Generating Weakly Chordal Graphs from Arbitrary Graphs

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Abstract

We propose a scheme for generating a weakly chordal graph from a randomly generated input graph, G = (V, E). We reduce G to a chordal graph H by adding fill-edges, using the minimum vertex degree heuristic. Since H is necessarily a weakly chordal graph, we use an algorithm for deleting edges from a weakly chordal graph that preserves the weak chordality property of H. The edges that are candidates for deletion are the fill-edges that were inserted into G. In order to delete a maximal number of fill-edges, we maintain these in a queue. A fill-edge is removed from the front of the queue, which we then try to delete from H. If this violates the weak chordality property of H, we reinsert this edge at the back of the queue. This loop continues till no more fill-edges can be removed from H. Operationally, we implement this by defining a deletion round as one in which the edge at the back of the queue is at the front. We stop when the size of the queue does not change over two successive deletion rounds and output H.

1 Introduction

A graph G = (V, E) is said to be weakly chordal if neither G nor its complement, \overline{G} , has an induced chordless cycle on five or more vertices (a hole). Figure 1 shows an example of a weakly chordal graph, G, and its complement, \overline{G} .

Weakly chordal graphs were introduced by Hayward in [6] as a generalization of chordal graphs, who showed that these graphs form a subclass of the perfect graphs. An alternate definition that does not refer to the complement graph is that G does not contain a hole or an anti-hole, which is the complement of a hole. Berry et al. [2] gave a very different and interesting definition of a weakly chordal graph as one in wich every edge is *LB*-simplicial. They also proposed the open problem of generating a weakly chordal graph from an arbitrary graph. A solution to this problem is the subject of this paper.



Figure 1: Weakly chordal graph [11]

Early work on graph generation foucussed on creating catalogues of graphs of small sizes. Cameron et al. [?], for instance, published a catalogue of all graphs on 10 vertices. The underlying motive was that such repositories were useful for providing counterexamples to old conjectures and coming up with new ones. Subsequent focus shifted to generating graphs of arbitrary size, labeled and unlabeled, uniformly at random. As such a generation method, involved solving a counting problem, research was focused to classes of graphs for which the counting problem could be solved and yielded polynomial time generation algorithms. Among these were graphs with prescribed degree sequence, regular graphs, special classes of graphs such as outerplanar graphs, maximal planar graphs. See [13] for a survey work prior to 1990.

As stated in [11], there are many situations where we would like to generate instances of these to test algorithms for weakly chordal graphs. For instance, in [10] the authors generate all linear layouts of weakly chordal graphs. A generation mechanism can be used to obtain test instances for this algorithm. It can do the same for optimization algorithms, like finding a maximum clique, maximum stable set, minimum clique cover, minimum coloring, for both weighted and unweighted versions, for weakly chordal graphs proposed in [7] and their improved versions in [8, 12].

If the input instances for a given algorithm are from a uniform distribution, a uniform random generation provides test instances to obtain an estimate of the average run-time of the algorithm. When the distribution is unknown, the assumption of uniform distribution might still help. Otherwise, we might look upon a generation algorithm as providing test-instances for an algorithm. With this motive, an algorithm for generating weakly chordal graphs by adding edges incrementally was recently proposed in [11]. An application of this generation algorithm would be to obtain test-instances for an algorithm for enumerating linear layouts of a weakly chordal graph proposed in [10].

The next section of the paper contains some common graph terminology, used subsequently. The following section contains details of our algorithms, beginning with a brief overview. In the concluding section, we summarize the salient aspects of the paper and suggest directions for further work.

2 Preliminaries

We will assume that G is a graph on n vertices and m edges, that is, |V| = n and |E| = m. The neighborhood N(v) of a vertex v is the subset of vertices $\{u \in V \mid (u, v) \in E\}$ of V. The degree $\deg(v)$ of a vertex v is equal to |N(v)|. A vertex v of G is simplicial if the induced subgraph on N(v) is complete (alternately, a clique). A path in a graph G is a sequence of vertices connected by edges. We use $P_k(k \geq 3)$ to denote a chordless path, spanning k vertices of G. For instance, a path on 3 vertices is termed as a P_3 and, similarly, a path on 4 vertices is termed as a P_4 . If a path starts and ends in the same vertex, the path is a cycle denoted by C_k , where k is the length of the cycle. A chord in a cycle is an edge between two non-consecutive vertices in the cycle.



