Inherently Balanced 4R Four-Bar Based Linkages

V. van der Wijk and J.L. Herder

Abstract Synthesis of mechanisms with their center of mass (CoM) at an invariant point on one of the elements is useful for the design of statically balanced and shaking-force balanced mechanisms and manipulators. For this purpose, a kinematic architecture based on a general 4R four-bar linkage is found by applying the method of principal vectors as a linkage together with a similar four-bar linkage. The balance conditions are obtained for an arbitrary mass distribution of each of the elements and a balanced grasper mechanism and a balanced two-degree-of-freedom manipulator are derived as practical examples.

Key words: Center of mass, four-bar linkage, shaking-force balancing, static balancing

1 Introduction

When the center of mass (CoM) of a mechanism (i.e. manipulator, robot) is at a stationary point with respect to the base, the mechanism is shaking-force balanced. This means that for all motion of the mechanism the resultant dynamic forces on the base are zero [7]. Shaking-force balance therefore is important for high speed mechanisms with minimal vibrations of the base. A mechanism with a stationary CoM is also statically balanced with respect to gravity. Then a mechanism can be maintained in each posture with minimal effort [3].

The CoM of a mechanism is stationary if it is an invariant point on at least one of the elements with this point or element being (part of) the base. An elementary way to describe the CoM with respect to its elements is with the method of principal vectors [1]. This method has been applied to derive such inherently balanced linkages considering general mass distributions of all elements [5, 6].

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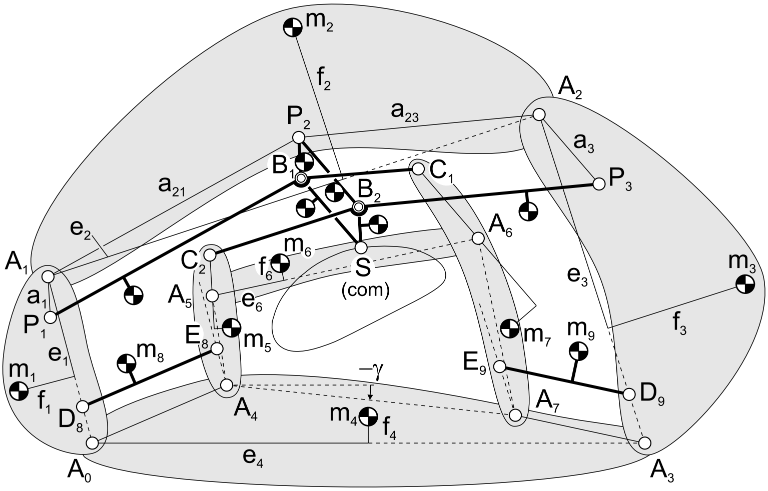

Fig. 1 Center of mass of four-bar linkage $A_0A_1A_2A_3$, similar linkage $A_4A_5A_6A_7$, the principal vector linkage $A_1P_1B_1SB_2P_3A_2P_2$, and links D_8E_8 and D_9E_9 is at invariant point S on links A_5A_6, B_1S, B_2S.

It was shown in [2, 8] that the CoM of a general 4R four-bar linkage also is an invariant point on the coupler link of a similar four-bar linkage moving synchronously. In addition, with the double contour method [4] similar linkages are found for the analysis of the CoM of more complex linkages. This method is based on principal vectors.

The goal of this paper is to combine and to apply the three mentioned approaches as linkages to obtain an inherently balanced 4R four-bar based kinematic architecture from which a wide variety of balanced mechanisms can be derived. First the kinematic architecture is found and subsequently its force balance conditions are derived.

