Frank Kammer ⊠©

THM, University of Applied Sciences Mittelhessen, Giessen, Germany

Andrej Sajenko 🖂 🗈

THM, University of Applied Sciences Mittelhessen, Giessen, Germany

— Abstract -

Let n be the size of a parameterized problem and k the parameter. We present kernels for FEEDBACK VERTEX SET, PATH CONTRACTION and CLUSTER EDITING/DELETION whose sizes are all polynomial in k and that are computable in polynomial time and with $O(\text{poly}(k) \log n)$ bits (of working memory). By using kernel cascades, we obtain the best known kernels in polynomial time with $O(\text{poly}(k) \log n)$ bits.

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Keywords and phrases path contraction, feedback vertex set, space-efficient algorithm, cluster editing / deletion, full kernel

Related Version This is the full version of the paper [30], which includes proofs and an additional result to cluster editing and deletion.

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1 Introduction

With the rise of big data the focus on algorithms that treat space as a valuable resource becomes increasingly important. Large inputs may cause "standard" solutions to fail their execution due to out-of-memory errors, or cause them to spend a significant amount of time for memory swapping due to cache faults.

Within the last ten years, there is a new research direction called *space-efficient algorithms* where one tries to solve a problem with as little space as possible while "almost" maintaining the same running time of a standard solution for the problem under consideration. Space-efficient algorithms are mostly designed for problems that already run in polynomial time, i.e., we have algorithms for connectivity problems [8, 18, 22], matching [14] and other graph problems [23, 29]. Further algorithms are known for, e.g., sorting [4, 33] and geometric problems [1, 3].

Our goal is to combine the research on space-efficient algorithms with parameterized algorithms. In the classical literature a parameterized problem $P \subseteq \Sigma^* \times \mathbb{N}$ is a language where Σ is a finite alphabet and the second part $k \in \mathbb{N}$ is called parameter. In addition, P is called fixed-parameter tractable (FPT) if there exists an algorithm \mathcal{A} (called FPT algorithm) and a computable function f such that, given an instance $(I, k) \in \Sigma^* \times \mathbb{N}$, the algorithm \mathcal{A} correctly decides whether $(I, k) \in P$ in a time bounded by $f(k) \operatorname{poly}(|I|)$. A popular way to find an FPT algorithm is to find a so-called kernelization algorithm. Given a parameterized problem P, a kernelization algorithm (or simply called kernelization) is a polynomial-time algorithm $\mathcal{A} : (\Sigma^* \times \mathbb{N}) \to (\Sigma^* \times \mathbb{N})$ such that, given an instance $(I, k) \in \Sigma^* \times \mathbb{N}$, then $(I', k') = \mathcal{A}(I, k)$ is another instance (called kernel) of the problem with the property that $(I', k') \in P$ if and only if $(I, k) \in P$ and $|I'|, k' \leq g(k)$ (where usually $k' \leq k$) for some function g. Then the kernel can usually be solved by a brute-force algorithm in O(f(g(k))) time, for some function f. Furthermore, we call (I, k) a yes-instance exactly if $(I, k) \in P$. Otherwise, we call it a no-instance. A yes- or no-instance of constant size is called trivial.

if, given input (I, k), it produces an output (I', k') such that (I', k') is a yes-instance if and only if (I, k) is a yes-instance.

In this paper, we describe space-efficient kernelizations, which we define as a kernelization \mathcal{A} (as above) with the additional property of using $O(h(k) \log |I|)$ bits for some computable function h. Following this definition, a space-efficient FPT algorithm is an FPT algorithm that runs within $O(h(k) \log |I|)$ bits of space. By using our kernel of g(k) vertices and edges as an intermediate kernel, which needs $O(g(k) \log |I|)$ bits to be stored, we then can apply the best known solution to further reduce the kernel size via kernel cascades (i.e., the consecutive application of a kernelization on the kernel). Alternatively, we can use the kernel to easily build a space-efficient FPT algorithm \mathcal{A} that produces an optimal solution S for the instance (I, k) in time $f(k) \operatorname{poly}(|I|)$ using $O((g(k) + h(k)) \log |I|)$ bits for some functions f and h. We focus on graph problems, i.e., an instance I is a graph G = (V, E) as well as |I| = |V| + |E|, and our space-efficient kernelizations are space-efficient graph kernelizations. For the remainder of the paper, let n and m be the vertices and edges, respectively, of the graph under consideration. Note that for simple graphs, $O(\log n) = O(\log m)$ holds.

We also recognize the need for *full kernels* [12], i.e., a kernel that contains the vertices/edges of all minimal solutions in a yes-instance (G, k). Such a full kernel allows us to enumerate all minimal solutions of size at most k. Those kernels are, e.g., necessary for the application of frameworks such as shown in [24] and for parameterized enumeration [12]. Our computation model is based on a read-only word-RAM with a word size of $w = \Omega(\log n)$ bits, enabling constant-time arithmetic operations $(+, -, \cdot, /, \text{ modulo})$ and bit-shift operations on w-bit sequences. The input is divided into three types: read-only *input memory*, write-only *output memory*, and read-write *working memory*. Space-efficient algorithm space bounds are typically in bits and refer to the working memory. When expressed in words, these space bounds include an extra factor of $\Theta(\log w)$ or $\Theta(\log n)$ depending on the specific implementation, making them less precise or more complex to describe. In contrast, we express kernel sizes in terms of vertices/edges (i.e., in words), following the conventional approach for describing kernel size.

Parameterized Space Complexity. In classical research, the focus is often on achieving minimal space bounds, but this comes at the cost of significantly increased running times, rendering them impractical. *Parameterized space complexity* is a research area where one mainly classifies problems based on the amount of memory required to solve them. Two important classes in this field are para–L (aka. logspace + advice) and XL (aka. slicewise logspace). para–L contains problems that can be solved with $f(k) + O(\log n)$ bits, while XL contains problems that can be solved with $O(f(k) \log n)$ bits [7].

We next give an overview over parameterized space complexity restricted to graph problems. An early work on parameterized space complexity is due to Cai et al. [7] who showed that vertex cover is in para–L. Flum and Grohe [21] presented that model-checking problems of first-order formulas of bounded degree graphs are also in para–L. Elberfeld et al. [17] showed that FEEDBACK VERTEX SET (FVS) is in XL. Bannach et al. [2] studied packing, covering and clustering problems and show (among other results) that TRIANGLE PACKING, (EXACT PARTIAL) VERTEX COVER, and MANY CLUSTER EDITING are in para–L (more precisely in a class called para–AC⁰ \subseteq para–L) and CLUSTER EDITING is in para–L (more precisely in para–TC⁰ \subseteq para–L). Fafianie and Kratsch [20] showed that several graph deletion problems where the target classes have finite forbidden sets are also in para–L. This result was recently generalized to infinite forbidden sets by Biswas et al. [6], who showed that deletion problems like deletion to linear forest and deletion to pathwidth 1 are also in para–L.

To our knowledge neither the membership of para–L nor a lower bound for FVS was discovered yet. While FVS can be formulated as a deletion problem, Biswas et al. [6] mentioned that the techniques required are not "easily" applicable to FVS (and other deletion problems). They attacked the problem by using different parameterizations and showed that (among other problems) FVS is in para–L if parameterized by vertex cover. Moreover, they presented a space-efficient FVS algorithm, parameterized by solution size k, that runs in $5^k n^{O(1)}$ time and with $O(k \log n)$ bits, an improvement over the previously known best space-efficient FVS algorithm of Elberfeld et al. [17], which runs in $O(k^k n^5)$ time and uses $O(k \log n)$ bits.

