SET OPERATOR DECOMPOSITION AND CONDITIONALLY TRANSLATION INVARIANT ELEMENTARY OPERATORS

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Abstract. In the first part, we recall the axiomatic definition of the elementary morphological operators (dilations, erosions, anti–dilations and anti–erosions) and their characterization in the case of Boolean lattices. This characterization is used to derive the set operator decompositions from the general decompositions of operators between complete lattices. In the second part, we define the notions of "conditionally translation invariant" (c.t.i.) and of "locally c.t.i." elementary operators. These operators are those usually implemented on digital computers. We show how any c.t.i. elementary operator can be decomposed in terms of locally ones.

Key words: set operator decomposition, conditionally translation invariant elementary operator, Boolean lattice.

1. Introduction

In Image Processing, to work with translation invariant (t.i.) operators we must assume that the image domain has a torus structure. Unfortunately, this assumption is not acceptable in many practical applications. For this reason, the linear operators or the morphological elementary operators commonly used in Image Processing, behave like t.i. operators only in the "central" area of the image domain.

In the first part of the paper, we recall the axiomatic definitions of the four classes of morphological elementary operators. When the domain or the range of these elementary operators is a Boolean lattice, we can characterize the pairs of elementary operators that form Galois connections. This characterization is useful for two reasons: on one hand, it gives the clue for the link between any operator and its left or right kernel, and, on the other hand, it can be used to characterize the elementary set operators in terms of structuring functions.

Based on theses results, we derive the set operator decompositions from the general operator decompositions introduced by Banon and Barrera (1993).

In the second part of the paper, we introduce the formal definition of the so–called conditionally translation invariant (c.t.i.) elementary operators. This definition corresponds to what is usually implemented on digital computers and it is based on the notion of c.t.i. structuring function. We present the collection of all the structuring elements that characterize the c.t.i. structuring functions.

Unfortunately, the c.t.i. elementary operators are not sufficient to represent any operator. For this reason, we introduce the notion of locally c.t.i. elementary operator and we give a constructive decomposition of any c.t.i. elementary operator. The decomposition of a c.t.i. dilation is given as an example to illustrate the theory.

2. Axiomatic Definition and Characterization of the Morphological Elementary Operators

Let $(\mathcal{L}, \leq)$, or simply $\mathcal{L}$, be a complete lattice (Birkhoff, 1967). We denote by $\bigvee$, $\bigwedge$ the union and intersection in $\mathcal{L}$. The *dual complete lattice of* $\mathcal{L}$, $(\mathcal{L}, \geq)$, is denoted $\widetilde{\mathcal{L}}$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two complete lattices. The class $\mathcal{L}_2^{\mathcal{L}_1}$ of the mappings ψ from $\mathcal{L}_1$ to $\mathcal{L}_2$, called *operators*, equipped with the *punctual ordering* $\underset{(\mathcal{L}_2)}{\leq}$

$$\psi \underset{(\mathcal{L}_2)}{\leq} \psi' \Leftrightarrow \psi(X) \underset{\mathcal{L}_2}{\leq} \psi'(X) \;\; (X \in \mathcal{L}_1)$$

is a complete lattice.

We define the morphological elementary operators axiomatically. The *dilations* (resp. *erosions*) from $\mathcal{L}_1$ to $\mathcal{L}_2$ are the operators in $\mathcal{L}_2^{\mathcal{L}_1}$ which commute with union (resp. intersection). The *anti–dilations* (resp. *anti–erosions*) from $\mathcal{L}_1$ to $\mathcal{L}_2$ are the dilations (resp. erosion) from $\mathcal{L}_1$ to $\widetilde{\mathcal{L}}_2$ (Banon & Barrera, 1993).

