Ramanujan Cayley graphs of the generalized quaternion groups and the Hardy-Littlewood conjecture

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Abstract

In this article, we investigate the bound of the valency of the Cayley graphs of the generalized quaternion groups which guarantees to be Ramanujan. As is the cases of the cyclic and dihedral groups in our previous studies, we show that the determination of the bound in a special setting is related to the classical Hardy-Littlewood conjecture for primes represented by a quadratic polynomial.

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1 Introduction

Expander graph is a sparse graph having strong connectivity properties. Because of its rich theory with many applications, it is widely studied in various fields of mathematics such as combinatorics, group theory, differential geometry and number theory (see [7, 9] for survey of the expander graphs). In particular, Ramanujan graph, which is an optimal expander graph in the sense of Alon Boppana's theorem and was first defined in [10], plays an important role in not only pure mathematics but also applied mathematics. Actually, because a graph is Ramanujan if and only if the associated Ihara zeta function satisfies the "Riemann hypothesis", it has a special interest for number theorists, especially who study zeta functions (see, e.g., [11]). Moreover, from the fact that a random walk on a Ramanujan graph quickly converges to the uniform distribution, it is used to construct a cryptography hash function [3]. From these reasons, it is worth finding or constructing Ramanujan graphs as many as possible, however, it is in general difficult.

In this paper, we consider the following problem on Ramanujan graphs. Naively, one easily imagines that, because a Ramanujan graph has a strong connectivity property, if we have a Ramanujan graph, then there expects to be another Ramanujan graph around it (cf. [1]). This means that, even if we get rid of some edges from the given Ramanujan graph anyhow, it may remain to be Ramanujan. Now our problem is to clarify how many edges we can freely remove from the given Ramanujan graph with remaining to be Ramanujan in a given family of graphs. In particular, as a first stage, we consider this problem starting from the trivial Ramanujan

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graph, that is, the complete graph, in a family of Cayley graphs of a fixed group. Notice that, in this setting, removing edges corresponds to reducing elements of a Cayley subset of the group. See the end of Section 2 for more precise mathematical formulation of our problem.

In [5], we first investigated this problem for the cyclic groups. Moreover, in [6], we studied it for the dihedral groups, which are non-abelian (simplest) extensions of the previous case (we actually consider this problem for groups in the class of the Frobenius groups in [6], which contains for example the semi-direct product of the cyclic groups and hence, especially, the dihedral groups). In both cases, we showed that the determination of the above maximal number of removable edges (it corresponds to \tilde{l} in our formulation) is related to the classical Hardy-Littlewood conjecture on analytic number theory, which asserts that every quadratic polynomials express infinitely many primes under some standard conditions, if the order of the group is odd prime (resp. twice odd prime) in the case of the cyclic group (resp. the dihedral group). In succession to these cases, in the present paper, we work the same problem for the generalized quaternion group Q_{4m} and actually obtain the similar result (Theorem 4.9) if we choose the set of Cayley graphs suitably. Notice that we indeed consider a wider class of groups in the sense that Q_{4m} can not be expressed as any semi-directed product of the cyclic groups. We also remark that our discussion may be applied to groups whose maximal degree of the irreducible representations is at most two.

We use the following notations in this paper. The set of all real numbers, integers and odd primes are denoted by \mathbb{R} , \mathbb{Z} and \mathbb{P} , respectively. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) denote the largest (resp. smallest) integer less (resp. greater) than or equal to x. We also note that the most of our numerical computations are performed by using Mathematica.

2 Preliminary

In this section, we prepare some definitions and notations of graph theory, which are necessary for our discussion (see, more precisely, [8]). Throughout this paper, all graphs are assumed to be finite, undirected, connected, simple and regular.

Let X be a k-regular graph with m-vertices. The adjacency matrix A_X of X is the symmetric matrix of size m whose entry is 1 if the corresponding pair of vertices are connected by an edge and 0 otherwise. We call the eigenvalues of A_X the eigenvalues of X. Let $\Lambda(X)$ be the set of all eigenvalues of X. We know that it can be written as $\Lambda(X) = \{k = \lambda_0 > \lambda_1 \ge \cdots \ge \lambda_{m-1}\} \subset [-k,k]$. Let $\lambda(X)$ be the largest non-trivial eigenvalue of X in the sense of absolute value; $\lambda(X) = \max\{|\lambda| \mid \lambda \in \Lambda(X), |\lambda| \ne k\}$. Then, X is called Ramanujan if the inequality $\lambda(X) \le 2\sqrt{k-1}$ holds. Here the constant $2\sqrt{k-1}$ is called the Ramanujan bound for X and is denoted by $\operatorname{RB}(X)$.

Let G be a finite group with the identity element 1. Let S be a Cayley subset of G, that is, S is a symmetric generating subset of G without 1. We denote by X(S) the Cayley graph of G with respect to the Cayley subset S. This is |S|-regular graph whose vertex set is G and edge set $\{(x, y) \in G^2 | x^{-1}y \in S\}$. Let S_G be the set of all Cayley subset of G. In what follows, for $S \in S$, we write $\Lambda(S) = \Lambda(X(S)), \lambda(S) = \lambda(X(S)), \operatorname{RB}(S) = \operatorname{RB}(X(S))$, and so on. It is well known that the eigenvalues of X(S) can be described in terms of the irreducible representations of G as follows.

Lemma 2.1 (cf. [2]). Let G be a finite group and Irr(G) the set of all equivalence classes of the

irreducible representations of G. Then, for $S \in S_G$, we have

$$\Lambda(S) = \bigcup_{\pi \in \operatorname{Irr}(G)} d_{\pi} \cdot \operatorname{Spec}(M_{\pi}(S)),$$

where, for $\pi \in \operatorname{Irr}(G)$, d_{π} is the degree of π , $M_{\pi}(S) = \sum_{s \in S} \pi(s)$ and $\operatorname{Spec}(M_{\pi}(S))$ is the set of all eigenvalues of $M_{\pi}(S)$. Here, we understand that an element in $\operatorname{Spec}(M_{\pi}(S))$ is counted d_{π} times in $d_{\pi} \cdot \operatorname{Spec}(M_{\pi}(S))$.

We here explain our problem on Ramanujan graphs. For a set $S \subset S_G$ of Cayley subsets of G, let $\mathcal{L} = \mathcal{L}_{G,S} = \{l(S) \mid S \in S\}$ where $l(S) = |G \setminus S| = |G| - |S|$ is the covalency of $S \in S$. Then, we have the decomposition $S = \bigsqcup_{l \in \mathcal{L}} S_l$ with $S_l = \{S \in S \mid l(S) = l\}$. Now our aim is to determine the bound

 $\tilde{l} = \tilde{l}_{G,S} = \max\{l \in \mathcal{L} \mid X(S) \text{ is Ramanujan for all } S \in \mathcal{S}_k \ (1 \le k \le l)\}.$

Remark that $\tilde{l} \geq 1$ if $G \setminus \{1\} \in S_1$ because $X(G \setminus \{1\})$ is the complete graph $K_{|G|}$ with |G| vertices, which is a (trivial) Ramanujan graph. Hence, in this case, roughly speaking, \tilde{l} represents the maximal number of removable edges from the complete graph $K_{|G|}$ keeping to be Ramanujan.

In this paper, we investigate l when G is the generalized quaternion group.