Approach and Contribution. Our approach departs from the conventional method of computing a kernel by applying reduction rules globally to the entire graph, a process that can be resource-intensive in terms of space. Aiming for a memory usage of $O(\operatorname{poly}(k) \log n)$ bits, we show a process that involves systematically modifying and condensing disjoint subgraphs of the input graph G = (V, E) while preserving $\operatorname{poly}(k)$ vertices and edges in the resulting kernel. The main idea here is to make use of a separator set U of size $\operatorname{poly}(k)$, which partitions graph G into disjoint subgraphs. Each subgraph undergoes a reduction process under consideration of the separator U to efficiently shrink it to $\operatorname{poly}(k)$ vertices and edges. Subsequently the reduced subgraph is then carefully integrated into an initial kernel G' = G[U] under construction. G' is then repeatedly reduced to ensure we stay within our space bound. To make our algorithm work we need to show that both, computing the separator and applying the reduction rules, must be implemented with $O(\operatorname{poly}(k) \log n)$ bits. This often means that we are not able to run all known reduction rules or to run the rules in a restricted setting.

In Section 2, we present a simple PATH-CONTRACTION kernelization that runs in $O(n \log k + poly(k))$ time using $O(poly(k) \log n)$ bits. To find a kernel for PATH CON-TRACTION, one usually searches for *bridges* (i.e., edges whose removal disconnect the graph) and merges the endpoints of such a bridge to a single vertex [25, 32]. Bridges are usually found by running a DFS—to the authors knowledge, all polynomial-time DFS need $\Omega(\sqrt{n})$ bits and a polynomial-time depth-first search (DFS) with $O(\sqrt{n})$ bits is due to Izumi and Otachi [28]. Instead of this reduction rule, we use a separator U (which is the queue of a breadth-first search (BFS)) and iteratively expand our kernel while shrinking *induced degree-2-chains* (paths whose vertices all have degree 2) as long as they consist of more than k+1 edges. To achieve our space bound we show that a ves-instance of PATH CONTRACTION cannot have a tree as an induced subgraph with more than k+2 leaves. This bounds the size of U by O(k) and makes it possible to construct a BFS algorithm that stores at most O(k) vertices at a time, allowing us to construct a kernel with $O(k^2)$ vertices and edges in $O((n + k^2) \log k)$ time using $O(k^2 \log n)$ bits. To get the current best kernel size we subsequently apply Li et al.'s [32] polynomial-time kernelization and so get a kernel of 3k + 4 vertices in $O(n \log k + poly(k))$ time and using $O(poly(k) \log n)$ bits. Li et al.'s kernelization for PATH CONTRACTION builds on Hegernes et al.'s kernelization [25] and uses $\Theta((n+m)\log n)$ bits due to searches for bridges and to store the modification in adherence to their reduction rules.

Our main result is a new kernelization for FEEDBACK VERTEX SET (FVS) in Section 4, prefaced by its own preliminaries in Section 3. Our idea is to compute an approximate minimum feedback vertex set U as a separator whose removal partitions the graph into several trees. We use a so-called Loop Rule and a restricted Flower Rule as well as a so-called Leaf Rule and Chain Rule. If the kernel is still too large, we follow ideas from Thomassé [34]. For details on these rules, see Section 4. We want to remark that a solution for PATH CONTRACTION of size k implies one for FVS of size 2k by simply taking the endpoints of

Authors	Time	Space [bits]
(randomized) Li and Nederlof [31]	$O(2.7^k(n+m))$	$\Omega((n+m)\log n)$
Iwata and Kobayashi [27]	$O(3.46^k n)$	$\Omega((n+m)\log n)$
Elberfeld et al. $[17]$	$O(k^k n^5)$	$O(k \log n)$
Biswas et al. [6]	$O(n^7 \operatorname{poly}(k) + 5^k n^{\Theta(1)})$	$O(k \log n)$
This paper $+$ Iwata and Kobayashi [27]	$O(n^5 \operatorname{poly}(k) + 3.5^k)$	$O(k^4 \log n)$

Table 1 A time and space comparison of FEEDBACK VERTEX SET algorithms.

the contracted edges. Our kernelization for FVS runs in $O(n^5 \operatorname{poly}(k))$ time, uses $O(k^4 \log n)$ bits and outputs a kernel of $n' = 2k^2 + k$ vertices. Note that FEEDBACK VERTEX SET has no kernel of size $O(k^{2-\epsilon})$ for any constant $\epsilon > 0$ unless NP \subseteq coNP/poly [15]. After computing our kernel we can use the deterministic algorithm of Iwata and Kobayashi to solve it in $O(3.46^k n') = O(3.5^k)$ time or the randomized algorithm of Li and Nederlof to solve it in $2.7^k \operatorname{poly}(n')$ time. In total, we can solve FVS in $O(n^5 \operatorname{poly}(k) + 3.5^k)$ time and with $O(k^4 \log n)$ bits.

Li and Nederlof and Iwata and Kobayashi do not focus on space efficiency in their search-tree algorithms for FEEDBACK VERTEX SET and thus do not state a space bound. Based in their description they assume either to be able to modify the input graph or to create at least one copy of it, which gives us a lower bound of $\Omega((n+m)\log n)$ bits. By assuming that one stores a copy of the reduced graph whenever descending in the search tree, the two search-tree algorithms have a space upper bound of $O(k(n+m)\log n)$ bits.

Compared to Elberfeld et al.'s algorithm [17], which solves FVS in time $O(k^k n^5)$, we are faster, but we use $O(k^4 \log n)$ instead of $O(k \log n)$ bits. Concurrently and independently to our result, Biswas et al. [6] presented an iterative compression algorithm (based on the Chen et al. [9] algorithm) that maintains the space bound of $O(k \log n)$ bits and has a runtime of $O(5^k n^{O(1)})$. The degree of the polynomial is not mentioned explicitly, presumably due to the fact that some of the used log-space auxiliary results do not mention their "exact" running time either. However, we and they use the $O(\log n)$ bit realizations of the so-called Leaf and Chain Rules (aka. Degree-2 Rule) from Elberfeld et al. [17]. The non space-efficient version of the rules removes vertices from the input graph that are not relevant for solving FVS, but due to the space restriction, the information of the graph resulting from the removal is computed on demand. By our analysis (Lemma 5) this results in a running time factor of $\Theta(n^5k \log k)$ for each vertex / edge access. Moreover, Biswas et al. have a nested loop where $\Theta(n^2)$ connectivity tests have to be performed, which increases the running time to $\Theta(n^7 \operatorname{poly}(k) + 5^k n^{\Theta(1)})$. Based on that our running time is faster in k and in n, but they use only $O(k \log n)$ bits. A summary is shown in Table 1.

In Section 5, we adapt a standard technique for CLUSTER EDITING/DELETION to compute a full kernel of $O(k^2)$ vertices in $O(nm \log k)$ time and within $O(k^2 \log n)$ bits. Here we use $O(k^2)$ times a rule searching for so-called *conflict triples*. This easily allows us to bound the size of a separator U by $O(k^2)$.

2 Path Contraction

Let G be an n-node m-edge graph and let C be a subset of edges of G. We write G/C for the graph obtained from G by contracting each edge in C. Contracting an edge is done by merging its endpoints and removing any loops or parallel edges afterwards. In the parameterized PATH CONTRACTION problem, a connected graph G = (V, E) is given together with a parameter

k and the task is to find a set $C \subseteq E$ with $|C| \leq k$ such that G/C is a path. In particular, G/C is a connected graph with $n' \in \mathbb{N}$ vertices and n' - 1 edges. One reduction rule used in Li et al.'s [32] and in Heggernes et al.'s [25] kernelization is an iterative contraction of a bridge for which no polynomial-time $O(\text{poly}(k) \log n)$ -bits algorithm is known. (Bridges are found by running a DFS and the best-known polynomial-time DFS with a minimum of space uses $\Omega(\sqrt{n})$ bits [28].) Instead of computing bridges, we introduce two new reduction rules below. In the following, a *subtree* T of G is a subgraph of G that is a tree. Moreover, let a degree-2 chain be a maximal simple path $P = v_1, \ldots, v_\ell$ ($\ell \in \mathbb{N}$) whose vertices v_1, \ldots, v_ℓ are all of degree 2. Observe, if P is not a cycle, then v_1 and v_ℓ must each have a neighbor that is not of degree 2.

