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Abstract

We present a sequent calculus for the weak Grzegorczyk logic **Go** allowing non-wellfounded proofs and obtain the cut-elimination theorem for it by constructing a continuous cut-elimination mapping acting on these proofs.

Keywords: non-well-founded proofs, weak Grzegorczyk logic, logic Go, cut-elimination, cyclic proofs.

1 Introduction

The logic Go, also known as the weak Grzegorzyk logic, is the smallest normal modal logic containing the axiom K and the axioms $\Box A \rightarrow \Box \Box A$ and $\Box (\Box (A \rightarrow \Box A) \rightarrow A) \rightarrow \Box A$. A survey of results on Go can be found in [5]. The logic is sound and complete with respect to the class of transitive frames with no proper clusters and infinite ascending chains [2], and it is a proper sublogic of both Gödel-Löb logic GL (also known as provability logic) and Grzegorzyk logic Grz.

Recently a new proof-theoretic presentation for the logic GL in the form of a sequent calculus allowing non-well-founded proofs was given in [10,4]. Later, the same ideas were applied to the modal Grzegorczyk logic Grz in [8,7], where it allowed to prove several proof-theoretic properties of this logic syntactically.

In this paper we use the same approach for the logic Go. We consider a sequent calculus allowing non-well-founded proofs Go_{∞} and present the cutelimination theorem for it. We consider the set of non-well-founded proofs of Go_{∞} and various sets of operations acting on theses proofs as ultrametric spaces and define our cut-elimination operator using the Prieß-Crampe fixedpoint theorem (see [6]), which is a strengthening of the Banach's theorem.

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In [3] Goré and Ramanayake remark that their method for cut elimination for the logic Go is more complex than the similar methods for the logics GL and Grz. This difference in complexity seems to be present in our approach as well. The proofs of cut-elimination for Go_{∞} and Grz_{∞} turn out to be almost the same, but the system Go_{∞} itself seems to be more complex (it includes rules of arbitrary arity, where Grz_{∞} has at most binary) and the translation from Go_{∞} to the standart system seems to require bigger induction measure.

2 Preliminaries

In this section we recall the weak Grzegorczyk logic **Go** and define an ordinary sequent calculus for it.

Formulas of Go, denoted by A, B, C, are built up as follows:

$$A ::= \bot \mid p \mid (A \to A) \mid \Box A ,$$

where p stands for atomic propositions.

The Hilbert-style axiomatization of Go is given by the following axioms and inference rules:

Axioms:

- (i) Boolean tautologies;
- (ii) $\Box(A \to B) \to (\Box A \to \Box B);$
- (iii) $\Box A \rightarrow \Box \Box A$;
- (iv) $\Box(\Box(A \to \Box A) \to A) \to \Box A$.

Rules: modus ponens, $A | \Box A$.

Now we define an ordinary sequent calculus for Go. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. For a multiset of formulas $\Gamma = A_1, \ldots, A_n$, we set $\Box \Gamma \coloneqq \Box A_1, \ldots, \Box A_n$.

The system $\mathsf{Grz}_{\mathsf{Seq}},$ is defined by the following initial sequents and inference rules:

$$\Gamma, A \Rightarrow A, \Delta$$
, $\Gamma, \bot \Rightarrow \Delta$,

$$\rightarrow_{\mathsf{L}} \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta}, \qquad \qquad \rightarrow_{\mathsf{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta},$$

$$\square_{\mathsf{Go}} \xrightarrow{\Box \Pi, \Pi, \Box (A \to \Box A) \Rightarrow A} \overline{\Gamma, \Box \Pi \Rightarrow \Box A, \Delta}$$

Fig. 1. The system Go_{Seq}

The cut rule has the form

$$\operatorname{cut} \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \xrightarrow{}$$

where A is called the *cut formula* of the given inference.

Lemma 2.1 $\operatorname{Go}_{\operatorname{Seq}} + \operatorname{cut} \vdash \Gamma \Rightarrow \Delta$ if and only if $\operatorname{Go} \vdash \wedge \Gamma \to \vee \Delta$.

Proof. Standard transformations of proofs.

Theorem 2.2 If $Go_{Seq} + cut \vdash \Gamma \Rightarrow \Delta$, then $Go_{Seq} \vdash \Gamma \Rightarrow \Delta$.

A syntactic cut-elimination for Go was obtained by R. Goré and R. Ramanayake in [3]. In this paper, we will give another proof of this cut-elimination theorem.

3 Non-well-founded proofs

Now we define a sequent calculus for Go allowing non-well-founded proofs.

Inference rules and initial sequents of the sequent calculus Go_∞ have the following form:

$$\begin{split} \Gamma, p \Rightarrow p, \Delta \ , & \Gamma, \bot \Rightarrow \Delta \ , \\ \rightarrow_{\mathsf{L}} \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}, & \stackrel{}{\rightarrow_{\mathsf{R}}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta} \ , \\ \hline \Box \Pi, \Pi \Rightarrow A_1, \dots, A_n, \Box A_1, \dots \Box A_n \quad \Box \Pi, \Pi \Rightarrow A_1 \quad \dots \quad \Box \Pi, \Pi \Rightarrow A_n \\ \hline \Gamma, \Box \Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Delta \end{split}$$

Fig. 2. The system Go_{∞}

The system $Go_{\infty} + cut$ is defined by adding the rule (cut) to the system Go_{∞} . The will refer to all but the leftmost premises of the rule (\Box) as "right".

An ∞ -proof in $\operatorname{Go}_{\infty}(\operatorname{Go}_{\infty} + \operatorname{cut})$ is a (possibly infinite) tree whose nodes are marked by sequents and whose leaves are marked by initial sequents and that is constructed according to the rules of the sequent calculus. In addition, every infinite branch in an ∞ -proof must pass through a right premise of the rule (\Box) infinitely many times. A sequent $\Gamma \Rightarrow \Delta$ is provable in $\operatorname{Go}_{\infty}(\operatorname{Go}_{\infty} + \operatorname{cut})$ if there is an ∞ -proof in $\operatorname{Go}_{\infty}(\operatorname{Go}_{\infty} + \operatorname{cut})$ with the root marked by $\Gamma \Rightarrow \Delta$.

For a multiset of formulas $\Gamma = A_1, \ldots, A_n$, we set

$$\boxtimes \Gamma \coloneqq A_1, \ldots, A_n, \Box A_1, \ldots, \Box A_n.$$

Then the rule (\Box) can be written as

$$\frac{\boxtimes \Pi \Rightarrow \boxtimes (A_1, \dots, A_n) \qquad \boxtimes \Pi \Rightarrow A_1 \qquad \dots \qquad \boxtimes \Pi \Rightarrow A_n}{\Gamma, \Box \Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Delta}.$$

Let us construct an ∞ -proof of the sequent $\Box(\Box(p \to \Box p) \to p) \Rightarrow \Box p$. Let $F = \Box(p \to \Box p) \to p$ and let ψ be the following proof:

$$Ax \\ \rightarrow_{\mathsf{R}} \frac{\boxtimes F, p \Rightarrow p, \Box p, \Box p, \Box (p \to \Box p)}{\boxtimes F \Rightarrow \boxtimes (p, p \to \Box p)}$$

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$$\rightarrow_{\mathsf{L}} \underbrace{\Box F, p \Rightarrow p} \qquad \Box F, \Box (p \to \Box p) \to p \Rightarrow p$$

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An ∞–proof of the sequent $\Box(\Box(p \to \Box p) \to p) \Rightarrow \Box p$ can be constructed as follows:

The *n*-fragment of an ∞ -proof is a finite tree obtained from the ∞ -proof by cutting every branch at the *n*th from the root right premise of a \Box -rule. The 1-fragment of an ∞ -proof is also called its main fragment. The local height $|\pi|$ of an ∞ -proof π is the length of the longest branch in its main fragment. An ∞ -proof only consisting of an initial sequent has height 0.