2 Kinematic Architecture with CoM at Invariant Link Point

Figure 1 shows a kinematic architecture of which the CoM of all elements is at invariant link point S. The architecture is based on a general 4R four-bar linkage $A_0A_1A_2A_3$. S is a point on the coupler link A_5A_6 of a similar four-bar linkage $A_4A_5A_6A_7$ [2, 8] and is also a point on a linkage of parallelograms $A_1P_1B_1P_2$, $P_2B_1SB_2$, and $A_2P_2B_2P_3$ of which the (principal) dimensions a_1, a_{21}, a_{23}, and a_3 are defined by the principal points P_i [1, 6]. In addition to their coinciding joint at S, these two linkages can be linked by parallelograms $SA_6C_1B_1$ and $SB_2C_2A_5$ of which B_1C_1 and B_2C_2 are part of elements $P_1B_1C_1$ and $P_3B_2C_2$, respectively. These parallelograms are found with the double contour method of which one solution could be

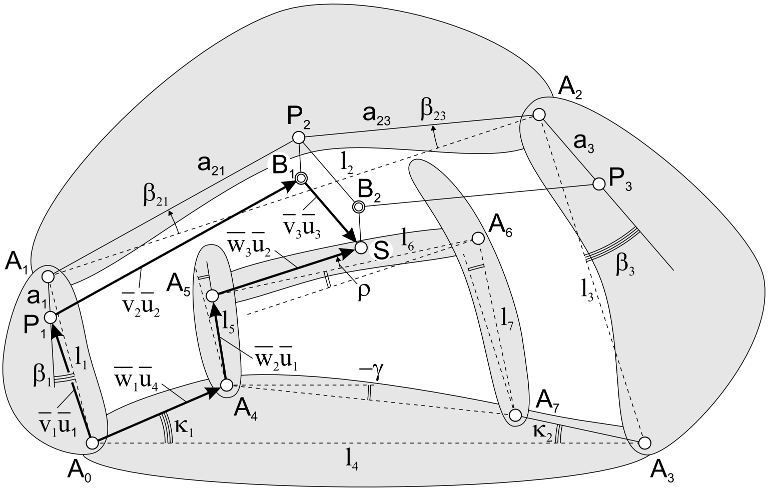

Fig. 2 By describing the CoM of all elements S along the principal vectors $\overline{v}_i$, the conditions $\overline{w}_i$ for which S is a coupler point of the similar linkage are found.

the linkage $P_1C_1A_6A_7A_4A_5$ [4]. Other possible links are D_8E_8 and D_9E_9 which are parallel to lines A_0A_4 and A_3A_7, respectively.

The conditions for which the CoM of all elements S is a coupler point of similar linkage $A_4A_5A_6A_7$ can be written as a function of the principal dimensions. To obtain the conditions, the position of S can be written with complex vectors as illustrated in Fig. 2 with

$$\overline{A_0S} = \overline{v}_1\overline{u}_1 + \overline{v}_2\overline{u}_2 + \overline{v}_3\overline{u}_3 = \overline{w}_1\overline{u}_4 + \overline{w}_2\overline{u}_1 + \overline{w}_3\overline{u}_2 \tag{1}$$

Vectors $\overline{u}_i$ are the time dependent vectors describing the relative positions of joints A_0, A_1, A_2, and A_3. Constant vectors $\overline{v}_i$ are the principal vectors describing the principal points P_i within each element. Vectors $\overline{w}_i$ are also constant and determine the size and pose of the similar linkage. These vectors can be written as

$$\overline{v}_1 = \frac{l_1 - a_1\cos\beta_1}{l_1} + \frac{a_1\sin\beta_1}{l_1}i \quad \overline{v}_2 = \frac{a_{21}\cos\beta_{21}}{l_2} + \frac{a_{21}\sin\beta_{21}}{l_2}i \quad \overline{v}_3 = \frac{a_3\cos\beta_3}{l_3} + \frac{a_3\sin\beta_3}{l_3}i$$
$$\overline{w}_1 = \kappa_1^R + \kappa_1^I i \qquad \overline{w}_2 = \gamma^R + \gamma^I i \qquad \overline{w}_3 = \overline{w}_2(\rho^R + \rho^I i)$$

