\eta)+\eta+i+1$ for every $i \in[\eta]$.

We next discuss why no path in $\mathcal{P}^{\prime}$ contains more than one chain connecting a pair $\left(a, a^{\prime}\right)$ of leaves, where $a$ and $a^{\prime}$ are leaves of $T_{s}$ and $T_{w}$, respectively. Assume that there is a path $P^{\prime} \in \mathcal{P}^{\prime}$ containing at least two chains connecting the pairs ( $a, a^{\prime}$ ) and $\left(b, b^{\prime}\right)$ of leaves, where $a, b$ and $a^{\prime}, b^{\prime}$ are leaves of $T_{s}$ and $T_{w}$, respectively, and vertex $a$ appears at smallest time step over all such leaves of $T_{s}$ in $P^{\prime}$ and $b^{\prime}$ appears at smallest time step over all such leaves of $T_{w}$ in $P^{\prime}$ (recall that $P^{\prime}$ must contain a leaf of $T_{s}$ at smaller time step than every leaf in $T_{w}$ ). By construction, $a^{\prime}$ and $b^{\prime}$ are the copies of $a$ and $b$ in $T_{w}$. Let $r$ denote the vertex in $T_{s}$ such that $r$ is the root of the subtree of minimum height in $T_{s}$ containing $a$ and $b$ as leaves. Let $r^{\prime}$ denote its copy in $T_{w}$. Observe that by construction, $r^{\prime}$ is the root of the subtree of minimum height in $T_{w}$ containing $a^{\prime}$ and $b^{\prime}$ as leaves. As $P^{\prime}$ starts at vertex $s$, and the path from $s$ to $a$ in $T_{s}$ is unique, $P^{\prime}$ contains the vertex $r$ at smallest time step among the vertices in $X:=\left\{r, a, a^{\prime}, r^{\prime}, b^{\prime}, b\right\}$. As vertex $a$ appears at smallest time step over all leaves of $T_{s}$ and $a$ is of degree two, $P^{\prime}$ contains the vertices $a$ and $a^{\prime}$ at second and third smallest time step, respectively, among the vertices in $X$. As in any tree, the unique path between every two leaves contains the root of the subtree of minimum height containing the leaves, $P^{\prime}$ contains the vertex $r^{\prime}$ at fourth smallest time step among the vertices in $X$. Finally, as vertex $b^{\prime}$ appears at smallest time step over all leaves of $T_{w}$ and $b^{\prime}$ is of degree two, $P^{\prime}$ contains the vertices $b^{\prime}$ and $b$ at fifth and sixth smallest time steps, respectively, among the vertices in $X$. In summary, the vertices in $X$ appear in $P^{\prime}$ in the order $\left(r, a, a^{\prime}, r^{\prime}, b^{\prime}, b\right)$. Now, observe that $\left\{r, r^{\prime}\right\}$ forms a $b-t$ separator in $G^{\prime}$, that is, there is no $b-t$ path in $G^{\prime}-\left\{r, r^{\prime}\right\}$. As $P^{\prime}$ contains $r$ and $r^{\prime}$ at smaller time steps than $b, P^{\prime}$ contains a vertex different to $t$ at last time step. This contradict the fact that $P^{\prime}$ is an $s$ - $t$ path in $G^{\prime}$. It follows that no path in $\mathcal{P}^{\prime}$ contains more than one chain connecting a pair consisting of leaf of $T_{s}$ and a leaf of $T_{w}$.

It follows that the paths in $\mathcal{P}^{\prime}$ contain the edge $\{w, t\}$ at the time steps $2 \log (\eta)+\eta+$ $i+1$ for each $i \in[\eta]$. Hence, the remaining $s$ - $t$ path containing the chain connecting $s$ with $v$, denoted by $P$, has to contain the edge $\{w, t\}$ at time step at most $2 \log (\eta)+$ $\eta+1$ or at least $2 \log (\eta)+2 \eta+2$. At the earliest $P$ can contain the edge $\{w, t\}$ at time step $2 \log (\eta)+\eta+(\eta-n)+3>2 \log (\eta)+\eta+1$, and thus, path $P$ has to contain
edge $\{w, t\}$ at time step $2 \log (\eta)+2 \eta+2$. This is only possible if $P$ forms a path in $H$ that visits each vertex in $H$, starting at $x_{1}$ and ending at vertex $x \in\left\{x_{2}, x_{3}\right\}$. As the edge $\left\{x_{1}, x\right\}$ is contained in $H$, it follows that $P$ restricted to $H$ forms a Hamiltonian cycle in $G$.

Proof of Theorem 4. We provide a many-one reduction from PCHC to Path-RCA on undirected graph via Construction 2 (for constant number $k$ of shared edges) on the one hand, and Construction 3 (for constant maximum degree $\Delta$ ) on the other. Let $(G)$ be an instance of PCHC with $n=|V(G)|$.
Via Construction 2. Let $\left(G^{\prime}, s, t, p, 0\right)$ be an instance of Path-RCA where $G^{\prime}$ is obtained from $G$ by applying Construction 2 and $p=n-1$. Note that ( $G^{\prime}, s, t, p, 0$ ) is constructed in polynomial time and, by Lemma $5, G$ is a yes-instance of PCHC if and only if ( $G^{\prime}, s, t, p, 0$ ) is a yes-instance of Path-RCA.

The case of constant $k>0$. Reduce $\left(G^{\prime}, s, t, p, 0\right)$ to an equivalent instance $\left(G_{k}^{\prime}, s^{\prime}, t, p, k\right)$ of PATH-RCA with $k>0$ as follows. Let $G_{k}^{\prime}$ denote the graph obtained from $G^{\prime}$ by the following modification: Add a chain of length $k$ to $G^{\prime}$, and identify one endpoint with $s$ and denote by $s^{\prime}$ the other endpoint. Set $s^{\prime}$ as the new source. Observe that any $s^{\prime}-t$ path in $G_{k}^{\prime}$ contains the $k$-chain appended on $s$, and hence, any solution introduces exactly $k$ shared edges.

The directed case. Direct the edges in $G^{\prime}$ as follows. Direct each chain connecting $s$ with $w$ from $s$ towards $t$. (In the case of $k>0$, also direct the chain from $s^{\prime}$ towards $s$.) Direct the edges $\{s, v\},\left\{v, x_{1}\right\},\left\{x_{2}, w\right\},\left\{x_{3}, w\right\}$, and $\{w, t\}$ as $(s, v),\left(v, x_{1}\right),\left(x_{2}, w\right)$, $\left(x_{3}, w\right)$, and ( $w, t$ ). Finally, replace each edge $\{a, b\}$ in $H$ by two (anti-parallel) arcs $(a, b),(b, a)$ to obtain the directed variant of $H$. The correctness follows from the fact that we consider paths that are not allowed to contain vertices more than once. Note that the planarity is not destroyed.
Via Construction 3. Let $\left(G^{\prime}, s, t, p, k\right)$ be an instance of Path-RCA where $G^{\prime}$ is obtained from $G$ by applying Construction $3, p=\eta+1$, and $k=\eta-2$. Note that $\left(G^{\prime}, s, t, p, k\right)$ is constructed in polynomial time and, by Lemma $6, G$ is a yesinstance of PCHC if and only if ( $G^{\prime}, s, t, p, k$ ) is a yes-instance of PATH-RCA.