Rule 1 If there exists a degree-2 chain P with more than k + 1 edges, contract all except k + 1 arbitrary edges of P.

Rule 2 After k applications of Rule 1, if G contains more than $k^2 + 4k + 1$ vertices, more than $(3k^2 + 13k)/2$ edges, or a subtree with more than k + 2 leaves, then output "no-instance".

The bound on the number of leaves in Rule 2 helps us to guarantee that our kernelization works in our space bound. We want to remark that we do not apply Rule 2 exhaustively; more precisely, we do not explicitly search for subtrees with more than k + 2 leaves, which is NP-hard.

▶ Lemma 1. Rule 1 and 2 are safe and produce a full kernel.

Proof. Rule 1 is safe. Observe that a solution either contracts the entire degree-2 chain P or none of it. Since contracting an entire degree-2 chain with more than k + 1 edges is not allowed, any solution does not contract any edge of P. Thus, Rule 1 is safe and we do not remove an edge of any minimal solution.

Rule 2 is safe. Vertex bound. Assume that (G, k) is a yes-instance. Let us define C as the subset of edges from E(G) that are requisite for contraction to arrive at a solution for G. Since C is a solution for G it is also a valid solution for G'. Consider the path P^* , defined by $P^* = G'/C$. Observe that P^* is an amalgamation of subpaths, each limited to k + 1 edges. These subpaths, when traced back to G', are individually connected by a unique vertex with a minimum degree of 3. Contraction on this specific vertex is obligatory to obtain the path P^* . Since one edge contraction reduces the vertex count by one, the path P^* would comprise a vertex count less than that of G' by at most k vertices. Consequently, there is an upper limit of k for such unique vertices in P^* . This means that P^* can accommodate at most k + 1 of these subpaths.

From the above deductions, it can be derived that the vertex count for P^* is capped at $(k+1)^2 + k$. Extending this reasoning, G' has a maximum vertex count of $(k+1)^2 + 2k$.

Edge bound. Let N(u) be the neighbors of a vertex u. Note that by definition, a contraction of an edge $\{u, v\}$ reduces the number of vertices by one, and reduces the number of edges by $|N(u) \cap N(v)| + 1$. Since G/C is a path where each vertex is of maximal degree 2, G cannot have a vertex u with $|N(u)| \ge k + 3$ since at least k + 1 contractions are needed to remove k + 1 neighbors of u to reach degree at most 2. Hence, the common neighborhood of two adjecent vertices is at most k + 1. Therefore, if we have $k' \le k$ contractions left, we can remove at most k' + 2 edges with one contraction. Thus, with k possible contraction we can remove at most $\sum_{i=1}^{k} (i+2) = 2k + \sum_{i=1}^{k} i = \frac{1}{2}(k^2 + k) + 2k$ edges. Hence, G cannot have more than $n' - 1 + \frac{1}{2}(k^2 + 5k)$ edges in total. Summarized, the kernel consists of $O(k^2)$ vertices and edges and is a full kernel since the only modification are done by Rule 1.

Bound on leaves in subtrees. Let us say that a *super vertex* w is a vertex obtained from a contraction of (*normal*) vertices u and v of G. For an easy intuition, we then say that

w contains u and v. Consider the following fact: if G' is obtained from G by contracting an arbitrary number of edges and (G, k) is a yes-instance, then (G', k) is also a yes-instance. By induction, it suffices to show the fact for graphs G' that are obtained by one contraction of an arbitrary edge $\{u, v\}$. Assume that $G' = G/\{\{u, v\}\}$. Consider an optimal solution C for (G, k) and the path P = G/C. We denote by S(u) and S(v) the super vertices in G/C containing u and v, respectively. Note that, either S(u) = S(v) or S(u) and S(v) are adjacent in P. If $S(u) \neq S(v)$, then $G'/C = (G/\{\{u, v\}\})/C = (G/C)/\{\{S(u), S(v)\}\} = P/\{\{S(u), S(v)\}\}$ is a path and thus, C is a solution for (G', k). If S(u) = S(v), then there is a u-v-path P' in G' consisting only of edges in C. Let e be an edge of P' and $C' = C \setminus \{e\}$. Then G'/C' = G/C is a path and C' is a solution L for (G', k-1) and thus for (G', k). To sum up, the fact holds.

Assume for a contradiction that G has a subtree T with $\ell > k + 2$ leaves. By contracting all edges of T in G without edges that have an endpoint in L we obtain a graph G' and by the fact above (G', k) is a yes-instance. However, G' has a subtree with $\ell > k + 2$ leaves—a contradiction being a yes-instance. Since we only reject no-instances the kernel remains a full kernel.

For the kernel construction we use a BFS. We shortly sketch a usual BFS and the construction of a so-called *BFS tree*. The BFS visits the vertices of an input graph roundwise. As a preparation of the first round it puts some vertex v into a queue Q, marks it as visited, and starts a round. In a round it dequeues every vertex u of Q, and marks u as visited. Moreover, it puts every unvisited neighbor $w \in N(u)$ of u into a queue Q' and marks it as visited. We then say that w was first discovered from u and add the edge $\{u, w\}$ to an initial empty BFS tree. If Q' is empty at the end of the round, the BFS finishes. Otherwise, it proceeds with the next round with Q := Q' and $Q' := \emptyset$.

During each BFS iteration, the BFS queue Q inherently acts as a separator, dividing the graph into two distinct categories: vertices already encountered by the BFS and those yet to be encountered. To understand this, envision a BFS queue Q established after a given round, prior to the initiation of the subsequent round. Let V_1 denote the vertices in G already encountered by the BFS, excluding those in Q, and V_2 represent vertices yet to be visited. If Q were not a separator, an edge (u, v) would exist such that u belongs to V_1 and v to V_2 , effectively suggesting that the BFS traversal overlooked an unvisited vertex v while exploring u's neighbors. This contradicts the BFS algorithm, as v should have been incorporated into Q. Therefore, Q functions as the desired separator U.

As the BFS progresses, its exploration adheres to the subtree structure of G. The BFS queue size is restricted to a maximum of k + 2 vertices; otherwise, we can determine a no-instance by Rule 2 and halt the BFS process. With regards to marking vertices as visited, since Q effectively delineates between previously visited and unvisited vertices after each BFS round, only the last BFS round's queue is necessary to verify the visited vertices. Notably, in this context, separator U consists of vertices in Q and is thus dynamic, adapting with each BFS iteration. So far this approach ensures that every yes-instances is traversed and any no-instance is identified in $O(n \log k)$ time utilizing only $O(k \log n)$ bits.

To realize Rule 1 we need additional information. Instead of storing just vertices v on the BFS queue we store quadruples that we use to identify degree-2 chains and apply Rule 1—see also Fig. 1. Each quadruple (v, p, i, v^*) consists of a vertex v and its predecessor p if v is not the root, the counter $i \in \{0, \ldots, n\}$ with i > 0 being v's position on a degree-2 chain, and the vertex v^* with $v^* \neq \text{null}$ being the (k + 1)th vertex on a degree-2 chain that contains v. So we can easily check Rule 1 as shown in the proof of Theorem 2. By Rule 2, we can guarantee our space bound by maintaining $O(k^2)$ vertices and edges.



Figure 1 Our adapted BFS starts from the leftmost vertex, removing dotted vertices on a degree-2 chain with over k + 1 predecessors and connecting the neighbors of removed vertices with bold edges. Dashed edges are skipped by the BFS.

▶ **Theorem 2.** Given an n-vertex instance (G, k) of PATH CONTRACTION, there is an $O((n+k^2)\log k)$ -time $O(k^2\log n)$ -bits kernelization that outputs a full kernel of $O(k^2)$ vertices and edges, or outputs that (G, k) is a no-instance. The result can be used to find a (possibly not full) kernel of at most 3k + 4 vertices in $O(n\log k + poly(k))$ time using $O(poly(k)\log n)$ bits.