with link lengths l_i and angles β_{ij} to describe the orientation of the principal dimensions a_{ij} with respect to the line connecting the joints. Vectors $\overline{w}_i$ are written with the real and imaginary parts of the orientation κ_1 of A_0A_4, the orientation γ of the similar linkage and the orientation ρ of A_5S with respect to A_5A_6. With the substitution of the loop equation $\overline{u}_1 + \overline{u}_2 + \overline{u}_3 = \overline{u}_4$ for $\overline{u}_4$, Eq. 1 can be rewritten as

$$(\overline{v}_1 - \overline{w}_1 - \overline{w}_2)\overline{u}_1 + (\overline{v}_2 - \overline{w}_1 - \overline{w}_3)\overline{u}_2 + (\overline{v}_3 - \overline{w}_1)\overline{u}_3 = 0 \tag{2}$$

and after substitution of the constant vectors it is written as

$$
\begin{aligned}
&\{(1-\frac{a_1\cos\beta_1}{l_1}-\kappa_1^R-\gamma^R)+(\frac{a_1\sin\beta_1}{l_1}-\kappa_1^I-\gamma^I)i\}\overline{u}_1+\\
&\{(\frac{a_{21}\cos\beta_{21}}{l_2}-\kappa_1^R-\gamma^R\rho^R+\gamma^I\rho^I)+(\frac{a_{21}\sin\beta_{21}}{l_2}-\kappa_1^I-\gamma^R\rho^I-\gamma^I\rho^R)i\}\overline{u}_2+\\
&\{(\frac{a_3\cos\beta_3}{l_3}-\kappa_1^R)+(\frac{a_3\sin\beta_3}{l_3}-\kappa_1^I)i\}\overline{u}_3=0
\end{aligned}
\tag{3}
$$

Since generally this equation must hold for all motion, i.e. for all independent values of $\overline{u}_i$ not being restricted to the relative motions of the 4R four-bar linkage, each of the six terms needs to be zero. The terms for $\overline{u}_3$ are zero when

$$
\kappa_1^R=\frac{a_3\cos\beta_3}{l_3},\quad \kappa_1^I=\frac{a_3\sin\beta_3}{l_3},\quad \kappa_1=\tan^{-1}\left(\frac{\kappa_1^I}{\kappa_1^R}\right)=\beta_3 \tag{4}
$$

from which κ_1 is found to be equal to β_3 as was shown in another way also in [8]. Subsequently, γ is found from the terms for $\overline{u}_1$ being zero resulting in

$$
\begin{aligned}
\gamma^R&=1-\frac{a_1\cos\beta_1}{l_1}-\frac{a_3\cos\beta_3}{l_3},\gamma^I=\frac{a_1\sin\beta_1}{l_1}-\frac{a_3\sin\beta_3}{l_3}\\
\gamma&=\tan^{-1}\left(\frac{\gamma^I}{\gamma^R}\right)=\tan^{-1}\left(\frac{\frac{a_1}{l_1}\sin\beta_1-\frac{a_3}{l_3}\sin\beta_3}{1-\frac{a_1}{l_1}\cos\beta_1-\frac{a_3}{l_3}\cos\beta_3}\right)\\
\eta&=\sqrt{(\gamma^R)^2+(\gamma^I)^2}=\sqrt{(1-\frac{a_1}{l_1}\cos\beta_1-\frac{a_3}{l_3}\cos\beta_3)^2+(\frac{a_1}{l_1}\sin\beta_1-\frac{a_3}{l_3}\sin\beta_3)^2}
\end{aligned}
\tag{5}
$$

