Sufficient conditions for the uniqueness of solution of the weighted norm minimization problem

K. Z. Najiya, Munnu Sonkar and C. S. Sastry
Department of Mathematics
Indian Institute of Technology, Hyderabad, 502285, India.
Email:{ma17resch01001, ma17resch11004 and csastry}@iith.ac.in

***Abstract*—Prior support constrained compressed sensing, achieved via the weighted norm minimization, has of late become popular due to its potential for applications. For the weighted norm minimization problem,**

$$min\|x\|_{p,w} \text{ subject to } y = Ax,\ p = 0, 1, \text{ and } w \in [0, 1],$$

uniqueness results are known when $w = 0, 1$. Here, $\|x\|_{p,w} = w\|x_T\|_p + \|x_{T^c}\|_p,\ p = 0, 1$ with T representing the partial support information. The work reported in this paper presents the conditions that ensure the uniqueness of the solution of this problem for general $w \in [0, 1]$.

I. Introduction

In Compressed Sensing (CS), a sparse signal $x \in \mathbb{R}^n$ can be recovered from a small set of measurements $y \in \mathbb{R}^m$ satisfying $y = Ax$ with $k \ll m$, where k is the number of nonzero elements in x. The results that guarantee the uniqueness of the recovery process depend on the restricted isometry property (RIP) of the sensing matrix A [3][4][8]. In many applications, one obtains some a priori information about the partial support of the sparse solution to be recovered. For instance, in applications involving recovering time-correlated signals [9], prior-support constrained sparse recovery attains importance. In recent years, compressed sensing with a priori support information has caught the attention of several researchers [9][5][6][10], to name a few. The weighted norm minimization aims at providing signals, satisfying the data constraint, that are sparse inside and sparsest outside a given prior support. In [9], the authors have modified the 1-norm by taking zero weights on the known partial support, minimizing thereby the terms in the complement of prior support set. The results in [9] have presented the uniqueness of solution of weighted norm minimization under the stated conditions. When all the weights are set to 1, the weighted 0-norm and the weighted 1-norm problems coincide respectively with their standard 0-norm and 1-norm counterparts, whose exact recovery conditions have been established in [1]. The authors of [5][7] have established the stability of recovery in noisy-setting for weighted 1-norm minimization problem. To the best of our knowledge, however, the uniqueness of the solution of the general weighted 0-norm and weighted 1-norm minimization problems has not been proposed to date. Motivated by this, the present work proposes sufficient conditions for the uniqueness of the solution of the weighted 0,1-norm minimization problems. We show that our conditions mostly coincide with those of known cases when the weights are 0, 1.

The paper is organized as 6 sections. In sections 2 and 3, we provide basic introduction to Compressed Sensing and existing uniqueness results respectively. In sections 4 and 5, we discuss the uniqueness results with general weights for 0-norm and 1-norm problems respectively. The paper ends with concluding remarks in section 6.

II. Compressed sensing

Compressive sensing (CS) [3] is a technique that reconstructs a signal, which is compressible or sparse in some domain, from a small set of linear measurements. Let $\sum_k^n := \{x \in \mathbf{R}^n : \|x\|_0 \leq k\}$ be the set of all k-sparse signals in $\mathbf{R}^n$. Here $\|x\|_0 = |\{i : x_i \neq 0\}|$ stands for the number of nonzero components in x. For simplicity in notation, we represent the set $\{1, 2, \ldots, n\}$ as $[n]$. For $A \in \mathbf{R}^{m\times n}$ with $m << n$, suppose $y = Ax$. One may recover the sparsest solution of this system from the following minimization problem :

$$(P_0)\ \ min\|x\|_0 \ \ \textit{subject to} \ \ y = Ax. \tag{1}$$

Since l_0 minimization problem becomes NP-hard as the dimension increases, the convex relaxation of l_0 problem has been proposed as

$$(P_1)\ \ min\|x\|_1 \ \ \textit{subject to} \ \ y = Ax. \tag{2}$$

The coherence $\mu(A)$ of a matrix A is the largest absolute normalized inner product between different columns of it, that is,

$$\mu(A) = \max_{1\leq i,j\leq n,\ i\neq j} \frac{|a_i^T a_j|}{\|a_i\|_2\|a_j\|_2},$$

where a_i denotes the i-th column in A.

