# Generalized Properties of Generalized Fuzzy sets GFScom and Its Application

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# Generalized Properties of Generalized Fuzzy sets GFScom and Its Application

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**Abstract**

We further continually develop the theory of a generalized fuzzy set with contradictory, opposite and medium negation (GFScom) . The generalized properties (including convexity and concavity) of GFScom are discussed. By introducing GFScom to Mamdani fuzzy systems, we propose new constructive approaches from any (infinite) input-output data pairs to approximate any continuous function on a compact set to a desired degree of accuracy, and investigate the approximation error bounds for the classes of the constructed fuzzy systems. Furthermore, the new better sufficient conditions for this class of fuzzy systems to be universal approximators are provided. For Mamdani fuzzy systems, the novel conditions require a smaller number of known fuzzy sets than all previously published classical conditions. Finally, we illustrate several cases and compare the new results to published error bounds through numerical cases.

**Keywords:** negation, generalized fuzzy sets GFScom, design of fuzzy systems, universal approximator

# 1 Introduction

Negative information plays an essential role in knowledge representation and commonsense inference(see Torres-Blanc et al (2019); Jiang et al (2021); Bustince et al (2022); Fernandez-Peralta et al (2022) and references therein). However, the notion of negation is often considered as a poorer form of meaning than affirmation Kassner et al (2020). In the past decades, some researchers suggested that uncertain information processing requires different forms of negations in various fields. Xiao and Zhu (1988); Zhu and Xiao (1988c,b) developed Medium Logics (ML) with the contradictory, opposite and fuzzy negation under the view of medium principle (i.e., the principle unconditionally recognizes that for any predicate $P$ and object $x$ it is not always true that either there exists $P(x)$ or the opposite side of $P(x)$, cf. Zhu and Xiao (1988a)), which has very sound, complete syntax and semantics Zou (1988, 1989). The medium algebra, which may be viewed as a generalization of the well-known De Morgan algebra, developed by Pan and Wu Pan and Wu (1990) is the algebraic abstract of Medium Propositional (MP) logic system in ML, i.e., the medium algebra can be viewed as the algebraic structure of MP. Wagner (2003); Analyti et al (2008) pointed out that there are (at least) two types of negation: a weak negation representing non-truth (e.g. "he does not like cat") and a strong negation denoting explicit falsity (e.g. "he dislikes cat"). Esteva et al (2000); Cintula et al (2010) extended the Strict Basic Logic (SBL; an extension of the well-known basic logic) with a unary connective $\sim$. The semantics of $\sim$ is any decreasing involution, i.e., the function $n : [0, 1] \to [0, 1]$ such that $n(n(x)) = x$ and $n(x) \leq n(y)$ whenever $x \geq y$. The SBL with an involutive negation is only the fuzzy logic with both negations (the other negation is the negation in Basic Logic (BL) proposed by Hájek (1998), namely, $\neg x = x \to 0$). Kaneiwa (2007) developed the description logic with classical and strong negations, where the classical negation expresses the negation of a statement, while the other is used to depict explicit negative information (or negative facts). Ferré (2006) proposed an epistemic extension of the concept of negation in Logical Concept Analysis, i.e., the extensional negation is the classical negation, such as "old/not old" and "pretty/not pretty", and the intentional negation can be interpreted as opposition, such as "big/small" and "fat/thin". Pan (2010, 2012, 2013) argued that there are three types of negations, namely, contradictory negation, opposite negation and medium negation, in fuzzy knowledge and its negative relationships, subsequently built up a novel fuzzy set referred to as the fuzzy sets with contradictory negation, opposite negation and medium negation (FScom). In order to provide one logic calculus tool for FScom, Pan (2013) and Zhang (2014) proposed fuzzy logic with three kinds of negations from the axiomatization and natural calculus reasoning points of view, respectively. Murinová and Novák (2014) studied the formal theory of generalized Aristotelian square of opposition with intermediate quantifiers(expressions such as *most*, *many*, *a lot of*, *a few*, *large part of*, *small part of*), and gave the formal definitions of contradictories, contraries and subcontraries. A first comprehensive research focusing on commonsense implications of negation and contradiction is presented in Jiang et al (2021).

In Novák (2001) and Novák (2008), Novák proposed a formal theory of the trichotomous evaluative expressions which are a subclass of evaluative expressions (expressions such as "very small","quite big","more or less medium", etc.) containing evaluative trichotomy of the type "*small-medium-big*". On the basis of fuzzy type logic (a higher order fuzzy logic, cf. Novák (2005)), Novák presented formal representation of fundamental evaluative trichotomy, i.e., the form of "*small-medium-big*". In the design procedure of fuzzy systems, on the one hand, the fuzzy distribution needs to be provided which covers the input and output spaces, i.e, the membership function of each fuzzy set must be constructed appropriately. However, in Novák (2001) and Novák (2008), this concrete constructive approach is not provided. On the other hand, the linguistic variables of the forms are commonly used, such as "positive big", "positive medium", "positive small", "zero", "negative small", "negative medium", "negative big" and so on, to represent the fuzzy sets in the input and output spaces. For such fuzzy sets, the method of how to determine their membership functions is not provided in Novák (2001) and Novák (2008). As we stated below, from the logical negations point of view, *big* and *small* are regarded as a pair of opposite negations, while *medium* may be viewed as the medium negation of *small* (or *big*). In general, we are willing to establish the membership function of *small* (or *big*) in a certain context and afterwards infer reasonably membership functions of the others.