3 Cayley graphs of the generalized quaternion groups

For a positive integer m, the generalized quaternion group Q_{4m} is defined by

$$Q_{4m} = \langle x, y \mid x^{2m} = 1, \ x^m = y^2, \ y^{-1}xy = x^{-1} \rangle.$$

This is non-commutative unless m = 1 and can not be expressed as a semi-direct product of any pair of subgroups of Q_{4m} . One easily sees that the order of Q_{4m} is 4m because it has the expression

$$Q_{4m} = \{x^k y^l \mid 0 \le k \le 2m - 1, \ l = 0, 1\} = \langle x \rangle \sqcup \langle x \rangle y,$$

where $\langle x \rangle = \{x^k \mid 0 \le k \le 2m - 1\}$ and $\langle x \rangle y = \{x^k y \mid 0 \le k \le 2m - 1\}$. Notice that $(x^k)^{-1} = x^{2m-k}$ and $(x^k y)^{-1} = x^{m+k} y$.

To calculate the eigenvalues of the Cayley graph of Q_{4m} , we need the information about the conjugacy classes and the irreducible representations of Q_{4m} . For $z \in Q_{4m}$, let C(z) be the conjugacy class of Q_{4m} containing z. Then, the following exhausts all conjugacy classes of Q_{4m} ; $C(1) = \{1\}, C(x^k) = \{x^k, x^{2m-k}\} \ (1 \le k \le m-1), C(x^m) = \{x^m\}, C(y) = \{x^{2k}y \mid 0 \le k \le m-1\}$ and $C(xy) = \{x^{2k+1}y \mid 0 \le k \le m-1\}$. Moreover, the irreducible representations of Q_{4m} are given as follows; $\chi_1 = \mathbf{1}$ (the trivial character), χ_2 , χ_3 and χ_4 which are of degree 1 and $\varphi_j \ (1 \le j \le m-1)$ of degree 2. We give the values of these representations in Table 1. Here, $\omega = e^{\frac{2\pi i}{2m}}$.

From now on, we let $S = S_{Q_{4m}}$ be the set of all Cayley subsets of Q_{4m} . Let us calculate the eigenvalues of the Cayley graph X(S) for $S \in S$. Put $S_1 = S \cap \langle x \rangle$ and $S_2 = S \cap \langle x \rangle y$ so that we can write

$$S = S_1 \sqcup S_2.$$

Moreover, put $l_1(S) = 2m - |S_1|$ and $l_2(S) = 2m - |S_2|$ so that $l(S) = l_1(S) + l_2(S)$. Notice that $S_1 \neq \langle x \rangle$ since $1 \notin S$ and hence $l_1(S) > 0$ and $S_2 \neq \emptyset$ because S generates Q_{4m} and therefore

	x^k $x^k y$				x^k	x^ky	
χ_1	1	1		χ_1	1	1	
χ_2	1	-1		χ_2	1	-1	
χ_3	$(-1)^k$	$i(-1)^k$		χ_3	$(-1)^k$	$(-1)^{k}$	
χ_4	$(-1)^k$	$i(-1)^{k+1}$		χ_4	$(-1)^k$	$(-1)^{k+1}$	
φ_j	$\begin{bmatrix} \omega^{jk} & 0\\ 0 & \omega^{-jk} \end{bmatrix}$	$\begin{bmatrix} 0 & \omega^{jk} \\ (-1)^j \omega^{-jk} & 0 \end{bmatrix}$		φ_j	$\begin{bmatrix} \omega^{jk} & 0\\ 0 & \omega^{-jk} \end{bmatrix}$	$\begin{bmatrix} 0 & \omega^{jk} \\ (-1)^j \omega^{-jk} & 0 \end{bmatrix}$	

Table 1: The tables of the values of the irreducible representations of Q_{4m} : the left one is the case of odd m and the right one is of even m.

 $l_2(S) < 2m$. One sees that, because S is symmetric, both S_1 and S_2 are also symmetric. This implies that they are respectively expressed as

(3.1)
$$S_{1} = \bigsqcup_{\substack{x^{k_{1} \in S} \\ 1 \le k_{1} \le m-1}} \left\{ x^{k_{1}}, x^{2m-k_{1}} \right\} \sqcup \{x^{m}\}^{\delta}$$
$$S_{2} = \bigsqcup_{\substack{x^{k_{2}} y \in S \\ 0 \le k_{2} \le m-1}} \left\{ x^{k_{2}}y, x^{m+k_{2}}y \right\},$$

where $\delta = \delta(S) = 1$ if $x^m \in S$ and 0 otherwise. Here, we understand that $A^0 = \emptyset$ and $A^1 = A$ for any set A. From these expressions, we have $l_2(S) \equiv 0 \pmod{2}$ and $l_1(S) \equiv l(S) \equiv \delta \pmod{2}$. Based on the above expression, we obtain the following decomposition of S;

(3.2)
$$S = \bigsqcup_{l \in \mathcal{L}} S_l = \bigsqcup_{l \in \mathcal{L}} \bigsqcup_{l \in \mathcal{L}} (l_1, l_2) \in \mathcal{L}_l S_{l_1, l_2},$$

where

$$\mathcal{L}_{l} = \left\{ (l_{1}, l_{2}) \in \mathbb{Z}^{2} \mid \begin{array}{c} 0 < l_{1} \leq 2m, \ l_{1} \equiv l \pmod{2}, \\ 0 \leq l_{2} < 2m, \ l_{2} \equiv 0 \pmod{2}, \end{array} \right. l_{1} + l_{2} = l \right\}$$

and $S_{l_1,l_2} = \{ S \in S | l_1(S) = l_1, l_2(S) = l_2 \}$ for $l \in \mathcal{L}$ and $(l_1, l_2) \in \mathcal{L}_l$. Put

$$\begin{split} \sigma_1^{\mathbf{e}} &= \sigma_1^{\mathbf{e}}(S) = \#\{k_1 \in \mathbb{Z} \mid 1 \le k_1 \le m-1, \ x^{k_1} \in S, \ k_1 \equiv 0 \pmod{2}\},\\ \sigma_1^{\mathbf{o}} &= \sigma_1^{\mathbf{o}}(S) = \#\{k_1 \in \mathbb{Z} \mid 1 \le k_1 \le m-1, \ x^{k_1} \in S, \ k_1 \equiv 1 \pmod{2}\},\\ \sigma_2^{\mathbf{e}} &= \sigma_2^{\mathbf{e}}(S) = \#\{k_2 \in \mathbb{Z} \mid 0 \le k_2 \le m-1, \ x^{k_2}y \in S, \ k_2 \equiv 0 \pmod{2}\},\\ \sigma_2^{\mathbf{o}} &= \sigma_2^{\mathbf{o}}(S) = \#\{k_2 \in \mathbb{Z} \mid 0 \le k_2 \le m-1, \ x^{k_2}y \in S, \ k_2 \equiv 1 \pmod{2}\}. \end{split}$$

Using these notations, it can be written as $|S_1| = 2(\sigma_1^e + \sigma_1^o) + \delta$ and $|S_2| = 2(\sigma_2^e + \sigma_2^o)$. Note that

$$0 \le \sigma_1^{\rm e} \le \frac{m-1}{2}, \ \ 0 \le \sigma_1^{\rm o} \le \frac{m-1}{2}, \ \ 0 \le \sigma_2^{\rm e} \le \frac{m+1}{2}, \ \ 0 \le \sigma_2^{\rm o} \le \frac{m-1}{2}$$

if m is odd and

$$0 \le \sigma_1^{\rm e} \le \frac{m}{2} - 1, \ \ 0 \le \sigma_1^{\rm o} \le \frac{m}{2}, \ \ 0 \le \sigma_2^{\rm e} \le \frac{m}{2}, \ \ 0 \le \sigma_2^{\rm o} \le \frac{m}{2}$$

otherwise.

