Loewy Decomposition of Linear Differential Equations

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Springer

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ISSN 0943-853X
ISBN 978-3-7091-1285-4 ISBN 978-3-7091-1286-1 (eBook)
DOI 10.1007/978-3-7091-1286-1
Springer Wien Heidelberg New York Dordrecht London

Library of Congress Control Number: 2012945905

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To Birgit

Preface

The aim of this book is to communicate some results on solving linear differential equations that have been achieved in the last two decades. The key concept is the factorization of a differential equation or the corresponding differential operator, and the resulting decomposition into unique objects of lower order. Although more than 100 years old, these results had been forgotten for almost a century before they were reawakened.

Several new developments have entailed novel interest in this subject. On the one hand, methods of differential algebra lead to a better understanding of the basic problems involved. Instead of dealing with individual equations, the corresponding differential operators are considered as elements of a suitable ring where they generate an ideal. This proceeding is absolutely necessary if partial differential equations and operators are investigated. In particular the concept of a Janet basis for the generators of an ideal and the Loewy decomposition of the ideal corresponding to the given equations are of fundamental importance. In order to apply these results for solving concrete problems, the availability of computer algebra software is indispensable due the enormous size of the calculations usually involved.

Proceeding along these lines, for large classes of linear differential equations – ordinary as well as partial – a fairly complete theory for obtaining its solutions in closed form has been achieved. Whenever feasible, constructive methods for algorithm design are given, and the possible limits of decidability are indicated. This proceeding may serve as a model for dealing with other problems in the area of differential equations.

I am grateful to Dima Grigoriev, Ziming Li, Michael Singer and Sergey Tsarev for numerous discussions and suggestions. Thanks are due to Hans-Heinrich Aumüller for carefully reading the final version of the manuscript, and to Winfried Neun at the Zuse Institut in Berlin for keeping the ALLTYPES website running. The ideal working environment at the Fraunhofer Institut SCAI is gratefully acknowledged.

Sankt Augustin, Germany — Fritz Schwarz
July 2012

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Introduction

Factoring algebraic polynomials into irreducible components of lowest degree has been of basic importance for a long time. By default, a polynomial is called irreducible if it does not factor into components of lower order without enlarging its coefficient domain which is usually the rational number field $\mathbb{Q}$. For univariate polynomials, the relevance of this proceeding for determining the solutions of the corresponding algebraic equation is obvious. In the multivariate case, it is the basis for understanding the structure of the algebraic manifold associated with any such polynomial. Good introductions into the subject may be found, e.g., in the books by Cox et al. [13], Adams and Loustaunau [2] or Greuel and Pfister [20].

It turns out that factoring differential operators and the corresponding differential equations is of fundamental importance as well. Originally the problem of factoring linear *ordinary* differential equations (lode's) has been considered by Beke [3], Schlesinger [58] and Loewy [46]. Thereafter it has been forgotten for almost a century until it was reassumed by Grigoriev [21] and Schwarz [59], where essentially computational and complexity issues were considered; see also Bronstein [6] and van Hoeij [72]. A good survey of these results is given in the book by van der Put and Singer [71].

Due to the non-commutativity of differential operators, some new phenomena appear compared to algebraic polynomials. In the first place, left and right factors have to be distinguished. Correspondingly there are left and right least common multiples and left and right greatest common divisors. For the topics discussed in this monograph, the least common left multiple $Lclm$ and the greatest common right divisor $Gcrd$ are of paramount importance. The term *factor* without further specification means always *right factor*. An operator that does not have any right factor of a certain kind is called *irreducible*, otherwise it is called *reducible*. A more detailed specification of these terms may be found on page 26.

Furthermore, Loewy [46] realized that a new concept is needed beyond reducibility which he baptized *complete reducibility* of a given operator. For ordinary differential operators it may simply be expressed by saying that an operator, or the corresponding differential equation, is completely reducible if it may be represented as the least common left multiple of its irreducible right factors, the greatest common right divisor of which is trivial. This amounts to saying that the solution space of the originally given equation is the direct sum of the solution spaces of its right factors. Applying this concept repeatedly, Loewy obtained a unique decomposition of any ordinary differential operator into completely reducible components of highest possible order.

