A Hybrid Quantum Secret Sharing Scheme based on Mutually Unbiased Bases

Dan-Li Zhi, Zhi-Hui Li*, Li-Juan Liu, and Zhao-Wei Han

Shaanxi Normal University, xi'an, Shaanxi, 710119, China

Abstract. With the advantages of both classical and quantum secret sharing, many practical hybrid quantum secret sharing have been proposed. In this paper, we propose a hybrid quantum secret sharing scheme based on mutually unbiased bases and monotone span program. First, the dealer sends the shares in the linear secret sharing to the participants in the authorization set via a secure channel. Then, the dealer and participants perform unitary transformation on a *d*-dimensional quantum state sequentially, and the dealer publishes the measurement result confidentially to the participants in the authorization set to recover the secret. The verifiability of the scheme is guaranteed by the Hash function. Next, the correctness and security of the scheme are proved and our scheme is secure against the general eavesdropper attacks. Finally, a specific example is employed to further clarify the flexibility of the scheme and the detailed comparison of similar quantum secret sharing schemes also shows the superiority of our proposed scheme.

Keywords: Quantum secret sharing · Mutually unbiased bases · Verifiability · Access structure.

1 Introduction

As a combination of cryptography and quantum mechanics, quantum cryptography plays an important role in cryptography. Compared with classical cryptography on the basis of computational complexity, quantum cryptography based on the laws of quantum physics can achieve unconditional security. Many branches of quantum cryptography have been developed, such as quantum key distribution(QKD)[1,2], quantum key agreement(QKA)[3-5], quantum secure direct communication(QSDC)[6,7], quantum teleportation[8,9], quantum signature[10,11], quantum authentication[12-14], quantum secret sharing(QSS)[15-30] and so on.

Quantum secret sharing (QSS) is an important research field in quantum cryptography, which means that the dealer divides a secret into several shadows and sends them to multiple participants. Only the participants in authorized sets can recover the secret, and the participants in unauthorized sets can not recover the secret. Since Hillery et al. [15] proposed the first quantum secret

^{*} lizhihui@snnu.edu.cn

sharing scheme by using GHZ state in 1999, a growing number of QSS schemes [16-30] have been proposed. For example, Williams et al. [22] described and experimentally demonstrated a three-party quantum secret sharing protocol using polarization-entangled photon pairs. Tsai et al. [23] used the entanglement property of W-state to propose the first three-party SQSS protocol. Song et al. [24] demonstrated a (t, n) threshold d-level quantum secret sharing scheme. A verifiable (t, n) threshold quantum secret sharing scheme was proposed using the *d*-dimensional Bell state and the Lagrange interpolation by Yang et al. in Ref. [25]. Hao et al. [26] put forward a secret sharing scheme using the mutually unbiased bases on the p^2 -dimensional quantum system. Bai et al. [27] proposed the concept of decomposition of quantum access structure to design a quantum secret sharing scheme. In Ref. [28], Liu et al. study the local distinguishability of the 15 kinds of seven-qudit quantum entangled states and then proposed a (k, n) threshold quantum secret sharing scheme. A new improving quantum secret sharing scheme was proposed by Xu et al. [29], in which more quantum access structures can be realized by the scheme than the one proposed by Nascimento et al. [30].

Although many schemes have been proposed, the verifiability and the flexibility of the schemes are also important issues worth of consideration. In this paper, we propose a hybrid and verifiable quantum secret sharing scheme based on mutually unbiased bases and the monotone span program, which focuses on transmitting a *d*-dimensional quantum state among the dealer Alice and participants and the application of the linear secret sharing. Each participant in a authorization set can perform a unitary transformation on the received particle and send it to the next one until the last one sends it to Alice. They can recover the secret by the linear secret sharing and the measurement value sent by Alice. Verifiability ensures that the secret recovered in each authorization set is the original one, and it also ensures that once a dishonest participant appears, he will be found. Compared with the threshold scheme, the quantum secret sharing scheme based on the access structure realizes the different influences of participants in the process of recovering secrets, thereby achieving the flexibility of the scheme.

By comparison, our scheme shows all the advantages of the previous QSS and the unique advantages, such as,

- (1) It uses a qudit state instead of a qubit state.
- (2) The participants can check the authenticity of the recovered secret.
- (3) It needs fewer quantum resources and quantum operations.
- (4) It has general access structure.
- (5) It reduces the communication costs and computation complexity.

This paper is organized as follows. In section 2, we illustrate the preliminary knowledge related to the proposed scheme. The new proposed scheme is introduced in section 3. Section 4 give a proof of the correctness, verifiability and security of the proposed scheme. In section 5, we give an example to further illustrate our proposed scheme. Finally, the comparison and conclusion is given in section 6 and section 7.

3

$\mathbf{2}$ **Preliminaries**

In this section, we introduce the preliminary knowledge of our scheme.