The local height of the ∞ -proof constructed for the sequent $\Box(\Box(p \rightarrow \Box p) \rightarrow p) \Rightarrow \Box p$ equals to 4 and its main fragment has the form

$$\rightarrow_{\mathsf{R}} \underbrace{ \begin{array}{c} \Box F, p \Rightarrow p, \Box p, \Box p, \Box (p \to \Box p) \\ \Box F \Rightarrow p, p \to \Box p, \Box p, \Box (p \to \Box p) \\ \hline \Box F \Rightarrow p, p \to \Box p, \Box p, \Box (p \to \Box p) \\ \hline \Box F \Rightarrow \Box (p \to \Box p), \Box p, p \\ \hline \hline \Box F, \Box (p \to \Box p) \to p \Rightarrow p, \Box p \\ \hline \hline \Box (\Box (p \to \Box p) \to p) \Rightarrow \Box p \end{array} }$$

We denote the set of all ∞ -proofs in the system $Go_{\infty} + cut$ by \mathcal{P} .

For $\pi, \tau \in \mathcal{P}$, we write $\pi \sim_n \tau$ if *n*-fragments of these ∞ -proofs coincide. For any $\pi, \tau \in \mathcal{P}$, we also set $\pi \sim_0 \tau$.

For every ∞ -proof $\pi \in \mathcal{P}$ one of the following holds:

- (i) π consists of a single initial sequent.
- (ii) π has the form

$$\begin{array}{c} \pi_0 \\ \rightarrow_{\mathsf{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta} \end{array}$$

with $|\pi_0| < |\pi|$. We denote this by $\pi = \rightarrow_{\mathsf{R}(\mathsf{A} \to \mathsf{B})}(\pi_0)$.

(iii) π has the form

$$\rightarrow_{\mathsf{L}} \frac{ \begin{array}{c} \pi_{0} \\ \Gamma, B \Rightarrow \Delta \end{array} }{ \Gamma, A \to B \Rightarrow \Delta } \Gamma \Rightarrow A, \Delta$$

with $|\pi_0|, |\pi_1| < |\pi|$. We denote this by $\pi = \rightarrow_{\mathsf{L}(\mathsf{A} \to \mathsf{B})}(\pi_0, \pi_1)$.

(iv) π has the form

$$\operatorname{cut} \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta},$$

with $|\pi_0|, |\pi_1| < |\pi|$. We denote this by $\pi = \operatorname{cut}_A(\pi_0, \pi_1)$.

(v) π has the form

with $|\pi_0| < |\pi|$. We denote this by $\pi = \Box_{\Gamma;\Delta}(\pi_0, \pi_1, \ldots, \pi_n)$.

Now we define two translations that connect ordinary and non-well-founded sequent calculi for the logic Go.

Lemma 3.1 We have $Go_{\infty} \vdash \Gamma, A \Rightarrow A, \Delta$ for any sequent $\Gamma \Rightarrow \Delta$ and any formula A.

Proof. Standard induction on the structure of A.

Lemma 3.2 We have $Go_{\infty} \vdash \Box(\Box(A \rightarrow \Box A) \rightarrow A) \Rightarrow \Box A$ for any formula A.

Proof. Consider an example of ∞ -proof for the sequent $\Box(\Box(p \to \Box p) \to p) \Rightarrow \Box p$ given above. We transform this example into an ∞ -proof for $\Box(\Box(A \to \Box A) \to A) \Rightarrow A$ by replacing p with A and adding required ∞ -proofs instead of initial sequents using Lemma 3.1. \Box

Recall that an inference rule is called admissible (in a given proof system) if, for any instance of the rule, the conclusion is provable whenever all premises are provable.

Lemma 3.3 The rule

wk
$$\frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}$$

is admissible in the systems Go_{Seq} and $Go_{\infty} + cut$.

The rule

$$\operatorname{ctr} \frac{\Gamma, \Pi, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}$$

is admissible in the system Go_{Seq} .

Proof. Standard induction on the structure (local height) of a proof of $\Gamma \Rightarrow \Delta$.

Theorem 3.4 If $Go_{Seq} + cut \vdash \Gamma \Rightarrow \Delta$, then $Go_{\infty} + cut \vdash \Gamma \Rightarrow \Delta$.

Proof. Assume π is a proof of $\Gamma \Rightarrow \Delta$ in $\mathsf{Go}_{\mathsf{Seq}} + \mathsf{cut}$. By induction on the size of π we prove $\mathsf{Go}_{\infty} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$.

If $\Gamma \Rightarrow \Delta$ is an initial sequent of $\mathsf{Go}_{\mathsf{Seq}} + \mathsf{cut}$, then it is provable in $\mathsf{Go}_{\infty} + \mathsf{cut}$ by Lemma 3.1. Otherwise, consider the last application of an inference rule in π .

The only non-trivial case is when the proof π has the form

$$\Box_{\mathsf{Go}} \xrightarrow{\Box \Pi, \Pi, \Box(A \to \Box A) \Rightarrow A} \Sigma, \Box \Pi \Rightarrow \Box A, \Lambda$$

where $\Sigma, \Box \Pi = \Gamma$ and $\Box A, \Lambda = \Delta$. By the induction hypothesis there is an ∞ -proof ξ of $\boxtimes \Pi, \Box(A \to \Box A) \Rightarrow A$ in $\mathsf{Go}_{\infty} + \mathsf{cut}$.

The required ∞ -proof for $\Sigma, \Box \Pi \Rightarrow \Box A, \Delta$ has the form:

$$\begin{array}{c} \xi \\ \rightarrow_{\mathsf{R}} & \underline{\boxtimes}\Pi, \Box(A \rightarrow \Box A) \Rightarrow A \\ \hline & \mathsf{wk} & \underline{\boxtimes}\Pi \Rightarrow F \\ \Box & \underline{\boxtimes}\Pi \Rightarrow \boxtimes F \\ \hline & \Box & \underline{\Sigma, \Box}\Pi \Rightarrow \Box F, \Box A, \Lambda \\ \hline & \mathsf{cut} & \underline{\Sigma, \Box}\Pi \Rightarrow \Box F, \Box A, \Lambda \\ \end{array} \\ \begin{array}{c} \psi \mathsf{k} & \underline{\square} & \varphi \\ \hline & \nabla, \Box \Pi \Rightarrow \Box A, \Lambda \\ \hline & \Sigma, \Box \Pi \Rightarrow \Box A, \Lambda \\ \end{array}$$

where $F = \Box(A \to \Box A) \to A$ and χ is an ∞ -proof of $\Box F \Rightarrow \Box A$, which exists by Lemma 3.2.

The cases of other inference rules being last in π are straightforward, so we omit them.

For a sequent $\Gamma \Rightarrow \Delta$, let $Sub(\Gamma \Rightarrow \Delta)$ be the set of all subformulas of the formulas from $\Gamma \cup \Delta$. For a finite set of formulas Λ , let Λ^* be the set $\{A \rightarrow \Box A \mid A \in \Lambda\}$.

Lemma 3.5 If $Go_{\infty} \vdash \Gamma \Rightarrow \Delta$, then $Go_{Seq} \vdash \Box(\Lambda_1^*), \Lambda_2^*, \boxtimes \Omega, \Gamma \Rightarrow \Delta$ for any finite sets of formulas Λ_1, Λ_2 , and Ω such that $\Lambda_2 \subset \Lambda_1$.

Proof. Assume π is an ∞ -proof of the sequent $\Gamma \Rightarrow \Delta$ in Go_{∞} and Λ_1 , Λ_2 , and Ω are finite sets of formulas, such that $\Lambda_2 \subset \Lambda_1$.