Figure 2: Complement of a chordal graph with a chordless 4-cycle [11]



Figure 3: Complement of a five cycle is also a five cycle

G is chordal if it has no induced chordless cycles of size four or more. However, as Figure 2 shows, the complement of a chordal graph G can contain an induced chordless cycle of size four. The complement cannot contain a five cycle though, as the complement of a five cycle is also a five cycle (see Figure 3). The above example makes it clear why chordal graphs are also weakly chordal.

In this paper, we propose an algorithm that generates a weakly chordal graph from an arbitrary input graph. It is built on top of a subroutine that maintains the weak chordality of a graph G, under edge deletion.

3 Arbitrary Graph to Weakly Chordal Graph

3.1 Overview of the Method

We start by generating a random graph G on n vertices and m edges. In a preprocessing step we check if G is weakly chordal, using the LB-simpliciality recognition algorithm due to [3]. If G is weakly chordal, we stop. Otherwise, we proceed as follows. We first reduce G to a chordal graph H by introducing additional edges, called fill-edges, using the minimum degree vertex (mdv, for short) heuristic [5]. The mdv heuristic adds edges so that a minimum degree vertex in the current graph is simplicial. Each fill-edge is also entered into a queue, termed a fill-edge queue, FQ. These fill-edges are potential candidates for subsequent deletion from H. Since H is chordal, it is necessarily weakly chordal. We propose an algorithm for deleting edges from this weakly chordal graph to remove fill-edges, maintaining the weak chordality property. A fill-edge is deleted only if does not create a hole or an anti-hole in the resulting graph and we have developed criteria for detecting this. A fill-edge is removed from the front of the queue, which we then try to delete. If we do not succeed we put it at the back of the queue. We keep doing this until no more fill edges can be removed. Operationally, we implement this by defining a deletion round as one in which the fill-edge at the back of the queue is at the front. We stop when the size of the queue does not change over two successive deletion rounds. Figure 4 is a pictorial illustration of the flow of control.



Figure 4: Overview of process

3.2 Random Arbitrary Graph

To generate a random graph, we invoke an algorithm by Keith M. Briggs, called 'dense_gnm_random_graph'. This algorithm, based on Knuth's Algorithm S (Selection sampling technique, see section 3.4.2 of [9]), takes the number of vertices, n and the number of edges, m, as input and produces a random graph. For a given n, we set m to a random value lying in the range between n-1 and $\frac{n(n-1)}{2}$. The output graph may be disconnected, in which case we connect the disjoint components, using additional edges.

3.3 LB-simpliciality test

In [3] Berry et al. proved the following result:

Theorem 3.1 [3] A graph is weakly chordal if and only if every edge is LB-simplicial.

We apply this recognition algorithm to the random graph generated by the previous step and continue with the next steps only if the recognition algorithm fails. Otherwise, we return G.

3.4 Arbitrary Graph to Chordal Graph

The arbitrary graph G is embedded into a chordal graph H by the addition of edges and the process is known as triangulation or fill-in. Desirable triangulations are those in which a minimal or a minimum number of edges is added. A triangulation $H = (V, E \cup F)$ of G = (V, E) is minimal if $(V, E \cup F')$ is non-chordal for every proper subset F' of F. In a minimum triangulation the number of edges added is the fewest possible. Berry at al. [1] proposed an algorithm, known as LB-Triangulation, for the minimal fill-in problem. LB-Triangulation works on any ordering α of the vertices, and produces a fill that is provably exclusion-minimal. In our algorithm, we have used the mdv heuristic [5], as our experiments have shown that this adds fewer fill-edges as compared to LB-Triangulation. We explain this heuristic in the next section.