2.1Access structure

Defination 1 Let $\mathcal{P} = \{P_1, P_2, \cdots, P_n\}$ be a set of participants, an access structure $\Gamma \subset 2^{\mathcal{P}}$ is a family of authorized sets of participants.

Defination 2 If Γ is the access structure on \mathcal{P} , then any set in Γ is called the authorization subset on \mathcal{P} , which is called the authorization set for short. If $A \in \Gamma, A \subseteq B \subseteq \mathcal{P}$, then $B \in \Gamma$. The family of the unauthorized sets is called an adversary structure, that is to say, $\Gamma^c = \Delta$.

Example 1 Let $\mathcal{P} = \{P_1, P_2, P_3, P_4\}, \Gamma = \{A_1, A_2, A_3\}, \text{where } A_1 = \{P_1, P_2, P_3\}, P_4 = \{P_1, P_2, P_4\}, P_4 = \{P_1, P_2, P_4\}, P_4 = \{P_1, P_4, P_4\}, P_4 = \{P_1, P$ $A_2 = \{P_1, P_2, P_4\}, A_3 = \{P_1, P_2, P_3, P_4\}.$ So

$$\Delta = \left\{ \begin{array}{l} \emptyset, \{P_1\}, \{P_2\}, \{P_3\}, \{P_4\}, \{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \\ \{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_4\}, \{P_1, P_3, P_4\}, \{P_2, P_3, P_4\} \end{array} \right\}.$$

$\mathbf{2.2}$ Monotone span program

MSP was introduced in Ref. [31] by Karchmer and Wigderson as a model of computation to design the linear secret sharing scheme.

Defination 3 $\mathcal{M}(\mathcal{F}, M, \psi, \boldsymbol{\xi})$ is a monotone span program(MSP), where M is a $k \times l$ matrix over a finite field $\mathcal{F}, \psi : \{1, 2, \cdots, k\} \to \mathcal{P}$ is a surjective labeling map, $\boldsymbol{\xi} = (1, 0, \dots, 0)^T \in \mathcal{F}^l$ is defined as the target vector. For any $A \subseteq \mathcal{P} =$ $\{P_1, P_2, \cdots, P_n\}$, there is a corresponding eigenvector $\boldsymbol{\delta}_A = (\delta_1, \delta_2, \cdots, \delta_n) \in$ $\{0,1\}^n$ if and only if $P_i \in A, \delta_i = 1$. The Boolean function $f : \{0,1\}^n \to \{0,1\}^n$ $\{0,1\}, f(\delta_A) = 1$ represents the corresponding ε rows of M, where $\psi(\varepsilon) \in$ $A, \varepsilon \in \{1, 2, \cdots, k\}.$

Defination 4 A monotone span program (MSP) is called a MSP for access structure Γ , if it can be satisfied that $\forall A \in \Gamma, \exists \lambda_A \in \mathcal{F}^k \Rightarrow M_A^T \lambda_A = \boldsymbol{\xi}$, and $\forall A \in \Delta, \exists \mathbf{h} = (1, h_2, \cdots, h_l) \in \mathcal{F}^l \Rightarrow M_A \mathbf{h} = \mathbf{0} \in \mathcal{F}^m.$

Example 2 $\mathcal{M}(\mathcal{F}, M, \psi, \xi)$ is an MSP of access structure Γ as shown in example

1, where $\mathcal{F} = \mathcal{Z}_5, \psi(i) = Bob_i, i \in \{1, 2, 3, 4\}, \boldsymbol{\xi} = (1, 0, 0, 0)^T, M = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 2 & 1 \\ 3 & 4 & 1 & 0 \\ 1 & 2 & 4 & 0 \end{pmatrix}$.

Therefore, $\lambda_{A_1} = (1, 1, 0)^T$, $\lambda_{A_2} = (1, 1, 0)^T$, $\lambda_{A_3} = (1, 1, 3, 4)^T$.

2.3 Linear secret sharing

Monotone span program is utilized to design the linear secret sharing scheme, which is aimed that the dealer Alice shares a secret s among k shareholders $Bob_1, Bob_2, \dots, Bob_k$ according to the MSP for access structure Γ . It includes the following two phases as follows.

Distribution phase

Alice prepares a random vector $\boldsymbol{\rho} = (s, \rho_2, \dots, \rho_l)^T \in {}^l$ and computes $\boldsymbol{s} = M\boldsymbol{\rho} = (s_1, \dots, s_k)^T$. Then, she sends s_i to $\psi(i)$ via a secure channel. Reconstruction phase

Let s_A be indicated the vector for the authorized set A. The participants in A restore the secrets cooperatively as follows.

$$\boldsymbol{s}_{A}^{T}\boldsymbol{\lambda}_{A} = \left(M_{A}\boldsymbol{\rho}\right)^{T}\boldsymbol{\lambda}_{A} = \boldsymbol{\rho}^{T}\left(M_{A}^{T}\boldsymbol{\lambda}_{A}\right) = \boldsymbol{\rho}^{T}\boldsymbol{\xi} = s.$$
(1)

2.4 Necessary quantum properties

Defination 5 Mutually unbiased base is defined that two sets of standard orthogonal bases $A_1 = \{ |\varphi_1\rangle, |\varphi_2\rangle, \dots, |\varphi_d\rangle \}$ and $A_2 = \{ |\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_d\rangle \}$, which defined over a *d*-dimensional complex space C^d in Ref.[32,33], if the following relationship is satisfied

$$|\langle \varphi_i | \psi_i \rangle| = \frac{1}{\sqrt{d}}.$$
(2)

If any two of the set of standard orthogonal bases $\{A_1, A_2, \dots, A_m\}$ in space are unbiased, then this set is called an unbiased bases set. Besides, it can be found d+1 mutually unbiased bases if d is an odd prime number.