We prove that $\mathsf{Go}_{\mathsf{Seq}} \vdash \Box(\Lambda_1^*), \Lambda_2^*, \boxtimes \Omega, \Gamma \Rightarrow \Delta$ by quadruple induction: by induction on the number of elements in the finite set $Sub(\Gamma \Rightarrow \Delta) \setminus \Lambda_1$ with a

subinduction on the number of elements in the finite set $Sub(\Gamma \Rightarrow \Delta) \setminus \Lambda_2$, subinduction on the number of elements in the finite set $Sub(\Gamma \Rightarrow \Delta) \setminus \Omega$, and with subinduction on $|\pi|$.

If $|\pi| = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent. We see that the sequent $\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes \Omega, \Gamma \Rightarrow \Delta$ is an initial sequent and it is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Otherwise, consider the last application of an inference rule in π .

Case 1. Suppose that $\pi = \rightarrow_{\mathsf{R}(\mathsf{A}\to\mathsf{B})}(\pi_0)$. Notice that $|\pi_0| < |\pi|$. By the induction hypothesis for Λ_2 , Λ_1 , Ω , and π_0 , the sequent $\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes \Omega, \Gamma, A \Rightarrow B, \Sigma$, where $A \to B, \Sigma = \Delta$, is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Applying the rule (\to_{R}) to it, we obtain that the sequent $\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes \Omega, \Gamma \Rightarrow \Delta$ is provable in $\mathsf{Go}_{\mathsf{Seq}}$.

Case 2. Suppose that $\pi = \rightarrow_{\mathsf{L}(\mathsf{A}\to\mathsf{B})}(\pi_0,\pi_1)$ We see that $|\pi_0| < |\pi|$. By the induction hypothesis for Λ_2 , Λ_1 , Ω , and π_0 , the sequent $\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes \Omega, \Sigma, B \Rightarrow \Delta$, where $\Sigma, A \to B = \Gamma$, is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Analogously, we have $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Box(\Lambda_1^*), \Lambda_2^*, \boxtimes \Omega, \Sigma \Rightarrow A, \Delta$. Applying the rule (\to_{L}) , we obtain that the sequent $\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes \Omega, \Gamma \Rightarrow \Delta$ is provable in $\mathsf{Go}_{\mathsf{Seq}}$.

Case 3. Suppose that π has the form

$$\begin{array}{cccc}
\pi_{0} & \pi_{1} & \pi_{n} \\
\underline{\otimes}\Pi \Rightarrow \boxtimes(A_{1}, \dots, A_{n}) & \boxtimes\Pi \Rightarrow A_{1} & \dots & \boxtimes\Pi \Rightarrow A_{n} \\
\hline
\Phi, \Box\Pi \Rightarrow \Box A_{1}, \dots, \Box A_{n}, \Sigma
\end{array}$$

where $\Phi, \Box \Pi = \Gamma$ and $\Box A_1, \ldots, \Box A_n, \Sigma = \Delta$.

Subcase 3.1: For some *i*, we have $A_i \notin \Lambda_1$. We have that the number of elements in $Sub(\Box\Pi, \Pi \Rightarrow A) \setminus (\Lambda_1 \cup \{A_i\})$ is strictly less than the number of elements in $Sub(\Phi, \Box\Pi \Rightarrow \Box A_1, \ldots, \Box A_n, \Sigma) \setminus \Lambda_1$. By the induction hypothesis for $\Lambda_1 \cup \{A_i\}, \Lambda_1, \emptyset$ and π_i , the sequent $\Box(\Lambda_1^*), \Box(A_i \Rightarrow \Box A_i), \Lambda_1^*, \Box\Pi, \Pi \Rightarrow A$ is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Then we have

$$\Box_{\mathsf{Go}} \frac{\Box(\Lambda_1^*), \Box(A_i \to \Box A_i), \Lambda_1^*, \Box \Pi, \Pi \Rightarrow A_i}{\Lambda_2^*, \Box(\Lambda_1^*), \boxtimes \Omega, \Phi, \Box \Pi \Rightarrow \Box A_1, \dots, \Box A_i, \dots, \Box A_n, \Sigma}$$

Subcase 3.2: For all i, we have $A_i \in \Lambda_1$, but there is i, such that $A_i \notin \Lambda_2$. We have that the number of elements in $Sub(\Box\Pi, \Pi \Rightarrow A_i) \setminus \Lambda_1$ is strictly less than the number of elements in $Sub(\Phi, \Box\Pi \Rightarrow \Box A_1, \ldots, \Box A_n, \Sigma) \setminus \Lambda_2$. By the induction hypothesis for $\Lambda_1, \Lambda_1, \emptyset$ and π_i , the sequent $\Box(\Lambda_1^*), \Lambda_1^*, \Box\Pi, \Pi \Rightarrow A_i$ is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Then we have

$$\Box_{\mathsf{Go}} \frac{ \underset{\Box(\Lambda_1^*),\Lambda_1^*,\Box\Pi,\Pi\Rightarrow A_i}{\Box(\Lambda_1^*),\Lambda_1^*,\Box\Pi,\Pi,\Box(A_i\to\Box A_i)\Rightarrow A_i}}{\Lambda_2^*,\Box(\Lambda_1^*),\boxtimes\Omega,\Phi,\Box\Pi\Rightarrow\Box A_1,\ldots,\Box A_i,\ldots,\Box A_n,\Sigma} \,.$$

Subcase 3.3: For all i, we have $A_i \in \Lambda_2 \subset \Lambda_1$, but there is a formula F in Π , such that $F \notin \Omega$. We have that the number of elements in $Sub(\Box\Pi,\Pi \Rightarrow A_1) \setminus (\Omega \cup \Pi)$ is strictly less than the number of elements in $Sub(\Phi,\Box\Pi \Rightarrow \Box A_1,\ldots,\Box A_n,\Sigma) \setminus \Omega$. By the induction hypothesis for $\Lambda_1, \Lambda_1,$ $\Omega \cup \Pi$ and π_1 , the sequent $\Box(\Lambda_1^*), \Lambda_1^*, \boxtimes \Omega, \boxtimes(\Pi \setminus \Omega), \boxtimes \Pi \Rightarrow A_1$ is provable in Go_{Seq} . Then we have

$$\begin{split} & \underset{\mathsf{Ctr}}{\overset{\mathsf{IC}}{\xrightarrow{}}} \frac{ \begin{array}{c} \Box(\Lambda_1^*), \Lambda_1^*, \boxtimes \Omega, \boxtimes(\Pi \setminus \Omega), \boxtimes \Pi \Rightarrow A_1 \\ \hline \Box(\Lambda_1^*), \Lambda_1^*, \boxtimes \Omega, \boxtimes(\Pi \setminus \Omega), \boxtimes \Pi, \Box(A \to \Box A) \Rightarrow A_1 \\ \hline \\ & \underset{\mathsf{ctr}}{\xrightarrow{}} \frac{ \begin{array}{c} \Lambda_2^*, \Box(\Lambda_1^*), \Box \Omega, \Omega, \Phi, \Box(\Pi \setminus \Omega), \Box(\Pi \setminus \Omega), \Box(\Pi \cap \Omega) \Rightarrow \Box A_1, \dots, \Box A_n, \Sigma \\ \hline \\ & \begin{array}{c} \Lambda_2^*, \Box(\Lambda_1^*), \boxtimes \Omega, \Phi, \Box\Pi \Rightarrow \Box A_1, \dots, \Box A_n, \Sigma \end{array} \end{split}$$