From Lemma 2.1 together with the expression (3.1) of the Cayley subset, one can explicitly calculate the eigenvalues of X(S) as follows.

Lemma 3.1. For $S \in S$, we have

$$\Lambda(S) = 1 \cdot \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \cup \bigcup_{j=1}^{m-1} 2 \cdot \left\{\mu_j^+, \mu_j^-\right\}.$$

Here, for $1 \le j \le m-1$, $\mu_j^{\pm} = \mu_j^{\pm}(S)$ is given by $\mu_j^{\pm} = z_j \pm |w_j|$ with

$$z_{j} = z_{j}(S) = \sum_{\substack{x^{k_{1} \in S} \\ 0 \le k_{1} \le 2m-1}} \omega^{jk_{1}} = \sum_{\substack{x^{k_{1} \in S} \\ 1 \le k_{1} \le m-1}} (\omega^{jk_{1}} + \omega^{-jk_{1}}) + \delta(-1)^{j},$$
$$w_{j} = w_{j}(S) = \sum_{\substack{x^{k_{2}} y \in S \\ 0 \le k_{2} \le 2m-1}} \omega^{jk_{2}} = (1 + (-1)^{j}) \sum_{\substack{x^{k_{2}} y \in S \\ 0 \le k_{2} \le m-1}} \omega^{jk_{2}},$$

and, for $1 \leq i \leq 4$, $\lambda_i = \lambda_i(S)$ are respectively given as follows.

(1) When m is odd,

$$\lambda_1 = 2(\sigma_1^{\mathrm{e}} + \sigma_1^{\mathrm{o}}) + \delta + 2(\sigma_2^{\mathrm{e}} + \sigma_2^{\mathrm{o}}),$$

$$\lambda_2 = 2(\sigma_1^{\mathrm{e}} + \sigma_1^{\mathrm{o}}) + \delta - 2(\sigma_2^{\mathrm{e}} + \sigma_2^{\mathrm{o}}),$$

$$\lambda_3 = 2(\sigma_1^{\mathrm{e}} - \sigma_1^{\mathrm{o}}) - \delta,$$

$$\lambda_4 = 2(\sigma_1^{\mathrm{e}} - \sigma_1^{\mathrm{o}}) - \delta.$$

(2) When m is even,

$$\lambda_{1} = 2(\sigma_{1}^{e} + \sigma_{1}^{o}) + \delta + 2(\sigma_{2}^{e} + \sigma_{2}^{o}),$$

$$\lambda_{2} = 2(\sigma_{1}^{e} + \sigma_{1}^{o}) + \delta - 2(\sigma_{2}^{e} + \sigma_{2}^{o}),$$

$$\lambda_{3} = 2(\sigma_{1}^{e} - \sigma_{1}^{o}) + \delta + 2(\sigma_{2}^{e} - \sigma_{2}^{o}),$$

$$\lambda_{4} = 2(\sigma_{1}^{e} - \sigma_{1}^{o}) + \delta - 2(\sigma_{2}^{e} - \sigma_{2}^{o}).$$

Proof. These are directly obtained from the above tables of the values of the irreducible representations of Q_{4m} . We notice that $\lambda_i = M_{\chi_i}(S)$ and $\{\mu_j^+, \mu_j^-\} = \operatorname{Spec}(M_{\varphi_j}(S)) = \operatorname{Spec}\left(\begin{bmatrix} z_j & w_j \\ \overline{w_j} & z_j \end{bmatrix}\right)$.

Remark that $\lambda_1 = |S|$, which corresponds to the trivial character, is the largest eigenvalue of X(S). We also notice that $z_i \in \mathbb{R}$ and $w_i = 0$ if j is odd.

The following lemma is useful in the case of estimating the eigenvalues of X(S) corresponding to λ_i .

Lemma 3.2. Fix $l \in \mathcal{L}$ and $(l_1, l_2) \in \mathcal{L}_l$. Let $S \in \mathcal{S}_{l_1, l_2}$. Then, we have $\lambda_1 = 4m - l$ and $\lambda_2 = -l + 2l_2$. Moreover,

- (1) when m is odd and
 - (i) l is odd, we have $-(l-l_2) \leq \lambda_3 = \lambda_4 \leq l-l_2-2$. The absolute values $|\lambda_3| = |\lambda_4|$ of λ_3 and λ_4 take the maximum value $l-l_2$ if and only if $(\sigma_1^e, \sigma_1^o) = (\frac{m+1}{2} \frac{l-l_2+1}{2}, \frac{m-1}{2})$.
 - (ii) l is even, we have $-(l-l_2-2) \leq \lambda_3 = \lambda_4 \leq l-l_2-2$. The absolute values $|\lambda_3| = |\lambda_4|$ of λ_3 and λ_4 take the maximum value $l-l_2-2$ if and only if $(\sigma_1^e, \sigma_1^o) = (\frac{m+1}{2} \frac{l-l_2}{2}, \frac{m-1}{2})$ or $(\frac{m-1}{2}, \frac{m+1}{2} \frac{l-l_2}{2})$.

(2) when m is even, we have $-l \leq \lambda_3 \leq l+2\delta - 4$ and $-l \leq \lambda_4 \leq l+2\delta - 4$. The absolute values $|\lambda_3|$ of λ_3 takes the maximum value l if and only if $(\sigma_1^{\rm e}, \sigma_1^{\rm o}) = (\frac{m}{2} - \frac{l-l_2+\delta}{2}, \frac{m}{2})$ and $(\sigma_2^{\rm e}, \sigma_2^{\rm o}) = (\frac{m}{2} - \frac{l_2}{2}, \frac{m}{2})$. Similarly, the absolute values $|\lambda_4|$ of λ_4 takes the maximum value l if and only if $(\sigma_1^{\rm e}, \sigma_1^{\rm o}) = (\frac{m}{2} - \frac{l-l_2+\delta}{2}, \frac{m}{2})$ and $(\sigma_2^{\rm e}, \sigma_2^{\rm o}) = (\frac{m}{2}, \frac{m}{2} - \frac{l-l_2+\delta}{2}, \frac{m}{2})$ and $(\sigma_2^{\rm e}, \sigma_2^{\rm o}) = (\frac{m}{2}, \frac{m}{2} - \frac{l-2}{2})$.

Proof. The claims on λ_1 and λ_2 are clear from Lemma 3.1 with the expressions $l_1 = 2m - (2(\sigma_1^{\rm e} + \sigma_1^{\rm o}) + \delta), l_2 = 2m - 2(\sigma_2^{\rm o} + \sigma_2^{\rm o})$ and $l_1 = l - l_2$. Now, let us consider the other cases.