The directed case. Direct the edges in $G^{\prime}$ as follows. Direct the edges in $T_{s}$ from $s$ towards the leaves, and the edges in $T_{w}$ from the leaves towards $w$. Direct each chain connecting $T_{s}$ with $T_{w}$ from $T_{s}$ towards $T_{w}$. Direct the edges $\left\{v, x_{1}\right\},\left\{x_{2}, u\right\},\left\{x_{3}, u\right\}$, and $\{w, t\}$ as $\left(v, x_{1}\right),\left(x_{2}, u\right),\left(x_{3}, u\right)$, and $(w, t)$. Direct the chain connecting $s$ with $v$ from $s$ towards $v$, and the chain connecting $u$ with $w$ from $u$ towards $w$. Finally, replace each edge $\{a, b\}$ in $H$ by two (anti-parallel) $\operatorname{arcs}(a, b),(b, a)$ to obtain the directed variant of $H$. The correctness follows from the fact that we consider paths that are not allowed to contain vertices more than once. Note that the planarity is not destroyed.

As the length of every s-t path is upper bounded by the number of vertices in the graph, we immediately obtain the following.

Corollary 3. Path-FRCA both on undirected planar and directed planar graphs is NP-complete, even if $k \geq 0$ is constant or $\Delta \geq 4$ is constant.


Figure 3: Graph $G^{\prime}$ obtained in Construction 4. The gray part represents the graph $H^{\prime}$.

## 5 Trail-RCA

We now show that Trail-RCA has the same computational complexity fingerprint as Path-RCA. That is, Trail-RCA (Trail-FRCA) is NP-complete on undirected and directed planar graphs, even if the number $k \geq 0$ of shared edges (arcs) or the maximum degree $\Delta \geq 5\left(\Delta_{\mathrm{i} / \mathrm{o}} \geq 3\right)$ is constant. The reductions are slightly more involved, because it is harder to force trails to take a certain way.

### 5.1 On Undirected Graphs

In this section, we prove the following.
Theorem 5. Trail-RCA on undirected planar graphs is NP-complete, even if $k \geq 0$ is constant or $\Delta \geq 5$ is constant.

We provide two constructions supporting the two subresults for constants $k, \Delta$. The reductions are again from Planar Cubic Hamiltonian Cycle (PCHC).

Construction 4. Let $G=(V, E)$ be an undirected planar cubic graph with $n=|V|$. Construct an undirected planar graph $G^{\prime}$ as follows (refer to Figure 3 for an illustration of the constructed graph). Initially, let $G^{\prime}$ be the empty graph. Add a copy of $G$ to $G^{\prime}$ and denote the copy by $H$. Subdivide each edge in $H$ and denote the resulting graph $H^{\prime}$. Note that $H^{\prime}$ is still planar. Consider a plane embedding $\phi\left(H^{\prime}\right)$ of $H^{\prime}$ and and let $x \in V\left(H^{\prime}\right)$ be a vertex incident to the outer face in the embedding. Next, add the vertex set $\{s, v, w, t\}$ to $G$. Add the edges $\{s, x\},\{s, v\},\{v, w\}$, and $\{w, t\}$ to $G$. Finally, add $n-1$ vertices $B=\left\{b_{1}, \ldots, b_{n-1}\right\}$ to $G$ and connect each of them with $s$ by two edges (in the following, we distinguish these edges as $\left\{s, b_{i}\right\}_{1}$ and $\left\{s, b_{i}\right\}_{2}$, for each $i \in[n-1]$ ). Note that the graph is planar (see Figure 3 for an embedding, where $H^{\prime}$ is embedded as $\phi\left(H^{\prime}\right)$ ) but not simple.

Lemma 7. Let $G$ and $G^{\prime}$ as in Construction 4. Then $G$ admits a Hamiltonian cycle if and only if $G^{\prime}$ admits $2 n$ s-t trails with no shared edge.

Proof. $(\Rightarrow)$ Let $G$ admit a Hamiltonian cycle $C$. Observe that $H^{\prime}$ allows for a cycle $C^{\prime}$ in $H^{\prime}$ that contains each vertex corresponding to a vertex in $H$ exactly once. We construct $2 n$ trails in $G^{\prime}$ as follows.

We group the trails in two groups. The first group of trails first visits some of the vertices $b_{1}, \ldots, b_{n-1}$ by each time first using the edge $\left\{s, b_{i}\right\}_{1}, i \in[n-1]$, and
then proceeding to $t$ via $v$. The second group of trails first visits some of the vertices $b_{1}, \ldots, b_{n-1}$ by each time first using the edge $\left\{s, b_{i}\right\}_{2}, i \in[n-1]$, and then proceeding via $x$, then following the cycle $C^{\prime}$, and finally again via $x$ towards $t$. Let $T_{1}^{i}, \ldots, T_{n}^{i}$ denote the trails of group $i \in\{1,2\}$. For each $j<n$, the trail $T_{j}^{i}$ first visits the vertices $b_{j}, \ldots, b_{n-1}$ in that order before proceeding as described above. The trails $T_{n}^{i}, i \in\{1,2\}$, do not contain any of the vertices $b_{1}, \ldots, b_{n-1}$, and directly approach $t$ as described above.

Observe that, within each of the two groups, no two trails share an edge. Between trails of different groups, only edge $\{w, t\}$ can possibly be shared. Note that any cycle in $H^{\prime}$ is of even length. Hence, the trails of group 1 contain the edge $\{w, t\}$ at each of the time steps $2 j+1$ for every $j \in[n]$. The trails of group 2 contain the edge $\{w, t\}$ at each of the time steps $2 n+2 j+1$ for every $j \in[n]$. Hence, no two trails share an edge.
$(\Leftarrow)$ Let $G^{\prime}$ admit $2 n s$ - $t$ trails with no shared edge. First note that $s$ has exactly $2 n$ incident edges. Observe that, for each $\beta=0, \ldots,|B|$, no more than two trails contain $\beta$ vertices of $B$, as otherwise any of the edges $\{s, v\}$ or $\{s, x\}$ would be shared. By the pigeon hole principle it follows that, for each $\beta=0, \ldots,|B|$, there are exactly two trails that contain $\beta$ vertices of $B$. Hence, for each even time step, there are two trails leaving $s$ via the edges $\{s, v\}$ and $\{s, x\}$, respectively. Observe that those trails that proceed towards $t$ via $v$ use the edges $\{w, t\}$ exactly at the time steps $2 j+1$ for every $j \in[n]$. Because each trial in $H^{\prime}$ that starts and ends at the same vertex has even length, those trials that proceed towards $t$ via $x$ can use $\{w, t\}$ only at odd time steps. Hence, since the edge $\{w, t\}$ is not shared, the trails proceeding towards $t$ via $x$ need to stay in $H^{\prime}$ for $2 n$ time steps. As $H^{\prime}-x$ has maximum degree three, no vertex in $H^{\prime}$ beside $x$ is contained more than once in all of these trails. As the length between every two vertices in $H^{\prime}$ corresponding to vertices in $H$, it follows that every of these trails visits the vertices in $H^{\prime}$ corresponding to the vertices in $H$. It follows that each of these trails forms a Hamiltonian cycle $C^{\prime}$ in $H^{\prime}$. As $C^{\prime}$ can easily turned into a Hamiltonian cycle $C$ in $G$ (consider the sequence when deleting all vertices that do not correspond to a vertex in $H$ ), the statement follows.