Subcase 3.4: For all i, we have $A_i \in \Lambda_2 \subset \Lambda_1$ and $\Pi \subset \Omega$. We see that $|\pi_0| < |\pi|$. By the induction hypothesis for Λ_1 , Λ_2 , Ω and π_0 the sequent $\Box(\Lambda_1^*), \Lambda_2^*, \boxtimes \Omega, \boxtimes \Pi \Rightarrow \boxtimes(A_1, \ldots, A_n)$ is provable in $\mathsf{Go}_{\mathsf{Seq}}$. Let $\Psi = \Box(\Lambda_1^*), \Lambda_2^*, \boxtimes \Omega$. Then we have

$$\begin{array}{c} \mathsf{Ax} \\ \overset{\Psi,\boxtimes\Pi,\square A_1 \Rightarrow \boxtimes (A_1,\dots,A_n)}{\operatorname{ctr}} \underbrace{\Psi,\boxtimes\Pi,\square A_1 \Rightarrow \boxtimes (A_1,\dots,A_n)} & \Psi,\boxtimes\Pi \Rightarrow \boxtimes (A_1,\dots,A_n) \\ \overset{(\Lambda_1^*),\Lambda_2^*,A_1 \rightarrow \square A_1,\boxtimes\Omega,\boxtimes\Pi \Rightarrow \square A_1,\boxtimes (A_2,\dots,A_n)}{\square(\Lambda_1^*),\Lambda_2^*,\boxtimes\Omega,\boxtimes\Pi \Rightarrow \square A_1,\boxtimes (A_2,\dots,A_n)} \\ & \vdots \\ \operatorname{ctr} \frac{\square(\Lambda_1^*),\Lambda_2^*,A_n \rightarrow \square A_n,\boxtimes\Omega,\boxtimes\Pi \Rightarrow \square A_1,\dots,\square A_n}{\square(\Lambda_1^*),\Lambda_2^*,\square\Omega,\Omega\backslash\Pi,\Pi,\square\Pi,\Pi \Rightarrow \square A_1,\dots,\square A_n} \\ \operatorname{weak} \frac{\square(\Lambda_1^*),\Lambda_2^*,\boxtimes\Omega,\Phi,\square\Pi \Rightarrow \square A_1,\dots,\square A_n}{\square(\Lambda_1^*),\Lambda_2^*,\boxtimes\Omega,\Phi,\square\Pi \Rightarrow \square A_1,\dots,\square A_n,\Sigma} \\ \end{array}$$

From Lemma 3.5 we immediately obtain the following theorem.

Theorem 3.6 If $Go_{\infty} \vdash \Gamma \Rightarrow \Delta$, then $Go_{Seq} \vdash \Gamma \Rightarrow \Delta$.

4 Ultrametric spaces

In this section we define ultrametrics on some spaces concerning ∞ -proofs. For some basic notions of the theory of ultrametric spaces (cf. [9]) and proofs please see Appendix.

Consider the set \mathcal{P} of all ∞ -proofs of the system $\mathsf{Go}_{\infty} + \mathsf{cut}$. An ultrametric $d_{\mathcal{P}}: \mathcal{P} \times \mathcal{P} \to [0,1]$ on \mathcal{P} is defined by

$$d_{\mathcal{P}}(\pi,\tau) = \inf\{\frac{1}{2^n} \mid \pi \sim_n \tau\}.$$

Proposition 4.1 $(\mathcal{P}, d_{\mathcal{P}})$ is a (spherically) complete ultrametric space.

For $m \in \mathbb{N}$, let \mathcal{F}_m denote the set of all non-expansive functions from \mathcal{P}^m to \mathcal{P} . For $\mathbf{a}, \mathbf{b} \in \mathcal{F}_m$, we write $\mathbf{a} \sim_{n,k} \mathbf{b}$ if $\mathbf{a}(\pi) \sim_n \mathbf{b}(\pi)$ for any $\pi \in \mathcal{P}^m$ and, in addition, $\mathbf{a}(\pi) \sim_{n+1} \mathbf{b}(\pi)$ whenever $\sum_{i=1}^m |\pi_i| < k$.² An ultrametric l_m on \mathcal{F}_m is defined by

$$l_m(\mathbf{a}, \mathbf{b}) = \frac{1}{2} \inf \{ \frac{1}{2^n} + \frac{1}{2^{n+k}} \mid \mathbf{a} \sim_{n,k} \mathbf{b} \}.$$

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² This definition is inspired by [1,Subsection 2.1].

Proposition 4.2 (\mathcal{F}_m, l_m) is a spherically complete ultrametric space.

Notice that any operator $U: \mathcal{F}_m \to \mathcal{F}_m$ is strictly contractive if and only if for any $a, b \in \mathcal{F}_m$, and any $n, k \in \mathbb{N}$ we have

$$a \sim_{n,k} b \Rightarrow U(a) \sim_{n,k+1} U(b).$$

Now we state a generalization of the Banach's fixed-point theorem for ultrametric spaces that will be used in the next sections.

Theorem 4.3 (Prieß-Crampe [6]) Let (M,d) be a non-empty spherically complete ultrametric space. Then every strictly contractive mapping $f: M \to M$ has a unique fixed-point.

5 Admissible Rules and Mappings

In this section, for the system $Go_{\infty} + cut$, we state admissibility of auxiliary inference rules, which will be used in the proof of the cut-elimination theorem.

Recall that the set \mathcal{P} of all ∞ -proofs of the system $Go_{\infty} + cut$ can be considered as an ultrametric space with the metric $d_{\mathcal{P}}$.

By \mathcal{P}_n we denote the set of all ∞ -proofs that do not contain applications of the cut rule in their *n*-fragments. We also set $\mathcal{P}_0 = \mathcal{P}$.

A mapping $\mathbf{u} : \mathcal{P}^m \to \mathcal{P}$ is called *adequate* if for any $n \in \mathbb{N}$ we have $\mathbf{u}(\pi_1, \ldots, \pi_n) \in \mathcal{P}_n$, whenever $\pi_i \in \mathcal{P}_n$ for all $i \leq n$.

In $Go_{\infty} + cut$, we call a single-premise inference rule *strongly admissible* if there is a non-expansive adequate mapping $u: \mathcal{P} \to \mathcal{P}$ that maps any ∞ -proof of the premise of the rule to an ∞ -proof of the conclusion. The mapping u must also satisfy one additional condition: $|u(\pi)| \leq |\pi|$ for any $\pi \in \mathcal{P}$.

In the following lemmata, non-expansive mappings are defined in a standard way by induction on the local heights of ∞ -proofs for the premises. So we omit further details.

Lemma 5.1 For any finite multisets of formulas Π and Σ , the inference rule

$$\mathsf{wk}_{\Pi;\Sigma} \xrightarrow{\Gamma \Rightarrow \Delta} \Pi, \Gamma \Rightarrow \Delta, \Sigma$$

is strongly admissible in $Go_{\infty} + cut$.

Lemma 5.2 For any formulas A and B, the rules

$$\begin{split} \mathsf{li}_{A \to B} & \frac{\Gamma, A \to B \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta} \qquad \mathsf{ri}_{A \to B} & \frac{\Gamma, A \to B \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} \\ \mathsf{i}_{A \to B} & \frac{\Gamma \Rightarrow A \to B, \Delta}{\Gamma, A \Rightarrow B, \Delta} \qquad \mathsf{i}_{\perp} & \frac{\Gamma \Rightarrow \bot, \Delta}{\Gamma \Rightarrow \Delta} \end{split}$$

are strongly admissible in $Go_{\infty} + cut$.

Lemma 5.3 For any atomic proposition p, the rules

$$\operatorname{acl}_p \frac{\Gamma, p, p \Rightarrow \Delta}{\Gamma, p \Rightarrow \Delta} \qquad \operatorname{acr}_p \frac{\Gamma \Rightarrow p, p, \Delta}{\Gamma \Rightarrow p, \Delta}$$

are strongly admissible in $Go_{\infty} + cut$.