We will denote by $\Delta(\mathcal{L}_1, \mathcal{L}_2)$, $\mathrm{E}(\mathcal{L}_1, \mathcal{L}_2)$, $\Delta^{a}(\mathcal{L}_1, \mathcal{L}_2)$ and $\mathrm{E}^{a}(\mathcal{L}_1, \mathcal{L}_2)$, respectively, the class of dilations, erosions, anti–dilations and anti–erosions from $\mathcal{L}_1$ to $\mathcal{L}_2$. We know (Banon & Barrera, 1993a) that these classes are complete lattices. Consequently, the Galois connections (α, β) between $\mathcal{L}_1$ and $\mathcal{L}_2$ (Birkhoff, 1967) form a complete lattice with respect the the partial ordering

$$(\alpha, \beta) \leq (\alpha', \beta') \Leftrightarrow \alpha \underset{(\mathcal{L}_2)}{\leq} \alpha' \text{ and } \beta \underset{(\mathcal{L}_1)}{\leq} \beta'.$$

If (α, β) is a Galois connection, then β is the *companion operator of* α and similarly α is the *companion operator of* β.

Let E be a non empty set, then $(\mathcal{P}(E), \subset)$, or simply $\mathcal{P}(E)$ or $\mathcal{P}$, the collection of all parts of E equipped with the inclusion $\subset$, is a complete Boolean lattice. We denote by X^{c} the set complement of a subset X of E.

Let $(\mathcal{L}, \leq)$ be a complete lattice. The set $\mathcal{L}^E$ of the functions a from E to $\mathcal{L}$ equipped with the punctual ordering $\underset{(\mathcal{L})}{\leq}$ is a complete lattice.

We now recall Corollary 4 of Achache (1982). A similar result (with $\mathcal{L} = \mathcal{P}(E)$) is in Serra (1988, Section 2.2).

Proposition 1 – The mapping $a \mapsto (\alpha, \beta)$ from the complete lattice $\mathcal{L}^E$ to the complete lattice of Galois connections between $\mathcal{L}$ and $(\mathcal{P}(E), \subset)$ defined by

$$\alpha(X) = \{y \in E : X \underset{\mathcal{L}}{\leq} a(y)\} \;\; (X \in \mathcal{L}) \quad \text{and} \quad \beta(Y) = \bigwedge_{y \in Y} a(y) \;\; (Y \in \mathcal{P}(E))$$

is a lattice isomorphism. Its inverse $(\alpha, \beta) \mapsto a$ is given by $a(y) = \beta(\{y\})$ $(y \in E)$. □

We know (Achache, 1982, Lemma 1) that $\alpha \in \Delta^{a}(\mathcal{L}, \mathcal{P})$ and $\beta \in \Delta^{a}(\mathcal{P}, \mathcal{L})$. From Proposition 1 we can derive the following proposition.

Proposition 2 – The mapping $a \mapsto (\alpha, \beta)$ from the complete lattice $\mathcal{L}^E$ to the complete lattice of Galois connections between $\mathcal{L}$ and $(\mathcal{P}(E), \supset)$ defined by

$$\alpha(X) = \{y \in E : X \underset{\mathcal{L}}{\leq} a(y)\}^c \ (X \in \mathcal{L}) \quad \text{and} \quad \beta(Y) = \bigwedge_{y \in Y^c} a(y) \ (Y \in \mathcal{P}(E))$$

is a lattice isomorphism. Its inverse $(\alpha, \beta) \mapsto a$ is given by $a(y) = \beta(\{y\}^c)$ $(y \in E)$. □

In this case, $\alpha \in \Delta(\mathcal{L}, \mathcal{P})$ and $\beta \in \mathrm{E}(\mathcal{P}, \mathcal{L})$. In Propositions 1 and 2 the function a is called the *structuring function of the elementary operators α and β.*

From Propositions 1 and 2 we can derive the elementary operators characterization given in Table 1 and the next corollary where $\mathcal{P}_2$ stands for $\mathcal{P}(E_2)$.

TABLE 1
Elementary operators characterization.