To deal with the parallel edges in graph $G^{\prime}$ in Construction 4, we now subdivide edges, maintaining an equivalent statement as in Lemma 7.

Lemma 8. Let $G$ be an undirected graph (not necessarily simple) with two distinct vertices s and $t$. Obtain graph $G^{\prime}$ from $G$ by replacing each edge $\{u, v\} \in E$ in $G$ by a path of length three, identifying its endpoints with $u$ and $v$. Then $G$ admits $p \in \mathbb{N}$ $s$-t trails with no shared edge if and only if $G^{\prime}$ admits $p$ s-t trails with no shared edge.

Proof. For each edge $e \in E(G)$ in $G$ denote by $P(e)$ the corresponding path of length three in $G^{\prime}$. By definition, for all $e, f \in E\left(G^{\prime}\right)$ it holds that $e \neq f$ if and only if $P(e) \neq P(f)$.
$(\Rightarrow)$ Let $\mathcal{P}=\left\{P_{i} \mid i \in[p]\right\}$ be a set of $p$ s-t trails in $G$ with no shared edge. Each $P_{i}$ as an edge sequence representation $P_{i}=\left(e_{1}^{i}, \ldots, e_{\ell_{i}}^{i}\right)$, where $\ell_{i}$ is the number of edges in $P_{i}$. For each $P_{i}$, consider the corresponding trail $P_{i}^{\prime}=\left(P\left(e_{1}^{i}\right), \ldots, P\left(e_{\ell_{i}}^{i}\right)\right)$ in $G^{\prime}$, and the set $\mathcal{P}^{\prime}=\left\{P_{i}^{\prime} \mid i \in[p]\right\}$. Suppose that two trails $P_{i}^{\prime}$ and $P_{j}^{\prime}$ share an edge.


Figure 4: Graph $G^{\prime}$ obtained in Construction 5. The gray part represents the graph $H^{\prime}$.

Then the shared edge is contained in subpaths $P\left(e_{x}^{i}\right)$ and $P\left(e_{x}^{j}\right)$. As $P\left(e_{x}^{i}\right)$ and $P\left(e_{x}^{j}\right)$ are not edge-disjoint (as they share an edge), it follows that $e_{x}^{i}=e_{x}^{j}$, and hence $P_{i}$ and $P_{j}$ share the edge $e_{x}^{i}$ in $G$. This contradicts the fact that $\mathcal{P}=\left\{P_{i} \mid i \in[p]\right\}$ is a set of $p s$ - $t$ trails in $G$ with no shared edge. It follows that $\mathcal{P}^{\prime}$ is a set of $p s$ - $t$ trails in $G^{\prime}$ with no shared edge.
$(\Leftarrow)$ Let $\mathcal{P}^{\prime}=\left\{P_{i}^{\prime} \mid i \in[p]\right\}$ be a set of $p s$ - $t$ trails in $G^{\prime}$ with no shared edge. Observe that, by the construction of $G^{\prime}$, each $s$ - $t$ trail in $G^{\prime}$ is composed of paths of length three with endpoints corresponding to vertices in $G$. Hence, for each $i \in[p]$, let $P_{i}^{\prime}$ be represented as $P_{i}^{\prime}=\left(P\left(e_{1}^{i}\right), \ldots, P\left(e_{\ell_{i}}^{i}\right)\right)$, where $3 \cdot \ell_{i}$ is the number of edges in $P_{i}^{\prime}$. For each $P_{i}^{\prime}$, consider the corresponding trail $P_{i}=\left(e_{1}^{i}, \ldots, e_{\ell_{i}}^{i}\right)$ in $G$, and the set $\mathcal{P}=\left\{P_{i} \mid\right.$ $i \in[p]\}$. Suppose that two trails $P_{i}$ and $P_{j}$ share an edge, that is, there is an index $x$ such that $e_{x}^{i}=e_{x}^{j}$. It follows that $P\left(e_{x}^{i}\right)=P\left(e_{x}^{j}\right)$. Let $e_{x}^{i}=\{v, w\}=$ : $e$. If both trails $P_{i}^{\prime}$ and $P_{j}^{\prime}$ traverse $P(e)$ in the same "direction", i.e. either from $v$ to $w$ or from $w$ to $v$, then $P_{i}^{\prime}$ and $P_{j}^{\prime}$ share at least three edges (all edges in $P(e)$ ). This contradicts the definition of $\mathcal{P}^{\prime}$. Consider the case that the trails $P_{i}^{\prime}$ and $P_{j}^{\prime}$ traverse $P(e)$ in opposite "directions", i.e. one from $v$ to $w$ and the other from $w$ to $v$. As $P(e)$ is of length three, the edge in $P(e)$ with no endpoint in $\{v, w\}$ is then used by $P_{i}^{\prime}$ and $P_{j}^{\prime}$ at the same time step, yielding that the edge is shared. This contradicts the definition of $\mathcal{P}^{\prime}$. It follows that $\mathcal{P}=\left\{P_{i} \mid i \in[p]\right\}$ is a set of $p s$ - $t$ trails in $G$ with no shared edge.

We now show how to modify Construction 4 for maximum degree five, giving up, however, a constant upper bound on the number of shared edges.

Construction 5. Let $G=(V, E)$ be an undirected planar cubic graph with $n=|V|$. Construct an undirected planar graph $G^{\prime}$ as follows (see Figure 4 for an illustration of the constructed graph). Let initially $G^{\prime}$ be the graph obtained from Construction 4. Subdivide each edge in $H^{\prime}$ and denote the resulting graph by $H^{\prime \prime}$. Observe that the distance in $H^{\prime \prime}$ between any two vertices in $V\left(H^{\prime \prime}\right) \cap V\left(H^{\prime}\right)$ is divisible by four. Next, delete all edges incident with vertex $s$. Connect $s$ with $v$ via a $2 n$-chain, and connect $s$ with $x$ via a $2 n$-chain. Connect $s$ with $b_{1}$ via two $P_{2}$ 's. Denote the two vertices on the $P_{2}$ 's by $\ell_{1}$ and $u_{1}$. Finally, for each $i \in[n-2]$, connect $b_{i}$ with $b_{i+1}$ via two $P_{2}$ 's. For each $i \in[n-2]$, denote the two vertices on the $P_{2}$ 's between $b_{i}$ and $b_{i+1}$ by $\ell_{i+1}$ and $u_{i+1}$. For an easier notation, we denote vertex $s$ also by $b_{0}$.

Lemma 9. Let $G$ and $G^{\prime}$ as in Construction 5. Then $G$ admits a Hamiltonian cycle if and only if $G^{\prime}$ has $2 n s$-t trails with at most $2 n-4$ shared edges.

Proof. $(\Leftarrow)$ Let $G^{\prime}$ admit a set $\mathcal{P}$ of $2 n s$ - $t$ trails with at most $2 n-4$ shared edges. At each time step, at most two trails leave $s$ towards $v$ and $x$. Otherwise, all the edges in at least one of the $2 n$-chains connecting $s$ with $v$ and $s$ with $x$ are shared, contradicting the fact that the trails in $\mathcal{P}$ share at most $2 n-4$ edges. Note that every $s$ - $t$ trail contains vertex $s$ at the first time step and at most once more at time step $4 j+1$, for some $j \in \mathbb{N}$ (indeed, we will show that $j \in[n-1]$ ). This follows on the one hand from the fact that $s$ has degree four and hence every trail can contain $s$ at most twice, and on the other hand from the fact that for each $i \in[n-1]$, every $s-b_{i}$ path is of even length.