η is the scaling factor of the similar linkage and equals $\eta=l_5/l_1=l_6/l_2=l_7/l_3=\|\overline{A_4A_7}\|/l_4$. ρ is found when the terms for $\overline{u}_2$ are zero, which is for

$$
\begin{aligned}
\rho^R&=\frac{(1-\frac{a_1\cos\beta_1}{l_1}-\frac{a_3\cos\beta_3}{l_3})(\frac{a_{21}\cos\beta_{21}}{l_2}-\frac{a_3\cos\beta_3}{l_3})-(\frac{a_1\sin\beta_1}{l_1}-\frac{a_3\sin\beta_3}{l_3})(\frac{a_{21}\sin\beta_{21}}{l_2}-\frac{a_3\sin\beta_3}{l_3})}{(\frac{a_1\sin\beta_1}{l_1}-\frac{a_3\sin\beta_3}{l_3})^2+(1-\frac{a_1\cos\beta_1}{l_1}-\frac{a_3\cos\beta_3}{l_3})^2}\\
\rho^I&=\frac{(1-\frac{a_1\cos\beta_1}{l_1}-\frac{a_3\cos\beta_3}{l_3})(\frac{a_{21}\sin\beta_{21}}{l_2}-\frac{a_3\sin\beta_3}{l_3})-(\frac{a_1\sin\beta_1}{l_1}-\frac{a_3\sin\beta_3}{l_3})(\frac{a_{21}\cos\beta_{21}}{l_2}-\frac{a_3\cos\beta_3}{l_3})}{(\frac{a_1\sin\beta_1}{l_1}-\frac{a_3\sin\beta_3}{l_3})^2+(1-\frac{a_1\cos\beta_1}{l_1}-\frac{a_3\cos\beta_3}{l_3})^2}\\
\rho&=\tan^{-1}\left(\frac{\rho^I}{\rho^R}\right),\quad \tau=\sqrt{(\rho^R)^2+(\rho^I)^2}
\end{aligned}
\tag{6}
$$

From polygon $A_0A_3A_7A_4$ and with κ_1 and γ known, angle κ_2 can be derived as

$$
\kappa_2^R=1-\kappa_1^R-\gamma^R=\frac{a_1\cos\beta_1}{l_1}\quad \kappa_2^I=\kappa_1^I+\gamma^I=\frac{a_1\sin\beta_1}{l_1}\quad \kappa_2=\tan^{-1}\left(\frac{\kappa_2^I}{\kappa_2^R}\right)=\beta_1 \tag{7}
$$

Herewith the similar linkage has been fully defined with parameters based on the principal dimensions and the dimensions of the four-bar linkage $A_0A_1A_2A_3$ solely.

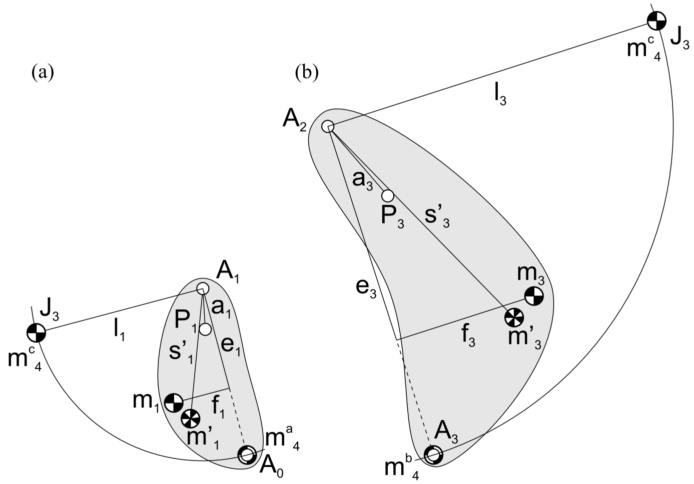

Fig. 3 The mass of link 4 is distributed on the other three links. Shown are (a) link 1 and (b) link 3 on which a mass m_4^a at A_0, a mass m_4^b at A_3, and a mass m_4^c at J_3 on each of the links are modeled.