Defination 6 The computation base is expressed as $\{|k\rangle | k \in D\}$, and the remaining groups can be expressed as:

$$\left|v_{l}^{(j)}\right\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k(l+jk)} \left|k\right\rangle,\tag{3}$$

where $|v_l^{(j)}\rangle$ represents the *l*-th vector in the *j*-th bases, $w = e^{\frac{2\pi i}{d}}l, j \in D, D = \{0, 1, \cdots, d-1\}$. These mutually unbiased bases satisfy the following conditions:

$$\left|\left\langle v_{l}^{(j)} \middle| v_{l}^{(j')} \right\rangle\right| = \frac{1}{\sqrt{d}}, j \neq j'.$$

$$\tag{4}$$

Defination 7 In Ref.[34], the two unitary transformations X_d and Y_d that we need to use in this paper can be expressed as:

$$X_{d} = \sum_{m=0}^{d-1} w^{m} |m\rangle \langle m|, Y_{d} = \sum_{m=0}^{d-1} w^{m^{2}} |m\rangle \langle m|.$$
(5)

Implementing (5) on $\left| v_{l}^{(j)} \right\rangle$ in turn, we can obtain:

$$X_d^x Y_d^y \left| v_l^{(j)} \right\rangle = \left| v_{l+x}^{(j+y)} \right\rangle.$$
(6)

For the convenience of expression, $X_d^x Y_d^y$ is denoted as $U_{x,y}$, that is,

$$U_{x,y}\left|v_{l}^{(j)}\right\rangle = \left|v_{l+x}^{(j+y)}\right\rangle.$$

$$\tag{7}$$

 $\mathbf{5}$

3 Proposed scheme

In this section, we construct a verifiable quantum secret sharing scheme that includes a dealer Alice and n shareholders $Bob_1, Bob_2, \dots, Bob_n$. The access structure Γ can be expressed as $\Gamma = \{A_1, A_2, \dots, A_r\}$, where $A_i (i = 1, 2, \dots, r)$ is a authorization set. For the convenience of description, the authorization set is recorded as $A_i = \{Bob_1^{(i)}, Bob_2^{(i)}, \dots, Bob_m^{(i)}\}$, $(1 \le m \le n)$. Without losing generality, it is assumed that the participants in the authorization set $A_i = \{Bob_1^{(i)}, Bob_2^{(i)}, \dots, Bob_m^{(i)}\}$ want to recover the secret s. The specific steps of the scheme are as follows.

3.1 Distribution phase

Alice implements the following steps.

3.1.1 Select a random vector $\vec{\rho} = (S_i, \rho_2, \rho_3 \cdots, \rho_l)^T$ according to authorization set A_i .

3.1.2 Calculate $\boldsymbol{s} = M_{n \times l} \boldsymbol{\rho} = \left(s_1^{(i)}, s_2^{(i)}, \dots, s_n^{(i)}\right)^T, i = 1, 2, \dots, r$ and send $\boldsymbol{s}^{(i)}$ to $\boldsymbol{s}^{(i)} = Bob_i(i = 1, 2, \dots, n)$ through the quantum secure channel

 $s_j^{(i)}$ to $\psi(j) = Bob_j (j = 1, 2, \dots, n)$ through the quantum secure channel. **3.1.3** Compute and publish $H_1 = h(S_i), H_2 = h(s)$, where h() is a public Hash function.

3.1.4 Prepare a quantum state $|\phi\rangle = |\varphi_0^0\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle$ and perform a unitary operation $U_{p_0^{(i)}, q_0^{(i)}}$ to get the quantum state $|\phi\rangle_0^{(i)} = U_{p_0^{(i)}, q_0^{(i)}} |\varphi_0^0\rangle = |\varphi_{p_0^{(i)}}^{q_0^{(i)}}\rangle$, where $p_0^{(i)} = s$ is the secret, $q_0^{(i)}$ is a secret value known only to Alice. Then, she sends the quantum state $|\phi\rangle_0^{(i)}$ performed by the unitary operation to the first participant $Bob_1^{(i)}$ in the authorization set A_i .

3.2 Reconstruction phase

Participants in $A_i = \left\{ Bob_1^{(i)}, Bob_2^{(i)}, \cdots, Bob_m^{(i)} \right\}, (1 \le m \le n)$ can recover the secret by the following steps.