We show that at each time step $4 j+1,0 \leq j \leq n-1$, exactly one $s$ - $t$ trial leaves $s$ towards $v$ and exactly one $s$ - $t$ trial leaves $s$ towards $x$. First, observe that $\left|\left\{b_{i}, u_{i}, \ell_{i} \mid i \in[n-1]\right\}\right|=3(n-1)$ and each trail can contain each vertex in $\left\{u_{i}, \ell_{i} \mid\right.$ $i \in[n-1]\} \cup\left\{b_{n-1}\right\}$ at most once (as each vertex in this set is of degree two) and each vertex $b_{i}, i \in[n-2]$, at most twice (as they are of degree four). Hence, any trail starting on $s$ and returning to $s$ after visiting the vertices in $B$ contains at most $3(n-1)+(n-2)+2=4(n-1)+1$ vertices. It follows that every $s$ - $t$ trail contains $s$ at the first time step and at most once more at time step $4 j+1$ for some $j \in[n-1]$. As there are $2 n s-t$ trails and at each time step at most two trails leave $s$ towards $v$ and $x$, together with the pigeon hole principle it follows that exactly one $s$ - $t$ trial leaves $s$ towards $v$ and exactly one $s$ - $t$ trial leaves $s$ towards $x$ at each time step $4 j+1$, $0 \leq j \leq n-1$. Moreover, note that each $b_{i}, i \in[n-1]$, appears in at least two $s$ - $t$ trails.

We claim that there are exactly $2 n-4$ shared edges and that every shared edge is incident with a vertex in $\left\{u_{i}, \ell_{i} \mid i \in[n-1]\right\}$. This follows from the fact that at least three trails going at the same time from $b_{i}$ to $b_{i+1}, 0 \leq i \leq n-3$, share at least two edges. As four trails contain $b_{n-2}$ (those four which leave $s$ towards $v$ and $x$ at the time steps $4 j+1$ with $j \in\{n-2, n-1\})$, it follows that at least $2(n-2)$ edges are shared. Hence, no two trails share an edge after they have left $s$ for $v$ or $x$.

There is a trail $P \in \mathcal{P}$ that contains the vertex $x$ and that contains vertex $s$ only once at the first time step, because at each time step, two trails leave $s$ for $v$ or $x$. Observe that vertex $w$ is contained in the trails containing $v$ at the time steps $4 j+2 n+2$ for all $j \in[n-1] \cup\{0\}$ whence edge $\{w, t\}$ is occupied at time steps $4 j+2 n+3$ for all $j \in[n-1] \cup\{0\}$. Hence, the edge $\{w, t\}$ is contained at time step $2 n+2$ in a trail different to $P$ and, thus, trail $P$ contains at least one vertex in $H^{\prime \prime}$. Furthermore, $P$ can contain $x$ a second time only at time steps of the form $4 j+2 n+1,3 \leq j \leq n$, because each path in $H^{\prime \prime}$ between two vertices that correspond to vertices in $G$ has length four. However, as mentioned, $\{w, t\}$ is occupied at time steps $4 j+2 n+3$, $j \in[n-1]$. Hence, $P$ has to stay in $H^{\prime \prime}$ for $4 n$ time steps. Recall that $G$ is cubic, and hence no vertex in $H^{\prime \prime}-\{x\}$ corresponding to a vertex in $G$ appears more than once in any trail. That is, $P$ follows a cycle in $H^{\prime \prime}$ containing each vertex corresponding to vertex in $G$ exactly once. It follows that $G$ admits a Hamiltonian cycle.
$(\Rightarrow)$ Let $G$ admit a Hamiltonian cycle $C$. Let $C^{\prime}$ be $C$, ordered such that $x$ is the first and last vertex in $C^{\prime}$. Let $C^{\prime \prime}$ denote the cycle in $H^{\prime \prime}$ following the order of the
vertices in $C^{\prime}$. We construct $2 n s-t$ trails in $G^{\prime}$ sharing at most $2 n-4$ edges as follows. We denote the trails by $P_{i}^{x}, 0 \leq i \leq n-1, x \in\{u, \ell\}$. The trails are divided into two groups according to their superscript $x \in\{u, \ell\}$. For $i \geq 1$, the trails $P_{i}^{u}$ and $P_{i}^{\ell}$ start with the sequences

$$
\begin{aligned}
P_{i}^{u}: & \left(s, \ell_{1}, b_{1}, \ldots, \ell_{i-1}, b_{i-1}, u_{i}, b_{i}, \ell_{i}, b_{i-1}, \ell_{i-1}, \ldots, b_{1}, \ell_{1}, s\right), \\
P_{i}^{\ell}: & \left(s, \ell_{1}, b_{1}, \ldots, \ell_{i-1}, b_{i-1}, \ell_{i}, b_{i}, u_{i}, b_{i-1}, \ell_{i-1}, \ldots, b_{1}, \ell_{1}, s\right) .
\end{aligned}
$$

Trails $P_{0}^{u}$ and $P_{0}^{\ell}$ do not visit any vertex in $B$ and simply start at $s$. Then, for each $i=0, \ldots, n-1$, trail $P_{i}^{u}$ follows the chain to $v$, the edge to $w$, and then to $t$. For each $i=0, \ldots, n-1$, trail $P_{i}^{\ell}$ follows the chain to $x$, then the cycle $C^{\prime \prime}$ in $H^{\prime \prime}$, then the edge from $x$ to $w$, then to $t$. Observe that trail $P_{i}^{x}, x \in\{u, \ell\}$, contains $s$ at time step one and $4 i+1$. Hence, $P_{i}^{u}$ contains the vertex $w$ at time step $4 i+2 n+2$. Moreover, $P_{i}^{\ell}$ contains the vertex $x$ at time steps $4 i+2 n+1$ and $4 i+2 n+4 n+1$. From the latter it follows that $P_{i}^{\ell}$ contains the vertex $w$ at time step $4 i+2 n+4 n+2$. Altogether, the edge $\{w, t\}$ is not shared by any pair of trails.