When *m* is odd, since $0 \leq \sigma_1^{\rm e} \leq \frac{m-1}{2}$ and $0 \leq \sigma_1^{\rm o} \leq \frac{m-1}{2}$, we see that $\sigma_1^{\rm e} + \sigma_1^{\rm o} = m - \frac{l-l_2+\delta}{2}$ implies that $-\frac{l-l_2+\delta-2}{2} \leq \sigma_1^{\rm e} - \sigma_1^{\rm o} \leq \frac{l-l_2+\delta-2}{2}$. This shows that $-(l-l_2-2) - 2\delta \leq \lambda_3 = \lambda_4 \leq l-l_2 - 2$. Now $|\lambda_3| = |\lambda_4|$ takes maximum value $l-l_2$ if *l* is odd, which is indeed realized when $-\frac{l-l_2+\delta-2}{2} = \sigma_1^{\rm e} - \sigma_1^{\rm o}$ (with $\delta = 1$), and $l-l_2 - 2$ otherwise, which is realized when $\sigma_1^{\rm e} - \sigma_1^{\rm o} = \pm \frac{l-l_2+\delta-2}{2}$ (with $\delta = 0$).

When *m* is even, since $0 \le \sigma_1^{\mathrm{e}} \le \frac{m}{2} - 1$, $0 \le \sigma_1^{\mathrm{o}} \le \frac{m}{2}$, $0 \le \sigma_2^{\mathrm{e}} \le \frac{m}{2}$ and $0 \le \sigma_2^{\mathrm{o}} \le \frac{m}{2}$, we see that $\sigma_1^{\mathrm{e}} + \sigma_1^{\mathrm{o}} = m - \frac{l - l_2 + \delta}{2}$ and $\sigma_2^{\mathrm{e}} + \sigma_2^{\mathrm{o}} = m - \frac{l_2}{2}$ imply that $-\frac{l - l_2 + \delta}{2} \le \sigma_1^{\mathrm{e}} - \sigma_1^{\mathrm{o}} \le \frac{l - l_2 + \delta - 4}{2}$ and $-\frac{l_2}{2} \le \sigma_2^{\mathrm{e}} - \sigma_2^{\mathrm{o}} \le \frac{l_2}{2}$, respectively. This shows that $-l \le \lambda_3 \le l + 2\delta - 4$ and $-l \le \lambda_4 \le l + 2\delta - 4$. Similarly as the above, $|\lambda_3|$ takes maximum value l if $-\frac{l - l_2 + \delta}{2} = \sigma_1^{\mathrm{e}} - \sigma_1^{\mathrm{o}}$ and $-\frac{l_2}{2} = \sigma_2^{\mathrm{e}} - \sigma_2^{\mathrm{o}}$ and $|\lambda_4|$ takes l if $-\frac{l - l_2 + \delta}{2} = \sigma_1^{\mathrm{e}} - \sigma_1^{\mathrm{o}}$ and $\sigma_2^{\mathrm{e}} - \sigma_2^{\mathrm{o}} = \frac{l_2}{2}$.

4 Main results

4.1 Trivial lower bound of *l*

We first show that a lower bound of l is obtained by using the trivial estimate of the eigenvalues of Cayley graphs.

Lemma 4.1. Assume $|S| \ge 2m$. Then, for all $\lambda \in \Lambda(S)$ with $|\lambda| \ne |S|$, we have $|\lambda| \le l(S)$.

Proof. The claim is clear for the cases $\lambda = \lambda_i$ for $2 \le i \le 4$. Actually, since $\lambda_i = \sum_{s \in S} \chi_i(s) = -\sum_{s \notin S} \chi_i(s)$, by the orthogonality of characters, it holds that $|\lambda_i| \le \min\{|S|, l(S)\} = l(S)$. We next consider the cases $\lambda = \mu_j^{\pm}$ for $1 \le j \le m - 1$. Let $|\mu_j| = \max\{|\mu_j^+|, |\mu_j^-|\}$. As is the case of the dihedral groups [6], we see that

(4.1)
$$|\mu_j| = |z_j| + |w_j|.$$

Hence, since

$$z_{j} = \sum_{\substack{x^{k_{1} \in S} \\ 0 \le k_{1} \le 2m-1}} \omega^{jk_{1}} = -\sum_{\substack{x^{k_{1} \notin S} \\ 0 \le k_{1} \le 2m-1}} \omega^{jk_{1}},$$
$$w_{j} = \sum_{\substack{x^{k_{2}} y \in S \\ 0 \le k_{2} \le 2m-1}} \omega^{jk_{2}} = -\sum_{\substack{x^{k_{2}} y \notin S \\ 0 \le k_{2} \le 2m-1}} \omega^{jk_{2}},$$

we have $|\mu_j| \leq \min\{|S_1|, l_1(S)\} + \min\{|S_2|, l_2(S)\}$. Now, it is easy to see that the right-hand side of the inequality is bounded above by l(S).

Proposition 4.2. Let $l_0 = \lfloor 4\sqrt{m} \rfloor - 2$. Then, we have $\tilde{l} \ge l_0$.

Proof. From Lemma 4.1, we see that if $l(S) \leq \operatorname{RB}(S) = 2\sqrt{|S| - 1} = 2\sqrt{(4m - l(S)) - 1}$, then X(S) is Ramanujan. Now one sees that this is equivalent to $l(S) \leq 4\sqrt{m} - 2$ and hence obtain the desired result. Remark that $l(S) \leq 4\sqrt{m} - 2$ implies that $l(S) \leq 2m$, that is, $|S| \geq 2m$ for all $m \geq 1$.

We call l_0 a trivial bound of \tilde{l} . Using Lemma 3.2, we can easily determine the bound \tilde{l} in the case of $S = S_{Q_{4m}}$.

Theorem 4.3. We have $\tilde{l} = l_0$.

Proof. Take any $S \in S_{l_0+1}$ with $l_2(S) = 0$, that is, $S \in S_{l_0+1,0}$. Then, from Lemma 3.2, we have $|\lambda_2| = l_0 + 1$ and hence, by the definition of l_0 , $|\lambda_2| > \text{RB}(S)$. This means that X(S) is not Ramanujan.

4.2 A modification

From Theorem 4.3, in the case of $S = S_{Q_{4m}}$, we may not expect a connection between our problem on Ramanujan graphs and a problem on analytic number theory, as our previous studies in the cases of the cyclic and dihedral groups [5, 6]. So, we next take another set of Cayley subsets of Q_{4m} , that is,

$$\mathcal{S}' = \{ S \in \mathcal{S}_{Q_{4m}} \mid l_2(S) \neq 0 \}.$$

Notice that $l_2(S) \neq 0$ is equivalent to $S_2 \neq \langle x \rangle y$. This means that the setting on \mathcal{S}' is reasonable in the sense that we do not consider the extreme case $S_2 = \langle x \rangle y$. Furthermore, put $\mathcal{L}' = \{l(S) \mid S \in \mathcal{S}'\}$ and $\mathcal{S}'_l = \mathcal{S}_l \cap \mathcal{S}'$. Now our new purpose is to determine the bound

 $\tilde{l}' = \max\{l \in \mathcal{L}' \mid X(S) \text{ is Ramanujan for all } S \in \mathcal{S}'_k \ (1 \le k \le l)\}.$

It is clear that

$$(4.2) l_0.$$

Moreover, it holds that

Theorem 4.4. When m is even, we have $\tilde{l}' = l_0$.

Proof. From Lemma 3.2 (2), we can find $S \in \mathcal{S}'_{l_0+1}$ with $l_2(S) \neq 0$ satisfying $|\lambda_3| = l_0 + 1 > \operatorname{RB}(S)$ (or $|\lambda_4| = l_0 + 1 > \operatorname{RB}(S)$). This immediately shows that X(S) is not Ramanujan.