Proof. In this proof we apply a modified BFS on the given graph, which adjustments are described below. The main structural adjustment is that the BFS maintains at most k + 2 quadruples instead of vertices in its queue and uses the queue of the previous round to identify already visited vertices, instead of marking all vertices of either visited or unvisited. Recall that each quadruple (v, p, i, v^*) consists of a vertex v and its predecessor p if v is not the root, a counter $i \in \{0, \ldots, n\}$ with i > 0 being v's position on a degree-2 chain, and the vertex v^* with $v^* \neq \text{null}$ being the (k + 1)th vertex on a degree-2 chain that contains v. During the run of the BFS we select vertices and edges that we can iteratively put into a kernel under construction G'.

Before we start to describe the adjustments, we want to point out that our approach works only if the BFS is started at a vertex of degree other than 2, which we can identify by simply iterating over all vertices. If there is no such vertex, then the graph is a simple cycle and we output G as the kernel if $m \leq k + 2$, otherwise we output "no-instance".

The BFS visits the vertices as usual and updates its quadruples as follows. For each quadruple (v, p, i, v^*) , it iterates over v's neighborhood and stores the quadruple (v', v, 1, null) in queue Q' for every unvisited neighbor v' if v is of degree other than two, and otherwise the quadruple $(v', v, i + 1, v^{**})$ where v^{**} is v if i = k + 1, otherwise $v^{**} = v^*$.

By Rule 2 we can bound the size of the BFS queue by k + 2 and the size of the kernel by $n' \leq k^2 + 4k + 1 = O(k^2)$ vertices and $m' \leq n' - 1 + \frac{1}{2}(k^2 + 5k) = O(k^2)$ edges. To ensure Rule 2, we can easily count the number of leaves in the BFS tree while executing the BFS.

We now describe how a kernel (G', k') can be constructed in adherence to Rule 1. Instead of contracting arbitrary edges we contract edges at the end of a degree-2 chain. The contraction is realized by not copying the inner vertices and edges at the end of a degree-2 chain while the BFS traverses the paths and connecting the (k + 1)st vertex with the last vertex of the path in the kernel. To avoid adding vertices into the queue that are already visited we maintain the vertices of the previous and the current queue inside two balanced heaps, respectively.

For the time being ignore a problem that two vertices in the BFS queue may be adjacent (i.e., the BFS starts to explore a degree-2 chain from both its endpoints). For each quadruple (v, p, i, v^*) , we add the vertex v into G' if $i \le k + 1$ and if additionally $p \ne \text{null}$, we also add the edge $\{v, p\}$ into G'—the condition ensures that we do not add the full degree-2 chain into the kernel. If the degree of v in G is not two, then v terminates a degree-2 chain and we add the edge $\{v^*, v\}$ if i > k + 1 (the bold edges in Fig. 1). We additionally add for every $u \in N(v)$ with u is in G', the edge $\{v, u\}$ into G' (in Fig. 1 they are shown dashed).

We now consider the case where two vertices v with (v, p, i, v^*) and v' with tuple (v', p', i', v'^*) on the current BFS queue are connected to each other in G and are both of degree two. If i + i' > k + 2 we move backwards on both paths until i + i' = k + 2 (but $i, i' \ge 0$) and modify the kernel by removing the vertices and edges used to move backwards. Add the edge $\{w, w'\}$ to the kernel where w and w' are the vertices at which we stopped our backward move.

It remains to show the space and time bounds of our kernelization. The size of the queues used for the BFS and our computation is bounded by O(k) vertices and, thus, cannot exceed $O(k \log n)$ bits. The kernel is bounded by $O(k^2)$ vertices and $O(k^2)$ edges and thus uses $O(k^2 \log n)$ bits. In total, we use $O(k^2 \log n)$ bits. Concerning the time bound note that a standard BFS runs in O(n + m) time. By Rule $2 m = O(n + k^2)$ or we stop. The algorithm has to check for each vertex if it is in a balanced heap (in the queue or in the kernel) of size at most $O(k^2)$, which takes $O(\log k)$ time per vertex. In total we have running time of $O((n + k^2) \log k)$. (Note that running backwards on degree-2 chains takes time linear to the length of the path and thus our asymptotic time bound remains the same.)

Our kernel is small enough to apply the polynomial-time kernelization of Li et al. and we obtain so a kernel of 3k + 4 vertices in $O(n \log n + poly(k))$ time using $poly(k) \log n$ bits.

3 Log-Space Tree Traversal and Cycle Check

For our result on FEEDBACK VERTEX SET we require the following two auxiliary lemmas to traverse trees and find a back edge in a graph. Cook and McKenzie [10] showed how this can be done in $O(n^2)$ and $O(n^3)$ time, respectively, by using $O(\log n)$ bits.

▶ Lemma 3. ([10, Theorem 2]) Given an n-vertex tree T and a node r of T as root there is a $O(n^2)$ -time $O(\log n)$ -bits algorithm that traverses all vertices of T in depth-first-search manner.

Proof. Let $r \in T$ be a root and p be the previously visited vertex. We use a known technique to traverse the graph in a special order, which main idea is as follows: Assume we visit a vertex v from a vertex p. The vertex v has several neighbors $v_1, v_2, \ldots, v_i = p, \ldots, v_{\deg(v)}$, from which one of it is p. Find the index i of p with p = A[v][i] by iterating throw v's adjacency array and visit $(A[v][(i+1) \mod \deg(v)])$. If we return to v, then from the vertex $p' = v_{(i+1)}$ and proceed with the next child $v_{(i+2) \mod \deg(v)}$ of v. With the modulo operation we so visit all children and leave the vertex via the back edge (v, p).

In detail we distinguish between the root (i.e., detectable via the check v = r) and the remaining nodes and treat them as follows. Let visit(v, p) be the procedure to visit all nodes in depth-first-search manner.

- **Treat root** If $v = r \wedge deg(v) = 0$, we output v and know that the tree consists only of one node, hence, we return. Otherwise and if $v = r \wedge p = null$ we know that its the first visit of the root, we output v and call visit(A[v][0]) to visit its first child. Otherwise, we have returned to the root after visiting the maximal subtree below its child p. We find the index i of p with p = A[v][i] by iterating throw v's adjacency array. Check if p was the last child, i.e., $(i = \deg(v)$ holds) and return, since the whole tree was traversed. Otherwise, we call visit(A[v][i + 1], v) to visit the next child.
- **Treat non-root** Output v. We find the index i of p with p = A[v][i] by iterating throw v's adjacency array and call visit $(A[v]](i+1) \mod \deg(v)]$).

Note, that the algorithm is actually defined recursively, however, it uses a tail-recursion which are known to be translateable to loops that do not need a stack. Hence, $O(\log n)$ bits

suffice to store the required information. Concerning the running-time note that all operations except the search for the next edge to follow require constant time. The search itself requires $O(\deg(v))$ time and has to be done for each visit of a vertex, i.e., $O(\deg(v))$ times. Hence, the total required time is $\sum_{v \in T} \deg(v)^2 \leq m(m+1) = O(n^2)$ [13, Theorem 3.7], where *m* is number of edges of a graph, which in case of a tree is bound by the handshaking lemma to O(n).

▶ Lemma 4. ([10, Theorem 2]) Given an n-vertex graph G and a vertex r, there is an $O(n^3)$ -time $O(\log n)$ -bits algorithm that either traverses the connected component with r if it is a tree, or otherwise, it returns a back edge of the DFS tree rooted at r.

Proof. Let us consider the scenario where the DFS, as described in Lemma 3, is on a path from r to u and is about to follow an edge $\{u, v\}$. To find out if $\{u, v\}$ "closes" a cycle or not, the DFS must know if v has not been previously discovered by the DFS or of v is the direct predecessor of u on the r-u path. Since the DFS has no knowledge on this, this condition is verified by a second DFS run until reaching u where we have to check for each discovered vertex v' if $v \neq v'$ and if not true, if v is not the direct predecessor of u in the second DFS run. Otherwise, this would imply a cycle constructed by the paths r-u and r-v (where the first is not a subpath of the second) and the edge $\{u, v\}$, implying that the connected component is not a tree and $\{u, v\}$ is a valid back edge.

