TRIANGULAR NORMS

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To

Bernhard, Sissy, Katharina,
Janka, Andrea,
and Danijela

Table of Contents

Part II

Appendix

Reference material

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Linz, Bratislava, and Novi Sad

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Introduction

The history of triangular norms started with the paper "Statistical metrics" [Menger 1942]. The main idea of Karl Menger was to construct metric spaces where probability distributions rather than numbers are used in order to describe the distance between two elements of the space in question. Triangular norms (t-norms for short) naturally came into the picture in the course of the generalization of the classical triangle inequality to this more general setting. The original set of axioms for t-norms was considerably weaker, including among others also the functions which are known today as triangular conorms.

Consequently, the first field where t-norms played a major role was the theory of probabilistic metric spaces (as statistical metric spaces were called after 1964). Berthold Schweizer and Abe Sklar in [Schweizer & Sklar 1958, 1960, 1961] provided the axioms of t-norms, as they are used today, and a redefinition of statistical metric spaces given in [Šerstnev 1962] led to a rapid development of the field. Many results concerning t-norms were obtained in the course of this development, most of which are summarized in the monograph [Schweizer & Sklar 1983].

Mathematically speaking, the theory of (continuous) t-norms has two rather independent roots, namely, the field of (specific) functional equations and the theory of (special topological) semigroups.

Concerning functional equations, t-norms are closely related to the equation of associativity (which is still unsolved in its most general form). The earliest source in this context seems to be [Abel 1826], further results in this direction were obtained in [Brouwer 1909, Cartan 1930, Aczél 1949, Hosszú 1954]. Especially János Aczél's monograph (both the German [Aczél 1961] and the English [Aczél 1966] version) had (and still has) a big impact on the development of t-norms. The main result based on this background was the full characterization of continuous Archimedean t-norms by means of additive generators in [Ling 1965] (for the case of strict t-norms see [Schweizer & Sklar 1961]). Further significant contributions are due to a group of Spanish researchers around Enric Trillas and Claudi Alsina.

Another direction of research was the identification of several parameterized families of t-norms as solutions of some (more or less) natural functional equations. The perhaps most famous result in this context has been proven in [Frank 1979], showing that the family of Frank t-norms and t-conorms (together with ordinal sums thereof) are the only solutions of the so-called Frank functional equation.

The study of a class of compact, irreducibly connected topological semigroups was initiated in [Faucett 1955], including a characterization of such semigroups, where the boundary points (at the same time annihilator and neutral element) are the only idempotent elements and where no nilpotent elements exist. In the language of t-norms, this provides a full representation of strict t-norms. In [Mostert & Shields 1957] all such semigroups, where the boundary points play the role of annihilator and neutral element, were characterized (see also [Paalman-de Miranda 1964]). Again in the language of t-norms, this provides a representation of all continuous t-norms [Ling 1965].

Several construction methods from the theory of semigroups, such as (isomorphic) transformations (which are closely related to generators mentioned above) and ordinal sums [Climescu 1946, Clifford 1954, Schweizer & Sklar 1963], have been successfully applied to construct whole families of t-norms from a few given prototypical examples [Schweizer & Sklar 1963].

Summarizing, starting with only three t-norms, namely, the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$, it is possible to construct all continuous t-norms by means of isomorphic transformations and ordinal sums [Ling 1965].

Many specific results, such as characterizations of the order or convergence theorems, are based on this general representation for continuous t-norms.

Non-continuous t-norms, such as the drastic product $T_{\mathbf{D}}$, have been considered from the very beginning [Schweizer & Sklar 1960]. In [Ling 1965] even an additive generator for this t-norm was given. However, a general classification of non-continuous t-norms is still not known.

For the construction of not necessarily continuous t-norms, several methods, which are more or less related to those already mentioned, have been proposed recently.

All these topics, together with an investigation of algebraic and analytical properties of t-norms and the relationship between these properties, constitute the content of Part I of this book. The exposition in this part is self-contained in the sense that all necessary concepts are precisely defined, and that the proofs of all stated results are given in full detail (with the only exception of the solution of the Frank functional equation). Also, several important parameterized families of t-norms and a large number of examples for the main properties are given, supported by 66 graphical illustrations.