Let us also define the mapping $\operatorname{clip}: \mathcal{P} \to \mathcal{P}$. Consider an ∞ -proof π . If the last rule application in π is not of the rule (\Box) then we put $\operatorname{clip}(\pi) = \pi$. If the ∞ -proof π has the form

$$\begin{array}{ccc} \pi_0 & \pi_1 & \pi_n \\ \underline{\boxtimes}\Pi \Rightarrow \underline{\boxtimes}(A_1, \dots, A_n) & \underline{\boxtimes}\Pi \Rightarrow A_1 & \dots & \underline{\boxtimes}\Pi \Rightarrow A_n \\ \hline \Gamma, \underline{\Box}\Pi \Rightarrow \underline{\Box}A_1, \dots, \underline{\Box}A_n, \Delta \end{array}$$

we define $\operatorname{clip}(\pi) = \pi$ to be

$$\begin{array}{ccc} \pi_0 & \pi_1 & \pi_n \\ \hline \boxtimes \Pi \Rightarrow \boxtimes (A_1, \dots, A_n) & \boxtimes \Pi \Rightarrow A_1 & \dots & \boxtimes \Pi \Rightarrow A_n \\ \hline \Box \Pi \Rightarrow \Box A_1, \dots, \Box A_n & \end{array} .$$

Clearly this mapping is non-expansive, adequate, and $|clip(\pi)| \leq |\pi|$ for any $\pi \in \mathcal{P}$.

6 Cut elimination

In this section, we construct a continuous function from \mathcal{P} to \mathcal{P} , which maps any ∞ -proof of the system $Go_{\infty} + cut$ to a cut-free ∞ -proof of the same sequent.

Let us call a pair of ∞ -proofs (π, τ) a *cut pair* if π is an ∞ -proof of the sequent $\Gamma \Rightarrow \Delta$, A and τ is an ∞ -proof of the sequent $A, \Gamma \Rightarrow \Delta$ for some Γ, Δ , and A. For a cut pair (π, τ) , we call the sequent $\Gamma \Rightarrow \Delta$ its *cut result* and the formula A its cut formula.

For a modal formula A, a non-expansive mapping \mathfrak{u} from $\mathcal{P} \times \mathcal{P}$ to \mathcal{P} is called *A*-removing if it maps every cut pair (π, τ) with the cut formula A to an ∞ -proof of its cut result. By \mathcal{R}_A , let us denote the set of all *A*-removing mappings.

Lemma 6.1 For each formula A, the pair (\mathcal{R}_A, l_2) is a non-empty spherically complete ultrametric space.

Proof. The proof of spherical completeness of the space (\mathcal{R}_A, l_2) is analoguos to the proof of the spherical completeness of (\mathcal{F}_m, L_m) .

We only need to check that the set \mathcal{R}_A is non-empty. Consider the mapping $\mathsf{u}_{cut}: \mathcal{P}^2 \to \mathcal{P}$ that is defined as follows. For a cut pair (π, τ) with the cut formula A, it joins the ∞ -proofs π and τ with an appropriate instance of the rule (cut). For all other pairs, the mapping u_{cut} returns the first argument.

Clearly, u_{cut} is non-expansive and therefore lies in \mathcal{R}_A .

In what follows, we use nonexpansive adequate mappings $\mathsf{wk}_{\Pi;\Sigma}$, $\mathsf{li}_{A\to B}$, $\mathsf{i}_{A\to B}$, $\mathsf{i}_{A\to B}$, i_{\bot} , acl_p , acr_p from Lemma 5.1, Lemma 5.2, and Lemma 5.3.

Lemma 6.2 For any atomic proposition p, there exists an adequate p-removing mapping re_p .

Proof.

Assume we have two ∞ -proofs π and τ . If the pair (π, τ) is not a cut pair or is a cut pair with the cut formula being not p, then we put $\operatorname{re}_p(\pi, \tau) = \pi$.

Otherwise, we define $\operatorname{re}_p(\pi, \tau)$ by induction on $|\pi|$. Let the cut result of the pair (π, τ) be $\Gamma \Rightarrow \Delta$.

If $|\pi| = 0$, then $\Gamma \Rightarrow \Delta, p$ is an initial sequent. Suppose that $\Gamma \Rightarrow \Delta$ is also an initial sequent. Then $\operatorname{re}_p(\pi, \tau)$ is defined as the ∞ -proof consisting only of the sequent $\Gamma \Rightarrow \Delta$. If $\Gamma \Rightarrow \Delta$ is not an initial sequent, then Γ has the form p, Φ , and τ is an ∞ -proof of the sequent $p, p, \Phi \Rightarrow \Delta$. Applying the non-expansive adequate mapping acl_p from Lemma 5.3, we put $\operatorname{re}_p(\pi, \tau) \coloneqq \operatorname{acl}_p(\tau)$.

Now suppose that $|\pi| > 0$. We define $\operatorname{re}_p(\pi, \tau)$ according to the last application of an inference rule in π :

$$(\rightarrow_{\mathsf{R}(\mathsf{B}\to\mathsf{C})}(\pi_0),\tau) \longmapsto \xrightarrow{\mathsf{re}_p(\pi_0,\mathsf{i}_{B\to C}(\tau))}_{\mathsf{R}} \xrightarrow{\Gamma,B \Rightarrow C,\Sigma}_{\Gamma \Rightarrow B \to C,\Sigma} ,$$

$$(\rightarrow_{\mathsf{L}(\mathsf{B}\to\mathsf{C})}(\pi_0,\pi_1),\tau) \longmapsto \begin{array}{c} \mathsf{re}_p(\pi_0,\mathsf{li}_{B\to C}(\tau)) & \mathsf{re}_p(\pi_1,\mathsf{ri}_{B\to C}(\tau)) \\ \underline{\Sigma,C \Rightarrow \Delta} & \underline{\Sigma\Rightarrow B,\Delta} \\ \underline{\Sigma,B \to C \Rightarrow \Delta} , \end{array}$$

$$(\operatorname{cut}_{\mathsf{B}}(\pi_{0},\pi_{1}),\tau) \longmapsto \operatorname{re}_{p}(\pi_{0},\operatorname{wk}_{\emptyset;B}(\tau)) \qquad \operatorname{re}_{p}(\pi_{1},\operatorname{wk}_{B;\emptyset}(\tau))$$
$$(\operatorname{cut}_{\mathsf{B}}(\pi_{0},\pi_{1}),\tau) \longmapsto \underbrace{\Gamma \Rightarrow B, \Delta}_{\Gamma \Rightarrow \Delta},$$

$$(\Box_{\Phi;\Sigma,\mathsf{p}}(\pi_0,\pi_1,\ldots,\pi_\mathsf{n}),\tau)\longmapsto \Box_{\Phi;\Sigma}(\pi_0,\pi_1,\ldots,\pi_\mathsf{n})$$

The mapping re_p is well defined, adequate and non-expansive.

Lemma 6.3 Given an adequate B-removing mapping re_B , there exists an adequate $\Box B$ -removing mapping $re_{\Box B}$.

Proof. Assume we have an adequate *B*-removing mapping re_B . The required $\Box B$ -removing mapping $re_{\Box B}$ is obtained as the fixed-point of a contractive operator $G_{\Box B}: \mathcal{R}_{\Box B} \to \mathcal{R}_{\Box B}$.

For a mapping $\mathbf{u} \in \mathcal{R}_{\Box B}$ and a pair of ∞ -proofs (π, τ) , the ∞ -proof $\mathsf{G}_{\Box B}(\mathbf{u})(\pi, \tau)$ is defined as follows. If (π, τ) is not a cut pair or a cut pair with the cut formula being not $\Box B$, then we put $\mathsf{G}_{\Box B}(\mathbf{u})(\pi, \tau) = \pi$.

Now let (π, τ) be a cut pair with the cut formula $\Box B$ and the cut result $\Gamma \Rightarrow \Delta$. If $|\pi| = 0$ or $|\tau| = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent. In this case, we define $\mathsf{G}_{\Box B}(\mathsf{u})(\pi, \tau)$ as the ∞ -proof consisting only of the sequent $\Gamma \Rightarrow \Delta$.