Since the algorithm can stop whenever a first cycle is found, the algorithm considers only O(n) edges. Thus the space and time bounds stated in the lemma hold.

4 Feedback Vertex Set

Given an *n*-vertex *m*-edge graph G = (V, E) a set $F \subseteq V$ is called *feedback vertex set* if the removal of the vertices of F from G turns G into an acyclic graph (also called *forest*). In the parameterized FEEDBACK VERTEX SET problem, a tuple (G, k) is given where G is a graph, and k is a parameter. We are searching for a feedback vertex set F of size at most k in G. In kernelization, it is common to identify vertices that must be in every minimal feedback vertex set of size at most k, remove them from the instance, and restart the kernelization. To avoid modifying the given instance, we simulate this by starting with an empty set F and adding vertices to F when we determine they must be in every solution of size at most k. Subsequently, we realize the graph $G[V \setminus F]$ by considering G and disregarding vertices in F.

Iwata showed a kernelization for FEEDBACK VERTEX SET that produces a kernel consisting of at most $2k^2 + k$ vertices and $4k^2$ edges and runs in $O(k^4m)$ time [26]. He mentions that all other kernelizations for FEEDBACK VERTEX SET exploit an exhaustive application of the three basic rules below and the so-called *v*-flower rule. A *v*-flower of order *d* is a set of *d* cycles pairwise intersecting exactly on vertex *v*.

Loop Rule. Remove a vertex v with a loop and reduce to (G - v, k - 1) and $F := F \cup \{v\}$. Leaf Rule. Remove a vertex v with $\deg(v) \leq 1$.

Chain Rule. Remove a vertex v that has only two incident edges $\{v, u\}$ and $\{v, w\}$ (possibly u = w), and add the edge $\{u, w\}$.

Flower Rule. Remove a vertex v if a v-flower of order k + 1 exists and reduce to (G - v, k - 1)and $F := F \cup \{v\}$.

By allowing $O(k \log n)$ bits for the algorithm, Elberfeld et al. also showed how to find a cycle of 2k vertices. To realize the flower rule at a vertex v we need to run along up to k+1 cycles and check if they intersect at vertices other than v. If the given graph is reduced

with respect to the Leaf and Chain Rule and does not contain vertices with self-loops, then it can be guaranteed that the smallest cycle is of length at most 2k (maximum girth of a graph with the mentioned restrictions, minimum degree 3 and a feedback vertex set of size at most k [16]). However, the length of the remaining cycles can be bound only by a function depending on n, not on k. So it seems to be hard to find and verify a flower with $O(\text{poly}(k) \log n)$ bits. As shown by Iwata, one does not need the Flower Rule to find a kernel for FVS. Instead he uses a so-called *s-cycle cover reduction* [26, Section 3] where he has to know which edges incident to a vertex s are bridges in the graph. (For space bounds to find bridges, recall Section 2.) Since we have no solution to find a v-flower or an *s*-cycle with $O(\text{poly}(k) \log n)$ bits, we show how to construct a kernel without using both rules exhaustively.

Thomassé [34] introduced the rule below to compute a kernel consisting of $4k^2$ vertices. As input we assume a simple graph. Since his rule introduces double edges, our kernel G' is a multi graph where every multi edge is a double edge.

- **Thomassé's Rule [34]** Let X be a set of vertices, let $x \in V \setminus X$ and let C be a set of connected components of $G \setminus (X \cup \{x\})$ (not necessarily all the connected components) such that
 - G is loopless, with degree ≥ 3 and all multi-edges are double-edges,
 - there is exactly one edge between x and every $C \in \mathcal{C}$,
 - every $C \in \mathcal{C}$ induces a tree, and
 - for every subset $Z \subseteq X$, the number of trees of \mathcal{C} having some neighbor in Z is at least 2|Z|.

Then reduce to (G', k) where G' is the graph obtained by joining x to every vertex of X by double edges and by removing the edges between x and the components of C.

Thomassé applies the rule to the whole graph G. This means, he has to store graph changes over the whole graph G. This is too expensive for us. As discussed in the introduction, we utilize a separator U to break down the graph into manageable components. This allows us to construct a kernel by processing and gradually incorporating these components. In the next subsection, we outline the construction of an approximate minimum feedback vertex set as separator U, so that the graph divides into trees. In the subsequent subsection, we show how to iterate over the trees in $\mathcal{T} = G[V \setminus U]$. In a third subsection, we iteratively add these trees into a graph G' (initially G' = G[U]) while upholding our desired space bound of G' having $O(k^4)$ vertices and edges, i.e., $O(k^4 \log n)$ bits. However, the size of the trees in \mathcal{T} is unbounded in k and we need to perform an on-the-fly tree size reduction of the tree. For this, we first make sure that every tree T has not too many edges to U (or we either find a vertex for the solution F and restart, or conclude a no-instance). Afterwards, we have to traverse T and put exactly those vertices of T into G' that are not removed by an exhaustive application of the Leaf and Chain Rule. To keep the size of G' within our space bound we show in a fourth subsection how to shrink G' again. To shrink the size of G' to $O(k^2)$ vertices, we apply Thomassé's Rule, which has a precondition requiring that G' has a minimum degree of 3. However, we cannot satisfy this precondition for vertices in U within G'. Nevertheless, we can demonstrate that violating the precondition only for the vertices in U still allows the rule to function if we adjust the bounds accordingly (see Lemma 10). Finally, we show that our construction of a kernel of $O(k^2)$ vertices and $O(k^3)$ edges runs in $O(n^5 \operatorname{poly}(k))$ -time and with $O(k^4 \log n)$ bits with this approach.

Separator U of size $3k^2$. Becker and Geiger [5] presented a 2-approximation algorithm for feedback vertex set in which they extend an $(2 \log d)$ -approximation algorithm (where d is

the maximum degree of the graph) by a phase that iteratively removes a vertex v from the computed feedback vertex set S if all cycles that intersect v in G also intersect $S \setminus \{v\}$. It is unlikely that this can be done with $O(\operatorname{poly}(k) \log n)$ bits or even with $O(f(k) \log n)$ bits for some function f since a cycle in G can consist of $\Theta(n)$ vertices and there can be $\Theta(n)$ cycles.

We instead present only an O(k)-approximation algorithm, but it runs with $O(\text{poly}(k) \log n)$ bits. For this we use the following well-known rule. Given a loopless graph G of minimum degree 3 and without self-loops, every FVS in G of size at most k contains at least one vertex of the 3k vertices of largest degree [11, Lemma 3.3]. A graph with such properties can be computed by an exhaustive application of the Loop, Leaf and Chain Rule. Elberfeld et al. [17, Theorem 4.13] showed how to implement the rules with $O(\log n)$ bits. The graph obtained does not actually have to be stored. Instead we compute the required information on demand with Lemma 5, which is similar to parts of the proof of [17, Theorem 4.13].

▶ Lemma 5. Assume that an n-vertex m-edge graph G = (V, E) and a set $U \subseteq V$ consisting of $k^{O(1)}$ vertices is given. Let G' be the graph obtained by an exhaustive application of the Loop, Leaf and Chain Rule on $G[V \setminus U]$. We can provide a structure that allows the iteration over the edges of every vertex of G' in $O(n^{5}k \log k)$ total time by using $O(\log n)$ bits. In particular, we do not store G'.