Zhang and Li (2017) proposed the notion of generalized fuzzy sets GFScom, and applied it to the table look-up scheme. Although we have given the notion of GFScom, there are still many important issues that have not been addressed. The first issue is to explore the generalized properties of GFScom. Considering the generalized triangular norm operations, namely $t$-norms and $s$-norms, what are the prominent properties of GFScom? What is the convexity of GFScom with respect to opposite negative operator, medium negative operator and contradictory negative operator? The second issue is to design the fuzzy systems from any (infinite) input-output data pairs that can approximate continuous function in some optimal fashion based on GFScom. The third issue is to check whether the designed fuzzy system is a universal approximator and what is the approximation bound for the above constructed fuzzy system. The fourth issue is to compare the novel conditions requiring the number of known fuzzy sets with all previously published classical conditions for Mamdani fuzzy systems constructed in this paper. We will give complete study to the above four issues in this paper.

The remainder of this paper is organized as follows. In Section 2, the notion of generalized fuzzy sets GFScom and its relative algebraic operations are given. In Section 3, we further investigate some interesting generalized properties (including convexity and concavity) of GFScom. In Section 4, on the basis of GFScom, we propose the design methods of the fuzzy system that can approximate a certain continuous function $g(x)$ in some optimal fashion.

In Section 5, we build up the approximation bounds for the classes of fuzzy systems constructed by this paper and give approximation accuracy analysis. The demonstrations and comparisons are given to illustrate the methods in Section 6 and conclusion and future work end in Section 7.

# 2 The Notion of GFScom

**On the necessity of extending fuzzy sets**. Fuzzy sets are very applicable for coping with vague and inaccurate phenomena. As we have stated above, from the philosophical point of view, we need to distinguish strictly the notions of contradictory, opposite and medium negation. However, in fuzzy sets, only one negation is considered, i.e., contradictory negation, usually defined by $\neg x = 1 - x$ for all $x \in [0, 1]$. Naturally, in order to deal with three types of negations, we need to extend fuzzy sets by introducing the notions of opposite negation and medium negation.

**The problem of symmetry**. The idea of contradictory negation and opposite negation is, somehow, symmetric. In other words, given two concepts $A$ and $B$ in a certain context, if $A$ is the contradictory negative concept *w.r.t.* $B$, in general, we would like to expect that $B$ is the contradictory negative concept *w.r.t.* $A$ too. The requirement for the opposite negation is identical. However, for the notion of medium negation, such requirement is not necessary since the medium negation only depicts a medium concept (state) negating two opposite sides.

Moreover, further requirement for the opposite negation is as follows: given a pair of opposite negative concepts $A$ and $B$ over the universe of discourse $U$, we hope the possibility distribution of $A$ looks like the "mirror image" of that of $B$. For example, we consider a pair of opposite negative concepts "tall stature" and "short stature" in a concrete district. If some person $x$ was viewed as "having tall stature", then there should exist another $y$ with short stature in this district, and vice versa.

In Zhang and Li (2017), the authors proposed the concept of GFScom, and applied it to construct the table look-up scheme. In what follows, for the integrity and ease of discussion, the notion of GFScom is represented, and $\mathcal{F}(U)$ denotes a set of all the fuzzy subsets in $U$.

**Definition 1** Given any universal discourse $U$ and finite numerical district $D$, namely, it has the following forms:$[a, b]$, $(a, b]$, $[a, b)$, $(a, b)$, or $\{a = x_1 < x_2 < \ < x_n = b\}$, where $a, b \in \mathbb{R}$, called the left and right end of $D$, respectively, we call the mapping $f : U \to D$ as (one dimensional) finite quantized district mapping.

**Definition 2** Suppose that$A$ belongs to $\mathcal{F}(U)$, $a, b$ are the left and right end of $U$, respectively, $\forall u \in U$, $\otimes$ be a $t$-norm, and $n$ be a complement.

(1) If a mapping $A^{\neg} : U \longrightarrow [0, 1]$ satisfying $A^{\neg}(u) = n(A(u))$, the fuzzy subset determined by $A^{\neg}(u)$ is said to be an $n$ contradictory negative set of $A$. Particularly, the fuzzy subset determined by $A^{\neg}(u) = n(A(u)) = 1 - A(u)$ is referred to as a contradictory negative set of $A$ when $n$ is the linear complement.

(2) If a mapping $A^{\dashv}: U \longrightarrow [0,1]$ satisfying $A^{\dashv}(u) = A(a+b-u)$ and $A^{\dashv}(u) + A(u) \leq 1$, the fuzzy subset determined by $A^{\dashv}$ is referred to as an opposite negative set of $A$.

(3) If a mapping $A^{\sim}: U \longrightarrow [0,1]$ satifying $A^{\sim}(u) = A^{\neg}(u) \otimes (A^{\dashv})^{\neg}(u) = n(A(u)) \otimes n(A^{\dashv}(u)) = n(A(u)) \otimes n(A(a+b-u))$, we call the fuzzy subset determined by $A^{\sim}$ a $\otimes$-$n$ medium negative set of $A$. Particularly, if $t$-norm $\otimes$ is a min-operator and $n$ a linear complement, the fuzzy subset satisfying $A^{\sim}(u) = \min\{1 - A(u), 1 - A(a+b-u)\}$ is referred to as a medium negative set of $A$.