3 Force Balance Conditions

To have S be the CoM of the complete kinematic architecture of Fig. 1, the principal dimensions a_1, a_{21}, a_{23}, and a_3 need to be calculated from the mass of each element and their positions. Since the principal dimensions are defined with respect to three elements of the four-bar linkage $A_0A_1A_2A_3$, the first step is to distribute the mass of the fourth element, m_4 of link 4, equivalently to the other elements. For links 1 and 3 this can be done by modeling a mass $m_4^a = m_4(1 - e_4/l_4)$ at A_0, a mass $m_4^b = m_4 e_4/l_4$ at A_3 and a mass $m_4^c = m_4 f_4/l_4$ at positions J_3 on both links 1 and 3 as indicated in Figs. 3a and b, respectively. m_4^a, m_4^b, and m_4^c also need to be modeled on link 2, which will be shown later on. For the analysis of link 1 it now has a total mass $m_1' = m_1 + m_4^a + m_4^c$ centered at s_1' from A_1 which is de CoM of the three masses. Similarly, for the analysis of link 3 it has a total mass $m_3' = m_3 + m_4^b + m_4^c$ centered at s_3' from A_2.

To include the masses of D_8E_8 and D_9E_9, also they can be distributed among the other elements in a similar way as with m_4. Unfortunately this paper leaves too little space to present this distribution in detail, for which they are not considered here.

With m_4 distributed, link 4 can be taken out resulting in the linkage of Fig. 4. This linkage is an extended composition of the linkage investigated in [5, 6] and the same method can be applied here to derive the principal dimensions. This means that P_i can be found independently from one another with linear momentum equations.

To find P_1, the linear momentum of the linkage for $\dot{\theta}_2 = \dot{\theta}_3 = 0$ (links 2 and 3 being immovable) can be written with respect to reference frame x_1y_1 aligned with a_1 as indicated in Fig. 4, to be equal to the total mass m_{tot} moving at S as

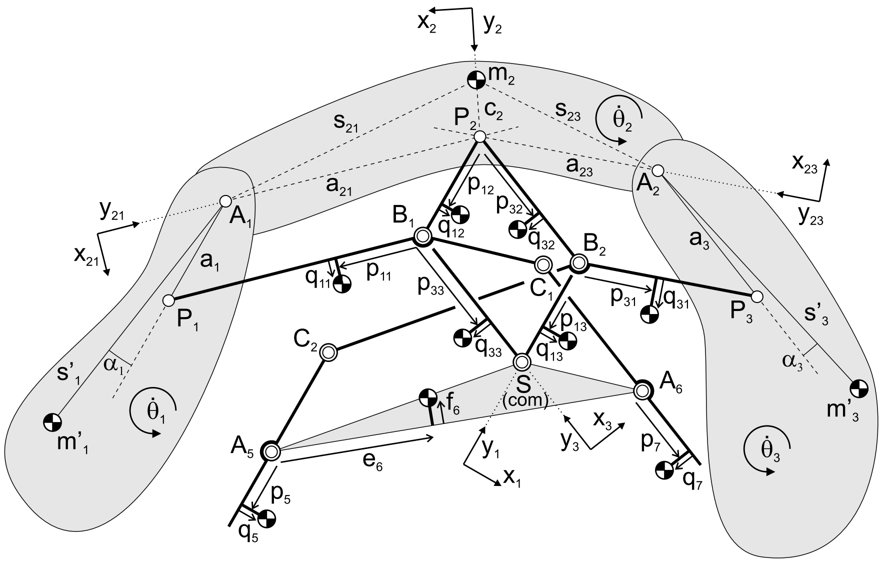

Fig. 4 Mass model of the kinematic architecture after distribution of the mass of link 4 to link 1 and link 2, and without considering links D_8E_8 and D_9E_9.