Proof. Take $G_U = G[V \setminus U]$. First of all, note that we can access G_U (e.g., run a DFS in G_U) as if G_U is given explicitly by accessing G and "ignoring" all vertices (edges leading to vertices) in U. More precisely, we define the neighbors $N_{G_U}(v) = N_G(v) \setminus U$. Note that a Loop can be easily identified and possibly we restart. Note further that the connected subgraphs that are removed from G_U by an exhaustive application of the Leaf Rule are trees, which we call *tree appendages*. We can identify each neighbor $u \notin U$ of a vertex v that is part of a tree appendage by running the algorithm of Lemma 4 on $G_U - v$ with r = u as input. If the algorithm returns that the connected component with r in $G_U - v$ is a tree and does not visit v, u is part of a tree appendage (not visiting v is important since otherwise, with v and the edge $\{v, u\}$ we have a cycle). Let Q_v be the set of neighbors of v that are not part of a tree appendage. Let G_1 be the graph obtained from G_U after an exhaustive application of the Leaf Rule. Then $\deg_{G_1}(v) = |Q_v|$ is the degree of v in G_1 , for every v with $|Q_v| > 1$. If $|Q_v| \le 1$, then v is itself part of a tree appendage. Otherwise, v is part of G_1 and we can output all vertices of Q_v as neighbors of v as required from the lemma.

Let G_2 be the graph obtained from G_1 after an exhaustive application of the Chain Rule. Observe that, if $\deg_{G_1}(v) = 2$, then v is part of a degree-2 chain in G_1 that is replaced by an edge in G_2 by the exhaustive application of the Chain Rule. A possibility is that the degree-2 chain connects two vertices u and w of G_1 (possibly u = w) that are not of degree two. Then v is not part of G_2 . However, we cannot simple assume that every vertex v with $\deg_{G_1}(v) = 2$ is not part of G_2 since there is a special case (\star): v may be part of simple cycle consisting of only degree-2 vertices and the Chain Rule may reduce the cycle to exactly one vertex z with a self-loop. The vertex z can be an arbitrary vertex of the cycle thus z = vis possible. We choose z always as the vertex with the smallest ID of the cycle and "ignore" the remaining vertices. More precisely, if v is such a vertex, then v is part of G_2 and has a self-loop. If $\deg_{G_1}(v) > 2$, then v is part of G_2 , but some of its edges in G_1 may connect vwith a degree-2 chain in G_1 that is replaced by an edge in G_2 .

To realize the lemma iterate over each vertex v of G_U : If $Q_v \leq 1$, "ignore" v. If $\deg_{G_1}(v) = 2$, we output v only if we are in the Special Case (\star) and v has the smallest ID on its cycle. We now may assume that $\deg_{G_1}(v) > 2$ and thus v is part of G_2 . For each neighbor u of v in G_1 follow the potentially empty degree-2 chain from u until a vertex w

with w = v or $w \neq v \wedge \deg_{G_1}(w) > 2$. If w = v, v has a self-loop thus, we output v as a neighbor of v. If $w \neq v \wedge \deg_{G_1}(w) > 2$, we output w as a neighbor of v.

Checking if v is part of G_1 and outputting its neighbors can be done in $O(\deg(v)n^3 \log k)$ time: Lemma 4 runs in $O(n^3)$ time per edge incident to v. However, since we have to ignore vertices and edges whose endpoints are in U an access to G_U needs an access to an heap of size |U| and thus runs in $\log k^{O(1)} = O(\log k)$ time. We must iterate over all n vertices. Thus, the total running time is $O(mn^3 \log k) = O(n^4 k \log k)$ (since m = O(nk)).

To check for the Chain Rule we have to additionally follow degree-2 chains. Since a chain can consists of $\Theta(n)$ vertices, this increases the running time by a factor of n, resulting in a total running time of $O(n^5k \log k)$ for the Chain Rule.

Since by Lemma 5 we have access to a loopless graph of minimum degree three, we iterate k-times over a graph $G[V \setminus U]$ (where initially $U = \emptyset$) and in each iteration select the 3k vertices of largest degree into U. We so get an O(k) approximate minimum feedback vertex set.

▶ **Theorem 6.** Given an n-vertex m-edge instance (G, k) of FEEDBACK VERTEX SET, there is an $O(n^5k^2 \log k)$ -time, $O(k^2 \log n)$ -bits algorithm that either returns a feedback vertex set U consisting of at most $3k^2$ vertices or answers that (G, k) is a no-instance.

Proof. Starting with (G, k) and initially $U = \emptyset$ as input we compute a graph G' by exhaustively applying the Leaf and Chain Rule on $G[V \setminus U]$. If G' is empty, we return U as a feedback vertex set for G. The Chain Rule may create self-loops and multi-edges. If a vertex with a self-loop exists, it must be part of the minimal feedback vertex set, thus, put it into F, reduce k by one, and restart. After an exhaustive application of the Leaf and Chain Rule the set consisting of 3k vertices of largest degree contains at least one vertex of the minimal feedback vertex set [11, Lemma 3.3]. Thus, take 3k vertices of largest degree of G' into U, reduce k by one, and restart. If at any point k < 0, output "no-instance". We so can compute a feedback vertex set U consisting of at most $3k^2$ vertices. Instead of storing G' we compute the required information with Lemma 5. We so iterate over each vertex v of G' and its edges to determine its degree and check for self-loops and compute U in k rounds in $k \cdot O(n^5k \log k) = O(n^5k^2 \log k)$ time and with $O(|U| \log n) = O(k^2 \log n)$ bits.

Iterations over Trees. We want to output every tree T of $G[V \setminus U]$ once. For this, we iterate over all $u \in U$ and, intuitively speaking, output those trees T adjacent to u, i.e., every T having a vertex v such that u and v are adjacent in G. Note that with such an iteration we will not iterate over components of G that have no edges to any vertex of U. However, since those components are trees in G, and thus cycle free, we can ignore them. Moreover, note that several vertices of U can have edges to the same tree. We show in the proof of the next lemma how to avoid outputting a tree multiple times. To distinguish the trees, we partition the trees \mathcal{T} as follows (also see Fig. 2). \mathcal{T}_0 is the set of trees in \mathcal{T} that have at most one edge to a single vertex of U. \mathcal{T}_1 is the set of trees in \mathcal{T} where each tree has at most one edge to at least two vertices of U. \mathcal{T}_2 is the set of the remaining trees in \mathcal{T} with least two edges to some vertex of U.

▶ Lemma 7. Given an n-vertex m-edge graph G = (V, E) and a set U of $O(k^2)$ vertices, there is an algorithm that outputs a single vertex w of T as a representative for each tree T in $\mathcal{T}_1 \cup \mathcal{T}_2$ and some trees of \mathcal{T}_0 once. The algorithm runs in $O(n^3k^3\log^2 k)$ time and with $O(\log n)$ bits.



Figure 2 Our partition of trees with edges to $U = \{u_1, u_2, u_3, ...\}$. The colored vertices are the vertices with the smallest ID in the trees.

Proof. Let a vertex be smaller than another vertex, if its vertex label is smaller. For each vertex u in U, iterate over its neighbors w not in U. For each such w, consider it as the root of a tree T. Traverse through T to find the smallest u' in U connected to T. If u' is not u, continue the iteration over U with the next vertex since T was previously acknowledged as adjacent to a smallest vertex in U. If not, output w as a representative for T.

We now focus on the performance of the algorithm. The iteration over U takes O(|U|) time. The iteration over the neighbors of U ignoring neighbors in U increases the time by a factor of $O(n \log |U|)$ to $O(|U|n \log |U|)$. Possibly each vertex is connected to a "large" tree, whose traversal with an adjusted Lemma 3 can be done in $O((n^2 \cdot nk) \log |U|)$ time increasing the time to $O(|U|n^3k \log^2 |U|)$. (The adjustment is necessary to ignore edges to U, which requires an $O(\log |U|)$ -time membership check in U. Furthermore, since we have to consider edges to U the time increases in each step at most by the possible number of edges, which is bounded by O(nk) for every yes-instance of FVS.) Note that the time to find the smallest vertex in U is already included in the time since we already have to deal with vertices in U by performing membership checks. Summarized, the running time is $O(|U|n^3k \log^2 |U|) = O(n^3k^3 \log^2 k)$. Considering the space, we need a constant amount of local variables and apply Lemma 3 which sums up to $O(\log n)$ bits.

Observe that we do not need to add trees of \mathcal{T}_0 to G' since they can be removed by the Leaf Rule anyway. If we identify such a tree, we skip over it.