The above defined fuzzy sets are called Generalized Fuzzy Set with Contradictory, Opposite and Medium negation, written as GFScom for short.

**Definition 3** In GFScom, the operations such as containment, equivalency, union and intersection between a pair of arbitrary fuzzy subsets are identical to the counterparts in Zadeh fuzzy sets.

# 3 Properties of GFScom

## 3.1 Generalized Properties of GFScom

In this subsection, the generalized properties of GFScom will be explored.

**Definition 4** Klement et al (2000); Klir and Yuan (1995) Given a universe of discourse $U$, $A, B, C \in \mathcal{F}(U)$. Let $\otimes, \oplus$ be a $t$-norm, $s$-norm, respectively, then

(1) If $C(x) = A(x) \oplus B(x)$, $\forall x \in U$, denoted by $C = A \bigcup_{\oplus} B$, $C$ is called the module union of $A$ and $B$;

(2) If $C(x) = A(x) \otimes B(x)$, $\forall x \in U$, written as $C = A \bigcap_{\otimes} B$, $C$ is called the module intersection of $A$ and $B$.

**Theorem 1** *Let $\otimes, \oplus, n$ be a $t$-norm, $s$-norm and complement, respectively. For any universal discourse $U$ with the left end $a$ and the right end $b$, $A, B$ and $C$ are any GFScom on $U$, we have*

*(1) (i) $A^{\neg\neg} = A$ (law of double contradictory), (ii) $A^{\dashv\dashv} = A$ (law of double opposition), (iii) $A^{\sim} = A^{\dashv\sim}$;*

*(2) (i) $A \bigcup_{\oplus} B = B \bigcup_{\oplus} A$, (ii) $A \bigcap_{\oplus} B = B \bigcap_{\oplus} A$;*

*(3) (i) $(A \bigcup_{\oplus} B) \bigcup_{\oplus} C = A \bigcup_{\oplus} (B \bigcup_{\oplus} C)$,*

*(ii) $(A \bigcap_{\otimes} B) \bigcap_{\otimes} C = A \bigcap_{\otimes} (B \bigcap_{\otimes} C)$;*

*(4) If $A \subseteq B$, then $\forall C \in \mathcal{F}(U)$, $A \bigcup_{\oplus} C \subseteq B \bigcup_{\oplus} C$, $A \bigcap_{\otimes} C \subseteq B \bigcap_{\otimes} C$;*

*(5) $A \bigcup_{\oplus} \emptyset = A$, $A \bigcap_{\otimes} \emptyset = \emptyset$, $A \bigcup_{\oplus} U = U$, $A \bigcap_{\otimes} U = A$;*

*(6) If $\otimes, \oplus$ are dual with respect to the complement $n$, and $(A \bigcup_{\oplus} B)^{\dashv}$ and $(A \bigcap_{\otimes} B)^{\dashv}$ are defined, i.e., $\forall u \in U$, $(A \bigcup_{\oplus} B)^{\dashv}(u) + (A \bigcup_{\oplus} B)(u) \leq 1$, $(A \bigcap_{\otimes} B)^{\dashv}(u) + (A \bigcap_{\otimes} B)(u) \leq 1$, then we have*

*(i) $(A \bigcup_{\oplus} B)^{\neg} = A^{\neg} \bigcap_{\otimes} B^{\neg}$, $(A \bigcap_{\otimes} B)^{\neg} = A^{\neg} \bigcup_{\oplus} B^{\neg}$,*

*(ii) $(A \bigcup_{\oplus} B)^{\dashv} = A^{\dashv} \bigcup_{\oplus} B^{\dashv}$, $(A \bigcap_{\otimes} B)^{\dashv} = A^{\dashv} \bigcap_{\otimes} B^{\dashv}$,*

*(iii) $A^{\sim} = A^{\neg} \bigcap_{\otimes} A^{\dashv\neg}$, $(A \bigcup_{\oplus} B)^{\sim} = A^{\sim} \bigcap_{\otimes} B^{\sim}$;*

*(7) $A \bigcap_{\otimes} B \subseteq A \bigcap B \subseteq A \bigcup B \subseteq A \bigcup_{\oplus} B$;*

*(8) (i) $A \bigcup_{\oplus} (\bigcup_{k=1}^{n} A_k) = \bigcup_{k=1}^{n} (A \bigcup_{\oplus} A_k)$,*

*(ii)* $A\bigcap_{\otimes}(\bigcup_{k=1}^{n} A_k) = \bigcup_{k=1}^{n}(A\bigcap_{\otimes} A_k)$,
*(iii)* $A\bigcup_{\oplus}(\bigcap_{k=1}^{n} A_k) = \bigcap_{k=1}^{n}(A\bigcup_{\oplus} A_k)$,
*(iv)* $A\bigcap_{\otimes}(\bigcap_{k=1}^{n} A_k) = \bigcap_{k=1}^{n}(A\bigcap_{\otimes} A_k)$;