identifying $\mathcal{L}$ to	in Prop.	leads to	and	with
$\widetilde{\mathcal{L}}_1$	1	$\alpha \in \mathrm{E}(\mathcal{L}_1, \mathcal{P}_2)$ $\beta \in \Delta(\mathcal{P}_2, \mathcal{L}_1)$	$\alpha = \varepsilon_a$ $\beta = \delta_a$	$\varepsilon_a(X) = \{y \in E_2 : a(y) \leq X\}$ $\delta_a(Y) = \bigvee_{y \in Y} a(y)$
$\mathcal{L}_1$	1	$\alpha \in \Delta^a(\mathcal{L}_1, \mathcal{P}_2)$ $\beta \in \Delta^a(\mathcal{P}_2, \mathcal{L}_1)$	$\alpha = {}_b\delta^a$ $\beta = \delta^a{}_b$	${}_b\delta^a(X) = \{y \in E_2 : X \leq b(y)\}$ $\delta^a{}_b(Y) = \bigwedge_{y \in Y} b(y)$
$\widetilde{\mathcal{L}}_1$	2	$\alpha \in \mathrm{E}^a(\mathcal{L}_1, \mathcal{P}_2)$ $\beta \in \mathrm{E}^a(\mathcal{P}_2, \mathcal{L}_1)$	$\alpha = \varepsilon^a{}_a$ $\beta = {}_a\varepsilon^a$	$\varepsilon^a{}_a(X) = \{y \in E_2 : a(y) \leq X\}^c$ ${}_a\varepsilon^a(Y) = \bigvee_{y \in Y^c} a(y)$
$\mathcal{L}_1$	2	$\alpha \in \Delta(\mathcal{L}_1, \mathcal{P}_2)$ $\beta \in \mathrm{E}(\mathcal{P}_2, \mathcal{L}_1)$	$\alpha = {}_b\delta$ $\beta = {}_b\varepsilon$	${}_b\delta(X) = \{y \in E_2 : X \leq b(y)\}^c$ ${}_b\varepsilon(Y) = \bigwedge_{y \in Y^c} b(y)$

Corollary 3 – Let a and b be two functions from E_2 to $\mathcal{L}_1$. Then we have:
(1) if a is the structuring function of $\delta \in \Delta(\mathcal{P}_2, \mathcal{L}_1)$ (i.e., $a(y) = \delta(\{y\})$ $(y \in E_2)$) then its companion erosion is ε_a;
(2) if b is the structuring function of $\delta^a \in \Delta^a(\mathcal{P}_2, \mathcal{L}_1)$ (i.e., $b(y) = \delta^a(\{y\})$ $(y \in E_2)$) then its companion anti–dilation is ${}_b\delta^a$;
(3) if a is the structuring function of $\varepsilon^a \in \mathrm{E}^a(\mathcal{P}_2, \mathcal{L}_1)$ (i.e., $a(y) = \varepsilon^a(\{y\}^c)$ $(y \in E_2)$) then its companion anti–erosion is $\varepsilon^a{}_a$;
(4) if b is the structuring function of $\varepsilon \in \mathrm{E}(\mathcal{P}_2, \mathcal{L}_1)$ (i.e., $b(y) = \varepsilon(\{y\}^c)$ $(y \in E_2)$) then its companion dilation is ${}_b\delta$. □

3. Operator Decomposition in terms of Elementary Operators

In order to specialize to set operators the general decomposition theorem (Banon & Barrera, 1993) we need to derive one more corollary from Proposition 1. In this section $\mathcal{P}_1$ and $\mathcal{P}_2$ stands, respectively, for $\mathcal{P}(E_1)$ and $\mathcal{P}(E_2)$. Let ψ be an operator from $\mathcal{P}_1$ to $\mathcal{P}_2$. We recall (Banon & Barrera, 1993) that the mappings $\cdot\mathcal{K}(\psi)$ and $\mathcal{K}\cdot(\psi)$ from $\mathcal{P}_2$ to $\mathcal{P}(\mathcal{P}_1)$ given by, for any $Y \in \mathcal{P}_2$,

$$\cdot\mathcal{K}(\psi)(Y) = \{X \in \mathcal{P}_1 : Y \subset \psi(X)\} \text{ and } \mathcal{K}\cdot(\psi)(Y) = \{X \in \mathcal{P}_1 : \psi(X) \subset Y\}$$

are called, respectively, the *left* and *right kernel of ψ*.