While the first part is devoted exclusively to the development of the theory of triangular norms, in Part II we present some fields where t-norms play a significant role. To keep this part readable and to avoid lengthy introductions to the notions and notations of these fields, we concentrate on the main concepts and present only the most important results highlighting the usefulness of t-norms. In particular, no proofs of the theorems and propositions are given, but full references are always included, encouraging the interested reader to learn more about the subjects in question.

The first of these fields, dealing with distribution functions, leads us back to the origins of t-norms. We discuss probabilistic metric spaces [Schweizer & Sklar 1983] and related topics.

A very fast developing field is that of general (not necessarily associative) aggregation operators [Zimmermann & Zysno 1980, Yager 1988, Fodor & Roubens 1994, Grabisch 1995]. Among the many examples we pick those which have a close relationship with t-norms, including uninorms [Yager & Rybalov 1996, Klement *et al.* 1996] and nullnorms [Calvo *et al.* 200x].

Based on the seminal work of Jan Łukasiewicz and Kurt Gödel in the twenties and thirties, an extensive theory of many-valued logic has been developed during the past few decades. The crucial role of t-norms in this context is presented in the monographs [Gottwald 1989, Hájek 1998b, Cignoli *et al.* 2000].

Already in his first paper [Zadeh 1965] on fuzzy sets, Lotfi A. Zadeh suggested to use the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and, in a restricted sense, the Łukasiewicz t-conorm $S_{\mathbf{L}}$. The use of general t-norms and t-conorms for modeling the intersection and the union of fuzzy sets (see, e.g., [Kruse *et al.* 1994a, Nguyen & Walker 1997]) apparently goes back to some seminars held by Enric Trillas and to suggestions given by Ulrich Höhle during some conferences in the late seventies (see also [Dubois & Prade 1980a, Barro *et al.* 1998]). Fuzzy sets recently found many practical applications, in particular in the context of intelligent control (see [Mamdani & Assilian 1975, Takagi & Sugeno 1985] and [Sugeno 1985a, 1985b]).

Finally we discuss a generalized theory of measures and integrals. Here t-norms and t-conorms generalize the standard set operations [Butnariu & Klement 1993], on the one hand, or standard arithmetic operations, on the other hand [Kampé de Fériet & Forte 1967, Sugeno 1974, Weber 1984] (compare also [Sion 1973]).

This last topic, in some sense, was also a starting point of the present book since the original intention of the authors was to write a monograph on measures with some generalized additivities. The usefulness, even indispensability of t-norms and t-conorms in that context, on the one hand, and the lack of a source with complete information of the state of the art of t-norms, on the other hand, were the main reasons for changing our mind and for starting the preparation of the present book. During this process we realized more and more that t-norms form a diversified and challenging topic of its own (including many applications in a variety of other mathematical fields), and we experienced a rapid growth of the volume of this manuscript. Needless to say, we benefitted a lot from numerous stimulating discussions with many colleagues, especially during the annual Linz Seminars on Fuzzy Set Theory.

The reader of this book (especially of Part I) is expected to have some basic knowledge (on the level of a graduate student) in algebra, logic and analysis. Although we sincerely hope that each chapter is of sufficient interest for everybody, here are some suggestions how to maximize the profit for the reader. To extract a basic impression combined with a reasonable amount of information, we suggest to concentrate in a first reading on Chapter 1, Sections 2.1, 3.2 and 3.3, Chapter 4, and Sections 5.1 and 5.3 (without proofs). Readers who want to locate the proper place of t-norms within the field of algebra at large are referred to the complete Chapter 2, those who want to work also with non-continuous t-norms, to Sections 3.1 and 3.4. Section 5.4 lists some of the most prominent functional equations related to t-norms. Finally, Chapters 6–8 present some additional results (partially unpublished so far) on comparison, values, discretization, and convergence of t-norms, and they should be studied when this type of information is required.

The chapters in Part II provide, on the one hand, a feeling of the importance and the role of t-norms in various theoretical and applied fields. On the other hand, they should attract the readers to these particular topics and invite them to obtain a deeper knowledge by studying the main references given there.