From this theorem, we may assume in what follows that m is odd. Remark that, in this case, from Lemma 3.2 again, we have $|\lambda_i| < l$ for $2 \leq i \leq 4$ for any $l \in \mathcal{L}'$ and $S \in \mathcal{S}'_l$.

4.3 An upper bound of l'

As is the case of \mathcal{S} , it is convenient to decompose \mathcal{S}' as follows;

(4.3)
$$\mathcal{S}' = \bigsqcup_{l \in \mathcal{L}'} \mathcal{S}'_l = \bigsqcup_{l \in \mathcal{L}'} \bigsqcup_{(l_1, l_2) \in \mathcal{L}'_l} \mathcal{S}'_{l_1, l_2}$$

where

$$\mathcal{L}'_{l} = \left\{ (l_{1}, l_{2}) \in \mathbb{Z}^{2} \mid \begin{array}{c} 0 < l_{1} \leq 2m, \ l_{1} \equiv l \pmod{2}, \\ 0 < l_{2} < 2m, \ l_{2} \equiv 0 \pmod{2}, \end{array} \right. l_{1} + l_{2} = l \left. \right\}$$

and $\mathcal{S}'_{l_1,l_2} = \mathcal{S}_{l_1,l_2} \cap \mathcal{S}'$ for $l \in \mathcal{L}'$ and $(l_1,l_2) \in \mathcal{L}'_l$.

The aim of this subsection is to show the following result.

Proposition 4.5. For $m \ge 65$, we have $\tilde{l}' = l_0$ or $\tilde{l}' = l_0 + 1$.

Let $l \in \mathcal{L}'$. To prove Proposition 4.5, we first construct $S^{(l_1,l_2)} \in \mathcal{S}'_{l_1,l_2}$ for each $(l_1,l_2) \in \mathcal{L}'_l$ such that $X(S^{(l_1,l_2)})$ may have the maximal eigenvalue (in the sense of absolute value) among X(S) with $S \in \mathcal{S}'_{l_1,l_2}$. Let $(l_1,l_2) \in \mathcal{L}'_l$. We define $S^{(l_1,l_2)} = S_1^{(l_1)} \sqcup S_2^{(l_2)} \in \mathcal{S}'_{l_1,l_2}$ by

$$\begin{split} S_1^{(l_1)} &= \langle x \rangle \setminus \{1, x^{\pm 1}, \dots, x^{\pm \frac{l_1 - 2 + \delta}{2}}\} \cup \{x^m\}^{1 - \delta}, \\ S_2^{(l_2)} &= \langle x \rangle y \setminus \{y, xy, \dots, x^{\frac{l_2}{2} - 1}y, x^m y, x^{m+1}y, \dots, x^{m + \frac{l_2}{2} - 1}y\}, \end{split}$$

where $\delta = 1$ if l is odd and 0 otherwise. We respectively write z_j , w_j and $|\mu_j|$ as $z_j^{(l_1,l_2)}$, $w_j^{(l_1,l_2)}$ and $|\mu_j^{(l_1,l_2)}|$ when $S = S^{(l_1,l_2)}$. Recall that $w_j^{(l_1,l_2)} = 0$ when j is odd. On the other hand when j is even, we have

$$w_j^{(l_1,l_2)} = -2\sum_{k_2=0}^{\frac{l_2}{2}-1} e^{\frac{2\pi i j k_2}{2m}} = -2e^{\frac{\pi i j (l_2-2)}{4m}} \frac{\sin \frac{\pi j l_2}{4m}}{\sin \frac{\pi j}{2m}}$$

Moreover, $z_j^{(l_1,l_2)}$ is calculated as

$$z_j^{(l_1,l_2)} = -\left(\sum_{k_1=-\frac{l_1-2+\delta}{2}}^{\frac{l_1-2+\delta}{2}} e^{\frac{2\pi i j k_1}{2m}} + (1-\delta)(-1)^j\right)$$
$$= -\left(\frac{\sin\frac{\pi j (l_1-1+\delta)}{2m}}{\sin\frac{\pi j}{2m}} + (1-\delta)(-1)^j\right).$$

Hence we have

(4.4)
$$|\mu_j^{(l_1, l_2)}| = \left(\frac{\sin\frac{\pi j(l_1 - 1 + \delta)}{2m}}{\sin\frac{\pi j}{2m}} + (1 - \delta)(-1)^j\right) + \delta_j \left(2\frac{\sin\frac{\pi j l_2}{4m}}{\sin\frac{\pi j}{2m}}\right),$$

where $\delta_j = 1$ if j is even and 0 otherwise. We now focus on the case of j = 2.

Lemma 4.6. Let $l \in \mathcal{L}'$. When $l \equiv r \pmod{6}$ for $0 \leq r \leq 5$, we have

(4.5)
$$\max\left\{ |\mu_2^{(l_1,l_2)}| \mid (l_1,l_2) \in \mathcal{L}'_l \right\} = |\mu_2^{(\check{l}_1,\check{l}_2)}|$$

where $(\check{l}_1, \check{l}_2) = (\frac{l+a_r}{3}, \frac{2l-a_r}{3}) \in \mathcal{L}'_l$ with

$$a_1 = 2$$
, $a_3 = 0$, $a_5 = -2$, $a_0 = 0$, $a_2 = 4$, $a_4 = 2$.

Proof. It holds that

$$\frac{\partial}{\partial l_2} |\mu_2^{(l_1, l_2)}| = \frac{\frac{2\pi}{m}}{\sin\frac{\pi}{m}} \sin\frac{\pi (2(l-1+\delta)-l_2)}{4m} \sin\frac{\pi (2(l-1+\delta)-3l_2)}{4m}.$$

Hence, noting that l, l_1 and l_2 are small enough rather than m, we see that, as a continuous function of l_2 , $\frac{\partial}{\partial l_2} |\mu_2^{(l_1,l_2)}| = 0$ on [1,l] if and only if $l_2 = \frac{2(l-1+\delta)}{3}$, which means that $|\mu_2^{(l_1,l_2)}|$ is monotone increasing on $[1, \frac{2(l-1+\delta)}{3}]$ and decreasing on $[\frac{2(l-1+\delta)}{3}, l]$. Let us find $(l_1^{\pm}, l_2^{\pm}) \in \mathcal{L}'_l$ such that l_2^- is the maximum and l_2^+ the minimum integer satisfying $l_2^- \leq \frac{2(l-1+\delta)}{3} \leq l_2^+$ (notice

that l_1^{\pm} are automatically determined from l_2^{\pm} by $l_1^{\pm} + l_2^{\pm} = l$). If we write l = 6k + r, then one sees that these are respectively given as follows:

Now the result follows from the facts $|\mu_2^{(l_1^-, l_2^-)}| > |\mu_2^{(l_1^+, l_2^+)}|$ for $r = 1, 2, |\mu_2^{(l_1^-, l_2^-)}| = |\mu_2^{(l_1^+, l_2^+)}|$ for r = 3, 4 and $|\mu_2^{(l_1^-, l_2^-)}| < |\mu_2^{(l_1^+, l_2^+)}|$ for r = 5, 0. Namely, $(\check{l}_1, \check{l}_2) = (l_1^-, l_2^-)$ for $r = 1, 2, (l_1^-, l_2^-) = (l_1^+, l_2^+)$ for r = 3, 4 and (l_1^+, l_2^+) for r = 5, 0.