Next, we count the number of edges shared between the two visits of $s$. Denote by $X \subseteq E\left(G^{\prime}\right)$ the set $\left\{\left\{b_{i}, \ell_{i}\right\} \mid 1 \leq i \leq n-2\right\} \cup\left\{\left\{b_{i}, \ell_{i+1}\right\} \mid 0 \leq i<n-2\right\}$. Observe that $|X|=n-2+n-2=2(n-2)$. We claim that the edges in $X$ are the only shared edges by the trails $P_{i}^{x}, 0 \leq i \leq n-1, x \in\{u, \ell\}$. As $P_{n-1}^{\ell}$ and $P_{n-1}^{u}$ contain the set $X$ at the same time steps, every edge in $X$ is shared. For each $i \geq 1$, the edges $\left\{u_{i}, b_{i-1}\right\}$ and $\left\{u_{i}, b_{i}\right\}$ are only contained in the trails $P_{i}^{x}, x \in\{u, \ell\}$. Recall that $P_{i}^{u}$ and $P_{i}^{\ell}$ contain $b_{i}$ exactly once and at the same time step. The subsequence around $b_{i}$ of $P_{i}^{\ell}$ and $P_{i}^{u}$ is $\left(b_{i-1}, \ell_{i}, b_{i}, u_{i}, b_{i-1}\right)$ and $\left(b_{i-1}, u_{i}, b_{i}, \ell_{i}, b_{i-1}\right)$, respectively. It follows that both edges $\left\{u_{i}, b_{i-1}\right\}$ and $\left\{u_{i}, b_{i}\right\}$ appear at two different time steps in $P_{i}^{u}$ and $P_{i}^{\ell}$. The same argument holds for the edges $\left\{b_{n-2}, \ell_{n-1}\right\}$ and $\left\{b_{n-1}, \ell_{n-1}\right\}$ as $P_{n-1}^{u}$ and $P_{n-1}^{\ell}$ are the only trails containing the two edges.

Altogether, it follows that $X$ is the set of shared edges of the $s-t$ trails $P_{i}^{x}, x \in$ $\{u, \ell\}$, and the claim follows. Finally, as $|X|=2 n-4$, the statement follows.

Proof of Theorem 5. We provide a many-one reduction from Planar Cubic Hamilton Circuit (PCHC) to Trail-RCA on undirected graph via Construction 4 on the one hand, and Construction 5 on the other. Let $(G=(V, E))$ be an instance of PCHC and let $n:=|V|$ vertices.
Via Construction 4. Let ( $G^{\prime}, s, t, p, 0$ ) an instance of Trail-RCA where $G^{\prime}$ is obtained from $G$ by applying Construction 4 and $p=2 n$. Note that ( $G^{\prime}, s, t, p, 0$ ) can be constructed in polynomial time and by Lemma $7,(G)$ is a yes-instance of PCHC if and only if ( $G^{\prime}, s, t, p, 0$ ) is a yes-instance of Trail-RCA. However, $G^{\prime}$ is not simple in general. Hence, replace each edge $\{u, v\} \in E\left(G^{\prime}\right)$ in $G^{\prime}$ by a path of length three and identify its endpoints with $u$ and $v$. Denote by $G^{\prime \prime}$ the obtained graph. Due to Lemma 8, ( $G^{\prime \prime}, s, t, p, 0$ ) is a yes-instance of Trail-RCA if and only if ( $G^{\prime}, s, t, p, 0$ ) is a yes-instance of Trail-RCA.

The case of constant $k>0$ works analogously as in the proof of Theorem 4.
Via Construction 5. Let ( $\left.G^{\prime}, s, t, p, k\right)$ an instance of Trail-RCA where $G^{\prime}$ is obtained from $G$ by applying Construction 5 and $p=2 n$. Instance ( $G^{\prime}, s, t, p, k$ ) can be


Figure 5: Sketch of the graph $G^{\prime}$ obtained in Construction 6. The enclosed gray part represents the graph $H$. Dashed lines represent directed chains.
constructed in polynomial time. By Lemma $9,(G)$ is a yes-instance of PCHC if and only if $\left(G^{\prime}, s, t, p, k\right)$ is a yes-instance of Trail-RCA.

As the length of each $s$ - $t$ trail is upper bounded by the number of edges in the graph, we immediately obtain the following.

Corollary 4. Trail-FRCA on undirected planar graphs is NP-complete, even if $k \geq$ 0 is constant or $\Delta \geq 5$ is constant.

### 5.2 On Directed Graphs

We know that Trail-RCA and Trail-FRCA are NP-complete on undirected graphs, even if the number of shared edges or the maximum degree is constant. In what follows, we show that this is also the case for Trail-RCA and Trail-FRCA on directed graphs.

Theorem 6. Trail-RCA on directed planar graphs is NP-complete, even if $k \geq 0$ is constant or $\Delta_{\mathrm{i} / \mathrm{o}} \geq 3$ is constant.

To prove Theorem 6, we reduce from the following NP-complete [17] problem.
Directed Planar 2/3-In-Out Hamiltonian Circuit (DP2/3HC)
Input: A directed, planar graph $G=(V, A)$ such that, for each $v \in V$, $\max \{\operatorname{outdeg}(v), \operatorname{indeg}(v)\} \leq 2$ and outdeg $(v)+\operatorname{indeg}(v) \leq 3$.
Question: Is there a directed Hamiltonian cycle in $G$ ?
Construction 6. Let $G=(V, A)$ be a directed, planar graph where for each vertex $v \in V$ holds max $\{\operatorname{outdeg}(v), \operatorname{indeg}(v)\} \leq 2$ and outdeg $(v)+\operatorname{indeg}(v) \leq 3$, and $n=|V|$. Construct a directed graph $G^{\prime}$ as follows (refer to Figure 5for an illustration of the constructed graph). Initially, let $G^{\prime}$ be the empty graph. Add a copy of the graph $G$ to $G^{\prime}$ and denote the copy by $H$. Add the vertex set $\{s, t, v, w\}$ to $G^{\prime}$. Consider a plane embedding $\phi(G)$ and choose a vertex $x \in V(H)$ incident to the outer face. Add the $\operatorname{arcs}(s, v),(v, x),(x, w)$, and $(w, t)$ to $G^{\prime}$. Moreover, add $n$ chains connecting $s$ with $v$ of lengths $3,4, \ldots, n+2$ respectively to $G^{\prime}$, and direct the edges from $s$ towards $v$. Note that $G^{\prime}$ is planar (see Figure 5 for an embedding where $H$ is embedded as $\phi(H)$ ).


Figure 6: Sketch of the graph $G^{\prime}$ obtained in Construction 7. The enclosed gray part represents the graph $H$. Dashed lines represent directed chains. $T_{s}^{\uparrow}$ and $T_{w}^{\downarrow}$ refer to the complete binary (directed) trees with $\eta$ leaves and rooted at $s$ and $w$, respectively.

Lemma 10. Let $G$ and $G^{\prime}$ as in Construction 6. Then $G$ admits a Hamiltonian cycle if and only if $G^{\prime}$ admits $n+1$ s-t trails with no shared arc.

Proof. $(\Rightarrow)$ Let $G$ admit a Hamiltonian cycle $C$. We construct $n s$ - $t$ trails in $G^{\prime}$, each using a chain connecting $s$ with $w$, where no two use the same chain. By construction, the $n s$ - $t$ trails do not introduce any shared arc. Moreover, the trails contain the $\operatorname{arc}(w, t)$ at every time step in $\{4, \ldots, n+3\}$. The remaining trail contains no chain, but the vertices $v, x, w$ as well as $C$. As $C$ is a Hamiltonian cycle, the trail uses the $\operatorname{arc}(w, t)$ at time step $n+4$.
$(\Leftarrow)$ Let $G^{\prime}$ admit a set $\mathcal{P}$ of $n+1$ s- $t$ trails with no shared arc. As $s$ has outdegree $n+1$, in $\mathcal{P} n$ trails contain a chain connecting $s$ with $w$, where no two contain the same chain. As there is no shared arc, the remaining trail cannot use the arc $(w, t)$ before time step $n+3$. As the shortest $s$ - $w$ path containing $v$ is of length three, the remaining trail has to contain $n$ arcs in the copy $H$ of $G$. As for each vertex $v \in V(G)$ holds that $\max \{\operatorname{outdeg}(v), \operatorname{indeg}(v)\} \leq 2$ and $\operatorname{outdeg}(v)+\operatorname{indeg}(v) \leq 3$, no vertex despite $x$ is visited twice by the trail. Hence, the trail restricted to the copy $H$ of $G$ forms an Hamiltonian cycle in $G$.