Corollary 4 – The left and right kernel of a set operator from $\mathcal{P}_1$ to $\mathcal{P}_2$ are, respectively, an anti–dilation and an erosion from $\mathcal{P}_2$ to $\mathcal{P}(\mathcal{P}_1)$. □

Proof – For the left kernel, the result follows from Proposition 1 by identifying E, $\mathcal{L}$, a and α to, respectively, $\mathcal{P}_1$, $\mathcal{P}_2$, ψ and $\cdot\,\mathcal{K}(\psi)$. For the right kernel, the result follows from Proposition 1 by identifying E, $\mathcal{L}$, a and α to, respectively, $\mathcal{P}_1$, $\widetilde{\mathcal{P}}_2$, ψ and $\mathcal{K}\cdot(\psi)$. □

Let $\alpha, \beta \in {\mathcal{P}_1}^{\mathcal{P}_2}$ and let $[\alpha, \beta]$ be the *interval function* from $\mathcal{P}_2$ to $\mathcal{P}(\mathcal{P}_1)$ *with extremities* α *and* β (Banon & Barrera, 1993).

Lemma 5 – Let ψ be a set operator from $\mathcal{P}_1$ to $\mathcal{P}_2$.

(1) If $\alpha \in \Delta(\mathcal{P}_2, \mathcal{P}_1)$ and $\beta \in \Delta^{a}(\mathcal{P}_2, \mathcal{P}_1)$ then, for any $Y \in \mathcal{P}_2$,

$$[\alpha,\beta](\{y\}) \subset \cdot\,\mathcal{K}(\psi)(\{y\}) \ (y \in Y) \ \Rightarrow [\alpha,\beta](Y) \subset \cdot\,\mathcal{K}(\psi)(Y).$$

(2) If $\alpha \in \mathrm{E}^{a}(\mathcal{P}_2, \mathcal{P}_1)$ and $\beta \in \mathrm{E}(\mathcal{P}_2, \mathcal{P}_1)$ then, for any $Y \in \mathcal{P}_2$,

$$[\alpha,\beta](\{y\}^c) \subset \mathcal{K}\cdot(\psi)(\{y\}^c) \ (y \in Y^c) \ \Rightarrow [\alpha,\beta](Y) \subset \mathcal{K}\cdot(\psi)(Y). \qquad \square$$

Proof – Let us prove part (1). For any $Y \in \mathcal{P}_2$,

$$\begin{aligned}
\text{TRUE} &\Leftrightarrow [\alpha,\beta](Y) \subset [\alpha,\beta](\{y\}) \ (y \in Y) && (\alpha \text{ is isotone and } \beta \text{ is antitone})\\
&\Leftrightarrow [\alpha,\beta](Y) \subset \bigcap_{y \in Y} [\alpha,\beta](\{y\}) && (\text{property of the intersection})\\
&\Rightarrow [\alpha,\beta](Y) \subset \bigcap_{y \in Y} \cdot\,\mathcal{K}(\psi)(\{y\}) && ([\alpha,\beta](\{y\}) \subset \cdot\,\mathcal{K}(\psi)(\{y\}) \ (y \in Y))\\
&\Leftrightarrow [\alpha,\beta](Y) \subset \cdot\,\mathcal{K}(\psi)(\bigcup_{y \in Y} \{y\}) && (\text{by Corollary 4, } \cdot\,\mathcal{K}(\psi) \text{ is an anti–dilation})\\
&\Leftrightarrow [\alpha,\beta](Y) \subset \cdot\,\mathcal{K}(\psi)(Y). && (Y \text{ representation by singletons})
\end{aligned}$$

The proof of part (2) is similar to the proof of part (1). □

Let AB be the set defined by

$$\mathrm{AB} = \{(a,b) \in {\mathcal{P}_1}^{E_2} \times {\mathcal{P}_1}^{E_2} : \forall y \in E_2, \ (a(y) \subset b(y)) \text{ or } (a(y) = E_1 \text{ and } b(y) = \emptyset)\}$$

Let $a, b \in {\mathcal{P}_1}^{E_2}$. We denote by $[a,b]$ the *interval function* from E_2 to $\mathcal{P}(\mathcal{P}_1)$ *with extremities* a *and* b. We denote the punctual ordering $\underset{(\subset)}{\leq}$ on $\mathcal{P}(\mathcal{P}_1)^{E_2}$ simply $\leq$.