Using Lemma 4.6, we give a proof of Proposition 4.5.

Proof of Proposition 4.5. It is sufficient to show that there exists $S \in \mathcal{L}'_{l_0+2}$ such that X(S) is not Ramanujan. Actually, let $l_0 = \lfloor 4\sqrt{m} \rfloor - 2 \equiv r \pmod{6}$ for $0 \leq r \leq 5$. Take $S^{(\check{l}_1,\check{l}_2)} \in \mathcal{S}'_{l_0+2}$ with $(\check{l}_1,\check{l}_2) = (\frac{l_0+2+a_{r+2}}{3}, \frac{2(l_0+2)-a_{r+2}}{3}) \in \mathcal{L}'_{l_0+2}$. Here the index of a_r is considered modulo 6. Then, noticing that $4\sqrt{m} - 1 < l_0 + 2 \leq 4\sqrt{m}$, we have

$$\begin{split} |\mu_{2}^{(\tilde{l}_{1},\tilde{l}_{2})}| &- \operatorname{RB}(S^{(\tilde{l}_{1},\tilde{l}_{2})}) \\ = \frac{\sin \frac{\pi(\tilde{l}_{1}-1+\delta)}{\sin \frac{\pi}{m}} + (1-\delta) + 2\frac{\sin \frac{\pi\tilde{l}_{2}}{2m}}{\sin \frac{\pi}{m}} - 2\sqrt{4m - (\tilde{l}_{1}+\tilde{l}_{2}) - 1} \\ &= \frac{\sin \left(\frac{\pi}{m}(\frac{l_{0}+2+a_{r+2}}{3}-1+\delta)\right)}{\sin \frac{\pi}{m}} + (1-\delta) + 2\frac{\sin \left(\frac{\pi}{2m}\frac{2(l_{0}+2)-a_{r+2}}{3}\right)}{\sin \frac{\pi}{m}} \\ &- 2\sqrt{4m - (l_{0}+2) - 1} \\ &> \frac{\sin \frac{\pi}{m}(\frac{4\sqrt{m}-1+a_{r+2}}{3}-1+\delta)}{\sin \frac{\pi}{m}} + (1-\delta) + 2\frac{\sin \frac{\pi}{2m}\frac{2(4\sqrt{m}-1)-a_{r+2}}{3}}{\sin \frac{\pi}{m}} \\ &- 2\sqrt{4m - (4\sqrt{m}-1) - 1} \\ &= 1 - \frac{64\pi^{2}-27}{54}m^{-\frac{1}{2}} + O(m^{-1}) \end{split}$$

as $m \to \infty$. This shows that $|\mu_2^{(\check{l}_1,\check{l}_2)}| > \operatorname{RB}(S^{(\check{l}_1,\check{l}_2)})$ for sufficiently large m and hence concludes that the corresponding Cayley graph $X(S^{(\check{l}_1,\check{l}_2)})$ is not Ramanujan. Actually, one can check that the inequality holds for $m \ge 105$. Moreover, we can numerically see that $|\mu_2^{(\check{l}_1,\check{l}_2)}| - \operatorname{RB}(S^{(\check{l}_1,\check{l}_2)}) >$ 0 for $65 \le m \le 103$ (however it does not hold when m = 63).

4.4 A characterization of exceptional primes

From now on, we concentrate on the case where m = p is odd prime (we can perform the similar discussion for general m as in [5], though it may be complicated). We know from Proposition 4.5 that it can be written as $\tilde{l}' = l_0 + \varepsilon$ for some $\varepsilon = \varepsilon_p \in \{0, 1\}$. As is the case of the cyclic and dihedral graphs [5, 6], we call p exceptional if $\varepsilon = 1$ and ordinary otherwise. Now our task is to clarify which $p \in \mathbb{P}$ is exceptional.

For $l \in \mathcal{L}'$, let $\lambda(l) = \max\{\lambda(S) | S \in \mathcal{S}'_l\}$ and $\operatorname{RB}(l) = 2\sqrt{4p - l - 1}$, which is nothing but the Ramanujan bound of X(S) for $S \in \mathcal{S}'_l$. From the definition, p is exceptional if and only if $\lambda(l_0 + 1) \leq \operatorname{RB}(l_0 + 1)$.

Lemma 4.7. Let $l \in \mathcal{L}'$. For $(l_1, l_2) \in \mathcal{L}'_l$, let $\lambda(l_1, l_2) = \max\{\lambda(S) \mid S \in \mathcal{S}'_{l_1, l_2}\}$. Then, we have $\lambda(l_1, l_2) = |\mu_2^{(l_1, l_2)}|$ for sufficiently large p.

Proof. Take any $S \in S'_{l_1,l_2}$. When j is odd, since $w_j = 0$, we have $|\mu_j| = |z_j| \le |z_1^{(l_1,l_2)}| = |\mu_1^{(l_1,l_2)}|$ because p is prime. On the other hand when j is even, since jk is always even modulo 2p for any k, it holds that $|\mu_j| \le |\mu_2^{(l_1,l_2)}|$ by the same reason as above. Moreover, since

$$\begin{aligned} |\mu_1^{(l_1,l_2)}| &= \frac{\sin\frac{\pi(l_1-1+\delta)}{2p}}{\sin\frac{\pi}{2p}} - (1-\delta) = (-2+2\delta+l_1) + O(p^{-2}),\\ |\mu_2^{(l_1,l_2)}| &= \frac{\sin\frac{\pi(l_1-1+\delta)}{p}}{\sin\frac{\pi}{p}} + (1-\delta) + 2\frac{\sin\frac{\pi l_2}{2p}}{\sin\frac{\pi}{p}} = (l_1+l_2) + O(p^{-2}), \end{aligned}$$

we see that $|\mu_2^{(l_1,l_2)}| - |\mu_1^{(l_1,l_2)}| = l_2 + 2 - 2\delta + O(p^{-2})$ as $p \to \infty$. Hence, under the condition $l_2 > 0$, we have $|\mu_2^{(l_1,l_2)}| > |\mu_1^{(l_1,l_2)}|$ for sufficiently large p. Combining this together with the fact $\max\{|\lambda_i| \mid 2 \le i \le 4, S \in S'_{l_1,l_2}\} = l_1 + l_2 - 2 < |\mu_2^{(l_1,l_2)}|$ for sufficiently large p, one obtains the claim.

Proposition 4.8. Let $p \ge 67$. When $l_0 \equiv r \pmod{6}$ for $0 \le r \le 5$, we have

$$\lambda(l_0 + 1) = |\mu_2^{(\check{l}_1, \check{l}_2)}|,$$

where $(\check{l}_1, \check{l}_2) = (\frac{l_0 + 1 + a_{r+1}}{3}, \frac{2(l_0 + 1) - a_{r+1}}{3}) \in \mathcal{L}'_{l_0 + 1}.$

Proof. This follows immediately from Lemma 4.6 and 4.7. Remark that the inequality $|\mu_2^{(\tilde{l}_1, \tilde{l}_2)}| - |\mu_1^{(\tilde{l}_1, \tilde{l}_2)}| > 0$ in fact holds for $p \ge 67$.