We provide another construction where the obtained graph has constant maximum in- and out-degree.
Construction 7. Let $G=(V, A)$ be a directed, planar graph where for each vertex $v \in V$ holds $\max \{\operatorname{outdeg}(v), \operatorname{indeg}(v)\} \leq 2$ and $\operatorname{outdeg}(v)+\operatorname{indeg}(v) \leq 3$, and let $n:=|V|$. Construct a directed graph $G^{\prime}$ as follows (refer to Figure 6 for an illustration of the constructed graph). Let initially $G^{\prime}$ be the graph obtained from Construction 6. Remove $s, w$, and the directed chains connecting $s$ with $w$ from $G^{\prime}$. Let $\eta$ be the smallest power of two larger than $n$ (note that $n \leq \eta \leq 2 n-2$ ). Add a complete binary tree $T_{s}$ with $\eta$ leaves to $G^{\prime}$, and denote its root by $s$. Denote the leaves by $a_{1}, \ldots, a_{\eta}$, ordered by a post-order traversal on $T_{s}$. Next add a copy of $T_{s}$ to $G^{\prime}$, and denote the copy by $T_{w}$ and its root by $w$. For each leaf $a_{i}$ of $T_{s}$, denote by $a_{i}^{\prime}$
its copy in $T_{w}$. Next, for each $i \in[\eta]$, connect $a_{i}$ and $a_{i}^{\prime}$ via a $(\eta+i)$-chain. Direct all edges in $T_{s}$ away from $s$ towards the leaves of $T_{s}$. Direct all edges in $T_{w}$ away from the leaves of $T_{w}$ towards $w$. Next, direct all chains connecting the leaves of $T_{s}$ and $T_{w}$ from the leaves of $T_{s}$ towards the leaves of $T_{w}$. To complete the construction of $G^{\prime}$, connect $s$ with $v$ via a $(\log (\eta)+\eta+(\eta-n))$-chain, and connect $x$ with $w$ via a $\log (\eta)$-chain. Note that $T_{s}$ and $T_{w}$ allow plane drawings, and the chains connecting the leaves can be aligned as illustrated in Figure 2. It follows that $G^{\prime}$ allows for an plane embedding.

Lemma 11. Let $G$ and $G^{\prime}$ as in Construction 7. Then $G$ admits a Hamiltonian cycle if and only if $G^{\prime}$ admits $\eta+1$ s-t trails with at most $\eta-2$ shared arcs.

Proof. $(\Rightarrow)$ Let $G$ admit a Hamiltonian cycle $C$. First, we construct $\eta s$ - $t$ trails in $G^{\prime}$ as follows. For each leaf of $T_{s}$, there is a trail containing the unique path from $s$ to the leaf in $T_{s}$. This part introduces $\eta-2$ shared arcs. Next, each trail follows the chain connecting the leaf of $T_{s}$ with a leaf of $T_{w}$, then the unique path from the leaf of $T_{w}$ to $w$, and finally the arc $(w, t)$. Observe that the trails contain the leaves of $T_{w}$ at different time steps $\log (\eta)+\eta+i+1, i \in[\eta]$, and hence no shared arc is introduced in this part. Moreover, the arc $(w, t)$ appears in the time steps $2 \log (\eta)+\eta+i+1$, $i \in[\eta]$. The remaining trail $P$ contains the chain connecting $s$ with $v$, the edge $\{v, x\}$, follows the cycle $C$ in $H$, starting and ending at vertex $x$. Trail $P$ then contains the chain connecting $x$ with $w$, and the arc $(w, t)$. Observe that the arc $(w, t)$ appears in $P$ at time step $2 \log (\eta)+2 \eta+2$ (recall that $C$ is a Hamiltonian cycle in $H$ ), and hence $(w, t)$ is not shared.
$(\Leftarrow)$ Let $G^{\prime}$ admit $\eta+1 s$ - $t$ trails with at most $\eta-2$ shared arcs.
First observe that the chain connecting $s$ with $v$ is not contained in more than one $s$ $t$ trail. Hence, at least $\eta$ trails leave $s$ through $T_{s}$. Note that no chain connecting the leaves of $T_{s}$ with the leaves of $T_{w}$ is contained in more than one $s$ - $t$ trail. It follows that exactly $\eta s$ - $t$ trails (denote the set by $\mathcal{P}^{\prime}$ ) leave $s$ via $T_{s}$ and each contains a different leaf of $T_{s}$. Herein, $\eta-2 \operatorname{arcs}$ are shared by the trails in $\mathcal{P}^{\prime}$. Note that the path from a leaf of $T_{s}$ to $t$ is unique, each trail in $\mathcal{P}^{\prime}$ follows the unique path to $t$. The $\operatorname{arc}(w, t)$ appears in the trails in $\mathcal{P}^{\prime}$ at time steps $2 \log (\eta)+\eta+i+1$ for every $i \in[\eta]$.

The remaining $s$ - $t$ trail $P \notin \mathcal{P}$ contains the chain connecting $s$ with $v$. Note that $\operatorname{arc}(w, t)$ is not shared, as all shared arcs are contained in $T_{s}$. As the shortest $s$ $t$ path via $x$ is of length $2 \log (\eta)+\eta+(\eta-n)+2 \geq 2 \log (\eta)+\eta+2$, trail $P$ has to contain a cycle $C$ in $H$. As the $\operatorname{arc}(w, t)$ is not shared and appears in the trails in $\mathcal{P}^{\prime}$ at the time steps $2 \log (\eta)+\eta+i+1$, for every $i \in[\eta]$, the cycle $C$ must be of length $n$. As for each vertex $v \in V(G)$ holds that $\max \{\operatorname{outdeg}(v), \operatorname{indeg}(v)\} \leq 2$ and $\operatorname{outdeg}(v)+\operatorname{indeg}(v) \leq 3$, no vertex in $H$ despite $x$ is visited twice by the trail $P$. Hence, trail $P$ restricted to the copy $H$ of $G$ forms a Hamiltonian cycle in $G$.

Proof of Theorem 6. We provide a many-one reduction from DP2/3HC to TrailRCA on directed graphs via Construction 6 on the one hand, and Construction 7 on the other. Let $(G)$ be an instance of DP2/3HC where $G$ consists of $n$ vertices.
Via Construction 6. Let $\left(G^{\prime}, s, t, p, 0\right)$ an instance of Trail-RCA where $G^{\prime}$ is obtained from $G$ by applying Construction 6 and $p=n+1$. Note that $\left(G^{\prime}, s, t, p, 0\right)$ is
constructed in polynomial time and by Lemma $10,(G)$ is a yes-instance of DP2/3HC if and only if ( $G^{\prime}, s, t, p, 0$ ) is a yes-instance of Trail-RCA.