Notations used in this book

For the logical operations conjunction, disjunction, negation, and implication, we shall write $\wedge$, $\vee$, $\neg$, and $\rightarrow$, respectively.

The intersection and the union of two sets A and B are denoted by $A \cap B$ and $A \cup B$, respectively, $A \setminus B = \{x \in A \mid x \notin B\}$ stands for the difference of the sets A and B, and $\complement A$ for the complement of A. The symbol $A \subseteq B$ means that A is a subset of B (where $A = B$ is possible), and $A \times B = \{(a, b) \mid a \in A, b \in B\}$ is the Cartesian product of A and B. The empty set is denoted by $\emptyset$, the power set of X, i.e., the set of all subsets of X, by $\mathcal{P}(X)$, and the topological closure of A, i.e., the smallest closed set containing A, by $\mathrm{cl}(A)$. For the set of all functions from a set X into a set Y we shall write Y^X. The cardinality of a set A is denoted by $\mathrm{card}(A)$.

For a subset A of X, its characteristic function $\mathbf{1}_A : X \longrightarrow \{0, 1\}$ is given by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

and the membership function of a fuzzy subset A of X will be denoted by $\mu_A : X \longrightarrow [0, 1]$ (see Definition 12.1).

The symbol $\mathbb{N}$ is used for the set of positive integers, i.e., $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}$ for the set of integers, i.e., $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Q}$ for the set of rational numbers, and $\mathbb{R}$ for the set of real numbers. The extended real line, i.e., the set $\mathbb{R} \cup \{-\infty, \infty\}$, is consistently denoted by $[-\infty, \infty]$. For a closed interval we shall write $[a, b]$, for an open interval $]a, b[$, and for half-open intervals $[a, b[$ and $]a, b]$, respectively.

A sequence of elements a_n of a set X is denoted $(a_n)_{n \in \mathbb{N}}$, and a family of elements a_i of X with index set I by $(a_i)_{i \in I}$ (observe that these are elements of $X^{\mathbb{N}}$ and X^I, respectively).

If $f : X \longrightarrow Y$ is a function, then instead of X we sometimes write $\mathrm{Dom}(f)$, and the range of f is given by $\mathrm{Ran}(f) = \{f(x) \mid x \in X\}$. For $A \subseteq X$ and $B \subseteq Y$ the image $f(A)$ and the preimage $f^{-1}(B)$ are given by, respectively,

$$f(A) = \{f(x) \mid x \in A\},$$
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Note that this means $\mathrm{Ran}(f) = f(X)$. The identity function $\mathrm{id}_X : X \longrightarrow X$ is defined by $\mathrm{id}_X(x) = x$. The composition $g \circ f : X \longrightarrow V$ of two functions $f : X \longrightarrow Y$ and $g : U \longrightarrow V$ with $Y \subseteq U$ is given by $g \circ f(x) = g(f(x))$, and the restriction $f|_A : A \longrightarrow Y$ of $f : X \longrightarrow Y$ to a subset A of X by $f|_A = f \circ \mathrm{id}_A$. If a function $f : X \longrightarrow Y$ is bijective then $f^{-1} : Y \longrightarrow X$ denotes its inverse function which satisfies $f^{-1}(y) = x$ if and only if $f(x) = y$ or, equivalently, both $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. For a function $f : X \longrightarrow [0, \infty]$ its support is given by $\mathrm{supp}(f) = f^{-1}(]0, \infty])$.

If $f : X \times Y \longrightarrow Z$ is a function in two variables then, for each $x_0 \in X$, the vertical section $f(x_0, .) : Y \longrightarrow Z$ and, for each $y_0 \in Y$, the horizontal section $f(., y_0) : X \longrightarrow Z$ (which are one-place functions) are given by, respectively,

$$\begin{aligned} f(x_0, .) : Y &\longrightarrow Z \\ y &\longmapsto f(x_0, y), \\[1ex] f(., y_0) : X &\longrightarrow Z \\ x &\longmapsto f(x, y_0). \end{aligned}$$

For the floor of a real number x, i.e., the greatest integer which is smaller than or equal to x, we shall write $\lfloor x \rfloor$, and for the (natural) logarithm (i.e., the logarithm with respect to e) of $x \in]0, \infty[$ simply $\log x$, whereas, for each number $\lambda \in]0, 1[\cup]1, \infty[$, $\log_\lambda x$ denotes the logarithm of x with respect to the basis λ.