Suppose that $|\pi| > 0$ and $|\tau| > 0$. We define $G_{\Box B}(\mathbf{u})(\pi, \tau)$ according to the last application of an inference rule in π :

$$\left(\rightarrow_{\mathsf{R}(\mathsf{C}\to\mathsf{D})}(\pi_{0}), \tau \right) \longmapsto \rightarrow_{\mathsf{R}} \frac{\mathsf{u}(\pi_{0}, \mathsf{i}_{C}\to D(\tau))}{\Gamma \Rightarrow C \to D, \Sigma},$$

$$\left(\rightarrow_{\mathsf{L}(\mathsf{C}\rightarrow\mathsf{D})}(\pi_{0},\pi_{1}),\tau \right) \longmapsto \underset{\rightarrow_{\mathsf{L}}}{\overset{\mathsf{u}(\pi_{0},\mathsf{li}_{C}\rightarrow_{D}(\tau))}{\xrightarrow{\Sigma,D\Rightarrow\Delta}} \underbrace{ \begin{array}{c} \mathsf{u}(\pi_{1},\mathsf{ri}_{C\rightarrow D}(\tau)) \\ \mathsf{u}(\pi_{1},\mathsf{ri}_{C\rightarrow D}(\tau)) \\ \xrightarrow{\Sigma,D\Rightarrow\Delta} \\ \overbrace{\Sigma,C\rightarrow D\Rightarrow\Delta} \\ \end{array} \right),$$

$$(\operatorname{cut}_{\mathsf{C}}(\pi_{0},\pi_{1}),\tau) \longmapsto \underset{\operatorname{cut}}{\overset{\mathsf{u}}(\pi_{0},\operatorname{wk}_{\varnothing;C}(\tau))} \overset{\mathsf{u}(\pi_{1},\operatorname{wk}_{C;\varnothing}(\tau))}{\Gamma \Rightarrow C,\Delta} \xrightarrow{\Gamma,C \Rightarrow \Delta},$$

$$(\Box_{\Phi;\Sigma,\Box\mathsf{B}}(\pi_0,\pi_1,\ldots,\pi_n),\tau)\longmapsto \Box_{\Phi;\Sigma}(\pi_0,\pi_1,\ldots,\pi_n)$$

Consider the case when π has the form

We define $G_{\square B}(u)(\pi, \tau)$ according to the last application of an inference rule in τ .

$$\begin{aligned} & \left(\pi, \rightarrow_{\mathsf{R}(\mathsf{C}\to\mathsf{D})}(\tau_{0})\right) \longmapsto \xrightarrow{\mathsf{u}(\mathsf{i}_{C\to D}(\pi), \tau_{0})}_{\mathsf{R} \xrightarrow{\Gamma, C \Rightarrow D, \Sigma}}, \\ & \left(\pi, \rightarrow_{\mathsf{L}(\mathsf{C}\to\mathsf{D})}(\tau_{0}, \tau_{1})\right) \longmapsto \xrightarrow{\mathsf{u}(\mathsf{l}_{C\to D}(\pi), \tau_{0})}_{\mathsf{L} \xrightarrow{\Sigma, D \Rightarrow \Delta}} \underbrace{\mathsf{u}(\mathsf{r}_{C\to D}(\pi), \tau_{0})}_{\Sigma, C \to D \Rightarrow \Delta} \underbrace{\mathsf{u}(\mathsf{w}_{\mathcal{K};\mathcal{C}}(\pi), \tau_{0})}_{\Sigma, C \to D \Rightarrow \Delta}, \\ & \left(\pi, \mathsf{cut}_{\mathsf{C}}(\tau_{0}, \tau_{1})\right) \longmapsto \underbrace{\mathsf{u}(\mathsf{w}_{\mathcal{K};\mathcal{C}}(\pi), \tau_{0})}_{\mathsf{cut} \xrightarrow{\Gamma \Rightarrow C, \Delta}} \underbrace{\mathsf{u}(\mathsf{w}_{\mathcal{K};\mathcal{C}}(\pi), \tau_{0})}_{\Gamma \Rightarrow \Delta}, \end{aligned}$$

 $(\pi, \Box_{\Phi', \Box B; \Sigma'}(\tau_0, \tau_1, \dots, \tau_k)) \longmapsto \Box_{\Phi'; \Sigma'}(\tau_0, \tau_1, \dots, \tau_k)$

It remains to consider the case when τ has the form

$$\Box \xrightarrow{\tau_0} \tau_1 \\ \Box \xrightarrow{\forall \Lambda, \boxtimes B \Rightarrow \boxtimes (C_1, \dots, C_k)} \boxtimes \Lambda, \boxtimes B \Rightarrow C_1 \\ \Phi', \Box \Lambda, \Box B \Rightarrow \Box C_1, \dots, \Box C_k, \Sigma' \\ \hline \end{array}$$

Notice that the sequent $\Gamma \Rightarrow \Delta$, the sequent $\Phi, \Box \Pi \Rightarrow \Box A_1, \ldots, \Box A_n, \Sigma$, and the sequent $\Phi', \Box \Lambda \Rightarrow \Box C_1, \ldots, \Box C_k, \Sigma'$ are the same.

Let $I := \{i \mid A_i \notin C_1, \ldots, C_k\}$ and $J := \{i \mid C_i \notin A_1, \ldots, A_n\}$, let π'_B be the followind ∞ -proof:

and condsider the following ∞ -proofs:

$$\psi_{1} \coloneqq \mathsf{u}(\mathsf{wk}_{\Lambda \setminus \Pi, \Box \Lambda \setminus \Pi; \{\boxtimes C_{i}\}_{i \in J}}(\pi_{0}), \mathsf{wk}_{\Pi \cup \Lambda, \Box(\Pi \setminus \Lambda); \{\boxtimes A_{i}\}_{i \in I}, \{C_{i}\}}(\mathsf{clip}(\tau))),$$

$$\psi_{2} \coloneqq \mathsf{u}(\mathsf{wk}_{\Pi \cup \Lambda, \Box \Lambda \setminus \Pi; \{\boxtimes C_{i}\}_{i \in J}, \{A_{i}\}}(\mathsf{clip}(\pi)), \mathsf{wk}_{\Pi \setminus \Lambda, \Box(\Pi \setminus \Lambda); \{\boxtimes A_{i}\}_{i \in I}}(\tau_{0})),$$

$$\phi \coloneqq \mathsf{u}(\mathsf{wk}_{\Pi \cup \Lambda, \Box \Lambda \setminus \Pi; \{\boxtimes C_{i}\}_{i \in J}, \{A_{i}\}}(\mathsf{clip}(\pi)), \mathsf{wk}_{\Pi \setminus \Lambda, \Box(\Pi \setminus \Lambda); \{\boxtimes A_{i}\}_{i \in I}}(\tau_{0})))$$

$$\phi_j \coloneqq \mathsf{u}(\mathsf{wk}_{\boxtimes(\Lambda \setminus \Pi);C_j}(\pi'_B), \mathsf{re}_{\mathsf{B}}(\mathsf{wk}_{\boxtimes(\Lambda \setminus \Pi, \square B; \varnothing)}(\pi_B), \mathsf{wk}_{\boxtimes(\Pi \setminus \Lambda; \varnothing)}(\tau_j))).$$

We define $G_{\Box B}(\mathbf{u})(\pi, \tau)$ as

Now the operator $G_{\Box B}$ is well-defined. By the case analysis according to the definition of $G_{\Box B}$, we see that $G_{\Box B}(u)$ is non-expansive and belongs to $\mathcal{R}_{\Box B}$ whenever $u \in \mathcal{R}_{\Box B}$.