The case of constant $k>0$ works analogously as in the proof of Theorem 4.
Via Construction 7. Let ( $\left.G^{\prime}, s, t, p, k\right)$ an instance of Trail-RCA where $G^{\prime}$ is obtained from $G$ by applying Construction $7, p=\eta+1$, and $k=\eta-2$. Note that $\left(G^{\prime}, s, t, p, k\right)$ is constructed in polynomial time and $\max _{v \in V\left(G^{\prime}\right)}\{\operatorname{outdeg}(v)+\operatorname{indeg}(v)\} \leq 5$. By Lemma 11, $(G)$ is a yes-instance of $\mathrm{DP} 2 / 3 \mathrm{HC}$ if and only if $\left(G^{\prime}, s, t, p, k\right)$ is a yes-instance of TRAIL-RCA.

As the length of each $s$ - $t$ trail is upper bounded by the number of edges in the graph, we immediately obtain the following.

Corollary 5. Trail-FRCA on directed planar graphs is NP-complete, even if $k \geq 0$ is constant or $\Delta_{\mathrm{i} / \mathrm{o}} \geq 3$ is constant.

## 6 Walk-RCA

Regarding their computational complexity fingerprint, Path-RCA and Trail-RCA are equal. In this section, we show that WALK-RCA differs in this aspect. We prove that the problem is solvable in polynomial time on undirected graphs (Section 6.1) and on directed graphs if $k \geq 0$ is constant(Section 6.2).

### 6.1 On Undirected Graphs

On a high level, the tractability on undirected graphs is because a walk can alternate arbitrarily often between two vertices. Hence, we can model a queue on the source vertex $s$, where at distinct time steps the walks leave $s$ via a shortest path towards $t$. However, if the time of staying in the queue is upper bounded, that is, if the lengthrestricted variant WALK-FRCA is considered, the problem becomes NP-complete.

Theorem 7. Walk-RCA on undirected graphs is solvable in linear time.
Proof. Let $\mathcal{I}:=(G, s, t, p, k)$ be an instance of WALK-RCA with $G$ being connected. Let $P$ be a shortest $s$ - $t$ path in $G$. We assume that $p \geq 2$, since otherwise $P$ witnesses that $\mathcal{I}$ is a yes-instance. We can assume that the length of $P$ is at least $k+1$, otherwise we can output that $\mathcal{I}$ is a yes-instance. Let $\{s, v\}$ be the edge in $P$ incident to the endpoint $s$. We distinguish the two cases whether $k$ is positive or $k=0$. In this proof, we represent a walk as a sequence of edges.
Case $k>0$ : We can construct $p s$ - $t$ walks $P_{1}, \ldots, P_{p}$ sharing at most one edge as follows. We set $P_{1}:=P$ and $P_{i}=(\underbrace{\{s, v\}, \ldots,\{s, v\}}_{2 i \text {-times }}, P)$ for $i \in[p]$, that is, the $s$ $t$ walk $P_{i}$ alternates between $s$ and $v i$ times. We show that the set $\mathcal{P}:=\left\{P_{1}, \ldots, P_{p}\right\}$ share exactly edge $\{s, v\}$. It is easy to see that $\{s, v\}$ is shared by all of the walks. Let us consider an arbitrary edge $e=\{x, y\} \neq\{s, v\}$ in $P$, which appears in $P$ at time step $\ell>1$ (ordered from $s$ to $t$ ). By construction, $e$ appears in $P_{i}$ at position $2 i+\ell$. Thus, no two walks in $\mathcal{P}$ contain an edge in $P$, that is on time step $\ell>1$ in $P$, at the same
time step. Since each walk in $\mathcal{P}$ only contains edges in $P$, it follows that $\{s, v\}$ is exactly the shared edges by all walks in $\mathcal{P}$. As $k \geq 1$, it follows that we can output that $\mathcal{I}$ is a yes-instance.
Case $k=0$ : Let $v_{1}, \ldots, v_{\ell}$ be the neighbors of $s$, and suppose that $v_{1}=v$ (that is, the vertex incident to $s$ appearing in $P$ ). If $\ell<p$, then we can immediately output that $\mathcal{I}$ is a no-instance, as by the pigeon hole principle at least one edge has to appear in at least two walks at time step one in any set of $p s-t$ walks in $G$. If $\ell \geq p$, we construct $\ell s$ - $t$ walks $P_{1}, \ldots, P_{\ell}$ that do not share any edge in $G$ as follows. We set $P_{1}=P$, and $P_{i}=(\underbrace{\left\{s, v_{i}\right\}, \ldots,\left\{s, v_{i}\right\}}_{2 i \text {-times }}, P)$ for $i \in[\ell]$, that is, the $s$ - $t$ walk $P_{i}$ alternates between $s$ and $v_{i} i$ times. Following the same argumentation as in preceding case, it follows that no edge is shared by the constructed walks $P_{1}, \ldots, P_{p}$.

In summary, if $k>0$, then we can output that $\mathcal{I}$ is a yes-instance. If $k=0$, then we first check the degree of $s$ in linear time, and then output that $\mathcal{I}$ is a yes-instance if $\operatorname{deg}(s) \geq p$, and that $\mathcal{I}$ is a no-instance, otherwise.

The situation changes for WALK-FRCA, that is, when restricting the length of the walks.

Theorem 8. WALK-FRCA on undirected graphs is NP-complete and $\mathrm{W}[2]$-hard with respect to $k+\alpha$.

Given a directed graph $G$, we call an undirected graph $H$ the undirected version of $G$ if $H$ is obtained from $G$ by replacing each arc with an undirected edge.

Proof. We give a (parameterized) many-one reduction from $\operatorname{Set} \operatorname{Cover.~Let~}(U, \mathcal{F}, \ell)$ be an instance of SET Cover. Let $G^{\prime}$ the graph obtained from applying Construction 1 given $(U, \mathcal{F}, \ell)$. Moreover, let $G$ be the undirected version of $G^{\prime}$. Let $(G, s, t, p, k, \alpha)$ be the instance of WALK-FRCA, where $p=n+m, k=\ell$, and $\alpha=\ell+3$. Note that any shortest $s$ - $t$ path in $G$ is of length $\alpha$, and hence every $s$ - $t$ walk of length at most $\alpha$ behaves as in the directed acyclic case. That is, each walk contains a vertex of $V_{\mathcal{F}}$ at time step $k+2$. The correctness follows then analogously as in the proof of Lemma 4.

It remains open whether WALK-FRCA is NP-complete when $k$ is constant.

### 6.2 On Directed Graphs

Due to Theorems 2 and 3, we know that WALK-RCA is NP-complete on directed graphs and is solvable in polynomial time on directed acyclic graphs when $k=0$, respectively. In this section, we prove that if $k \geq 0$ is constant, then WALK-RCA remains tractable on directed graphs (this also holds true for Walk-FRCA). Note that for Path-RCA and Trail-RCA the situation is different, as both become NPcomplete on directed graphs, even if $k \geq 0$ is constant.

Theorem 9. WALK-RCA and WALK-FRCA on directed $n$-vertex m-arc graphs is solvable in $O\left(m^{k+1} \cdot n \cdot(p \cdot n)^{2}\right)$ time and $O\left(m^{k+1} \cdot n \cdot \alpha^{2}\right)$ time, respectively.