The k-th restricted isometry property (k-RIP) constant δ_k of a matrix A is the smallest real number such that

$$(1 - \delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k)\|x\|_2^2,$$

for all x such that $\|x\|_0 \leq k < n$. The restricted orthogonality constant $\theta_{s,\tilde{s}}$ of a matrix A is the smallest real number such that

$$|\eta' A_T' A_{\tilde{T}}\tilde{\eta}| \leq \theta_{s,\tilde{s}}\|\eta\|_2\|\tilde{\eta}\|_2,$$

for all disjoint sets T and $\tilde{T}$ with $|T| \leq s$ and $|\tilde{T}| \leq \tilde{s}$ such that $s + \tilde{s} \leq n$ and for all vectors $\eta \in \mathbb{R}^{|T|}$ and $\tilde{\eta} \in \mathbb{R}^{|\tilde{T}|}$. Here, A_T denotes the restriction of the matrix A to the columns corresponding to the indices in $T \subseteq [n]$. For simplicity, we denote $\theta_s := \theta_{s,s}$. In [1], E. Candes and T. Tao have given the conditions for the exact recovery of x from the pair (A, y) in terms of Restricted Isometry Constant (RIC) for (1) and (2). These results, stated in our notation, are as follows:

Theorem 1. *(E. Candes et. al. [1]): Suppose that $s \geq 1$ is such that*

$$\delta_{2s} < 1$$

and let $N \subseteq [n]$ be such that $|N| \leq s$. Let $y := Ax$, where x is an arbitrary vector supported on N. Then x is the unique minimizer to (1) so that x can be reconstructed from knowledge of the vector y (and a_i 's). □

Theorem 2. *(E. Candes et. al. [1]): Suppose that $s \geq 1$ is such that*

$$\delta_s + \theta_{s,s} + \theta_{2s,s} < 1$$

and let x be a real vector supported on a set $N \subseteq [n]$ obeying $|N| \leq s$. Put $y := Ax$. Then x is unique minimizer to (2). □

D. Donoho and X. Huo [2] have shown the exact recovery condition for P_1 in terms of mutual coherence. If x is a k sparse vector and matrix A is k-RIP compliant, $k < \frac{1}{2}\big(1 + \frac{1}{\mu}\big)$ is an exact recovery condition for P_1 problem. The following result is relevant to the objective of present work.

Lemma 3. *(E. Candes et. al. [1]): Let $s \geq 1$ be such that $\delta_s + \theta_{s,2s} < 1$, and c be a real vector supported on $N \subseteq [n]$ obeying $|N| \leq s$. Then there exists a vector $\gamma \in \mathbb{R}^n$ such that $\gamma' a_i = c_i$ for all $i \in N$ where a_i is the i^{th} column of a matrix $A \in \mathbb{R}^{m \times n}$. Furthermore, γ obeys*

$$| < \gamma, a_i > | \leq \frac{\theta_s}{(1 - \delta_s - \theta_{s,2s})\sqrt{s}}.\|c\|, \ \forall i \notin N. \quad (3)$$

□

III. Compressed sensing with partial support constraint

It may be noted that the reconstruction method given by P_1 in (2) is nonadaptive as no information about x is used in P_1. It can, however, be made partially adaptive by imposing constraints on the support of the solution to be obtained. In [9][5][7] (and the references therein) the authors have modified the cost function of P_1 problem by incorporating the partial support information into the reconstruction process as detailed below.

Consider that T is the known partial support information of signal x. Here T is considered in general sense that it can have an error part which corresponds to the complement of support of x. In [9], the authors have modified the P_0 problem by considering zero weights in T and posed it as follows:

$$min\|x_{T^c}\|_0 \ \ \textit{subject to } y = Ax. \quad (4)$$

This problem recovers a signal that satisfies the data constraint and whose support is sparsest outside T. The following result in [9] establishes the uniqueness of (4).