We claim that $G_{\Box B}$ is contractive. It sufficient to check that for any $u, v \in \mathcal{R}_{\Box B}$ and any $n, k \in \mathbb{N}$ we have

$$\mathsf{u} \sim_{n,k} \mathsf{v} \Rightarrow \mathsf{G}_{\Box \mathsf{B}}(\mathsf{u}) \sim_{n,k+1} \mathsf{G}_{\Box \mathsf{B}}(\mathsf{v}),$$

which we prove by case analysis.

Now we define the required $\Box B$ -removing mapping $\mathsf{re}_{\Box B}$ as the fixed-point of the the operator $\mathsf{G}_{\Box B}: \mathcal{R}_{\Box B} \to \mathcal{R}_{\Box B}$, which exists by Lemma 6.1 and Theorem 4.3.

It remains to check that the mapping $\mathbf{re}_{\Box B}$ is adequate. For some $n, k \in \mathbb{N}$, let us call a mapping $\mathbf{u} \in \mathcal{R}_{\Box B}$ (n, k)-adequate if it satisfies the following two conditions: $\mathbf{u}(\pi, \tau) \in \mathcal{P}_i$ for any $i \leq n$ and any $\pi, \tau \in \mathcal{P}_i$; $\mathbf{u}(\pi, \tau) \in \mathcal{P}_{n+1}$ whenever $\pi, \tau \in \mathcal{P}_{n+1}$ and $|\pi| + |\tau| < k$.

By case analysis we establish that $G_{\Box B}(\mathbf{u})$ is (n, k + 1)-adequate for any (n, k)-adequate $\mathbf{u} \in \mathcal{R}_{\Box B}$. Notice that if a mapping \mathbf{u} is (n, k)-adequate for all $k \in \mathbb{N}$, then it is also (n + 1, 0)-adequate. Now by induction on n with a subinduction on k, we immediately obtain that the mapping $\mathbf{re}_{\Box B}$, which is a fixed-point of $G_{\Box B}$, is (n, k)-adequate for all $n, k \in \mathbb{N}$. Therefore the mapping $\mathbf{re}_{\Box B}$ is adequate.

Lemma 6.4 For any formula A, there exists an adequate A-removing mapping re_A .

Proof.

We define re_A by induction on the structure of the formula A.

Case 1: A has the form p. In this case, re_p is defined by Lemma 6.2.

Case 2: A has the form \perp . Then we put $\operatorname{re}_{\perp}(\pi, \tau) \coloneqq i_{\perp}(\pi)$, where i_{\perp} is a non-expansive adequate mapping from Lemma 5.2.

Case 3: A has the form $B \to C$. Then we put

$$\operatorname{re}_{B \to C}(\pi, \tau) \coloneqq \operatorname{re}_{C}(\operatorname{re}_{B}(\operatorname{wk}_{\emptyset, C}(\operatorname{ri}_{B \to C}(\tau)), \operatorname{i}_{B \to C}(\pi)), \operatorname{li}_{B \to C}(\tau))$$

where $r_{B\to C}$, $i_{B\to C}$, $l_{B\to C}$ are non-expansive adequate mappings from Lemma 5.2 and $wk_{\emptyset,C}$ is a non-expansive adequate mapping from Lemma 5.1.

Case 4: A has the form $\Box B$. By the induction hypothesis, there is an adequate *B*-removing mapping re_B . The required $\Box B$ -removing mapping $\operatorname{re}_{\Box B}$ exists by Lemma 6.3.

A mapping $u: \mathcal{P} \to \mathcal{P}$ is called *root-preserving* if it maps ∞ -proofs to ∞ -proofs of the same sequents. Let \mathcal{T} denote the set of all root-preserving non-expansive mappings from \mathcal{P} to \mathcal{P} .

Lemma 6.5 The pair (\mathcal{T}, l_1) is a non-empty spherically complete ultrametric space.

Theorem 6.6 (cut-elimination) If $\operatorname{Grz}_{\infty} + \operatorname{cut} \vdash \Gamma \Rightarrow \Delta$, then $\operatorname{Grz}_{\infty} \vdash \Gamma \Rightarrow \Delta$.

Proof. We obtain the required cut-elimination mapping ce as the fixed-point of a contractive operator $F: \mathcal{T} \to \mathcal{T}$.

For a mapping $u \in \mathcal{T}$ and an ∞ -proof π , the ∞ -proof $F(u)(\pi)$ is defined as follows. If $|\pi| = 0$, then we define $F(u)(\pi)$ to be π .

Otherwise, we define $F(u)(\pi)$ according to the last application of an inference rule in π :

Now the operator F is well-defined. By the case analysis according to the definition of F, we see that F(u) is non-expansive and belongs to $\mathcal T$ whenever $u\in \mathcal T.$

We claim that F is contractive. It sufficient to check that for any $u, v \in \mathcal{T}$ and any $n, k \in \mathbb{N}$ we have

$$u \sim_{n,k} v \Rightarrow F(u) \sim_{n,k+1} F(v).$$

which is done by case analysis.

Now we define the required cut-elimination mapping ce as the fixed-point of the the operator $F: \mathcal{T} \to \mathcal{T}$, which exists by Lemma 6.5 and Theorem 4.3.

For some $n, k \in \mathbb{N}$, let us call a mapping $\mathbf{u} \in \mathcal{T}$ (n, k)-free if it satisfies the following two conditions: $\mathbf{u}(\pi) \in \mathcal{P}_n$ for any $\pi \in \mathcal{P}$; $\mathbf{u}(\pi) \in \mathcal{P}_{n+1}$ whenever $|\pi| < k$.

By case analysis we established that F(u) is (n, k+1)-free for any (n, k)-free $u \in \mathcal{T}$. Notice that if a mapping u is (n, k)-free for all $k \in \mathbb{N}$, then it is also (n+1,0)-free. Now by induction on n with a subinduction on k, we immediately obtain that the mapping ce, which is a fixed-point of F, is (n, k)-free for all $n, k \in \mathbb{N}$. Therefore, for any ∞ -proof π , the ∞ -proof $ce(\pi)$ does not contain instances of the rule (cut).

Now assume $\operatorname{Grz}_{\infty} + \operatorname{cut} \vdash \Gamma \Rightarrow \Delta$. Take an ∞ -proof of the sequent $\Gamma \Rightarrow \Delta$ in the system $\operatorname{Grz}_{\infty} + \operatorname{cut}$ and apply the mapping ce to it. We obtain an ∞ -proof of the same sequent in the system $\operatorname{Grz}_{\infty}$.

Theorem 2.2 is now established as a direct consequence of Theorem 3.4, Theorem 6.6, and Theorem 3.6.

7 Conclusions and Future Work

We have proven the cut elimination theorem for the logic Go syntacticly by constructing a continuous cut eliminating mapping for proofs in a system allowing non-well-founded proofs. This method seems to provide uniform approach to cut-elimination in different logics. Indeed — we can write systems for logics K4, GL, and Grz by slightly modifying the system Go_{Seq} , define appropriate ultrametrics on them, and leave the proof of cut elimilation virtually unchanged. The next step is to take this method to more complicated logics like the logic of transitive closure.

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Appendix

Here we recall basic notions of the theory of ultrametric spaces (cf. [9]) and consider several examples concerning ∞ -proofs.

An *ultrametric space* (M, d) is a metric space that satisfies a stronger version of the triangle inequality: for any $x, y, z \in M$

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}.$$

For $x \in M$ and $r \in [0, +\infty)$, the set $B_r(x) = \{y \in M \mid d(x, y) \leq r\}$ is called the *closed ball* with *center* x and *radius* r. Recall that a metric space (M, d)is *complete* if any descending sequence of closed balls, with radii tending to 0, has a common point. An ultametric space (M, d) is called *spherically complete* if an arbitrary descending sequence of closed balls has a common point.