3.2.1 After receiving the quantum state $|\phi\rangle_0$, the first participant $Bob_1^{(i)}$ performs unitary operates $U_{p_1^{(i)},q_1^{(i)}}$ on it and gets the quantum state $|\phi\rangle_1^{(i)} = U_{p_1^{(i)},q_1^{(i)}}$ $\left|\varphi_{p_0^{(i)}}^{q_0^{(i)}}\right\rangle = \left|\varphi_{p_0^{(i)}+p_1^{(i)}}^{q_0^{(i)}+q_1^{(i)}}\right\rangle$. Next, the quantum state $|\phi\rangle_1^{(i)}$ is sent to the second participant $Bob_2^{(i)}$ in the authorization set A_i , where $p_1^{(i)} = \lambda_1^{(i)}s_1^{(i)}, q_1^{(i)} = \lambda_1^{(i)}$.

3.2.2 The other participants $Bob_j^{(i)}$ $(j = 2, 3, \dots, m)$ in the authorization set A_i perform the same operation as in step 3.2.1, which means that after receiving

the quantum state $|\phi\rangle_{j-1}^{(i)}$, $Bob_j^{(i)}$ performs unitary operation $U_{p_j^{(i)},q_j^{(i)}}$ on it and gets the quantum state $|\phi\rangle_j^{(i)} = U_{p_j^{(i)},q_j^{(i)}} \left| \begin{array}{c} \sum\limits_{k=0}^{j-1} q_k^{(i)} \\ \varphi_{j-1}^{k-2} \\ \sum\limits_{k=0}^{j-1} p_k^{(i)} \end{array} \right\rangle = \left| \begin{array}{c} \sum\limits_{k=0}^{j} q_k^{(i)} \\ \sum\limits_{k=0}^{j} p_k^{(i)} \\ \sum\limits_{k=0}^{j} p_k^{(i)} \end{array} \right\rangle$, and then sends

it to the next participant $Bob_{j+1}^{(i)}$, $(j = 2, 3, \dots m - 1)$ until the last participant $Bob_m^{(i)}$ in the authorization set A_i completes the operation and sends the final quantum state to Alice, where $p_j^{(i)} = \lambda_j^{(i)} s_j^{(i)}$, $q_j^{(i)} = \lambda_j^{(i)}$. For the authorization set A_i , when all the participants act and transmit, the final quantum state is

$$|\phi\rangle_{m}^{(i)} = \prod_{k=0}^{m} U_{p_{k}^{(i)}, q_{k}^{(i)}} \left|\varphi_{0}^{0}\right\rangle = \left|\varphi_{m}^{\sum\limits_{k=0}^{m} q_{k}^{(i)}}\right\rangle.$$
(8)

3.2.3 When Alice receives the final quantum state $|\phi\rangle_m^{(i)}$, she can know that it satisfies the following condition on account of $q_0^{(i)}, q_1^{(i)}, \cdots, q_m^{(i)}$,

$$q_0^{(i)} + q_1^{(i)} + \dots + q_m^{(i)} = q_i.$$
 (9)

She selects the measurement bases $M_{q_i} = \left\{ \left| \varphi_j^{(q_i)} \right\rangle | j \in D \right\}$ to measure it, and then infers the following condition should be established in the authorization set A_i

$$p_0^{(i)} + p_1^{(i)} + \dots + p_m^{(i)} = p_0^{(i)} + S_i = r_i$$
(10)

If it is established, Alice checks whether H_1 of the participants are equal to the published one. If so, the measurement results r_i will be sent to all participants in the authorization set A_i through the secure channel and then it move to the next step. If not, the scheme is terminated.

3.2.4 In order to reconstruct the secret, each participant in authorization set A_i can recover the secret by calculating $s = p_0 = r_i - \sum_{i=1}^m p_i = r_i - S_i$.

4 Correctness, verifiability and security

In this section, the provability of the correctness, verifiability and security of our scheme is given.

4.1 Correctness

Theorem 1 If a *d*-dimensional quantum state in mutually unbiased bases is $\left|v_{l}^{(j)}\right\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} w^{k(l+jk)} |k\rangle$, and a unitary operation $U_{x,y} = X_{d}^{x}Y_{d}^{y}$ is performed

on it, then it will become another state $|v_{l+x}^{(j+y)}\rangle$, that is, $U_{x,y}|v_l^{(j)}\rangle = |v_{l+x}^{(j+y)}\rangle$. **Proof** When implementing Y_d^y, X_d^x on $|v_l^{(j)}\rangle$ in turn, we can obtain,

$$\begin{split} X_{d}^{x}Y_{d}^{y} \left| v_{l}^{(j)} \right\rangle &= X_{d}^{x} \left(\sum_{m=0}^{d-1} w^{ym^{2}} \left| m \right\rangle \left\langle m \right| \right) \left(\frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k(l+jk)} \left| k \right\rangle \right) \\ &= \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} w^{xm} \left| m \right\rangle \left\langle m \right| \sum_{k=0}^{d-1} w^{k(l+(j+y)k)} \left| k \right\rangle \\ &= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k[(l+x)+(j+y)k]} \left| k \right\rangle \\ &= \left| v_{l+x}^{(j+y)} \right\rangle. \end{split}$$
(11)

This completes the proof.