Our proof of Theorem 9 follows the same strategy as our proof of Theorem 1. That is, we try to guess the shared arcs, make them infinite capacity in some way, and then solve the problem with zero shared arcs via a network flow formulation in the timeexpanded graph. The crucial difference is that here we do not have at first an upper bound on the length of the walks in the solution.

Theorem 10. If $k=0$, then WALK-RCA on directed $n$-vertex $m$-arc graphs is solvable in $O\left(n \cdot m \cdot(p \cdot n)^{2}\right)$ time.

Lemma 12. Every yes-instance ( $G, s, t, p, k$ ) of WALK-RCA on directed graphs admits a solution in which the longest walk is of length at most $p \cdot d_{t}$, where $d_{t}=$ $\max _{v \in V}: \operatorname{dist}_{G}(v, t)<\infty \operatorname{dist}_{G}(v, t)$.

Observe that $d_{t}$ is well-defined on every yes-instance of WALK-RCA. Moreover, it holds that $d_{t} \leq|V(G)|$. In the subsequent proof, we use the following notation: For two walks $P_{1}=\left(v_{1}, \ldots, v_{\ell}\right)$ and $P_{2}=\left(w_{1}, \ldots, w_{\ell^{\prime}}\right)$ with $v_{\ell}=w_{1}$, denote by $P_{1} \circ P_{2}$ the walk $\left(v_{1}, \ldots, v_{\ell}, w_{2}, \ldots, w_{\ell^{\prime}}\right)$ obtained by the concatenation of the two walks.

Proof of Lemma 12. Let $\mathcal{P}$ be a solution to $(G, s, t, p, k)$ with $|\mathcal{P}|=p$ where the sum of the lengths of the walks in $\mathcal{P}$ is minimum among all solutions to $(G, s, t, p, k)$. Suppose towards a contradiction that the longest walk $P^{*} \in \mathcal{P}$ is of length $\left|P^{*}\right|>p \cdot d_{t}$. Then, by the pigeon hole principle, there is an $i \in[p]$ such that there is no walk in $\mathcal{P}$ of length $\ell$ with $(i-1) \cdot d_{t}<\ell \leq i \cdot d_{t}$.

Let $v=P^{*}\left[(i-1) \cdot d_{t}+1\right]$, that is, $v$ is the $\left((i-1) \cdot d_{t}+1\right)$ th vertex on $P^{*}$, and let $S$ be a shortest $v-t$ path. Observe that the length of $S$ is at most $d_{t}$. Consider the walk $P^{\prime}:=P^{*}\left[1,(i-1) \cdot d_{t}+1\right] \circ S$, that is, we concatenate the length- $\left((i-1) \cdot d_{t}\right)$ initial subpath of $P^{*}$ with $S$ to obtain $P^{\prime}$. Observe that $(i-1) \cdot d_{t}<\left|P^{\prime}\right| \leq i \cdot d_{t}$. If $\mathcal{P} \backslash P^{*} \cup P^{\prime}$ forms a solution to $(G, s, t, p, k)$, then, since $\left|P^{\prime}\right|<\left|P^{*}\right|, \mathcal{P} \backslash P^{*} \cup P^{\prime}$ is a solution of smaller sum of the lengths of the walks, contradicting the choice of $\mathcal{P}$. Otherwise, $P^{\prime}$ introduce additional shared arcs and let $A^{\prime} \subseteq A(G)$ denote the corresponding set. Observe that $A^{\prime}$ is a subset of the arcs of $S$. Let $a=(x, y) \in A^{\prime}$ be the shared arc such that $\operatorname{dist}_{S}(y, t)$ is minimum among all shared $\operatorname{arcs}$ in $A^{\prime}$, and let $P^{\prime}[j]=y$. Let $P \in \mathcal{P}$ be a walk sharing the arc with $P^{\prime}$. Then $P^{\prime \prime}:=P[1, j] \circ P^{\prime}\left[j+1,\left|P^{\prime}\right|\right]$ is a walk of shorter length than $P$.Moreover, $\mathcal{P} \backslash P \cup P^{\prime \prime}$ is a solution to $(G, s, t, p, k)$. As $\left|P^{\prime \prime}\right|<|P|$, $\mathcal{P} \backslash P \cup P^{\prime \prime}$ is a solution of smaller sum of the lengths of the walks, contradicting the choice of $\mathcal{P}$. As either case yields a contradiction, it follows that $\left|P^{*}\right| \leq p \cdot d_{t}$.

The subsequent proof of Theorem 10 relies on time-expanded graphs. Due to Lemma 12, we know that the time-horizon is bounded polynomially in the input size.

Proof of Theorem 10. Let $(G, s, t, p, 0)$ be an instance of Walk-RCA where $G=$ $(V, A)$ is an directed graph. We first compute $d_{t}$ in linear time. Let $\tau:=p \cdot d_{t}$. Next, we compute the $\tau$-time-expanded (directed) graph $H=\left(V^{\prime}, A^{\prime}\right)$ of $G$ with $p$ additional $\operatorname{arcs}\left(t^{i-1}, t^{i}\right)$ for each $i \in[\tau]$. We compute in $O\left(\tau^{2} \cdot(|V| \cdot|A|) \subseteq O\left(p^{2} \cdot\left(|V|^{3} \cdot|A|\right)\right.\right.$ time the value of a maximum $s^{0}-t^{\tau}$ flow in $H$. Due to Lemma 12 together with Lemma 2, the theorem follows.

Restricting to $\alpha$-time-expanded graphs yields the following.

Corollary 6. If $k=0$, WALK-FRCA on directed $n$-vertex $m$-arc graphs is solvable in $O\left(n \cdot m \cdot \alpha^{2}\right)$ time.

Proof of Theorem 9. Let $(G=(V, E), s, t, p, k)$ be an instance of WALK-RCA with $G$ being a directed graph. For each $k$-sized subset $K \subseteq A$ of arcs in $G$, we decide the instance $(G(K, p), s, t, p, 0)$. The statement for WALK-RCA then follows from Lemma 3 and Theorem 10. The running time of the algorithm is in $O\left(|A|^{k} \cdot p^{2} \cdot\left(|V|^{3} \cdot|A|\right)\right)$. The statement for WALK-FRCA then follows from Lemma 3 and Corollary 6.

## 7 Conclusion and Outlook

Some of our results can be seen as a parameterized complexity study of RCA focusing on the number $k$ of shared edges. It is interesting to study the problem with respect to other parameters. Herein, the first natural parameterization is the number of routes. Recall that the Minimum Shared Edges problem is fixed-parameter tractable with respect to the number of path [6]. A second parameterization we consider as interesting is the combined parameter maximum degree plus $k$. In our NP-completeness results for Path-RCA and Trail-RCA it seemed difficult to achieve constant $k$ and maximum degree at the same time.

Another research direction is to further investigate on which graph classes PathRCA and Trail-RCA become tractable. We proved that that both problems remain NP-complete even on planar graphs. Do both Path-RCA and Trail-RCA remain NP-complete on graphs of bounded treewidth? Recall that the Minimum Shared Edges problem is tractable on this graph class [20, 1].

Finally, we proved that on undirected graphs, WALK-RCA is solvable in polynomial time while Walk-FRCA is NP-complete. However, we left open whether WalkFRCA on undirected graphs is NP-complete or polynomial-time solvable when $k$ is constant.

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