3.4.1 The Minimum Degree Vertex Heuristic

Let $H = (V, E \cup F)$ be the graph obtained from G = (V, E), where F is set of fill-edges, following these steps. We first assign G to H and then prune from G all vertices of degree 1. From the remaining vertices of G we choose a vertex v of minimum degree (breaking ties arbitrarily) and turn the neighborhood N(v) of v into a clique by adding edges. These are fill-edges that we add to the edge set of H, as well as to the fill-queue, FQ. Finally, we remove from G, the vertex v and all the edges incident on it. We repeat this until G is empty. The graph H is now chordal and is identical with the initial graph G, sans degree 1 vertices, and with fill edges added. We illustrate this with an example.



Figure 5: Arbitrary graph to chordal graph

The initial graph G is shown in Fig. 5(a) and the graph H with all fill-edges added is shown in Fig. 5(b). In the initial graph G both v_1 and v_5 have minimum degree. We break tie in favour of v_5 . Since the induced subgraph on $N(v_5)$ is already a clique no fill-edges are added and G is set

to $G - \{v_5\}$. In the reduced graph G, v_1 is of minimum degree and the induced graph on $N(v_1)$ is turned into a clique by adding $\{v_3, v_4\}$ as a fill- edge, which is also added to H. Since the reduced graph $G - \{v_1\}$ is a clique, we can pick the vertices v_0, v_2, v_3, v_4 in an arbitrary order to reduce G to an empty graph, without introducing any further fill edges into H. The formal algorithm is described below:

Algorithm 1 ArbitraryToChordal

Input: An arbitrary graph G = (V, E)

Output: Returns a chordal graph $H = (V, E \cup F)$ and fill-edge queue FQ

- 1: $H \leftarrow G$
- 2: Delete all vertices of degree 1 from G
- 3: Sort V in ascending order of degrees
- 4: Choose a vertex v of minimum degree
- 5: Turn N(v) of v into a clique by adding edges, which are added to the edge set of H and to the fill-queue, FQ
- 6: Remove the vertex v from G and all the edges incident on it
- 7: Repeat steps 3 to 6 until G is empty

3.5 Chordal Graph to Weakly Chordal Graph

Since the chordal graph H obtained from the previous stage is also weakly chordal, we apply an edge deletion algorithm to H that preserves weak chordality. The edges that are candidates for deletion are the ones that have been added by the mdv heuristic. Each candidate edge is temporarily deleted from H, and we check if its deletion creates a hole or an anti-hole in H. If not, we delete this edge. The process is explained in details in the subsequent sections.

3.5.1 Fill-Edge Queue

As mentioned earlier, each edge added to convert an arbitrary input graph into chordal graph is called a fill-edge. In order to delete as many fill-edges as possible, we maintain a queue of fill-edges, FQ. A fill-edge is removed from the front of this queue, which we then try to delete from H. If we do not succeed because a hole or anti-hole is created, we put it at the back of the queue. We keep doing this until no more fill-edges can be removed from FQ.

3.5.2 Detecting Holes

To reiterate, a hole in a graph G is an induced chordless cycle on five or more vertices. Since, a graph is weakly chordal if it is (hole, anti-hole)-free [4], it is crucial to detect if any hole is formed by the deletion of an edge. For the class of weakly chordal graphs, since the biggest cycle allowed is of size four, the holes can be formed either by a combination of two P_4 's or a by a combination of a P_3 and a P_4 , as illustrated in Fig. 6.

To detect the formation of a hole in H, we pick an edge $e = \{u, v\}$ of H and temporarily delete it. Now, we check if this deletion creates a hole in H. To detect a hole, we perform a breadth-first



Figure 6: Detecting Holes

search in H with u as the source vertex and find all chordless P_3 and P_4 paths between u and v. A hole can be created in two distinct ways: (i) by a disjoint pair of P_4 , with six distinct vertices between them such that there exist no chord joining an internal vertex on one P_4 to an internal vertex on the other; this we call a hole on two P_{4s} ; (ii) by a disjoint pair of P_3 and P_4 between uand v, with five distinct vertices between them, such that there exist no chord joining an internal vertex on the P_4 to the internal vertex of the P_3 ; this we call a hole on a P_3 and a P_4 .

3.5.3 Antiholes

An anti-hole in a graph is, by definition, the complement of a hole [4]. An anti-hole configuration in a weakly chordal graphs has the structure shown in Fig. 7. This is an induced graph on six distinct vertices each of which is of degree three.