Lemma 1 In the secret sharing scheme, according to Theorem 1, the initial state selected by Alice is $|\phi\rangle = |\varphi_0^0\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle$, and the unitary operation $U_{p_k^{(i)},q_k^{(i)}} = X_d^{p_k^{(i)}} Y_d^{q_k^{(i)}}, k = 0, 1, \cdots, m$ is performed on the states sequentially by Alice and all the participants in the authorization set A_i , then the final state is $|\phi\rangle_m^{(i)} = \left(\prod_{u=0}^m U_{p_u,q_u}\right) |\phi\rangle$, that is, $|\phi\rangle_m^{(i)} = \prod_{k=0}^m U_{p_k^{(i)},q_k^{(i)}} |\varphi_0^0\rangle = \left|\varphi_{k=0}^{\sum p_k^{(i)}}\right|$. When Alice announces the measurement result r_i via the quantum secure channel to the participants in A_i , they can restore the secret $s = p_0 = r_i - \sum_{k=1}^m p_k^{(i)} = r_i - S_i$.

4.2 Verifiability

On one hand, before Alice sends the measurement result, she can check H_1 to ensure that the secret value recovered by linear secret sharing is correct, which provides a prerequisite for participants to recover the correct secret. On the other hand ,each participant can check

$$H_2 = h\left(s\right),\tag{12}$$

to ensure that the recovered secret is the original one.

4.3 Security

We analyze the security of our scheme against the general attacks here.

Entangle and measure attack We assume that eavesdropper Eve intercepts the particles sent among Alice and the participants and then uses a unitary operation U_E to entangle an ancillary state $|E\rangle$ on the transmitted particle. In order to steal secret information by measuring the ancillary state, Eve act the unitary operator U_E on $|E\rangle$ and the transmitted particle. To simplify the

description, we consider the bases corresponding to j = 0, namely, $\left| v_l^{(0)} \right\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{kl} \left| k \right\rangle$, so

$$U_E |k\rangle |E\rangle = \sum_{h=0}^{d-1} a_{kh} |h\rangle |e_{kh}\rangle, \qquad (13)$$

$$U_{E} \left| v_{l}^{(0)} \right\rangle \left| E \right\rangle = U_{E} \left(\frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{kl} \left| k \right\rangle \right) \left| E \right\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{kl} \left(\sum_{h=0}^{d-1} a_{kh} \left| h \right\rangle \left| e_{kh} \right\rangle \right)$$

$$= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \sum_{h=0}^{d-1} w^{kl} a_{kh} \left(\frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} w^{-hm} \left| v_{m}^{(0)} \right\rangle \right) \left| e_{kh} \right\rangle$$

$$= \frac{1}{d} \sum_{k=0}^{d-1} \sum_{h=0}^{d-1} \sum_{m=0}^{d-1} w^{kl-hm} a_{kh} \left| v_{m}^{(0)} \right\rangle \left| e_{kh} \right\rangle,$$
(14)

where $w = e^{\frac{2\pi i}{d}}$, $|E\rangle$ is the initial state of the auxiliary space, $|e_{kh}\rangle$ are pure ancillary states determined uniquely by the unitary operation U_E , so

$$\sum_{h=0}^{d-1} |a_{kh}|^2 = 1, k \in \{0, 1, \cdots, d-1\}.$$
(15)

For the sake of avoiding the rising error rate, Eve has to set $a_{kh} = 0, k, h \in \{0, 1, \dots, d-1\}, k \neq h$. Therefore, (11) And (12) can be simplified to

$$U_E \ket{k} \ket{E} = a_{kk} \ket{k} \ket{e_{kk}}, \qquad (16)$$

$$U_E \left| v_l^{(0)} \right\rangle \left| E \right\rangle = \frac{1}{d} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} w^{k(l-m)} a_{kk} \left| v_m^{(0)} \right\rangle \left| e_{kk} \right\rangle.$$
(17)

Similarly, to avoid the eavesdropping check, Eve has to set

$$\sum_{k=0}^{d-1} w^{k(l-m)} a_{kk} |e_{kk}\rangle = 0,$$
(18)

where $m \in \{0, 1, \dots, d-1\}$, $m \neq l$. For any $l \in \{0, 1, \dots, d-1\}$, we can obtain d equations

$$a_{00} |e_{00}\rangle = a_{11} |e_{11}\rangle = \dots = a_{d-1,d-1} |e_{d-1,d-1}\rangle.$$
 (19)

So, whatever quantum state Eve uses, he can only get the same information from the auxiliary particles. Similar analysis can be used for the other quantum states $\left|v_{l}^{(j)}\right\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k(l+jk)} \left|k\right\rangle$, so the entanglement measurement attack is invalid in our scheme.