Tree Size Reduction. We now want to shrink each tree T of $G[V \setminus U]$ so that we can add them to G' without exceeding our space bound of $O(k^4 \log n)$ bits. Due to Cook and McKenzie [10], $O(\log n)$ bits DFS exists which suffice to find out for each tree T to which set \mathcal{T}_0 , \mathcal{T}_1 or \mathcal{T}_2 it belongs.

By definition a tree $T \in \mathcal{T}_1$ can have at most one edge to every vertex of U. Thus T has at most $|U| = O(k^2)$ edges into U. By the lemma below, we can add T into G'.

▶ Lemma 8. Given U and an \bar{n} -vertex tree $T = (V_T, E_T)$ in $G[V \setminus U]$ such that T has ℓ edges to U. After applying the Leaf and Chain Rule to $G[V_T \cup U]$ while forbidding the removal of vertices of U, T has at most $O(\ell)$ vertices. This can be done in $O(\bar{n}^3)$ time using $O(\log n)$ bits.

Proof. Traverse the tree T with the algorithm of Lemma 3. Before visiting a new vertex v from a vertex u, check with Lemma 3 if a vertex of the subtree T' with root v is adjacent with some vertex of U. If not, skip over v and thereby the whole subtree T' (since T' can be removed by the Leaf Rule). Otherwise check if a chain starts at v and ends at w. If so, add only the edge $\{u, w\}$ to G' and continue at vertex w (v is removed by the Chain Rule). If not, add vertex v as well as edge $\{u, v\}$ and continue with the children of v. Let T' be the final tree.

Let \bar{n} be the vertices of T. Then the traversal over all \bar{n} subtrees T runs in $\bar{n} \cdot O(\bar{n}^2) = O(\bar{n}^3)$ time. To bound the vertices n' of T', note that T' has n' - 1 edges (total degree is 2n' - 2), but every vertex of T' must have degree at least three if we add the edges to U. Thus, $3n' <= 2n' - 2 + \ell$ and $n' <= \ell - 2$.

For a tree $T = (V_T, E_T) \in \mathcal{T}_2$, a vertex of U can be connected to multiple vertices of T. Hence we need to run the following two steps to bound the number of edges, where Step 1 is a precondition for Step 2. By Step 2 we get subtrees T' of T such that T' and U have at most $(k+1)^2$ edges in between. Thus, we can add T' with $O(k^2)$ vertices to G' by Lemma 8.

- Step 1: Consider a bipartite graph $Y = (U \cup V_T, E')$ where $\{u, w\} \in E'$ exactly if there are at least k + 2 internally vertex disjoint paths between u and w. (For the ongoing algorithm observe: To find a solution for feedback vertex set of size k in Y, we need a vertex cover of size at most k in Y. Furthermore, any vertex in any vertex cover of size kin Y must also be in any feedback vertex set of size k in G.) If a vertex $u \in U$ has degree at least k + 1 in Y, take u into our solution set F and restart. If Y has more than $k^2 + k$ vertices, output no-instance. Otherwise, define the common vertices of Y and V_T as set U'. Note that $|U'| \leq k^2$. Temporarily take $U \cup U'$ as separator. This splits T in several small trees T', which we will process iteratively in Step 2.
- **Step 2:** If a small tree T' has at least $(k+1)^2$ edges to a vertex $u \in U \cup U'$, take u into our solution F and restart. Otherwise add T' to our graph G'.

▶ Lemma 9. Steps 1 and 2 are safe and both steps run in $O(n^3 \log k)$ time and with $O(k^3 \log n)$ bits.

Proof. Consider Step 1. A vertex u of degree k + 1 in Y must be in any vertex cover of size k and, by construction, the vertex must be also in our feedback vertex set, i.e., it is safe to add u to F. Furthermore, if Y has more than $k^2 + k$ vertices, but degree bounded by k, then Y has no vertex cover of size k and thus we can not find a feedback vertex set of size k. To sum up, Step 1 is safe.

Now focus on Step 2. Let X be a set of the vertices of T that are adjacent to a fixed vertex $u \in U \cup U'$. Ignoring the parts of T that are not on a path between two vertices of X, we obtain a tree with maximum degree $\Delta \leq k + 1$ by Step 1. One can easily see that, given a tree $T = (V_T, E_T)$ with maximum degree Δ and $X \subseteq V_T$, we can find $\lfloor |X|/(\Delta + 1) \rfloor$ pairs of vertices in X such that the paths in T between each pair are vertex disjoint [19, Lemma 2.4]. Thus, if u has $|X| \geq (k+1)^2$ edges to T, we have an x-flower of order $|X|/(k+1) \geq k+1$. Thus, also Step 2 is safe.

We now turn our attention to the algorithmic details of Step 1. To track the number of internally vertex-disjoint paths between vertices, we employ a table C. Each vertex of U gets its own counter in this table.

As we traverse the tree T by Lemma 4, consider every vertex w once in T. Every neighbor of w, denoted as $v_i = v_1, v_2, \ldots, \deg(w)$, serves as the roots of a maximal subtree T - w. Observe here that if ℓ subtrees have edges leading to a vertex $u \in U$ in graph G, then ℓ internally vertex-disjoint paths exist between w and u.

Now, for each subtree rooted at v_i , traverse it. Whenever an edge leads to a vertex $u \in U$, set the corresponding value C'(u) to true in a temporary table C' (initialized with false). After completing the subtree traversal, increase the counter in C for every vertex in C' marked true.

If any vertex u in C sees its counter surpass k + 1, insert the edge $\{u, w\}$ into an initially empty graph Y. Whenever a vertex gets degree greater than k + 1 in Y put into F and

Concerning the running time, an applications of Lemma 4 in every tree runs in $O(n^3 \log |U|)$ total time. Since we then iterate over each tree for each vertex w once, i.e., at most O(n) times, it can be done by Lemma 3 in $O(n^2 \log |U|)$ total time where the extra factor of $\log |U|$ comes from membership tests in U. In the same time, we can compute U'. To sum up, Step 1 runs in $O(n^3 \log k)$ time. Concerning our space consumption, we can easily observe that Y has never more than $O(k^2)$ vertices and $O(k^3)$ edges, i.e., $O(k^3 \log n)$ bits suffice.

We finally consider the time and space necessary for Step 2. We can easily count the number of edges of a tree T' to U by Lemma 3 in neglitable time and space. By Lemma 8, we can simply add T' into G'.

Finally note that, by Step 2 above, we temporarily have a separator $U \cup U'$ of size $4k^2$. After adding all subtrees T' of a tree T in \mathcal{T}_2 to G', we can also add U' to G' and we are back to a separator of size $3k^2$.

Shrink the Kernel again. After adding several trees to G' we have to ensure that the size of G' does not exceed our space bound. We run into two issues.

- 1. A naive application of a reduction rule may "unsafely" remove vertices since only subgraphs are considered, e.g., a vertex of U could be considered as being a leaf in G' because several vertices that are connected to U were not added to G' yet.
- 2. Thomassé's Rule requires G' to be a loopless graph of minimum degree 3, and we cannot ensure that the vertices of U in G' are of minimum degree 3.

We address issue (1) with respect to all reduction rules. The application of the Loop and the Flower is still safe to use because whenever they apply, vertices are selected into a solution F for feedback vertex set and we restart. To deal with the Leaf and the Chain Rule we forbid that vertices of U are removed in G'. Thomassé's Rule does not remove vertices. Instead, it only removes existing edges and adds new ones within our subgraph G'. Moreover as stated in the rule, it does not have to consider all connected components, which makes the rule safe for usage in G'.

Concerning issue (2) we show in the next lemma that vertices of U can be exempted from fulfilling the property of being of minimum degree 3.

▶ Lemma 10. Let G = (V, E) be is a subgraph of G' where G' is a loopless graph and multi-edges are double edges, such that G has $n > 16k^2$ vertices and only a subset $U \subseteq V$ with $|U| \le 4k^2$ have degree 0, 1 or 2 in G. If G has a feedback vertex set of size k, then we can apply the Flower Rule or Thomassé's Rule in polynomial time and with $O(k^2 \log n)$ bits.