Theorem 6 – Any operator $\psi \in {\mathcal{P}_2}^{\mathcal{P}_1}$ can be decomposed in terms of a set of sup–generating or inf–generating operators and the constructive decompositions are

$$\psi = \bigvee_{(a,b) \in \mathrm{AB} \text{ and } [a,b] \leq \cdot\,\mathcal{K}(\psi)} (\varepsilon_a \wedge {}_b\delta^{a})$$

where $\cdot\,\mathcal{K}(\psi)(y) = \{X \in \mathcal{P}_1 : y \in \psi(X)\}$, for any $y \in E_2$, and

$$\psi = \bigwedge_{(a,b) \in \mathrm{AB} \text{ and } [a,b] \leq \mathcal{K}\cdot(\psi)} (\varepsilon^{a}{}_a \vee {}_b\delta)$$

where $\mathcal{K}\cdot(\psi)(y) = \{X \in \mathcal{P}_1 : y \notin \psi(X)\}$, for any $y \in E_2$. □

Proof – We can make a direct proof or, as we do below, derive the result from the general decomposition theorem of Banon & Barrera (1993). For any ψ from $\mathcal{P}_1$ to $\mathcal{P}_2$,

$$\psi = \bigvee_{(\alpha,\beta)\,\in\,\Delta\Delta^{a} \text{ and } [\alpha,\beta]\,\leq\,\cdot\,\mathcal{K}(\psi)} (\overline{\alpha\mathbf{I}} \wedge \overline{\mathbf{O}\beta})$$

(Theorem 6.1 of Banon & Barrera (1993a) or Theorem 1 of Banon & Barrera (1993b))

$$= \bigvee_{(\alpha,\beta)\,\in\,\Delta\Delta^{a} \text{ and } ([\alpha,\beta](y)\,\subset\,\cdot\,\mathcal{K}(\psi)(\{y\})\ (y\,\in\,E_2))} (\overline{\alpha\mathbf{I}} \wedge \overline{\mathbf{O}\beta}) \qquad \text{(Lemma 5)}$$

$$= \bigvee_{(a,b)\,\in\,\text{AB and } [a,b]\,\leq\,\cdot\,\mathcal{K}(\psi)} (\varepsilon_a \wedge {}_b\delta^{a}). \qquad \text{(Corollary 3)}$$

The proof of the second decomposition is similar to the proof of the first one. □

A direct proof of a similar result is given in Banon & Barrera (1990).

The sup–generating and inf–generating operators of Theorem 6 are, respectively, the operators $\varepsilon_a \wedge {}_b\delta^{a}$ and $\varepsilon^{a}{}_a \vee {}_b\delta$, where a and b are functions from E_2 to $\mathcal{P}_1$.

4. Conditionally Translation Invariant Elementary Operators

Let $(\mathbf{Z}^2, +)$ be the set of ordered pairs of integers equipped with the usual addition. Let u be a point of $\mathbf{Z}^2$, we denote by $B + u$ the translate by u of a subset B of $\mathbf{Z}^2$ and by B^{t} its transpose (Banon & Barrera 1991). From now on, we assume that the sets E_1 and E_2 of Section 3 are subsets of $\mathbf{Z}^2$ (for example "rectangles").

A function b from E_2 to $\mathcal{P}_1$ is *conditionally translation invariant* or a *ct–function* iff

$$\exists B \in \mathcal{P}(\mathbf{Z}^2),\ \forall y \in E_2,\ b(y) = (B + y) \cap E_1.$$

Let us consider the following subcollection $\mathcal{B}_{E_1,E_2}$, or simply $\mathcal{B}$, of $\mathcal{P}(\mathbf{Z}^2)$

$$\mathcal{B} = \{B \in \mathcal{P}(\mathbf{Z}^2) : \forall b \in B,\ \exists u \in E_2,\ b + u \in E_1\}.$$

Let $\oplus$ denote the Minkowski addition on $\mathcal{P}(\mathbf{Z}^2)$ (Hadwiger, 1950). We observe that $\mathcal{B}$ is an ideal and a complete sublattice of $\mathcal{P}(\mathbf{Z}^2)$, its greatest element is $E_1 \oplus E_2{}^{t}$ and $\mathcal{B} = \mathcal{P}(E_1 \oplus E_2{}^{t})$. Figure 1 shows an element B of $\mathcal{B}$ generated by two rectangles.