Theorem 4. *(N. Vasawani et. al. [9]): Given a sparse vector x with support $N = T \cup \Delta / \Delta_e$ where Δ and T are unknown and known disjoint supports respectively, and Δ_e is the error in known support such that $\Delta_e \subseteq T$. Consider reconstructing it from $y = Ax$ by solving (4). Then x is the unique minimizer of (4) if $\delta_{k+2u} < 1$,where $k := |T|$ and $u := |\Delta|$.* □

In [9], the authors have also considered the convex relaxation of (4) as

$$min\|x_{T^c}\|_1 \ \ \textit{subject to } y = Ax. \quad (5)$$

The uniqueness condition of (5) has been established by the following results.

Theorem 5. *(N. Vasawani et. al. [9]): Given a sparse vector x whose support $N = T \cup \Delta / \Delta_e$ where Δ and T are unknown and known disjoint supports respectively, and Δ_e is the error in known support such that $\Delta_e \subseteq T$. Consider reconstructing it from $y = Ax$ by solving (5). Then x is the unique minimizer of (5) if*

1) $\delta_{k+u} < 1$ *and* $\delta_{2u} + \delta_k + \theta^2_{k,2u} < 1$,
2) $\rho_k(2u, u) + \rho_k(u, u) < 1$, *with* $\rho_k(s, \tilde{s}) := \frac{\theta_{\tilde{s},s} + \frac{\theta_{\tilde{s},k}\theta_{s,k}}{1-\delta_k}}{1-\delta_s - \frac{\theta^2_{s,k}}{1-\delta_k}}$,

where $s := |N|$, $k := |T|$ and $u := |\Delta|$. □

Corollary 6. *(N. Vasawani et. al. [9]): Given a sparse vector, x, whose support $N = T \cup \Delta / \Delta_e$ where Δ and T are unknown and known disjoint supports respectively, and Δ_e is the error in known support such that $\Delta_e \subseteq T$. Consider reconstructing it from $y = Ax$ by solving (5). Then x is the unique minimizer of (5) if $u \leq k$ and $\delta_{k+2u} < \frac{1}{5}$.* □

Since sparsity of a signal inside T is unconstrained in (4), the recovered signal may not be sparse in T. In order to recover a signal, satisfying the data constraint, which is in general sparse inside T and sparsest outside T, one may choose general weights $w \in [0, 1]$ and propose the general weighted-zero-norm problem:

$$(P_{0,w}) \ \ min\|x\|_{0,w} \ \ \textit{subject to } \ y = Ax, \quad (6)$$

where $\|x\|_{0,w} = w\|x_T\|_0 + \|x_{T^c}\|_0$. It may be noted that when $w = 0$, $P_{0,w}$ coincides with (4) and when $w = 1$, it coincides with the standard P_0 problem (1). As stated in previous section, the uniqueness results in these two cases are established by Theorem 4 and Theorem 1 respectively. In [5], nevertheless, the authors have convexified this problem for a general weight vector $w \in [0, 1]$ and an arbitrary subset T of $[n]$ the following way:

$$(P_{1,w}) \ \ \ min\|x\|_{1,w} \ \ \textit{subject to } \ y = Ax, \quad (7)$$

where $\|x\|_{1,w} := \sum_i w_i|x_i|$ with $w_i = \begin{cases} w & \text{for } i \in T \\ 1 & \text{for } i \notin T \end{cases}$.

In general, in applications, T can be drawn from the estimate of the support of signal or from its largest coefficients. It has been shown in [5] that a signal x can be stably and robustly recovered from $P_{1,w}$ problem in noisy case if at least 50% of the partial support information is accurate. The uniqueness result in Theorem 5 holds in a case when w is set to 0 in $P_{1,w}$. In the case, where $w = 1$, however, $P_{1,w}$ coincides with P_1. To the best of our knowledge, the uniqueness of solution of $P_{p,w}$, with $p = 0, 1$, is not known for $w \in (0, 1)$. The present work aims at providing the stated uniqueness in the cases complementary to the known cases (viz, w = 0, 1).