Starting from Loewy's results, it appears self-evident trying a similar approach for partial differential operators; the goal is to develop more systematic methods for solving linear partial differential equations than those available in the literature [14, 17, 18, 28, 31, 33]. It turns out that for a special class of problems, i.e. those systems of pde's that have a finite-dimensional solution space, Loewy's original theory for decomposing ordinary equations may be generalized almost straightforwardly as has been shown by Li et al. [43].

Factorization problems for general pde's were considered first in the dissertation of Blumberg [5]. He described a factorization of a third-order operator in two variables which appeared to preclude a generalization of Loewy's result beyond the ordinary case. Twenty years after that Miller [49] discussed several factorizations of partial operators in his dissertation. After a long period of inactivity, a lot of interest in this field has arisen in the last two decades, starting with some articles by Grigoriev and Schwarz [22–25]; see also [67]. At this point another analogy with the theory of algebraic polynomials comes into play. Beyond a certain point, further progress is only possible if the underlying scope is broadened and more abstract algebraic concepts are applied. In commutative algebra this means that polynomials are considered as elements of a ring generating certain ideals. Then the results known from ring theory may be applied to the problems involving these polynomials. Only in this way progress in this field has been possible, e.g. generating primary decompositions of ideals and above all the theory of Gröbner bases by Buchberger [7] and its ubiquitous applications.

In the differential case a similar development takes place. Although the solutions of differential equations are the ultimate objects of interest, it is advantageous to consider their left-hand sides as the result of applying a differential operator to a differential indeterminate. Solving an equation means to assign those elements from a suitable function space to the indeterminate such that the full expression vanishes. The differential operators are considered as elements of a non-commutative ring, or as modules over such a ring, so-called $\mathscr{D}$-modules. The extensive theory for these non-commutative rings or modules may then be applied to study their properties. It turns out that only in this way satisfactory results may be obtained. Good introductions into the underlying differential algebra are the books by Kolchin [37], Kaplanski [35] and Coutinho [12], and the article by Buium and Cassidy [8]. Those aspects that are relevant for this monograph may also be found in the book by van der Put and Singer [71].

It turns out that Loewy's original theory for factoring, and more generally decomposing, an ordinary differential operator uniquely into lower-order components may be extended to partial differential operators if this algebraic language is applied. The following observations are of particular relevance. In the first place, the left intersection of two partial differential operators is not necessarily principal;

in general it is an ideal in the ring determined by the originally given operators that may be generated by any number of elements. Secondly, right divisors of an operator need not be principal either; they are overideals of the ideal generated by the given operators that may be generated by any number of elements as well. Taking these observations into account, the objections concerning the generalization of Loewy's results do not apply.

This extended concept of factoring partial differential operators comprises Laplace's method for solving equations of the form $z_{xy} + az_x + bz_y + cz = 0$ as a special case. It consists of an iterative procedure which is described in detail in Chap. 5 of the second volume of Goursat's books on second order differential equations [18], or in Chap. 2 of the second volume of Darboux's series on surfaces [14]; see also [69] and Appendix C of this monograph. By proper substitutions, new equations are generated from the given one until eventually an equation is obtained that may be solved, from the solution of which a partial solution of the originally given equation may be constructed. An equivalent procedure which is also described in the books by Goursat and Darboux consists of generating an additional equation that is *in involution* with the given one. The operators corresponding to the two equations may be considered as generators of an ideal that contains the principal ideal generated by the operator corresponding to the originally given equation. In this way the apparent ad hoc nature of the method of Goursat and Darboux disappears.