Write $l_0 = \lfloor 4\sqrt{p} \rfloor - 2$ as

$$l_0 = 24k + r$$

for $k \ge 0$ and $0 \le r \le 23$. In this case, we see that $p \in I_{r,k} \cap \mathbb{P}$ where

$$I_{r,k} = \left\{ t \in \mathbb{R} \left| \lfloor 4\sqrt{t} \rfloor - 2 = 24k + r \right\} \right.$$
$$= \left[36k^2 + 3(r+2)k + \frac{(r+2)^2}{16}, 36k^2 + 3(r+3)k + \frac{(r+3)^2}{16} \right).$$

In other words, p can be written as $p = f_{r,c}(k)$ for some integers $k \ge 0$ and $c \in \mathbb{Z}$ with $f_{r,c}(x)$ being a quadratic polynomial defined by

$$f_{r,c}(x) = 36x^2 + 3(r+3)x + c$$

and $-3k + \left\lceil \frac{(r+2)^2}{16} \right\rceil \le c \le \lfloor \frac{(r+3)^2}{16} \rfloor.$

For $0 \le r \le 23$, let $I_r = \bigsqcup_{k\ge 0} I_{r,k} \cap \mathbb{P}$ and $C_r = \{\lfloor \frac{(r+3)^2}{16} \rfloor + s \mid -5 \le s \le 0\}$. Moreover, let $C'_r = \{c \in C_r \mid f_{r,c}(x) \text{ is irreducible over } \mathbb{Z}\}$. Furthermore, for $c \in C'_r$, define $k_{r,c} \in \mathbb{Z}$ as in Table 2. The following is our main result, which gives a characterization for the exceptional primes.

Theorem 4.9. A prime $p \in I_r$ with $p \ge 67$ is exceptional if and only if it is of the form of $p = f_{r,c}(k)$ for some $c \in C'_r$ and $k \ge k_{r,c}$.

Proof. We first notice that, from the previous discussion with Proposition 4.8, p is exceptional if and only if $|\mu_2(\check{l}_1,\check{l}_2)| \leq \text{RB}(l_0+1)$. To clarify when this inequality holds, we introduce an interpolation function $F_r(t)$ of the difference between $|\mu_2(\check{l}_1,\check{l}_2)|$ and $\text{RB}(l_0+1)$ on $I_{r,k}$, that is,

$$F_r(t) = \frac{\sin\frac{\pi(8k + \frac{r+1+a_{r+1}}{3} - 1 + \delta)}{t}}{\sin\frac{\pi}{t}} + (1 - \delta) + 2\frac{\sin\frac{\pi(16k + \frac{2(r+1)-a_{r+1}}{3})}{2t}}{\sin\frac{\pi}{t}} - 2\sqrt{4t - (24k + r + 1) - 1}.$$

Notice that $(\check{l}_1, \check{l}_2) = \left(8k + \frac{r+1+a_{r+1}}{3}, 16k + \frac{2(r+1)-a_{r+1}}{3}\right)$ when $l_0 = 24k + r$. One can see that $F_r(t)$ is monotone decreasing on $I_{r,k}$ for sufficiently large k. Moreover, at $t = p = f_{r,c}(k) \in I_{r,k} \cap \mathbb{P}$, one has

$$F_r(p) = \frac{27(r+3)^2 - 432c - 256\pi^2}{1296}k^{-1} + O(k^{-2})$$

as $k \to \infty$ because

$$\begin{aligned} |\mu_2(\check{l_1},\check{l_2})| &= \frac{\sin\frac{\pi(8k+\frac{r+1+a_{r+1}}{3}-1+\delta)}{36k^2+3(r+3)k+c}}{\sin\frac{\pi}{36k^2+3(r+3)k+c}} + (1-\delta) + 2\frac{\sin\frac{\pi(16k+\frac{2(r+1)-a_{r+1}}{3})}{2(36k^2+3(r+3)k+c)}}{\sin\frac{\pi}{36k^2+3(r+3)k+c}} \\ &= 24k + (1+r) - \frac{16\pi^2}{81}k^{-1} + O(k^{-2}), \\ \operatorname{RB}(l_0+1) &= 2\sqrt{4(36k^2+3(r+3)k+c) - (24k+r+1) - 1} \\ &= 24k + (1+r) - \frac{(r+3)^2 - 16c}{48}k^{-1} + O(k^{-2}). \end{aligned}$$

This shows that $F_r(p) < 0$ for sufficiently large k if and only if $27(r+3)^2 - 432c - 256\pi^2 < 0$, in other words, $\lceil \frac{27(r+3)^2 - 256\pi^2}{432} \rceil \le c$. Here, we see that $\lceil \frac{27(r+3)^2 - 256\pi^2}{432} \rceil = \lfloor \frac{(r+3)^2}{16} \rfloor - 5$ for all $0 \le r \le 23$, which means that $c \in C_r$. Moreover, since $f_{r,c}(k)$ does not express any prime if $f_{r,c}(x)$ is not irreducible over \mathbb{Z} , c must be in C'_r . Furthermore, it is checked that, for each $0 \le r \le 23$ and $c \in C'_r$, the inequalities $f_{r,c}(k) \ge 67$ and $F_r(p) < 0$ for $p = f_{r,c}(k)$ hold if and only if $k \ge k_{r,c}$. This completes the proof of the theorem.

For $0 \leq r \leq 23$ and $c \in C'_r$, let

$$J_{r,c} = \{ p \mid p = f_{r,c}(k) \in I_r \text{ for some } k \ge k_{r,c} \}.$$

Namely, $J_{r,c}$ is the set of exceptional primes p of the form of $p = f_{r,c}(k)$. We show the first five such primes in Table 2 for each r and c.

The classical Hardy-Littlewood conjecture [4, Conjecture F] asserts that if a quadratic polynomial $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$ satisfies the conditions that a > 0, a, b and c are relatively prime, a + b and c are not both even and the discriminant $D(f) = b^2 - 4ac$ of f is not a square, then there are infinitely many primes represented by f(x) and, moreover, that

$$\pi(f;x) = \#\{p \le x \mid p = f(k) \in \mathbb{P} \text{ for some } k \ge 0\}$$

obeys the asymptotic behavior

$$\pi(f;x) \sim \frac{\varepsilon(f)C(f)}{\sqrt{a}} \prod_{\substack{p \mid a, p \mid b \\ p \ge 3}} \frac{p}{p-1} \cdot \frac{\sqrt{x}}{\log x}$$