*(9) If both $\otimes$ and $\oplus$ are continuous, for any index set $T$, we have*
*(i)* $A\bigcup_{\oplus}(\bigcup_{t\in T} A_t) = \bigcup_{t\in T}(A\bigcup_{\oplus} A_t)$,
*(ii)* $A\bigcap_{\otimes}(\bigcup_{t\in T} A_t) = \bigcup_{t\in T}(A\bigcap_{\otimes} A_t)$,
*(iii)* $A\bigcup_{\oplus}(\bigcap_{t\in T} A_t) = \bigcap_{t\in T}(A\bigcup_{\oplus} A_t)$,
*(iv)* $A\bigcap_{\otimes}(\bigcap_{t\in T} A_t) = \bigcap_{t\in T}(A\bigcap_{\otimes} A_t)$.

**Proof**. We only prove (1),(6) and (9), while others are analogous.

(1)(i)For arbitrary $u$ in $U$ , by Definition 2, one can see $A^{\neg\neg}(u) = n(n(A(u))) = A(u)$. Hence $A^{\neg\neg} = A$ follows.

(ii) For arbitrary $u$ in $U$ , by Definition 2, we have $A^{\dashv\dashv}(u) = A^{\dashv}(a+b-u) = A(a+b-(a+b-u)) = A(u)$, where $a, b$ is, respectively, the left and right end of $U$. Consequently, $A^{\dashv\dashv} = A$ holds.

(iii) $\forall u \in u$, $A^{\dashv\sim}(u) = A^{\dashv\neg}(u) \otimes A^{\dashv\dashv\neg}(u) = A^{\neg}(u) \otimes A^{\dashv\neg}(u) = A^{\sim}(u)$ follows from Definition 2 and (1)(ii). Hence, $A^{\sim} = A^{\dashv\sim}$ holds.

(6)(i) For any $u$ in $U$, since $\otimes$ and $\oplus$ are mutually dual with respect to $n$, one can see that $(A\bigcup_{\oplus} B)^{\neg}(u) = n((A\bigcup_{\oplus} B)(u)) = n(\oplus(A(u), B(u)))$ $= n(n(\otimes(n(A(u)), n(B(u))))) = \otimes(n(A(u)), n(B(u))) = A^{\neg}(u)\bigcap_{\otimes} B^{\neg}(u)$ by Definitions 2 and 3. Analogously, we can prove the other.

(ii) For any $u$ in $U$, by Definition 2 we get $(A\bigcup_{\oplus} B)^{\dashv}(u) = (A\bigcup_{\oplus} B)(a+b-u) = \otimes(A(a+b-u), B(a+b-u)) = A^{\dashv}(u)\bigcup_{\oplus} B^{\dashv}(u)$. Hence, the equality holds. The verification of the other equality is similar.

(iii) The first equality is trivial. Subsequently, we need only to prove the second equality, i.e., $(A\bigcup_{\oplus} B)^{\sim} = A^{\sim}\bigcap_{\otimes} B^{\sim}$.

By the above proved outcomes, we have

$$\begin{aligned}
(A\bigcup_{\oplus} B)^{\sim} &= (A\bigcup_{\oplus} B)^{\neg}\bigcap_{\otimes}(A\bigcup_{\oplus} B)^{\dashv\neg} \\
&= (A^{\neg}\bigcap_{\otimes} B^{\neg})\bigcap_{\otimes}(A^{\dashv}\bigcup_{\oplus} B^{\dashv})^{\neg} \\
&= (A^{\neg}\bigcap_{\otimes} B^{\neg})\bigcap_{\otimes}(A^{\dashv\neg}\bigcap_{\otimes} B^{\dashv\neg}) \\
&= (A^{\neg}\bigcap_{\otimes} A^{\dashv\neg})\bigcap_{\otimes}(B^{\neg}\bigcap_{\otimes} B^{\dashv\neg}) \\
&= A^{\sim}\bigcap_{\otimes} B^{\sim}.
\end{aligned}$$

(9) (i) For any $u$ in $U$, thanks to Definitions 2 and 3 and continuity of $\oplus$, we can get $(A\bigcup_{\oplus}(\bigcup_{t\in T} A_t))(u) = A(u) \oplus (\vee_{t\in T} A_t(u)) = \vee_{t\in T} A(u) \oplus A_t(u)$ $= (\bigcup_{t\in T}(A\bigcup_{\oplus} A_t))(u)$. Consequently, the equality follows. The proof for the rest is analogous and omitted.

## 3.2 Convexity and Concavity of GFScom

It is well known that convexity is an important concept for the quantitative and qualitative analysis in operation research which helps to optimize the solution of problems. The notion of convexity also forms one of the pillars of nonclassical analysis which is a novel branch of fuzzy mathematics. So, many scholars studied some properties of convex fuzzy sets (e.g., see Nourouzi and Aghajani (2008) and references therein). Therefore, in the design procedure of fuzzy systems, it is useful for us to analyze the constructed fuzzy system if the convex (or concave) fuzzy sets are used to construct the desired fuzzy system.

In this subsection, we suppose for concreteness that $U$ is a $n$-dimensional Cartesian product $D^n$, where $D$ is an interval of the form: $[a,b]$, $(a,b]$, $[a,b)$, $(a,b)$ such that $a,b \in \mathbb{R}$.