These remarks may be summarized as follows. To any given system of linear differential equations, there corresponds the ideal or module generated by the differential operators at their left-hand sides. Factorization means finding a divisor of this ideal. In general there may be more than a single divisor. The complete answer consists of the sublattice in the lattice of left ideals or modules which has the given one as the lowest element. By analogy with Cohn's [11] lattice of factors, it is called the *divisor lattice*. The simplicity of this theory for ordinary differential operators, or for modules of partial differential operators corresponding to systems of pde's with finite dimensional solution space, originates from the fact that they form a sublattice in the respective ring or module.

This monograph deals with linear ordinary differential equations and linear partial differential equations in two independent variables of order not higher than three. Their decompositions are described along the lines described above.

It turns out that many calculations that occur when concrete problems are considered cannot be performed by pencil and paper due to their size, i.e. the complexity of the respective procedure is too large. Therefore computer algebra software has been developed for this purpose. It is available gratis on the website www.alltypes.de after registration. There is a special demo showing the functionality of this software for the applications relevant for this monograph. To start this demo, go to the ALLTYPES website and click on the button `StartALLTYPES`, the interactive alltypes window opens, then submit

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Demo LoewyDecompopsitions;
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The demo starts. If an example has been completed and the result has been displayed in a separate window, the system asks `cont?`; in order to continue with the next problem submit `y`.

However, it should be emphasized that it is by no means assured that any factorization problem may be solved algorithmically. On the contrary, there are severe indications that this may not be the case. If it could be proved that these problems are in general undecidable, it would be interesting to identify special classes of problems for which this is not true as it is the case, e.g. for diophantine equations.

The contents of the individual chapters may be summarized as follows:

Chapter 1. Loewy's Results for Ordinary Differential Equations. The original results of Loewy for decomposing ordinary differential operators are presented. Its application to operators of order 2 and 3 is worked out in detail. It is shown how a nontrivial decomposition may be applied for solving the corresponding equation.

Chapter 2. Rings of Partial Differential Operators. Basic properties of the ring $\mathbb{Q}(x, y)[\partial_x, \partial_y]$ and its left ideals are described, with particular emphasis on the sum and intersection of ideals and how they determine a lattice structure in this ring. This is the foundation for the decompositions described later on.

Chapter 3. Equations with Finite-Dimensional Solution Space. Loewy's theory for linear ode's may be extended in a rather straightforward manner to systems of linear pde's with the property that their general solution involves only constants, i.e. if it is a finite-dimensional vector space. Systems of such equations of order 2 and 3 in two independent variables are discussed.

Chapter 4. Decomposing Second-Order Operators. Principal ideals correspond to individual linear pde's. There is an extensive literature on such equations in the nineteenth and early twentieth century. These results may be obtained systematically, avoiding any ad hoc procedure, by applying the algebraic theory described in Chap. 2.

Chapter 5. Solving Second-Order Equations. The results of the preceding chapter are applied for solving second-order homogeneous equations; at the end inhomogeneous equations are discussed because they are needed later on for solving certain third-order homogeneous equations.

Chapter 6. Decomposing Third-Order Operators. Similar results as in Chap. 4 are obtained for third-order operators.

Chapter 7. Solving Third-Order Equations. The results of the preceding chapter are applied for solving third-order homogeneous equations.

Chapter 8. Conclusions and Summary. The results of this monograph are summarized and several possible extensions are outlined.

Appendix A. Solutions to the exercises are given.

Appendix B. Many algorithms in the main part of the book rely on the solution of Riccati equations and certain generalizations, e.g. Riccati equations of higher order and partial Riccati equations in two independent variables. Their solutions are described in detail. Furthermore, first integrals of first-order ode's are discussed.

Appendix C. Laplace's solution procedure for certain second-order linear pde's in two variables is described.

Appendix D. Lie developed a solution theory for certain second-order linear pde's in two variables based on its symmetries. These results are reviewed; its relation to decomposition properties is briefly discussed,

Appendix E. On the website www.alltypes.de [62] userfunctions are provided that may be applied interactively for performing the voluminous calculations required for many problems in differential algebra. This appendix gives a short introduction to this website.