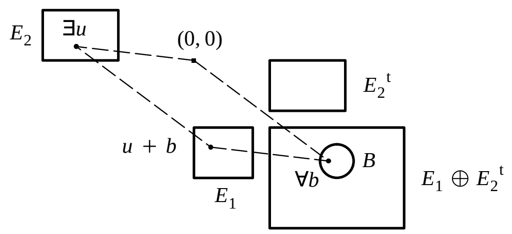

Fig. 1 – An element of an ideal generated by two rectangles.

The next proposition characterizes the ct–functions in $\mathcal{P}_1^{E_2}$ in terms of subsets of $E_1 \oplus E_2{}^{t}$.

Proposition 7 – The mapping $B \mapsto b_B$ from $\mathcal{B}$ to the set of ct–functions in $\mathcal{P}_1^{E_2}$ given by

$$b_B(y) = (B + y) \cap E_1 \quad (y \in E_2)$$

is a bijection. Its inverse $b \mapsto B_b$ is given by $B_b = \bigcup_{y\,\in\,E_2} (b(y) - y)$. □

Proof – (1) For any $B \in \mathcal{B}$, b_B is by construction a ct–function.

(2) For any ct–function b from E_2 to $\mathcal{P}(E_1)$

$$\begin{aligned} B_b &= \bigcup_{y \in E_2} (b(y) - y) && \text{(definition of } B_b\text{)} \\ &\subset \bigcup_{y \in E_2} (E_1 - y) && (b(y) \subset E_1) \\ &= E_1 \oplus E_2{}^{t}, && \text{(definition of } \oplus\text{)} \end{aligned}$$

That is, $B_b \in \mathcal{B}$.

(3) Let us prove that $B \mapsto b_B$ is "one to one". On one hand, for any $B \in \mathcal{B}$,

$$\begin{aligned} \text{TRUE} &\Leftrightarrow (B + y) \cap E_1 \subset (B + y) \quad (y \in E_2) && \text{(property of the intersection)} \\ &\Leftrightarrow ((B + y) \cap E_1) - y \subset B \quad (y \in E_2) && \text{(translation is isotone)} \\ &\Leftrightarrow \bigcup_{y \in E_2} (((B + y) \cap E_1) - y) \subset B && \text{(property of the union)} \\ &\Leftrightarrow B_{b_B} \subset B. && \text{(definitions of } b_B \text{ and } B_b\text{)} \end{aligned}$$

On the other hand, for any $B \in \mathcal{B}$ and $b \in \mathbf{Z}^2$,

$$\begin{aligned} b \in B &\Rightarrow b \in B \text{ and } \exists y \in E_2,\ b + y \in E_1 && \text{(definition of } \mathcal{B}\text{)} \\ &\Rightarrow \exists y \in E_2,\ b + y \in B + y \text{ and } y + b \in E_1 && \text{(property of the translation)} \\ &\Leftrightarrow \exists y \in E_2,\ b + y \in (B + y) \cap E_1 && \text{(definition of the intersection)} \\ &\Leftrightarrow \exists y \in E_2,\ b \in ((B + y) \cap E_1) - y && \text{(translation is isotone)} \\ &\Leftrightarrow b \in \bigcup_{y \in E_2} (((B + y) \cap E_1) - y) && \text{(definition of the union)} \\ &\Leftrightarrow b \in B_{b_B}. && \text{(definitions of } b_B \text{ and } B_b\text{)} \end{aligned}$$

That is, for any $B \in \mathcal{B}$, $B \subset B_{b_B}$. Therefore, $B \mapsto b_B$ is "one to one".