r	$c \in C'_r$	$f_{r,c}(x)$	$k_{r,c}$	$J_{r,c}$	$N_{r,c}$	$\frac{C(f_{r,c})}{2\delta_r}$
0	-5	$36x^2 + 9x - 5$	9	7177, 11821, 20947, 52321, 121621	9597	0.24501
0	-4	$36x^2 + 9x - 4$	2	347, 941, 1823, 4451, 6197	17722	0.45086
0	-2	$36x^2 + 9x - 2$	2	349, 6199, 8233, 16063, 19249	11061	0.28123
1	-1	$36x^2 + 12x - 1$	2	167, 359, 1367, 1847, 2399	24414	0.61666
2	-4	$36x^2 + 15x - 4$	9	4517, 16187, 22871, 30707, 44621	9685	0.24501
2	-2	$36x^2 + 15x - 2$	1	367, 1867, 3049, 4519, 6277	13501	0.34106
2	-1	$36x^2 + 15x - 1$	1	173, 2423, 11933, 14699, 28643	11181	0.28123
3	-1	$36x^2 + 18x - 1$	2	179, 647, 1889, 2447, 3779	31692	0.80725
3	1	$36x^2 + 18x + 1$	1	181, 379, 991, 3079, 7309	23288	0.59109
4	-1	$36x^2 + 21x - 1$	1	659,7349,9551,12041,33029	10633	0.26894
4	1	$36x^2 + 21x + 1$	1	661, 1423, 2473, 5437, 7351	15712	0.40086
4	2	$36x^2 + 21x + 2$	2	389, 1913, 6359, 13397, 16319	15405	0.39341
5	-1	$36x^2 + 24x - 1$	2	191, 1019, 1439, 1931, 5471	23332	0.59109
5	1	$36x^2 + 24x + 1$	2	193, 397, 673, 1021, 1933	27255	0.69166
6	1	$36x^2 + 27x + 1$	1	199, 1459, 2521, 9649, 33211	10609	0.26894
6	4	$36x^2 + 27x + 4$	1	67, 409, 1039, 3163, 4657	15494	0.39341
7	1	$36x^2 + 30x + 1$	2	1051, 3187, 7477, 9697, 13567	18210	0.46393
7	5	$36x^2 + 30x + 5$	1	71,419,701,1481,1979	23192	0.59109
8	2	$36x^2 + 33x + 2$	9	4721, 8597, 23327, 61871, 81077	9591	0.24501
8	4	$36x^2 + 33x + 4$	1	73, 1069, 1999, 3217, 4723	13526	0.34106
8	5	$36x^2 + 33x + 5$	1	1499, 7523, 9749, 12263, 29153	10933	0.28123
9	7	$36x^2 + 36x + 7$	1	79, 223, 439, 727, 1087	24281	0.61666
10	5	$36x^2 + 39x + 5$	9	5657, 7607, 18287, 65147, 99377	9537	0.24501
10	7	$36x^2 + 39x + 7$	1	229, 739, 5659, 12373, 15187	13322	0.34106
10	8	$36x^2 + 39x + 8$	1	83, 449, 1103, 4793, 6599	11175	0.28123
11	7	$36x^2 + 42x + 7$	2	457, 751, 1117, 2647, 3301	18110	0.46393
	11	$36x^2 + 42x + 11$	1	89, 239, 461, 1559, 2069	23297	0.59109
12	10	$36x^2 + 45x + 10$	1	2089, 3331, 4861, 6679, 16831	10588	0.26894
12	13	$36x^2 + 45x + 13$	1	169, 1579, 2677, 5737, 7699	15505	0.39341
13	11	$36x^2 + 48x + 11$	1	251, 479, 1151, 2111, 2699	23137	0.59109
13	13	$36x^2 + 48x + 13$	1	97,1153,1597,2113,3361	27257	0.69166
14	14	$36x^2 + 51x + 14$	1	101, 491, 3389, 4931, 6761	10559	0.26894
14	10	30x + 51x + 10 $36x^2 + 51x + 17$	1	$\begin{array}{c} 103, 1171, 2137, 3391, 4933 \\ \hline 263, 707, 1610, 2720, 4127 \\ \hline \end{array}$	15202	0.40080
15	17	$36x^2 + 54x + 17$	1		21695	0.03041
15	17	$36x^2 + 54x + 17$	1	107, 209, 505, 809, 1187 100, 271, 811, 2161, 4150	23208	0.80725
16	17	$36x^2 + 57x + 17$	2	2777 20827 34127 54167 72221	0606	0.00100
16	10	$36x^2 + 57x + 17$ $36x^2 + 57x + 19$	1	277 823 1657 7873 15559	13448	0.24501
16	20	$36x^2 + 57x + 20$	1	113, 3449, 5003, 11393, 17093	11096	0.28123
17	23	$36x^2 + 60x + 23$	1	839, 1223, 2207, 5039, 5927	24229	0.61666
18	 22	$36x^2 + 63x + 20$	8	9067. 11497 24097 27967 36571	9662	0.24501
18	23	$36x^2 + 63x + 23$	1	293, 1697, 4253, 10247, 12821	17614	0.45086
18	25	$36x^2 + 63x + 25$	1	853, 1699, 2833, 7963, 12823	10918	0.28123
19	25	$36x^2 + 66x + 25$	1	127.547.2251.2857.5107	18271	0.46393
19	29	$36x^2 + 66x + 29$	1	131, 1259, 1721, 2861, 3539	23270	0.59109
20	29	$36x^2 + 69x + 29$	1	311, 881, 15809, 34499, 43991	10567	0.26894
20	31	$36x^2 + 69x + 31$	1	313, 883, 1741, 2887, 6043	15875	0.40086
20	32	$36x^2 + 69x + 32$	1	137, 563, 1277, 5147, 7013	15649	0.39341
21	31	$36x^2 + 72x + 31$	1	139, 571, 1291, 1759, 5179	23262	0.59109
22	35	$36x^2 + 75x + 35$	1	911, 2939, 13049, 22571, 26321	10591	0.26894
22	37	$36x^2 + 75x + 37$	1	331, 1783, 6121, 10453, 15937	15764	0.40086
22	38	$36x^2 + 75x + 38$	1	149,587,11717,17489,20807	15460	0.39341
23	37	$36x^2 + 78x + 37$	2	337, 1327, 1801, 2347, 10501	18177	0.46393
23	41	$36x^2 + 78x + 41$	1	599, 929, 2351, 2969, 3659	23223	0.59109

as $x \to \infty$ where $\varepsilon(f)$ is 1 if a + b is odd and 2 otherwise and

$$C(f) = \prod_{\substack{p \nmid a \\ p \ge 3}} \left(1 - \frac{\left(\frac{D(f)}{p}\right)}{p-1} \right)$$

with $\left(\frac{D}{p}\right)$ being the Legendre symbol. The constant C(f) is called the Hardy-Littlewood constant of f. Because the polynomial $f_{r,c}(x)$ satisfies the above conditions, we can expect that it indeed represents infinitely many primes. In our case, it may hold that

$$\pi(f_{r,c};x) \sim \frac{C(f_{r,c})}{2\delta_r} \frac{\sqrt{x}}{\log x}, \quad C(f_{r,c}) = \prod_{p \ge 5} \left(1 - \frac{\left(\frac{(r+3)^2 - 16c}{p}\right)}{p-1}\right).$$

We also show both the numerical value of $\frac{C(f_{r,c})}{2\delta_r}$ and the exact number of $N_{r,c} = \#\{p \le 10^{12} | p = f_{r,c}(k) \in I_r$ for some $k \ge k_{r,c}\}$ in Table 2. Notice that $\frac{\sqrt{x}}{\log x} = 36191.20...$ when $x = 10^{12}$.

The following is immediate from Theorem 4.9.

Corollary 4.10. There exists infinitely many exceptional primes if the Hardy-Littlewood conjecture is true for at least one of $f_{r,c}(x)$ for $0 \le r \le 23$ and $c \in C'_r$.

We notice that, if we can show that there exists infinitely many exceptional primes (in the frame work of the graph theory) on the other hand, then it implies that at least one of $f_{r,c}(t)$ represents infinitely many primes.

We also remark that though we omit to show here but the existence of infinitely many ordinary primes is similarly verified by using Dirichlet's theorem on arithmetic progressions as [5, 6].

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