Figure 7: Antihole

3.5.4 Detecting Antiholes

To detect an anti-hole configuration, we pick an edge $\{u, v\}$ and temporarily delete it from the graph. Next, we check if deleting the edge $\{u, v\}$ creates an anti-hole configuration in the graph. To detect this, we do breadth-first search with u as the source vertex to find all chordless P_3 and P_4 paths between u and v. An anti-hole configuration is formed by a combination of two P_3 and one P_4 such that the induced graph on the six vertices that define these paths, are uniformly of degree

three and there exists a chord from the internal vertex of each P_3 to one of the internal vertices in the P_4 . For example, in Fig. 7, $\{v_1, v_2, v_5, v_4\}$ is a P_4 , $\{v_1, v_3, v_4\}$ and $\{v_1, v_6, v_4\}$ are two P_3 paths. There exists exactly one chord from v_2 to v_3 and exactly one from v_5 to v_6 and, in the induced graph on these six vertices, every vertex has degree three, making it an anti-hole configuration.

3.5.5 Proposed Algorithm

We use an algorithm for deleting edges from a weakly chord graph to remove fill edges, maintaining its weak chordality property. In order to delete as many fill-edges as possible, a fill-edge $\{u, v\}$ is removed from the front of the fill-queue, which we then try to delete from H. If we do not succeed, we put it at the back of the queue. We keep doing this until no more fill-edges can be removed. Operationally, we implement this by defining a deletion round as one in which the edge at the back of the queue is at the front. One deletion round comprises of picking an edge from the start of the queue and deleting it from H. Now we check if the deletion of $\{u, v\}$ creates a hole or an antihole in H. If so, we do not delete the edge $\{u, v\}$ and add it back to the fill-queue. Otherwise, we delete the edge from H and also remove it from te fill-queue FQ. We stop when the size of FQ does not change over two successive deletion rounds.

Algorithm 2 ChordalToWeaklyChordal

Input: A chordal graph $H = (V, E \cup F)$ with fill edge queue F **Output:** A weakly chordal graph G_w 1: $T \leftarrow H$ \triangleright Make a copy of H2: $FQ \leftarrow \text{fill-edges of } H$ 3: $prevSize \leftarrow 0$ 4: $newSize \leftarrow |FQ|$ 5: while $(prevSize \neq newSize \&\& newSize \neq 0)$ do \triangleright Check size of FQ over two deletion rounds $prevSize \leftarrow newSize$ 6: for $(each \ edge\{u, v\} \ infill-queue, FQ)$ do 7:8: Delete edge $\{u, v\}$ from T if (Hole or Antihole Detected) then 9: Do not delete edge from graph H, add edge back to temporary graph T, and to the 10: back of the queue FQelse 11: 12:Delete edge $\{u, v\}$ from graph H end if 13: end for 14: $newSize \leftarrow |FQ|$ 15:16: end while 17: $G_w \leftarrow H$ 18: return G_w

For example in Fig. 8, a random arbitrary graph on 6 vertices and 8 edges is obtained. It is converted into a chordal graph by inserting two additional edges. These two additional edges added are put in the fill-edge queue $[\{v_2, v_4\}, \{v_1, v_4\}]$. Maintain a temporary copy of chordal graph G in T. The

deletion algorithm begins by picking first edge $\{v_2, v_4\}$ in the fill-edge queue and temporarily deletes it from graph T to check for hole and antihole configurations. Since deleting $\{v_2, v_4\}$ does not give rise to any hole or anti-hole configurations, $\{v_2, v_4\}$ is permanently deleted from starting graph Hwhich is now a weakly chordal graph. Now the updated fill-edge queue is $[\{v_1, v_4\}]$. The deletion algorithm now picks the first edge in $\{v_1, v_4\}$ in the fill edge queue and temporarily deletes it from graph T to check for hole and antihole configurations. Since deleting $\{v_1, v_4\}$ gives rise to a hole configuration on one P_4 $\{v_1, v_2, v_3, v_4\}$ and one P_3 $\{v_1, v_5, v_4\}$, $\{v_1, v_4\}$ is not permanently deleted from H. Since the queue is now empty, the graph G_w returned by the algorithm is weakly chordal with a small subset of fill-edges added to the original graph G.