**Definition 5** *convexity* (*up convexity*). Let $A$ be any GFScom on $U$. $A$ is convex if and only if

$$A(\lambda x_1 + (1-\lambda)x_2) \geq A(x_1) \wedge A(x_2) = \min\{A(x_1), A(x_2)\} \qquad (1)$$

for all $x_1$ and $x_2$ in $U$ and all $\lambda$ in $[0,1]$.

In contrast to convexity, one can readily get the following notion.

**Definition 6** *concavity* (*down convexity*). Let $A$ be any GFScom on $U$. $A$ is concave if and only if

$$A(\lambda x_1 + (1-\lambda)x_2) \leq A(x_1) \vee A(x_2) = \max\{A(x_1), A(x_2)\} \qquad (2)$$

for all $x_1$ and $x_2$ in $U$ and all $\lambda$ in $[0,1]$. Specially, we call $A$ strongly concave if $A(\lambda x_1 + (1-\lambda)x_2) \leq A(x_1) \wedge A(x_2) = \min\{A(x_1), A(x_2)\}$ holds for all $x_1$ and $x_2$ in $U$ and all $\lambda$ in $[0,1]$.

Clearly, it is not hard to see that if $A$ is strongly concave, then it is concave; conversely, the result does not follow.

A basic property of convex (strongly-concave) GFScom is expressed by

**Theorem 2** *Let $A$ and $B$ be any GFScom on $U$, $\otimes$, $\oplus$ a $t$-norm, $s$-norm, respectively. Then we have*

*(1) If $A$ and $B$ are strongly concave, so are their module intersection $A \cap_{\otimes} B$ and module union $A \cup_{\oplus} B$;*

*(2) If $A$ and $B$ are convex, so is their intersection $A \cap B$.*

**Proof**. We only prove (1), (2) is similar.

(1) Let $C = A \cap_{\otimes} B$ be strongly concave. Then

$$C(\lambda x_1 + (1-\lambda)x_2) = A(\lambda x_1 + (1-\lambda)x_2) \cap_{\otimes} B(\lambda x_1 + (1-\lambda)x_2).$$

Now, since $A$ and $B$ are strongly concave, the following inequalities

$$A(\lambda x_1 + (1-\lambda)x_2) \leq A(x_1) \wedge A(x_2)$$
$$B(\lambda x_1 + (1-\lambda)x_2) \leq B(x_1) \wedge B(x_2)$$

hold, and hence

$$C(\lambda x_1 + (1-\lambda)x_2) \leq (A(x_1) \wedge A(x_2)) \cap_{\otimes} (B(x_1) \wedge B(x_2))$$

follows from the monotonicity of $t$-norms. For the right-hand side of the above inequality, one can get

$$\begin{aligned}
&(A(x_1) \wedge A(x_2)) \cap_{\otimes} (B(x_1) \wedge B(x_2)) \\
&= [(A(x_1) \wedge A(x_2)) \cap_{\otimes} B(x_1)] \wedge [(A(x_1) \\
&\qquad \wedge A(x_2)) \cap_{\otimes} B(x_2)] \\
&= (A(x_1) \cap_{\otimes} B(x_1)) \wedge (A(x_2) \cap_{\otimes} B(x_1)) \\
&\qquad \wedge (A(x_1) \cap_{\otimes} B(x_2)) \wedge (A(x_2) \cap_{\otimes} B(x_2)) \\
&\leq (A(x_1) \cap_{\otimes} B(x_1)) \wedge (A(x_2) \cap_{\otimes} B(x_2)) \\
&= (A \cap_{\otimes} B)(x_1) \wedge (A \cap_{\otimes} B)(x_2)
\end{aligned}$$

from Definition 6 and Theorem 1(8). Hence,

$$C(\lambda x_1 + (1-\lambda)x_2) \leq (A \cap_{\otimes} B)(x_1) \wedge (A \cap_{\otimes} B)(x_2)$$

follows. Thus, $C(\lambda x_1+(1-\lambda)x_2) \leq C(x_1) \wedge C(x_2)$ holds. That is to say, $A \cap_{\otimes} B$ is strongly concave. The proof of the other is analogous.

(2) It is immediate from Lemma 1 in Nourouzi and Aghajani (2008). The proof is finished.

In the following, we present the convex-concave connections of a fuzzy set and its three types of negative sets.

**Theorem 3** *Let $A$ be any GFScom on $U$. Then we have*

*(1) If $A$ is convex, then its opposite negative set $A^{\dashv}$ is also convex, and vice versa.*

*(2) If $A$ is concave, then its opposite negative set $A^{\dashv}$ is also concave, and vice versa.*

**Proof.** We only prove (1). The proof of (2) is analogous. If $A$ is convex, i.e., the inequality (1) follows, we then have $A(\lambda x_1 + (1-\lambda)x_2) \geq (A(x_1) \wedge A(x_2))$ for all $x_1$ and $x_2$ in $U$ and all $\lambda$ in $[0,1]$ . In the special case of $a+b-x_1$ , $a+b-x_2$ in $U$ , the above inequality follows, too, that is,

$$A(\lambda(a+b-x_1) + (1-\lambda)(a+b-x_2)) \geq A(a+b-x_1) \wedge A(a+b-x_2)$$

or equivalently

$$A(a+b-(\lambda x_1 + (1-\lambda)x_2)) \geq A(a+b-x_1) \wedge A(a+b-x_2)$$

and therefore $A^{\dashv}(\lambda x_1 + (1-\lambda)x_2) \geq A^{\dashv}(x_1) \wedge A^{\dashv}(x_2)$.