IV. Uniqueness of solution of weighted 0-norm problem

Our uniqueness result for weighted 0-norm minimization may be summarized in the form of following theorem, which is motivated by the results in [9].

Theorem 7. *Let x be a real sparse vector supported on $N \subseteq [n]$ with $|N| = s$ and $y = Ax$, where $A \in \mathbb{R}^{m\times n}$ with $m < n$. Let $T \subseteq [n]$, with $|T| = k$ and $\Delta_1 = T \cap N$ with $|\Delta_1| = t$ and $\Delta = T^c \cap N$ with $|\Delta| = u$. If*

$$\delta_{k+2u+\lceil wt\rceil} < 1, \tag{8}$$

then x is the unique minimizer to the $P_{0,w}$ problem in (6) for $0 \le w \le 1$.

Proof: Let $\tilde{x}$ be a minimizer of (6). Then, $\|\tilde{x}\|_{0,w} \le \|x\|_{0,w}$, which implies that $\|\tilde{x}_{T^c}\|_0 \le w\|x_T\|_0 + \|x_{T^c}\|_0 - w\|\tilde{x}_T\|_0 \le w\|x_T\|_0 + \|x_{T^c}\|_0 \le wt + u$. Hence, $\tilde{x}_{T^c}$ has at most $wt + u$ number of non-zero elements. Therefore $\tilde{x}$ remains supported on a subset of T of cardinality at most k and and on a set $\tilde{\Delta} \subseteq T^c$ of cardinality at most $wt + u$. Similarly x is also supported on a subset $\Delta_1 \subseteq T$ of cardinality $t \le k$ and on a set $\Delta \subseteq T^c$ of cardinality at most u. Then the support of $\tilde{x} - x$ remains contained in the union $T \cup \Delta \cup \tilde{\Delta}$, which is of cardinality at most $k + u + wt + u = k + 2u + wt$. Now $A(\tilde{x} - x) = 0$ reduces to $A_{T\cup\Delta\cup\tilde{\Delta}}(\tilde{x} - x) = 0$. As $0 < \delta_{k+2u+\lceil wt\rceil} < 1$, $A_{T\cup\Delta\cup\tilde{\Delta}}$ is a full rank matrix, which implies that $\tilde{x} = x$. $\square$

Remark 1. *Here the ceiling operation $\lceil wt\rceil$ is used to take the smallest integer greater than or equal to the real number wt.*

Remark 2. *When $w = 1$, the weighted 0-norm problem coincides with the standard 0-norm problem in (1) and $k + 2u+wt = k-t+2(t+u) = 2s+e$ with $e = |T\cap N^c|$. Further, if $T \subseteq N$ then $e = 0$. Hence $\delta_{k+2u+\lceil wt\rceil} < 1$ coincides with the uniqueness condition $\delta_{2s} < 1$ of the standard 0-norm problem in (1).*

When $w = 0$, the weighted 0-norm problem coincides with the 0-norm problem in (4) and the uniqueness condition in (8) of the weighted 0-norm problem coincides with $\delta_{k+2u} < 1$ of Theorem 4.

V. Uniqueness of solution of weighted 1-norm problem

Our uniqueness result for weighted 1-norm minimization is established with the help of following lemma:

Lemma 8. *Let $x \in \mathbb{R}^n$ be a real sparse vector supported on $N \subseteq [n]$ with $|N| = s$ and $A \in \mathbb{R}^{m\times n}$ with $m < n$. Let $c \in \mathbb{R}^n$ be such that*

$$c_i = \begin{cases} w.sgn(x_i) & \text{for } i \in T \\ sgn(x_i) & \text{for } i \in \Delta \\ 0 & \text{otherwise,} \end{cases}$$

where $T \subseteq [n]$ with $|T| = k$, $\Delta = T^c \cap N$ with $|\Delta| = u$ and $w \in [0, 1]$. If

$$\left(\sqrt{\frac{kw^2+u}{k+u}}\right)\theta_{k+u} + \delta_{k+u} + \theta_{k+u,2(k+u)} < 1, \tag{9}$$

then there exists a vector $\gamma \in \mathbb{R}^n$ such that

1) *$\gamma' a_i = w.sgn(x_i)$ for $i \in T$*
2) *$\gamma' a_i = sgn(x_i)$ for $i \in \Delta$*
3) *$|\gamma' a_i| < 1$ for $i \in (T \cup \Delta)^c$.*

Proof: Since $\delta_{k+u}+\theta_{k+u,2(k+u)} < 1$ follows from (9), Lemma 3 implies that there exists a vector $\gamma \in \mathbb{R}^n$ such that $\gamma' a_i = c_i$ for $i \in T \cup \Delta$, that is, $\gamma' a_i = w.sgn(x_i)$ for $i \in T$ and $\gamma' a_i = sgn(x_i)$ for $i \in \Delta$. Again, from (3) and (9), we have

$$\begin{aligned} |\gamma' a_i| &\le \frac{\theta_{k+u}\,\|c\|}{(1-\delta_{k+u}-\theta_{k+u,2(k+u)})\sqrt{k+u})} \\ &= \frac{\theta_{k+u}(\sqrt{kw^2+u})}{(1-\delta_{k+u}-\theta_{k+u,2(k+u)})\sqrt{k+u}} < 1. \quad \square \end{aligned}$$

The following result summarizes the uniqueness of solution of weighted 1-norm minimization problem, whose proof is motivated by the results in [1].

Theorem 9. *Let x be a real sparse vector supported on $N \subseteq [n]$ with $|N| = s$ and $y = Ax$, where $A \in \mathbb{R}^{m\times n}$ with $m < n$. Let $T \subseteq [n]$ with $|T| = k$ and $\Delta = T^c \cap N$ with $|\Delta| = u$. If*

$$\left(\sqrt{\frac{kw^2+u}{k+u}}\right)\theta_{k+u} + \delta_{k+u} + \theta_{k+u,2(k+u)} < 1, \tag{10}$$

then x is the unique minimizer to the $P_{1,w}$ problem in (7) for $0 \le w \le 1$.

Proof: By standard convex arguments, there exists one minimizer $\tilde{x}$ to the problem (7), which implies that $\|\tilde{x}\|_{1,w} \le \|x\|_{1,w}$. Note that $x_i = 0$ for $i \in (T \cup N)^c \subseteq N^c$. We have

$$\begin{aligned} \|\tilde{x}\|_{1,w} &= \sum_{i\in T} w|\tilde{x}_i| + \sum_{i\in T^c}|\tilde{x}_i| \\ &= \sum_{i\in T} w|\tilde{x}_i| + \sum_{i\in\Delta}|\tilde{x}_i| + \sum_{i\in(T\cup\Delta)^c}|\tilde{x}_i| \\ &= \sum_{i\in T} w|x_i+\tilde{x}_i-x_i| + \sum_{i\in\Delta}|x_i+\tilde{x}_i-x_i| \\ &\quad + \sum_{i\in(T\cup\Delta)^c}|\tilde{x}_i-x_i| \\ &\ge \sum_{i\in T} w.sgn(x_i)(x_i+\tilde{x}_i-x_i) \\ &\quad + \sum_{i\in\Delta} sgn(x_i)(x_i+\tilde{x}_i-x_i) + \sum_{i\in(T\cup\Delta)^c}(\tilde{x}_i-x_i) \\ &= \sum_{i\in T} w|x_i| + \sum_{i\in\Delta}|x_i| + \sum_{i\in T} w.sgn(x_i)(\tilde{x}_i-x_i) \\ &\quad + \sum_{i\in\Delta} sgn(x_i)(\tilde{x}_i-x_i) + \sum_{i\in(T\cup\Delta)^c}(\tilde{x}_i-x_i) \\ &\ge \|x\|_{1,w} + \sum_{i\in T}\gamma' a_i(\tilde{x}_i-x_i) + \sum_{i\in\Delta}\gamma' a_i(\tilde{x}_i-x_i) \\ &\quad + \sum_{i\in(T\cup\Delta)^c}\gamma' a_i(\tilde{x}_i-x_i) \\ &= \|x\|_{1,w} + \gamma' A(\tilde{x}-x) = \|x\|_{1,w}. \end{aligned} \tag{11}$$