Figure 8: Arbitrary graph to weakly chordal graph

For another example, consider Figure 9(a), a random arbitrary graph on 6 vertices and 9 edges is obtained. It is converted into a chordal graph H (see Figure 9(b)) by adding three additional edges. These three additional edges added are put in the fill-edge queue $[\{v_1, v_5\}, \{v_2, v_4\}, \{v_1, v_4\}]$. Maintain a temporary copy of the chordal graph H in T. The deletion algorithm begins by picking first edge $\{v_1, v_5\}$ in the fill edge queue and temporarily deletes it from graph T to check for hole and antihole configurations. Since deleting $\{v_1, v_5\}$ does not give rise to any hole or anti-hole configurations, $\{v_1, v_5\}$ is permanently deleted from starting graph H which is now a weakly chordal graph shown in Figure 9(c). Now the updated fill-edge queue is $[\{v_2, v_4\}, \{v_1, v_4\}]$. The deletion algorithm now picks the first edge in $\{v_2, v_4\}$ in the fill-edge queue and temporarily deletes it from graph T to check for hole and anti-hole configurations. Since deleting $\{v_2, v_4\}$ does not give rise to any hole or anti-hole configuration, $\{v_2, v_4\}$ is permanently deleted from starting graph H, which is now a weakly chordal graph. Now the updated fill-edge queue is $[\{v_1, v_4\}]$. The deletion algorithm now picks the first and only edge $\{v_1, v_4\}$ in the fill-edge queue and temporarily deletes it from graph T to check for a hole or an anti-hole configuration. Since deleting $\{v_1, v_4\}$ gives rise to an anti-hole configuration on two P_3 paths $\{v_1, v_3, v_4\}, \{v_1, v_6, v_4\}$ and one P_4 $\{v_1, v_2, v_5, v_4\}$, the edge $\{v_1, v_4\}$ is not permanently deleted from starting graph H. Since the queue is now empty, the graph G_w returned by the algorithm is weakly chordal with a small subset of fill-edges added to the original graph G as shown in Figure 9(d).



Figure 9: Arbitrary graph to weakly chordal graph

3.6 Complexity

The mdv heuristic can be implemented in $O(n^2m)$ time, while the time-complexity of the recognition algorithm based on LB-simpliciality is in O(nm).

To bound the query complexity of deleting an edge $\{u, v\}$ from the weakly chordal graph, we note that this is dominated by the task of finding multiple P_3 and P_4 paths between u and v and we have to consider these in pairs and run the breadth-first search. An upper bound on the number of pairs of P_3 and P_4 paths between u and v is $O(d_u^2 d_v^2)$, where d_u and d_v are the degrees of u and v respectively. For consider such a path from u to v (see Figure 10): x is one of the at most d_u vertices adjacent to u and y is one of the at most d_v vertices adjacent to v, so that we have at most $O(d_u d_v) P_4$ paths from u to v and thus $O(d_u^2 d_v^2)$ disjoint pairs of P_4 paths from u to v.



Figure 10: A P_4 -path from u to v

If |E| be the number of edges currently, in the weakly chordal graph, the complexity of running a breadth-first search is O(n+|E|). Since m is the number of edges in the final weakly chordal graph, an upper bound on the query complexity is $O(d_u^2 d_v^2 (n+m))$.

The deletion of an edge take constant time since we maintain an adjacency matrix data structure

to represent G.

4 Conclusion

We have proposed a simple method for generating a weakly chordal graph from an arbitrary graph. The proposed algorithm can also be used to generate weakly chordal graphs by deleting edges from input graphs that are known to be weakly chordal, such as complete graphs. Starting with complete graphs also helps in generating dense weakly chordal graphs. An interesting open problem is to establish if the proposed method to generate a weakly chordal graph from an arbitrary graph adds a minimal number of edges.

We have implemented our algorithm in Python. Some sample outputs are shown below in an appendix. In each of the figures, the purples edges in the chordal graph are edges that are candidates for deletion.

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5 Appendix



Figure 11: Arbitary graph to a weakly chordal one



Figure 12: Arbitary graph to a weakly chordal one



Figure 13: Arbitary graph to a weakly chordal one