Conversely, assume that $A^{\dashv}$ is convex. By the just above-proven procedure, one can readily see $A^{\dashv\dashv}$ is convex. Furthermore, obviously, $A^{\dashv\dashv} = A$ holds by the aforementioned Theorem 1(1). Hence, $A$ is convex. The proof is completed.

**Theorem 4** *Let $A$ be any GFScom on $U$, $n$ any complement. Then we have*

*(1) If $A$ is concave, then its $n$ contradictory negative set $A^{\neg}$ is convex, and vice versa.*

*(2) If $A$ is convex, then its $n$ contradictory negative set $A^{\neg}$ is concave, and vice versa.*

**Proof.** (1) If $A$ is concave, by inequality (2), we get

$$A(\lambda x_1 + (1-\lambda)x_2) \leq A(x_1) \vee A(x_2)$$

for all $x_1$ and $x_2$ in $U$ and all $\lambda$ in $[0,1]$. Furthermore, the following inequality

$$n(A(\lambda x_1 + (1-\lambda)x_2)) \geq n(A(x_1) \vee A(x_2))$$

holds, where $n$ is any complement, and therefore,

$$A^{\neg}(\lambda x_1 + (1-\lambda)x_2) \geq A^{\neg}(x_1) \wedge A^{\neg}(x_2).$$

Consequently, $A^{\neg}$ is convex.

Conversely, the $n$ contradictory negative set $A^{\neg}$ is convex, by inequality (1), we have

$$A^{\neg}(\lambda x_1 + (1-\lambda)x_2) \geq A^{\neg}(x_1) \wedge A^{\neg}(x_2)$$

for all $x_1$ and $x_2$ in $U$ and all $\lambda$ in $[0,1]$. Moreover, the inequality

$$n(A^{\neg}(\lambda x_1 + (1-\lambda)x_2)) \leq n(A^{\neg}(x_1) \wedge A^{\neg}(x_2))$$

follows, where $n$ is any complement. The equivalent inequality

$$n(n(A(\lambda x_1 + (1-\lambda)x_2))) \leq n(n(A(x_1))) \vee n(n(A(x_2)))$$

holds. Thus,

$$A(\lambda x_1 + (1-\lambda)x_2) \leq A(x_1) \vee A(x_2).$$

Therefore, $A$ is concave.

(2) It is analogous to the proof of Theorem 4(1).

**Theorem 5** *Let $A$ be any GFScom on $U$. If $A$ is concave, then its medium negative set $A^{\sim}$ is convex.*

**Proof.** If $A$ is concave, then $A^{\neg}$ is convex by Theorem 4(1) and $A^{\dashv}$ is concave by Theorem 3(2). Moreover, it is immediate that $A^{\dashv\neg}$ is convex from Theorem 4(1). According to Theorems 2(1) and 1(6), one can easily see that $A^{\sim} = A^{\neg} \cap A^{\dashv\neg}$ is convex.

Note that the only case is considered in Theorem 5 but two cases in Theorems 3 and 4, the reason is that, in generally, we can not obtain the convex-concave property of intersection of fuzzy sets $A$ and $B$ when $A$ and $B$ are concave. Thus, when $A$ is convex, the convex-concave property of $\otimes$-$n$ medium negative set of $A$ can not been determined according to Definition 6.

However, the special case of $\otimes$-$n$ medium negative set, i.e., the $t$-norm $\otimes$ is a min-operator and $n$ a linear complement, is convex whenever $A$ is concave.

# 4 Design of fuzzy systems based on GFScom

In this section, we assume that the analytic formula of nonlinear function: $g(x) : U \subset \mathbb{R}^n \to \mathbb{R}$ is unknown. But we can determine the input-output pairs $(x;\ g(x))$ for any $x \in U$. Based on the above GFScom, our task is to design a fuzzy system that can approximate $g(x)$ in an optimal manner.

## 4.1 Preliminary Concepts and Notations

**Definition 7** Wang (1997); Zeng and Singh (1996) *Pseudo-Trapezoid-Shaped Membership Functions (PTS).* Let $[a, d] \subseteq U \subset \mathbb{R}$ and $a \leq d$. A continuous function $A(x) = A(x;\ a, b, c, d, H)$ with $a \leq b \leq c \leq d$ is a PTS function given by

$$A(x;\ a, b, c, d, H) = \begin{cases} I(x), & \text{when } x \in [a, b) \\ H, & \text{when } x \in [b, c] \\ D(x), & \text{when } x \in (c, d] \\ 0, & \text{when } x \in U - [a, d] \end{cases}, \tag{3}$$

where $0 < H \leq 1$, $0 \leq I(x) \leq 1$ is strictly monotone increasing in $[a, b)$ and $0 \leq D(x) \leq 1$ is strictly monotone decreasing in $(c, d]$. When $H = 1$, it is simply denoted by $A(x) = A(x;\ a, b, c, d)$.