In the above chain of steps, the vector $\gamma \in \mathbb{R}^n$ is supposed to satisfy the following properties:

1) $\gamma' a_i = w.sgn(x_i)$ for $i \in T$
2) $\gamma' a_i = sgn(x_i)$ for $i \in \Delta$
3) $|\gamma' a_i| < 1$ for $i \in (T \cup \Delta)^c$.

In view of (10), the existence of such a vector γ is guaranteed by Lemma 8. From (11), it follows that $\|\tilde{x}\|_{1,w} = \|x\|_{1,w}$. Consequently, all the inequalities in (11) must be equalities. But then $\sum_{i \in (T\cup\Delta)^c} |\tilde{x}_i| = \sum_{i\in(T\cup\Delta)^c} (\gamma' a_i)\tilde{x}_i$ implies that $\tilde{x}_i = 0$ on $(T \cup \Delta)^c$ as $|\gamma' a_i| < 1$ on $(T\cup\Delta)^c$. Now $Ax = A\tilde{x}$ reduces to $A_{T\cup\Delta}(x - \tilde{x}) = 0$. By (10), we have $\delta_{k+u} < 1$ which implies that $\tilde{x}_i = x_i$ on $T \cup \Delta$. Thus $\tilde{x} = x$ as claimed. $\square$

Remark 3. *When $w = 1$, the weighted 1-norm problem coincides with the standard 1-norm problem in (2) and $k+u = t+u+k-t = s+e$, where $t = |T \cap N|$ and $e = |T \cap N^c|$. Further, if $T \subseteq N$ then $e = 0$. In this case, $k+u$ coincides with s and the uniqueness condition (10) of Theorem 9 coincides with the uniqueness condition $\theta_s + \delta_s + \theta_{s,2s} < 1$ of the standard 1-norm problem.*

When $w = 0$, the weighted 1-norm problem coincides with 1-norm problem (5), and the uniqueness condition gets reduces to

$$(\sqrt{\frac{u}{k+u}})\theta_{k+u} + \delta_{k+u} + \theta_{k+u,2(k+u)} < 1. \tag{12}$$

As such, it is not possible to compare the above condition to the uniqueness condition of Theorem 5. This is because, the proofs of both adopt different strategies. In order to deduce a condition from (12) in terms of RIC (that is akin to the condition in Corollary 6), we use the inequality $\theta_{s,\tilde{s}} \leq \delta_{s+\tilde{s}}$. Then, $\theta_{k+u} \leq \delta_{2(k+u)}$ and $\theta_{k+u,2(k+u)} \leq \delta_{3(k+u)}$. Again if $u \leq k$, then $\frac{u}{k+u} \leq \frac{1}{2}$. Hence, (12) holds if $(\frac{1}{\sqrt{2}}+2)\delta_{3(k+u)} < 1$, that is, $\delta_{3(k+u)} < \frac{\sqrt{2}}{1+2\sqrt{2}} \approx 0.369$.

VI. Conclusion

The current work has proposed the conditions that guarantee the uniqueness of solution of weighted 0-norm and weighted 1-norm minimization problems for $w \in [0, 1]$. It has been analyzed further that the uniqueness conditions match with their known counterparts in the particular cases where (i). $w = 0, 1$ with 0-norm, (ii). $w = 1$ with 1-norm. In the case where $w = 0$ with 1-norm, however, our RIC-condition does not exactly match with its corresponding known condition.

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