9

Intercept and resend attack The eavesdropper Eve intercepts the transmitted particles among Alice and the participants and resends some forged particles. For a simple description, we suppose that the eavesdropper Eve intercepts the quantum state $|\phi\rangle_k$ sent by $Bob_k^{(i)}$ to $Bob_{k+1}^{(i)}$. However, he does not know any information about the measurement bases and only chooses the correct measurement bases with the probability of $\frac{1}{4}$ to get measure outcome

$$p_0 + \sum_{i=1}^k p_i.$$
 (20)

Even if the result is measured with the probability of $\frac{1}{d}$, the secret information cannot be obtained because $p_i, i \in \{k + 1, \dots, m\}$ is unknown. If Alice shares n secret information, the probability that eavesdropper succeed will be $(\frac{1}{d})^n$. With the increase of the number of n, there will be $\lim_{n\to\infty} (\frac{1}{d})^n = 0$. The other is that Eve intercepts the s_i sent by Alice to the participants, but the s_i does not carry any information of the secret. In short, Eve cannot obtain the secret in intercept-and-resend attack.

Forgery attack If Alice shares a fake s_t to $Bob_t^{(i)}$, the secret s will not be restored by the participants in A_i . If one or some of the participants perform the false unitary operation, they will be found by Alice because the measurement result will be inconsistent with Alice's expectation. What's more, even if some dishonest participants performed the fake unitary transformation and Alice successfully measured the expected result, there is no use for this attack. Because the recovered secret s' with $H_2' = h(s') \neq H_2 = h(s)$ guaranteed. So, the forgery attack is useless.

Collusion attack If the participants in $B_i, B_i \subseteq A_i$ collude to restore the secret, they must obtain the $s_k^{(i)}$ and $\lambda_k^{(i)}$ of each participant in the authorization set A_i to recover S_i . When $B_i \subseteq A_i$, they can not get the other's secret share information, so this attack is unsuccessful.

5 Example

Here, we explain our scheme more clearly by giving an example.

Example 3 According to the MSP and the access structure Γ in the example 2, assuming Alice wants to share secret $s = 3 \in \mathbb{Z}_5$ among the four participants $Bob_1, Bob_2, Bob_3, Bob_4$, she prepares a random vector $\boldsymbol{\rho} = (4, 1, 0, 2)^T$ firstly and then computes $\boldsymbol{s} = M\boldsymbol{\rho} = (s_1, s_2, s_3, s_4)^T = (2, 2, 1, 1)^T$. Next, she sends s_i to Bob_i , (i = 1, 2, 3, 4) via a secure channel and publishes $H_1 = h(4), H_2 = h(3)$. Without losing generality, we assume that the participants in A_1 want to restore the secret. The dealer Alice prepares a state $|\phi\rangle = |\varphi_0^0\rangle = \frac{1}{\sqrt{5}} \sum_{i=0}^4 |i\rangle$ and performs

 $U_{p_0,q_0} = U_{3,2}$ on it to obtain $|\phi\rangle_0 = U_{3,2} |\varphi_0^0\rangle = |\varphi_3^2\rangle$, where $p_0 = s = 3$ is the secret, $q_0 = 2 \in \mathbb{Z}_5$ is a randomly selected secret value only known by Alice. Next she sends the quantum state $|\phi\rangle_0 = |\varphi_3^2\rangle$ to Bob_1 . After receiving $|\phi\rangle_0 = |\varphi_3^2\rangle$, Bob_1 performs the unitary operation $U_{\lambda_1 s_1,\lambda_1} = U_{2,1}$ to get $|\phi\rangle_1 = |\varphi_0^0\rangle$ and sends it to Bob_2 . When receiving $|\phi\rangle_1 = |\varphi_0^3\rangle$, Bob_2 performs the unitary operation $U_{\lambda_2 s_2,\lambda_2} = U_{2,1}$ to get $|\phi\rangle_2 = |\varphi_2^4\rangle$ and sends to Bob_3 . After receiving $|\phi\rangle_2 = |\varphi_2^4\rangle$, Bob_3 performs $U_{\lambda_3 s_3,\lambda_3} = U_{0,0}$ to get $|\phi\rangle_3 = |\varphi_2^4\rangle$ and sends it to Alice. For the authorization set A_1 , when all participants act and transmit particle, the final quantum state is

$$|\varphi\rangle_{final} = \left(\prod_{i=0}^{3} U_{p_i,q_i}\right) \left|\varphi_0^0\right\rangle = \left|\sum_{\substack{j=0\\3\\j\in 0}}^{3} q_i\right\rangle = \left|\varphi_2^4\right\rangle.$$
(21)