*Remark 1* Pseudo-trapezoid membership functions of the form Eq.(3) contain a number of commonly-employed membership functions as special cases. For instance, if we choose

$$I(x) = \frac{x-a}{b-a} \quad \text{and} \quad D(x) = \frac{x-d}{c-d}, \tag{4}$$

then the pseudo-trapezoid-shaped membership functions change into the *trapezoid membership functions*. If $b = c$, and $I(x)$ and $D(x)$ are defined as in Eq.(4), we can obtain the *triangular membership functions*. For normal triangular membership functions, we often denote them by the simpler notation $\triangle(x;\ a, b, d)$.

## 4.2 Design of Fuzzy System with First-Order Approximation Accuracy

Now, based on the above defined GFScom, we are ready to design a particular type of fuzzy systems that have some nice properties. We first specify the problem as follows.

**The Problem:** Let $g(x)$ be a function on the compact set $U = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \cdots \times [\alpha_n, \beta_n] \subset \mathbb{R}^n$ and the analytic expression of $g(x)$ be unknown. Assume that for any $x \in U$, we can determine $g(x)$. Our task is to design a fuzzy system that approximates $g(x)$ to any degree of accuracy using GFScom developed by this paper.

We now design such a fuzzy system step-by-step as follows.

**Step 1.** Define $\lceil \frac{N_i}{2} \rceil$ ($i = 1, 2, 3, \ldots, n$; $\lceil x \rceil$ denotes the smallest integer which is not less than $x$) fuzzy sets $A_1^i, A_2^i, \ldots, A^i_{\lceil \frac{N_i}{2} \rceil}$ in $[\alpha_i, \frac{\alpha_i+\beta_i}{2}]$ which are normal, consistent and complete.

Specially, we may, for example, take those fuzzy sets with PTS functions $A_1^i(x;\ a_i^1, b_i^1, c_i^1, d_i^1), \ldots, A^i_{\lceil \frac{N_i}{2} \rceil}(x;\ a_i^{\lceil \frac{N_i}{2} \rceil}, b_i^{\lceil \frac{N_i}{2} \rceil}, c_i^{\lceil \frac{N_i}{2} \rceil}, d_i^{\lceil \frac{N_i}{2} \rceil})$, and $A_1^i < A_2^i < \cdots < A^i_{\lceil \frac{N_i}{2} \rceil}$ with $a_i^1 = b_i^1 = \alpha_i$, and the arguments of $A^i_{\lceil \frac{N_i}{2} \rceil}$ on the domain $U_i = [\alpha_i, \beta_i]$ satisfy the following conditions: $c_i^{\lceil \frac{N_i}{2} \rceil} = d_i^{\lceil \frac{N_i}{2} \rceil} = \frac{\alpha_i+\beta_i}{2}$ whenever $N_i$ is odd; otherwise, $c_i^{\lceil \frac{N_i}{2} \rceil} = \alpha_i + \beta_i - d_i^{\lceil \frac{N_i}{2} \rceil} < d_i^{\lceil \frac{N_i}{2} \rceil} \le \alpha_i + \beta_i - c_i^{\lceil \frac{N_i}{2} \rceil}$ and $A^i_{\lceil \frac{N_i}{2} \rceil}(\frac{\alpha_i+\beta_i}{2}) \le 0.5$.

**Step 2**. By Definition 6, 1) compute $(A^i_{j_i})^{\dashv}(j_i = 1, 2, \ldots, \lceil \frac{N_i}{2} \rceil, i = 1, 2, \ldots, n)$ on $U_i = [\alpha_i, \beta_i]$ when $N_i$ is even; 2) when $N_i$ is odd, calculate $(A^i_{j_i})^{\dashv}(j_i = 1, 2, \ldots, \lceil \frac{N_i}{2} \rceil - 1, i = 1, 2, \ldots, n)$ on $U_i = [\alpha_i, \beta_i]$, and $A^{i'}_{\lceil \frac{N_i}{2} \rceil}(x) = A^i_{\lceil \frac{N_i}{2} \rceil}(\alpha_i + \beta_i - x)$ for any $x \in U_i$.