(4) Let us prove that $B \mapsto b_B$ is "onto". On one hand, for any ct–function b in $\mathcal{P}_1^{E_2}$ and $y \in E_2$,

$$\begin{aligned} b_{B_b}(y) &= ((\bigcup_{v \in E_2} (b(v) - v)) + y) \cap E_1 && \text{(definitions of } B_b \text{ and } b_B\text{)} \\ &\supset b(y) \cap E_1 && (v = y \text{ e property of the union}) \\ &= b(y). && (b(y) \subset E_1) \end{aligned}$$

On the other hand, for any ct–function b in $\mathcal{P}_1^{E_2}$ and $y \in E_2$,

$$\begin{aligned} \text{TRUE} &\Leftrightarrow ((\bigcup_{v \in E_2} (b(v) - v)) + y) \cap E_1 \subset (\bigcup_{v \in E_2} (b(v) - v)) + y && \text{(property of } \cap\text{)} \\ &\Leftrightarrow (((\bigcup_{v \in E_2} (b(v) - v)) + y) \cap E_1 \subset (b(v) - v) + y) \ (v \in E_2) && \text{(prop. of } \textstyle\bigcup\text{)} \\ &\Leftrightarrow (b_{B_b}(y) \subset (b(v) - v) + y) \ (v \in E_2) && \text{(definitions of } B_b \text{ and } b_B\text{)} \\ &\Rightarrow b_{B_b}(y) \subset b(v). && (v = y) \end{aligned}$$

Therefore, $B \mapsto b_B$ is "onto". □

We say that an elementary operator from $\mathcal{P}(E_1)$ to $\mathcal{P}(E_2)$ is *conditionally translation invariant* (*c.t.i.*) iff its structuring function from E_2 to $\mathcal{P}(E_1)$ is a ct–function. For any $B \in \mathcal{B}$, we denote by α_B (or ${}_B\alpha$) the c.t.i. elementary operator which has the structuring function b_B. In particular, we have $\varepsilon_B(X) = ((X \cup E_1{}^c) \ominus B) \cap E_2$ $(X \in \mathcal{P}(E_1))$ and $\delta_B(Y) = (Y \oplus B) \cap E_1$ $(Y \in \mathcal{P}(E_2))$ where $\ominus$ is the Minkowski subtraction on $\mathcal{P}(\mathbf{Z}^2)$ (Hadwiger, 1950).

5. Locally c.t.i. Elementary Operators

A function b from E_2 to $\mathcal{P}_1$ is said to be a *locally ct–function* iff there exist a subset M of E_2, called *mask*, and a ct–function b' from E_2 to $\mathcal{P}_1$, such that

$$b(y) = \begin{cases} b'(y) & \text{if } y \in M \\ \emptyset & \text{otherwise} \end{cases} \quad (y \in E_2).$$

We will now give a constructive function decomposition in terms of locally ct–function. Let b be a function from E_2 to $\mathcal{P}_1$. We define the binary relation $\mathcal{R}_b$ on E_2 by

$$y_1 \ \mathcal{R}_b \ y_2 \ \Leftrightarrow \exists B \in \mathcal{B}, \ (B + y_1) \cap E_1 = b(y_1) \text{ and } (B + y_2) \cap E_1 = b(y_2).$$

The relation $\mathcal{R}_b$ is an equivalence relation. We denote by $E_2/\mathcal{R}_b$ the resulting partition of E_2.

Proposition 8 – Any function b from E_2 to $\mathcal{P}_1$ can be decomposed in terms of locally ct–functions and the constructive decomposition is

$$b = \bigvee_{M \in E_2/\mathcal{R}_b} b_{B_{b/M},M}$$

where, for any $M \in \mathcal{P}(E_2)$ and $B \in \mathcal{B}$, $b_{B,M}$ is the function from E_2 to $\mathcal{P}_1$ given by

$$b_{B,M}(y) = \begin{cases} b_B(y) & \text{if } y \in M \\ \emptyset & \text{otherwise} \end{cases} \quad (y \in E_2)$$

and b/M denotes the restriction of b to M. □

Proof – For any $N \in E_2/\mathcal{R}_b$ and $y \in N$,

$$\begin{aligned} \Big(\bigvee_{M \in E_2/\mathcal{R}_b} b_{B_{b/M},M}\Big)(y) &= b_{B_{b/N}}(y) && \text{(definition and property of } b_{B,M}) \\ &= b/N(y) && \text{(Proposition 7 applied to the ct–function } b/N) \\ &= b(y). && \text{(definition of restriction)} \end{aligned}$$

□

The locally ct–functions of Proposition 8 are the functions $b_{B,M}$.