In this case, Alice selects $M_4 = \left\{ \left| \varphi_j^{(4)} \right\rangle | j \in \{0, 1, 2, 3, 4\} \right\}$ to measure $|\varphi\rangle_{final} = \left| \varphi_2^4 \right\rangle$ and records the measurement result r_1 . Afterwards, Alice checks whether $r_1 = 2$ and $H_1 = h(4)$ are true. If not, the scheme is terminated. If they are established, the measurement result r_1 is sent to each participant in A_1 through a quantum secure channel. After the participant receives it, the secret s can be recovered as

$$s = p_0^{(1)} = r_1 - p_1^{(1)} - p_2^{(1)} - p_3^{(1)} = r_1 - \lambda_1^{(1)} s_1^{(1)} - \lambda_2^{(1)} s_2^{(1)} - \lambda_3^{(1)} s_3^{(1)}.$$
 (22)

That is s = 2 - (2 + 2 + 0) = 3. Last but not least, they can check H_1 to make certain of the authenticity of the secret.

6 Comparison

In this section, we give a comparison among our scheme and other similar ddimensional QSS schemes [24,35,36] in terms of basic properties, computational complexity and communication costs. The schemes in Ref. [24, 36] are the threshold QSS, however the scheme in Ref. [35] and ours are the general access structure QSS. The general access structure makes the level and influence of the participants different, making the scheme more flexible. They all use the Hash function to make the verifiability of the d-dimensional QSS scheme. The scheme proposed by Song et al. [24] shared a classical secret by utilizing polynomials according to the Lagrange interpolation formula. The transformation of the particles includes some operations such as d-level CNOT, QTF, Inverse QTF, and generalized Pauli operator. However, the general access structure QSS is far more flexible and practical than the threshold one. In Ref. [35], Mashhadi proposed a hybrid secret sharing based on the quantum Fourier transform and monotone span program, in which the participants recover the secret by means of measuring the entangled state. The number of unitary operators is not much different in the premise, while the number of required quantum states and the number of measurement operations are greatly reduced, which consumes less quantum resources

and the scheme is more practical. Qin et al.[36] put forward a verifiable (t, n) threshold QSS using d-dimensional Bell state and they realize the authentication of quantum state transmission by adding some decoy particles. According to the Lagrange interpolation and the unitary operation, they can recover the secret with measuring the final Bell state. The Specific comparison of basic property among Ref.[24,35,36] and ours is given in Table 1. The comparison of the computational complexity and communication costs of the general access structure QSS[35] and the new is given in Table 2.

Table 1. Basic	comparison	among the	QSS	schemes
----------------	------------	-----------	-----	---------

Property	Song[24]	Mashhadi[35]	Qin[36]	New
Model	(t, n)threshold	General	(t, n)threshold	General
Verification	Hash function	Hash function	Hash function	Hash function
Secret	Classic	Classic	Classic	Classic
Dimension	d	d	d	d
Method	LI	MSP,LC	LI	MSP,MUB,LC
NQO	$QFT, QFT^{-1},$ Pauli	QFT,Pauli	UO	UT

Table 2. Comparison of communication costs and computational complexity

Property	Mashhadi[35]	Ours
Number of message particles	m-1	1
Unitary operation	m	m + 1
QTF	1	_
Measure operation	m	1
Hash function	2	2

Remark 1. LI: Lagrange interpolation, MSP: Monotone span program, MUB: Mutually unbiased bases, LC: Linear computation, NQO: necessary quantum operation, QTF: Quantum Fourier Transform, QFT^{-1} : Inverse Quantum Fourier Transform, UO: Unitary operation, UT: Unitary transformation.

7 Conclusions

The verifiable quantum secret sharing scheme based on the access structure is very useful in practice. In this paper, we construct a verifiable quantum secret

sharing scheme based on the property of the mutually unbiased base and the monotone span program. The dealer and participants in the authorization set can restore secret through the transformation and transmission of a *d*-dimensional quantum state as well as linear secret sharing. In addition, the correctness, verifiability and security analysis of the scheme have been proved. Finally, a specific example and a comparison are given to further clarify the advantages and practicality of our scheme.

For the future work, the verifiability of the scheme is analyzed from the view that the recovered secret is consistent with the original one. However, the issue of mutual authentication among the participants in the authorization set is still worth studying.