$$\frac{\overline{L}_1}{\dot{\theta}_1} = \begin{bmatrix} m_1' s_1' \cos\alpha_1 + (m_5+m_6+m_7+m_{11}+m_{33})a_1 + m_{12}p_{12} + m_{13}p_{13} + m_5 p_5 \\ m_1' s_1' \sin\alpha_1 - m_{12}q_{12} - m_{13}q_{13} - m_5 q_5 \end{bmatrix}$$
$$= \begin{bmatrix} m_{tot} a_1 \\ 0 \end{bmatrix} \tag{8}$$

with $m_{tot} = m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_{11}+m_{12}+m_{13}+m_{31}+m_{32}+m_{33}$. From these equations a_1 and α_1 are obtained with a_1 resulting in

$$a_1 = \frac{\sqrt{m_1'^2 s_1'^2 - (m_{12}q_{12}+m_{13}q_{13}+m_5q_5)^2} + m_{12}p_{12} + m_{13}p_{13} + m_5p_5}{m_{tot} - m_5 - m_6 - m_7 - m_{11} - m_{33}} \tag{9}$$

P_3 is found similarly by writing the linear momentum of the linkage for $\dot{\theta}_1 = \dot{\theta}_2 = 0$ with respect to frame x_3y_3 aligned with a_3 to be equal to m_{tot} moving at S as

$$\frac{\overline{L}_3}{\dot{\theta}_3} = \begin{bmatrix} m_3' s_3' \cos\alpha_3 + (m_5+m_6+m_7+m_{31}+m_{13})a_3 + m_{32}p_{32} + m_{33}p_{33} + m_7 p_7 \\ m_3' s_3' \sin\alpha_3 - m_{32}q_{32} - m_{33}q_{33} - m_7 q_7 \end{bmatrix}$$
$$= \begin{bmatrix} m_{tot} a_3 \\ 0 \end{bmatrix} \tag{10}$$

From these equations a_3 and α_3 are obtained with a_3 resulting in

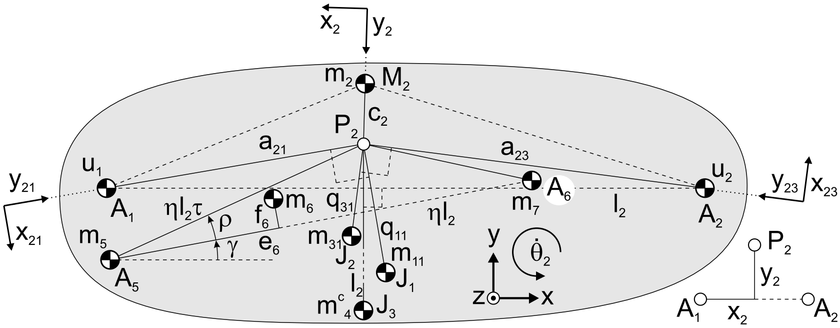

Fig. 5 Equivalent Linear Momentum System (ELMS) for $\dot{\theta}_2$ when $\dot{\theta}_1 = \dot{\theta}_3 = 0$ for which the masses of the moving elements are projected on link 2. P_2 is found as being the CoM of the ELMS.

$$a_3 = \frac{\sqrt{m_3'^2 s_3'^2 - (m_{32}q_{32} + m_{33}q_{33} + m_7 q_7)^2} + m_{32}p_{32} + m_{33}p_{33} + m_7 p_7}{m_{tot} - m_5 - m_6 - m_7 - m_{31} - m_{13}} \tag{11}$$

As in [5, 6], P_2 can be found by using an Equivalent Linear Momentum System (ELMS). This means that the mass of the moving elements for $\dot{\theta}_1 = \dot{\theta}_3 = 0$ (immovable parallelogram $P_2B_1SB_2$, link 2 rotating about P_2) are modeled on link 2 such that their linear momentum is equal to one of the reference frames $x_{21}y_{21}$, $x_{23}y_{23}$, and x_2y_2. Figure 5 shows the resulting ELMS with masses $u_1 = m_1 + m_4^a + m_{11}p_{11}/a_{21}$ and $u_2 = m_3 + m_4^b + m_{31}p_{31}/a_{23}$ at A_1 and A_2, respectively, masses m_5 and m_6 at A_5 and A_6, respectively, mass m_6 at distances e_6 and f_6 with respect to line A_5A_6, and masses m_{11} and m_{31} also placed at J_1 and J_3, respectively. u_1 and u_2 contain the distributed masses m_4^a and m_4^b of link 4 on link 2 and a mass m_4^c is placed at J_3. J_3 is located at a distance l_2 from P_2 normal to line A_1A_2 in indicated direction.