We say that an elementary operator from $\mathcal{P}_1$ to $\mathcal{P}_2$ is *locally c.t.i.* iff its structuring function in $\mathcal{P}_1^{E_2}$ is a locally ct–function. For any $M \in \mathcal{P}_2$ and $B \in \mathcal{B}$, we denote by $\alpha_{B,M}$ (or ${}_{B,M}\alpha$) the locally c.t.i. elementary operator which has the structuring function $b_{B,M}$.

6. Elementary Operator Decomposition in terms of Locally c.t.i. Elementary Operators

Theorem 9 – Any elementary operator δ (resp. ε, δ^a and ε^a) of the class $\Delta(\mathcal{P}_1, \mathcal{P}_2)$ (resp. $\mathrm{E}(\mathcal{P}_1, \mathcal{P}_2)$, $\Delta^a(\mathcal{P}_1, \mathcal{P}_2)$ and $\mathrm{E}^a(\mathcal{P}_1, \mathcal{P}_2)$) can be decomposed in terms of locally c.t.i. elementary operators of the same class and, if b is its structuring function, the constructive decomposition is $\delta = \bigwedge_M {}_{B_{b/M},M}\delta$ (respectively. $\varepsilon = \bigwedge_M \varepsilon_{B_{b/M},M}$, $\varepsilon^a = \bigvee_M \varepsilon^a{}_{B_{b/M},M}$ and $\delta^a = \bigvee_M {}_{B_{b/M},M}\delta^a$), where the union and intersection are taken over $E_2/\mathcal{R}_b$. □

Proof – The result is a consequence of Propositions 1, 2 and 8. The decomposition involves an union (resp. intersection) when the mapping $a \mapsto \alpha$ is an isomorphism (resp. a dual isomorphism). □

From Theorems 6 and 9, we see that any operator can be decomposed in terms of locally c.t.i. elementary operators.

Let E be a non empty subset of $\mathbf{Z}^2$ and $B \in \mathcal{B}_{E,E}$. We now consider the *example* of decomposition of δ_B, the *ct–dilation by B* defined from $\mathcal{P}(E)$ to $\mathcal{P}(E)$. We know that $\delta_B(X) = (X \oplus B) \cap E$. The left kernel (as defined in Theorem 6) of δ_B is given by

$$\cdot\,\mathcal{K}(\delta_B)(y) = \{X \in \mathcal{P} : (B^t + y) \cap X \neq \emptyset\} \quad (y \in E).$$

For any $y \in E$, if $(B^t + y) \cap E \neq \emptyset$, then $\cdot\,\mathcal{K}(\delta_B)(y) \neq \emptyset$ and the pairs (a, b) of interest in the decomposition of δ_B are such that the $a(y)$ contain at least one point in $(B^t + y) \cap E$; if $(B^t + y) \cap E = \emptyset$, then $\cdot\,\mathcal{K}(\delta_B)(y) = \emptyset$ and by convention $a(y) = E$ and $b(y) = \emptyset$.

Let consider the following simple case where $B = \{p\}$ with $p \in E \oplus E^t$ and let $Z = E \cap (E + p)$. The pairs (a, b) of interest leading to the greatest interval functions reduce to only one defined by $a(y) = \{y - p\}$ and $b(y) = E$ if $y \in Z$, and $a(y) = E$ and $b(y) = \emptyset$ if $y \in E - Z$.

Hence, by Theorem 6, $\delta_{\{p\}} = \varepsilon_a \wedge {}_b\delta^a$ where a and b are the above ct–functions. We observe that even $\delta_{\{p\}}$ being a ct–dilation ε_a is neither a ct–erosion nor a locally ct–erosion. Just ${}_b\delta^a$ is a locally ct–anti–dilation (with $M = Z$ and $B = E$). Nevertheless, by Theorem 9, we can decompose ε_a in terms of two locally ct–erosions: $\varepsilon_a = \varepsilon_{\{-p\},Z} \wedge \varepsilon_{E,Z^c}$ Finally, we get the following decomposition of the ct–dilation $\delta_{\{p\}}$ in terms of locally ct–erosions and ct–anti–dilation:

$$\delta_{\{p\}} = (\varepsilon_{\{-p\},Z} \wedge \varepsilon_{E,Z^c}) \wedge {}_{E,Z}\delta^a.$$

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