Proof. Let S be a feedback vertex set of G with at most k vertices. Then $G[V \setminus S]$ is a forest and has at most $|V \setminus S| - 1$ edges. Take $U' := V \setminus U$. Since all vertices in U' have degree at least 3, the total number of edges between S and $V \setminus S$ is at least $3|U' \setminus S| - 2|V \setminus S| - 1 = 3 \cdot ((6k^2 - 4k^2) - k) - 2 \cdot (6k^2 - k) - 1 = 36k^2 - 3k - 32k^2 + 2k - 1 > 4k^2 - k$. Thus, a vertex of S has degree at least 4k.

Now Thomassé concludes [34, Theorem 4.1] that we find in polynomial time a flower of order k + 1 or we can apply Thomassé's Rule to G.

Construct a kernel: By Theorem 6, we have access to a approximate minimum feedback vertex set U of size $3k^2$. U divides G into several trees, enabling us to construct our initial kernel G' = G[U] under construction. We iterate over each tree T of $\mathcal{T}_1 \cup \mathcal{T}_2$ using Lemma 7 and determine its type. If $T \in \mathcal{T}_1$, we add it to G' using Lemma 8. Otherwise, if $T \in \mathcal{T}_2$, it

must be integrated into G' by first breaking it down into smaller trees. To achieve this, we execute Step 1 and 2 and get another separator U'. Combining U' with U devides T into several subtrees T_1, T_2, \ldots . We then utilize Lemma 7 with $G[V \setminus (U \cup U')]$, but use U' to iterate over the trees $T_i = T_1, T_2, \ldots$ (since they are all adjacent to U') and add one tree at a time with Lemma 8. While adding a tree apply the Loop, Leaf, Chain Rule and shrink G' by the Flower and Thomassé's Rule. If all trees of T_i were added, add also U' into G'. After all trees of G have been added to G', we apply the best kernelization [26] to get a kernel of $2k^2 + k$ vertices.

▶ **Theorem 11.** Given an *n*-vertex instance (G, k) of FEEDBACK VERTEX SET, there is an $O(n^5 \operatorname{poly}(k))$ -time, $O(k^4 \log n)$ -bits kernelization that either outputs a kernel consisting of $2k^2 + k$ vertices or returns that (G, k) is a no-instance.

Proof. Since we shrink G' (initial size $|U| = O(k^2)$) repeatedly to $O(k^2)$ vertices after adding trees consisting of $O(k^4)$ vertices and edges we have a space bound of $O(k^4 \log n)$ bits. All other algorithms also run within this space bound.

Concerning the running time, the construction of U runs with Theorem 6 in $O(n^5k^2 \log k)$. Even if we restart at most k times we can reuse U by removing vertices from U whenever they become part of F.

An iteration over all trees by Lemma 7 runs in $O(n^3k^3\log^2 k)$ time. To identify the type of a tree, we have to traverse it with Lemma 3 which runs in $O(n^2\log k)$ time (while performing membership checks for U (and U')). To perform Step 1 und 2 for each tree we have to run Lemma 9, which can be done in $O(n^3\log k)$ time. Adding a tree to G' runs with Lemma 8 in $O(n^3)$ time. Applying the kernelization rules on G' runs in $O(\operatorname{poly}(k))$ time.

Since we have at most *n* trees, the total running time is $O(n^5k^2\log k) + O(n^3k^3\log^2 k) + O(n) \cdot (O(n^2\log k) + O(n^3\log k) + O(n^3) + O(poly(k)))$, which in total is $O(n^5\operatorname{poly}(k))$.

5 Cluster Editing and Cluster Deletion

The CLUSTER EDITING problem can be described as follows. Given a graph G = (V, E) with n vertices and m edges, and an integer parameter k, can we add or delete no more than k edges such that the modified graph comprises entirely of disjoint cliques? Recall that due to Bannach et al. [2] CLUSTER EDITING is in para-L (more precisely a subclass of it). Our goal is to address the need of Heeger et al. [24] for a space-efficient full kernel for CLUSTER EDITING, which is used in their framework in a temporal setting of the cluster editing problem. To solve the problem, it is important to find so-called *conflict triples* in G. Each conflict tripple is a subgraph formed by vertices $\{u, v, w\}$ with edges $\{u, v\}$ and $\{v, w\}$, but lacking an edge $\{w, u\}$. For each existing conflict triple, one should either remove one of the edges $\{u, v\}$ or $\{v, w\}$ or add a new edge $\{w, u\}$.

In the folklore technique for CLUSTER EDITING kernelization, we iterate over all conflict triples and, for every vertex pair u, v within a conflict triple, we maintain two global counters: $C_{u,v}$ for the occurrence of the edge $\{u, v\}$ and $C'_{u,v}$ for the conflict triples where the edge $\{u, v\}$ is missing. Following this iteration, we update the graph as follows: add an edge $\{u, v\}$ to the graph if it is missing in at least k + 1 conflict triples (i.e., for each pair u, v with $C'u, v \ge k + 1$ add the edge $\{u, v\}$), and remove all edges $\{u, v\}$ that are part of at least k + 1 conflict triples (i.e., all edges $\{u, v\}$ with $C_{u,v} \ge k + 1$). We then reset all counters and repeat the described iteration process.

After each iteration, we count the number of non-zero counters in C and C'. If this count exceeds $(k + 1)^2$, we conclude with a "no-instance" answer (since adding or deleting a single

edge $\{u, v\}$ can resolve at most k + 1 conflict triples in G). Otherwise, non-zero counters indicate exactly those vertices that are not part of a clique. For a full kernel, we include at most $(k + 1)^2$ conflict triples, encompassing both vertices and edges. If, across all iterations, an edge is introduced and then removed (or vice versa), we answer with "no-instance".

Typically, a counter is created for every conflict tripple. However, since only non-zero counters are of interest, we maintain only those counters and reject instances as "no-instance" as soon as they exceed $(k + 1)^2$ different counters.

▶ **Theorem 12.** Given an n-vertex m-edge instance (G, k) of CLUSTER EDITING, there is an $O(nm \log k)$ -time $O(k^2 \log n)$ -bits kernelization that either outputs a full kernel of $O(k^2)$ vertices/edges or returns that (G, k) is a no-instance.

Proof. We next describe and analyze an implementation of the algorithm above. First of all, we can iterate over all vertex triples $\{u, v, w\}$ by iterating over all edges and, for each edge $\{u, v\}$, over all vertices $w \in N(v) \cup N(u)$. We cannot modify G on the read-only word-RAM, instead we store O(k) modifications to G in a heap and whenever there is an access to G, we access the heap to check for the existence of an edge, for which we pay for with an extra factor in the running time logarithmic to the size of the heap. Thus, we can iterate over the vertex triples of the virtually modified graph G in $O(m \log k)$ time. The time needed to update all $O(k^2)$ counters in a heap is $O(k^2 \log k)$ —note that we can assume without loss of generality that k < m or we have a yes-instance. Since the iteration has to be repeated at most (k + 1) times, the final running time is $O(mk \log k)$. The time $(k^2 \log k)$ to create the full kernel by adding $(k + 1)^2$ vertex pairs and edges to an initially empty graph is included in the final running time.

We have to store the counters, but only the non-zero counters, the modifications to G in a heap and the kernel. Because the amount of non-zero counters is bounded by $O(k^2)$, the number of modifications is bounded by O(k), and the kernel contains $k(k+2) = O(k^2)$ vertices and $k(2k+1) = O(k^2)$ edges and therefore, our algorithms uses $O(k^2 \log n)$ bits. The full-kernel consists of k(k+3) vertices and k(2k+3) edges.

Note that to solve CLUSTER DELETION the only change is that we return a no-instance for (G, k) whenever a counter $C'_{u,v} \ge k + 1$ for an edge $\{u, v\}$.

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