If $f :]a, b[\longrightarrow [-\infty, \infty]$ is a real function then the values $f(a^+)$ and $f(b^-)$ denote the right- and left-hand limit of f in the points a and b, respectively, i.e.,

$$\begin{aligned} f(a^+) &= \lim_{x \searrow a} f(x), \\ f(b^-) &= \lim_{x \nearrow b} f(x). \end{aligned}$$

For a monotone function $f : [a, b] \longrightarrow [c, d]$, $f^{(-1)} : [c, d] \longrightarrow [a, b]$ will denote the pseudo-inverse of f (see Definition 3.2).

In order to obtain well-defined continuous functions on a closed subinterval of the extended real line $[-\infty, \infty]$ (with values in $[-\infty, \infty]$), we shall frequently use the subsequent continuous extension: given a continuous function $f :]a, b[\longrightarrow]c, d[$ (where $]a, b[$ and $]c, d[$ are open subintervals of $[-\infty, \infty]$) such that the two limits $f(a^+)$ and $f(b^-)$ exist in $[c, d]$, then we define the continuous function $\overline{f} : [a, b] \longrightarrow [c, d]$ via continuous extension, i.e.,

$$\overline{f}(x) = \begin{cases} f(a^+) & \text{if } x = a, \\ f(x) & \text{if } x \in]a, b[\,, \\ f(b^-) & \text{if } x = b. \end{cases}$$

To simplify the notation, the function f and its extension $\overline{f}$ are quite often identified.

For example, this means that, extending the function $f :]0,\infty[\longrightarrow]0,\infty[$ given by $f(x) = \frac{1}{x}$, we have

$$\frac{1}{0} = \infty, \qquad \frac{1}{\infty} = 0,$$

but also, extending the addition, for each $x \in \mathbb{R}$

$$x + \infty = \infty, \qquad x - \infty = -\infty,$$

as an extension of the multiplication, for each $x \in \mathbb{R} \setminus \{0\}$

$$x \cdot \infty = \begin{cases} \infty & \text{if } x > 0, \\ -\infty & \text{if } x < 0, \end{cases} \qquad \infty^x = \begin{cases} \infty & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

and, finally, extending several other important functions,

$$\begin{aligned} e^{-\infty} &= 0, & e^{\infty} &= \infty, \\ \log 0 &= -\infty, & \log(\infty) &= \infty, \\ \arctan(-\infty) &= -\tfrac{\pi}{2}, & \arctan(\infty) &= \tfrac{\pi}{2}. \end{aligned}$$

For expressions which cannot be well-defined via continuous extensions (such as $\frac{0}{0}$ or $\infty - \infty$) we sometimes use individual conventions which are mentioned where they are needed.

In a lattice $(L, \preceq)$, the meet (i.e., the greatest lower bound) and the join (i.e., the least upper bound) of a subset A of L are denoted $\bigwedge A$ (or $\inf A$) and $\bigvee A$ (or $\sup A$), respectively, for the meet and join of two elements $a, b \in L$ we simply write $a \wedge b$ and $a \vee b$, respectively. Observe that each element of L is both an upper and a lower bound of the empty set $\emptyset$, so $\inf \emptyset$ and $\sup \emptyset$ depend on the underlying set L. In particular, in the lattice $([a, b], \leq)$ (with $[a, b] \subseteq [-\infty, \infty]$ and where $\leq$ is the usual order on the (extended) real line) we obtain $\inf \emptyset = b$ and $\sup \emptyset = a$.

Basic references for algebra are, e.g., [Lidl & Pilz 1998] and, concerning ordered and topological semigroups, [Fuchs 1963, Carruth *et al.* 1983], for lattice theory we usually refer to [Birkhoff 1973], and for the necessary background in calculus and real analysis to [Apostol 1965, Hewitt & Stromberg 1965]. In the context of logic our exposition is based on [Rosser 1953], topological concepts and notions are understood in the sense of [Bourbaki 1966, Dugundji 1966], and our basic references in measure and probability theory are [Halmos 1950, Bauer 1981].