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Muhammad Akram · Musavarah Sarwar ·
Wieslaw A. Dudek

Graphs for the Analysis of Bipolar Fuzzy Information

Springer

Muhammad Akram
Department of Mathematics
University of the Punjab
Lahore, Pakistan

Musavarah Sarwar
Department of Mathematics
Government College Women University
Sialkot, Pakistan

Wieslaw A. Dudek
Faculty of Pure and Applied Mathematics
Wrocław University of Technology
Wrocław, Poland

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We dedicate this book to the memory of Professor Lotfi Zadeh!

Foreword

Graphs are considered as one of the fundamental structures studied in combinatorics. The subject started with the paper published by Euler on the seven bridges of Königsberg in 1736. Graph theory is highly applied in other areas of mathematics and other branches of science. However, "crisp graphs" are not appropriate for describing all of the existing relations between objects. Thus, Kauffman (1973) presented the first definition of fuzzy graphs based on Zadeh's fuzzy relations. In addition, Rosenfeld (1975) introduced "fuzzy analogs" of several basic graph-theoretic concepts.

In many real-world problems, "bipolarity" plays a role when it is important to distinguish between positive information representing satisfaction degree and negative information indicating dissatisfaction degree. Both types of information cannot be understood simultaneously by the fuzzy model, leading to the existence of bipolar information. This type of information cannot be well-represented by means of fuzzy graphs. In these situations, bipolar fuzzy set theory is applied to the graphs in order to describe the relationships. Muhammad Akram (2011) first introduced the notion of bipolar fuzzy graphs and published different papers with his colleagues and students on a wide variety of bipolar fuzzy graph-theoretic structures. This book presents some fundamental theories of such types of bipolar fuzzy graphs, complex bipolar fuzzy graphs, Cayley bipolar fuzzy graphs, bipolar fuzzy planar graphs, bipolar fuzzy matroids, double dominating energy of bipolar fuzzy graphs, bipolar single-valued neutrosophic competition graphs, and bipolar single-valued neutrosophic graph structures.

The authors are well-known researchers in the field of fuzzy graphs. They have presented all concepts of bipolar fuzzy graph theory with applications in a variety of diverse fields and have paid careful attention to algorithmic issues. Therefore, the book can contribute to the development of fuzzy graph theory for students and researchers.

Rajab Ali Borzooei
Shahid Beheshti University
Tehran, Iran

Preface

The motivation of this monograph can be traced back to the work of late Lotfi A. Zadeh (1921–2017). Professor Zadeh introduced fuzzy set theory in 1965, as a model for the representation of uncertainty and vagueness in real-world systems. A mathematical framework—fuzzy logic—was initiated in order to give formal support to the notion of partial truth between "absolute true" and "absolute false." This meant one of the most important, creative, and fruitful concepts introduced in science and technology during the second half of the twentieth century. One of its interactions with other branches of mathematics produced the concept of fuzzy graphs, which is based on Zadeh's fuzzy relations, originated with Kafmann. In 1975, Rosenfeld laid the foundations for fuzzy graph theory. Mordeson and Nair also made early contributions to that field.

Some aspects of data uncertainty are likely to appear in many contexts and scientific disciplines. Thus, different forms of vagueness have been recognized in applied fields. Some come from conflicting or incomplete information, as well as from multiple interpretations of some phenomena. In applications including expert systems, belief systems, and information fusion, it is not only convenient to use the one real value from the unit interval [0, 1] supported by the evidence, but also useful to consider the counter-membership against the evidence. Such 'bipolarity' is a core feature in many domains of information processing, both from the point of view of knowledge representation and from the perspective of processing and reasoning. Bipolarity appears whenever it is important to distinguish between (*i*) positive information that represents what is possible or preferred and (*ii*) negative information that represents what is impossible or forbidden. Both inputs cannot be simultaneously discerned by the fuzzy model. Motivated by this concern, Zhang introduced in 1994 a new component that expresses a negative degree of membership in addition to the structure of a fuzzy set. This idea launched the concept of bipolar fuzzy set that inspires the concept to which this monograph is devoted. With this work we aim at filling a gap, to wit, the lack of a mathematical approach towards bipolar information, that is, the coexistence of positive and negative information.

This volume is based on a number of papers by the authors that have been published in various scientific journals. They are part of a long-lasting research program that concerns decision-making methods under bipolar fuzzy graphical models. Now they are wrapped together in a book that may be useful for researchers in mathematics, computer scientists, and social scientists alike. Let us describe its content.

In Chap. 1, we review the notion of bipolar fuzzy sets. Then, we present the concept of bipolar fuzzy graphs and describe various methods of their construction. We discuss the concept of isomorphism of bipolar fuzzy graphs and investigate their fundamental properties. We present certain types of bipolar fuzzy graphs, complex bipolar fuzzy graphs, bipolar fuzzy walk, bipolar fuzzy bridge, strength of connectedness, and weak and strong bipolar fuzzy edges. We also endorse the importance of bipolar fuzzy digraphs with a number of real-world problems.