P_2 is found as being the CoM of the ELMS. With P_2 being located at a distance x_2 from A_1 along A_1A_2 and y_2 normal to A_1A_2 as indicated in Fig. 5, P_2 is found by solving the linear momentum equations of the ELMS

$$\begin{aligned}\frac{\overline{L}_2}{\dot{\theta}_2} = {} & u_1 \begin{bmatrix} y_2 \\ -x_2 \end{bmatrix} + v_1 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + u_2 \begin{bmatrix} y_2 \\ -(x_2 - l_2) \end{bmatrix} - v_2 \begin{bmatrix} x_2 - l_2 \\ y_2 \end{bmatrix} + m_2 \begin{bmatrix} y_2 - f_2 \\ -(x_2 - e_2) \end{bmatrix} + \\ & m_5 \eta l_2 \tau \begin{bmatrix} \sin(\gamma+\rho) \\ -\cos(\gamma+\rho) \end{bmatrix} + m_7 \eta l_2 \begin{bmatrix} \tau \sin(\gamma+\rho) - \sin(\gamma) \\ -\tau\cos(\gamma+\rho) + \cos(\gamma) \end{bmatrix} + \\ & m_6 \begin{bmatrix} \eta l_2 \tau \sin(\gamma+\rho) - e_6 \sin(\gamma) - f_6 \cos(\gamma) \\ -\eta l_2 \tau \cos(\gamma+\rho) + e_6 \cos(\gamma) - f_6 \sin(\gamma) \end{bmatrix} + m_4^c \begin{bmatrix} -l_2 \\ 0 \end{bmatrix} = \overline{0}\end{aligned} \tag{12}$$

with $v_1 = m_{11}q_{11}/a_{21}$ and $v_2 = m_{31}q_{31}/a_{23}$. No algebraic solution for P_2 was found, for which the equations have to be solved numerically. The principal dimensions defining P_2 are calculated as $a_{21} = \sqrt{x_2^2 + y_2^2}$ and $a_{23} = \sqrt{(l_2 - x_2)^2 + y_2^2}$ with which all principal dimensions are obtained.

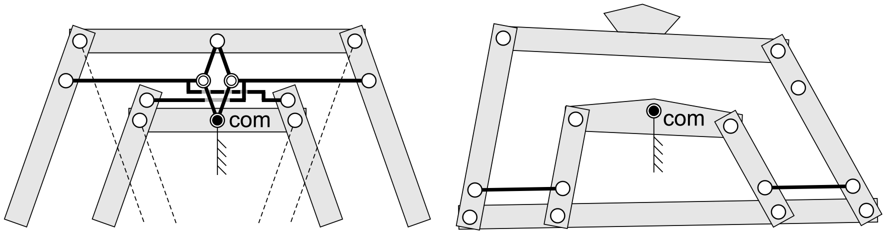

Fig. 6 Two examples of balanced mechanisms derived from Fig. 1 with the CoM being a joint with the base: (a) double grasper mechanism, (b) two-degree-of-freedom balanced manipulator.

4 Conclusion

An inherently balanced kinematic architecture of which the CoM is an invariant link point has been composed based on a general 4R four-bar linkage and by applying the method of principal vectors as a linkage together with a similar four-bar linkage. The conditions for the kinematic architecture were derived as a function of the principal dimensions. The principal dimensions were calculated from a generally defined mass and mass location of each element, resulting in the general force balance conditions of the kinematic architecture. Figure 6 shows examples of possible balanced devices that can be derived from the kinematic architecture such as a double grasper mechanism and a 2-DoF manipulator with end-effector considering an arbitrary mass distribution of each of the shown elements.

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