References

- 1. Shor P W, Preskill J. Simple Proof of Security of the BB84 Quantum Key Distribution Protocol. J. Physical Review Letters, 2000, 85(2): 441-444.
- Lo H, Ma X, Chen K, et al. Decoy state quantum key distribution. J. Physical Review Letters, 2005, 94(23): 230504-230504.
- Chong S K, Hwang T. Quantum key agreement protocol based on BB84. J. Optics Communications, 2010, 283(6): 1192-1195.
- Liu B, Gao F, Huang W, et al. Multiparty quantum key agreement with single particles. J. Quantum Information Processing, 2013, 12(4): 1797-1805.
- Shukla C, Alam N, Pathak A, et al. Protocols of quantum key agreement solely using Bell states and Bell measurement. J. Quantum Information Processing, 2014, 13(11): 2391-2405.
- Deng F, Long G. Secure direct communication with a quantum one-time pad. J. Physical Review A, 2004, 69(5).
- 7. Wang C, Deng F, Li Y S, et al. Quantum secure direct communication with highdimension quantum superdense coding. J. Physical Review A, 2005, 71(4).
- Furusawa A, Sorensen J, Braunstein S L, et al. Unconditional Quantum Teleportation. J. Science, 1998, 282(5389): 706-709.
- Bouwmeester D, Pan J, Mattle K, et al. Experimental quantum teleportation. J. Nature, 1997, 390(6660): 575-579.
- Lee H, Hong C, Kim H, et al. Arbitrated quantum signature scheme with message recovery. J. Physics Letters A, 2004, 321(5): 295-300.
- 11. Fei G, Sujuan Q, Fenzhuo G, et al. Cryptanalysis of the arbitrated quantum signature protocols. J. Physical Review A, 2011, 84(2).
- Li X, Barnum H. QUANTUM AUTHENTICATION USING ENTANGLED STATES. J. International Journal of Foundations of Computer Science, 2004, 15(04): 609-617.
- Naseri M. Revisiting Quantum Authentication Scheme Based on Entanglement Swapping. J. International Journal of Theoretical Physics, 2016, 55(5): 2428-2435.
- Naseri M. Revisiting Quantum Authentication Scheme Based on Entanglement Swapping. J. International Journal of Theoretical Physics, 2016, 55(5): 2428-2435.
- Hillery M, Bu?ek V, Berthiaume A, et al. Quantum secret sharing. J. Physical Review A, 1999, 59(3): 1829-1834.
- 16. Hsu L. Quantum secret-sharing protocol based on Grover's algorithm. J. Physical Review A, 2003, 68(2).

- 17. Xiao L, Long G, Deng F, et al. Efficient multiparty quantum-secret-sharing schemes. J. Physical Review A, 2004, 69(5).
- Sun Y, Wen Q, Gao F, et al. Multiparty quantum secret sharing based on Bell measurement. J. Optics Communications, 2009, 282(17): 3647-3651.
- 19. Hsu J, Chong S, Hwang T, et al. Dynamic quantum secret sharing. J. Quantum Information Processing, 2013, 12(1): 331-344.
- Rahaman R, Parker M G. Quantum secret sharing based on local distinguishability. J. Physical Review A, 2015, 91(2).
- Wang J, Li L, Peng H, et al. Quantum-secret-sharing scheme based on local distinguishability of orthogonal multiqudit entangled states. J. Physical Review A, 2017, 95(2).
- 22. Williams B P, Lukens J M, Peters N A, et al. Quantum secret sharing with polarization-entangled photon pairs. J. Physical Review A, 2019, 99(6).
- Tsai C, Yang C, Lee N, et al. Semi-quantum secret sharing protocol using W-state. J. Modern Physics Letters A, 2019, 34(27).
- Song X, Liu Y, Deng H, et al. (t, n) Threshold d-Level Quantum Secret Sharing. J. Scientific Reports, 2017, 7(1).
- Yang Y, Jia X, Wang H, et al. Verifiable quantum (k, n)-threshold secret sharing. J. Quantum Information Processing, 2012, 11(6): 1619-1625.
- Hao, N., Li, Z., Bai, H. et al. A New Quantum Secret Sharing Scheme Based on Mutually Unbiased Bases. Int J Theor Phys 58, 1249C1261 (2019).
- Bai, C., Li, Z., Si, M. et al. Quantum secret sharing for a general quantum access structure. Eur. Phys. J. D 71, 255 (2017).
- Liu, C., Li, Z., Bai, C. et al. Quantum-Secret-Sharing Scheme Based on Local Distinguishability of Orthogonal Seven-Qudit Entangled States. Int J Theor Phys 57, 428C442 (2018).
- Xu, T., Li, Z., Bai, C. et al. A New Improving Quantum Secret Sharing Scheme. Int J Theor Phys 56, 1308C1317 (2017).
- Nascimento A C, Muellerquade J, Imai H, et al. Improving Quantum Secret-Sharing Schemes. J. Physical Review A, 2001, 64(4).
- Karchmer M, Wigderson A. On span programs. C. structure in complexity theory annual conference, 1993: 102-111.
- Ivonovic I D. Geometrical description of quantal state determination. J. Journal of Physics A, 1981, 14(12): 3241-3245.
- Wootters W K , Fields B D . Optimal state-determination by mutually unbiased measurements. J. Annals of Physics, 1989, 191(2):363-381.
- 34. Tavakoli A, Herbauts I, Zukowski M, et al. Secret sharing with a single d -level quantum system. J. Physical Review A, 2015, 92(3).
- 35. Mashhadi S. General secret sharing based on quantum Fourier transform. J. Quantum Information Processing, 2019, 18(4).
- Qin H, Dai Y. Verifiable (t,n) threshold quantum secret sharing using ddimensional Bell state. J. Information Processing Letters, 2016, 116(5): 351-355.