For example, consider the set \mathcal{P} of all ∞ -proofs of the system $\mathsf{Go}_{\infty} + \mathsf{cut}$. We can define an ultrametric $d_{\mathcal{P}}: \mathcal{P} \times \mathcal{P} \to [0,1]$ on \mathcal{P} by putting

$$d_{\mathcal{P}}(\pi,\tau) = \inf\{\frac{1}{2^n} \mid \pi \sim_n \tau\}.$$

We see that $d_{\mathcal{P}}(\pi,\tau) \leq 2^{-n}$ if and only if $\pi \sim_n \tau$. Thus, the ultrametric $d_{\mathcal{P}}$ can be considered as a measure of similarity between ∞ -proofs.

Proposition 1 (4.1) $(\mathcal{P}, d_{\mathcal{P}})$ is a (spherically) complete ultrametric space.

Consider the following characterization of spherically complete ultrametric spaces. Let us write $x \equiv_r y$ if $d(x, y) \leq r$. Trivially, the relation \equiv_r is an equivalence relation for any ultrametric space and any number $r \geq 0$.

Proposition 2 An ultametric space (M,d) is spherically complete if and only if for any sequence $(x_i)_{i\in\mathbb{N}}$ of elements of M, where $x_i \equiv_{r_i} x_{i+1}$ and $r_i \ge r_{i+1}$ for all $i \in \mathbb{N}$, there is a point x of M such that $x \equiv_{r_i} x_i$ for any $i \in \mathbb{N}$.

Proof. (\Rightarrow) Assume (M, d) is a spherically complete ultrametric space. Consider a sequence $(x_i)_{i\in\mathbb{N}}$ of elements of M such that $x_i \equiv_{r_i} x_{i+1}$ and $r_i \ge r_{i+1}$ for all $i \in \mathbb{N}$. Then the sequence $(B_{r_i}(x_i))$ is a descending sequence of closed balls, and therefore by spherical completeness has a common point x. Trivially, the point x satisfies the desired conditions.

(\Leftarrow) Assume there is a descending sequence of closed balls $(B_{r_i}(x_i))$. We have that $x_0 \equiv_{r_0} x_1 \equiv_{r_1} \ldots$ and $r_i \ge r_{i+1}$ for all $i \in \mathbb{N}$. So there is an element $x \in M$ such that $x \equiv_{r_i} x_i$, meaning it lies in all the balls.

In an ultrametric space (M, d), a function $f: M \to M$ is called *non-expansive* if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in M$. For ultrametric spaces (M, d_M) and (N, d_N) , the Cartesian product $M \times N$ can be also considered as an ultrametric space with the metric

$$d_{M \times N}((x_1, y_1), (x_2, y_2)) = \max\{d_M(x_1, x_2), d_N(y_1, y_2)\}.$$

Let us consider another example. For $m \in \mathbb{N}$, let \mathcal{F}_m denote the set of all non-expansive functions from \mathcal{P}^m to \mathcal{P} . Note that any function $u: \mathcal{P}^m \to \mathcal{P}$ is

non-expansive if and only if for any tuples π and π' , and any $n \in \mathbb{N}$ we have

$$\pi_1 \sim_n \pi'_1, \ldots, \pi_m \sim_n \pi'_m \Rightarrow \mathsf{u}(\boldsymbol{\pi}) \sim_n \mathsf{u}(\boldsymbol{\pi}').$$

Now we introduce an ultrametric for \mathcal{F}_m . For $\mathbf{a}, \mathbf{b} \in \mathcal{F}_m$, we write $\mathbf{a} \sim_{n,k} \mathbf{b}$ if $\mathbf{a}(\boldsymbol{\pi}) \sim_n \mathbf{b}(\boldsymbol{\pi})$ for any $\boldsymbol{\pi} \in \mathcal{P}^m$ and, in addition, $\mathbf{a}(\boldsymbol{\pi}) \sim_{n+1} \mathbf{b}(\boldsymbol{\pi})$ whenever $\sum_{i=1}^m |\pi_i| < k$.³ An ultrametric l_m on \mathcal{F}_m is defined by

$$l_m(\mathsf{a},\mathsf{b}) = \frac{1}{2} \inf\{\frac{1}{2^n} + \frac{1}{2^{n+k}} \mid \mathsf{a} \sim_{n,k} \mathsf{b}\}.$$

We see that $l_m(\mathbf{a}, \mathbf{b}) \leq 2^{-n-1} + 2^{-n-k-1}$ if and only if $\mathbf{a} \sim_{n,k} \mathbf{b}$.

Proposition 3 (4.2) (\mathcal{F}_m, l_m) is a spherically complete ultrametric space.

Proof. Assume we have a series $a_0 \sim_{n_0,k_0} a_1 \sim_{n_1,k_1} \ldots$, where the sequence $r_i = 2^{-n_i} + 2^{-n_i-k_i-1}$ is non-increasing. From Proposition 2, it is sufficient to find a function $a \in \mathcal{F}_m$ such that $a \sim_{n_i,k_i} a_i$ for all $i \in \mathbb{N}$.

Suppose $\lim_{i\to\infty} r_i = 0$. Consider a tuple $\pi \in \mathcal{P}^m$. We have that $\lim_{i\to\infty} n_i = +\infty$ and $a_0(\pi) \sim_{n_0} a_1(\pi) \sim_{n_1} \ldots$. By Proposition 1, there is an ∞ -proof τ such that $\tau \sim_{n_i} a_i(\pi)$ for all $i \in \mathbb{N}$. We define $a(\pi) = \tau$. We need to check that the mapping a is non-expansive. If for tuples π and π' we have $\pi_1 \sim_n \pi'_1, \ldots, \pi_m \sim_n \pi'_m$ for some $n \in \mathbb{N}$, then we can choose i such that $n_i > n$. We have $a(\pi) \sim_n a_i(\pi) \sim_n a_i(\pi') \sim_n a(\pi')$. Therefore $a(\pi) \sim_n a(\pi')$ and the mapping a is non-expansive.

If $\lim_{i\to\infty} r_i > 0$, then $\lim_{i\to\infty} n_i = n$ for some $n \in \mathbb{N}$. We have two cases: either $\lim_{i\to\infty} k_i = k$ for a number $k \in \mathbb{N}$, or $\lim_{i\to\infty} k_i = +\infty$. In the first case, there is $j \in \mathbb{N}$ such that $(n_i, k_i) = (n_j, k_j)$ for all i > j, and we can take a_j as a. Here the mapping \mathbf{a} is obviously non-expansive. In the second case, for a tuple $\boldsymbol{\pi}$ we define $\mathbf{a}(\boldsymbol{\pi})$ to be $\mathbf{a}_j(\boldsymbol{\pi})$, where $j = \min\{i \in \mathbb{N} \mid n_i = n \text{ and } \sum_{s=1}^m |\pi_s| < k_i\}$. For all tuples $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ we have $\mathbf{a}(\boldsymbol{\pi}) \sim_0 \mathbf{a}(\boldsymbol{\pi}')$. If for tuples $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ and some $n \ge 1$ we have $\pi_1 \sim_n \pi'_1, \ldots, \pi_m \sim_n \pi'_m$, then $\sum_{s=1}^m |\pi_s| = \sum_{s=1}^m |\pi'_s| = t$ and $\mathbf{a}(\boldsymbol{\pi}) = \mathbf{a}_j(\boldsymbol{\pi}) \sim_n \mathbf{a}_j(\boldsymbol{\pi}') = \mathbf{a}(\boldsymbol{\pi}')$, where $j = \min\{i \in \mathbb{N} \mid n_i = n \text{ and } t < k_i\}$. Therefore $\mathbf{a}(\boldsymbol{\pi}) \sim_n \mathbf{a}(\boldsymbol{\pi}')$ and the mapping \mathbf{a} is non-expansive.

In an ultrametric space (M, d), a function $f: M \to M$ is called *(strictly)* contractive if d(f(x), f(y)) < d(x, y) when $x \neq y$.

³ This definition is inspired by [1,Subsection 2.1].