In Chap. 2, we discuss certain properties of distance functions in bipolar fuzzy graphs. We establish the formulae of distance in complete bipolar fuzzy graphs, complete bipartite bipolar fuzzy graphs and products of bipolar fuzzy graphs. We present an algorithm for computing the distance matrix, eccentricity of the vertices, radius, and diameter in bipolar fuzzy graphs. We study the concepts of self-centered bipolar fuzzy graphs and antipodal bipolar fuzzy graphs. We describe applications of bipolar fuzzy graphs in product manufacturing.

In Chap. 3, we present certain types of bipolar fuzzy graphs with various methods of construction. We discuss the concept of degree and isomorphism of irregular bipolar fuzzy graphs. We describe the concepts of bipolar fuzzy line graphs and the Cayley bipolar fuzzy graphs.

In Chap. 4, we present different types of concepts concerning bipolar fuzzy competition graphs and their extensions. Various methods for the construction of bipolar fuzzy competition graphs of certain products of bipolar fuzzy digraphs are studied with theoretical proofs. We study certain types of competition graphs under complex bipolar fuzzy environment. We describe applications of bipolar fuzzy competition graphs in different domains.

In Chap. 5, we apply the powerful technique of bipolar fuzzy sets to multigraphs and planar graphs. We present the notions of bipolar fuzzy multigraphs, bipolar fuzzy planar graphs, and bipolar fuzzy dual graphs. Chapter 6 considers various types of dominations in bipolar fuzzy graphs and their importance in decision-making models.

In Chap. 7, we study the theory of matroids and circuits in bipolar fuzzy environment, soft environment, and bipolar fuzzy soft environment. We present certain applications of bipolar fuzzy matroids in decision support systems, secret sharing problems, and network analysis.

Chapter 8 is concerned with various energy-like quantities of bipolar fuzzy graphs. In particular, we study Laplacian energy, signless Laplacian energy, dominating energy, and double dominating energy of bipolar fuzzy graphs. Two case studies illustrate these concepts.

In Chap. 9, we present the concepts of bipolar single-valued neutrosophic competition graphs, bipolar neutrosophic single-valued p-competition graphs, and m-step bipolar single-valued neutrosophic competition graphs. We describe some applications of bipolar single-valued neutrosophic competition graphs and m-step bipolar single-valued neutrosophic competition graphs in economics, organizational designations, and business competition.

In Chap. 10, we apply the concept of bipolar single-valued neutrosophic set to graph structures and study a framework to handle bipolar neutrosophic information by combining bipolar neutrosophic sets with graph structures. We study several operations on bipolar single-valued neutrosophic graph structures with interesting properties. We describe the isomorphism properties of various types of bipolar single-valued neutrosophic graph structures with φ-complement. We explain the importance of bipolar single-valued neutrosophic graph structures with real-world applications in international relations, psychology, and global terrorism.

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Lahore, Pakistan — Muhammad Akram
Sialkot, Pakistan — Musavarah Sarwar
Wrocław, Poland — Wieslaw A. Dudek

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About the Authors

Muhammad Akram is Professor at the Department of Mathematics, University of the Punjab, Lahore, Pakistan. He previously served at Punjab University College of Information Technology as Assistant Professor and Associate Professor. He earned his Ph.D. in Fuzzy Mathematics from Government College University, Lahore, Pakistan. His research interests include numerical algorithms, fuzzy graphs, fuzzy algebras, and fuzzy decision support systems. He has published 9 books and over 365 research articles in peer-reviewed international journals and supervised 12 Ph.D. students.

Musavarah Sarwar is Assistant Professor at Government College Women University, Sialkot, Pakistan. A gold medalist in academic excellence, Dr. Sarwar received her Ph.D. in Mathematics from the University of the Punjab, Lahore, Pakistan. Her research interests include graph theory, fuzzy graphs and fuzzy decision support systems/decision-making models. She has published 18 research articles in international peer-reviewed journals.

Wieslaw A. Dudek is Professor at the Department of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wrocław, Poland. He completed his Ph.D. in Mathematics at the Department of Higher Algebra, State University of Chisinau, Republic of Moldova, under the supervision of Prof. V. D. Belousov. He received his Degree of Habilitated Doctor from the Warsaw University of Technology. The main areas of his scientific interests include polyadic groups, quasigroups, fuzzy algebras, fuzzy graphs and their applications. Professor Dudek has authored 2 monographs, 4 textbooks, and published 200 research articles in international peer-reviewed journals. He is Editor-in-Chief of the journal *Quasigroups and Related Systems* and on the editorial board of several international journals. He has promoted 4 doctors of mathematics.

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