Specifically, whenever $N_i$ is even, let $A^i_{N_i}(x;\ a_i^{N_i}, b_i^{N_i}, c_i^{N_i}, d_i^{N_i}) = (A_1^i)^{\dashv}$, $\ldots$, $A^i_{\lceil \frac{N_i}{2} \rceil+1}(x;\ a_i^{\lceil \frac{N_i}{2} \rceil+1}, b_i^{\lceil \frac{N_i}{2} \rceil+1}, c_i^{\lceil \frac{N_i}{2} \rceil+1}, d_i^{\lceil \frac{N_i}{2} \rceil+1}) = (A^i_{\lceil \frac{N_i}{2} \rceil})^{\dashv}$, where $a_i^{N_i} = \alpha_i + \beta_i - d_i^1$, $b_i^{N_i} = \alpha_i + \beta_i - c_i^1$, $c_i^{N_i} = \alpha_i + \beta_i - b_i^1$, $d_i^{N_i} = \alpha_i + \beta_i - a_i^1$; $\ldots$; $a_i^{\lceil \frac{N_i}{2} \rceil+1} = \alpha_i + \beta_i - d_i^{\lceil \frac{N_i}{2} \rceil}$, $b_i^{\lceil \frac{N_i}{2} \rceil+1} = \alpha_i + \beta_i - c_i^{\lceil \frac{N_i}{2} \rceil}$, $c_i^{\lceil \frac{N_i}{2} \rceil+1} = \alpha_i + \beta_i - b_i^{\lceil \frac{N_i}{2} \rceil}$, $d_i^{\lceil \frac{N_i}{2} \rceil+1} = \alpha_i + \beta_i - a_i^{\lceil \frac{N_i}{2} \rceil}$. Whenever $N_i$ is odd, let $A^i_{N_i} = (A_1^i)^{\dashv}, \ldots, A^i_{\lceil \frac{N_i}{2} \rceil+1} = (A^i_{\lceil \frac{N_i}{2} \rceil-1})^{\dashv}$, $A^i_{\lceil \frac{N_i}{2} \rceil} = A^i_{\lceil \frac{N_i}{2} \rceil} \bigcup A^{i'}_{\lceil \frac{N_i}{2} \rceil}$(Here $A^i_{\lceil \frac{N_i}{2} \rceil} \bigcup A^{i'}_{\lceil \frac{N_i}{2} \rceil}$ is a new fuzzy set, also written as $A^i_{\lceil \frac{N_i}{2} \rceil}$ for the sake of simplicity), where $\bigcup$ represents fuzzy union, i.e., max operator.

From Theorem 6 and the above two steps , one can see that $A_1^i, A_2^i, \ldots, A^i_{N_i}$ are normal, consistent and complete GFScom on $U_i = [\alpha_i, \beta_i]$, and $A_1^i < A_2^i < \ldots < A^i_{N_i}$.

**Step 3**. Define $e_j^1 = \alpha_j$, $e_j^{N_j} = \beta_j$ and $e_j^{i_j} \in [b_j^{i_j}, c_j^{i_j}]$ (e.g. $e_j^{i_j} = \frac{1}{2}(b_j^{i_j} + c_j^{i_j})$) for $i_j = 2, \ldots, N_j - 1$; $j = 1, 2, \ldots, n$.

**Step 4**. Construct $m = N_1 \times N_2 \times \cdots N_n = \prod_{i=1}^n N_i$ fuzzy IF-THEN rules in the following form:

$$R_{i_1 i_2 \ldots i_n} : IF\ x_1\ is\ A^1_{i_1}\ and\ \cdots\ and\ x_n\ is\ A^n_{i_n},$$
$$THEN\ y\ is\ C_{i_1 i_2 \ldots i_n}$$

where $i_1 = 1, \ldots, N_1, \ldots, i_n = 1, \ldots, N_n$ and the point in $\mathbb{R}$ at which the fuzzy set $C_{i_1 i_2 \ldots i_n}$ achieves its maximum value, denoted as $\bar{y}_{i_1 i_2 \ldots i_n}$ (when $C_{i_1 i_2 \ldots i_n}$ is a normal fuzzy set, $C_{i_1 i_2 \ldots i_n}(\bar{y}_{i_1 i_2 \ldots i_n}) = 1$; in this paper, we always assume that $C_{i_1 i_2 \ldots i_n}$ is a normal fuzzy set), is chosen as

$$\bar{y}_{i_1 i_2 \ldots i_n} = g(e_1^{i_1}, e_2^{i_2}, \ldots, e_n^{i_n}). \tag{5}$$

**Step 5**. Construct the fuzzy system $f(x)$ from the ${}_{i=1} N_i$ generated by Step 4 using singleton fuzzifier Wang (1997); Zeng and Singh (1996), product inference engine Wang (1997); Zeng and Singh (1996), center average defuzzifier Wang (1997); Zeng and Singh (1996), i.e., taking "and" as product operator, fuzzy implication as Mamdani product implication (i.e., $a \to b = ab, \forall a, b \in [0, 1]$) as follows:

$$\begin{aligned} y = f(x) &= \frac{\sum_{i_n=1}^{N_n} \cdots \sum_{i_1=1}^{N_1} A_{i_1 i_2 \ldots i_n}(x) \bar{y}_{i_1 i_2 \ldots i_n}}{\sum_{i_n=1}^{N_n} \cdots \sum_{i_1=1}^{N_1} A_{i_1 i_2 \ldots i_n}(x)} \\ &= \sum_{i_n=1}^{N_n} \cdots \sum_{i_1=1}^{N_1} B_{i_1 i_2 \ldots i_n}(x) \bar{y}_{i_1 i_2 \ldots i_n} \end{aligned}, \tag{6}$$

where the crisp input value $x = (x_1, x_2, \ldots, x_n) \in U$, $A_{i_1 i_2 \ldots i_n}(x) = A^1_{i_1}(x_1) A^2_{i_2}(x_2) \cdots A^n_{i_n}(x_n)$, and

$$\begin{aligned} B_{i_1 i_2 \ldots i_n}(x) &= \frac{A_{i_1 i_2 \ldots i_n}(x)}{\sum_{i_n=1}^{N_n} \cdots \sum_{i_1=1}^{N_1} A_{i_1 i_2 \ldots i_n}(x)} \\ &= \frac{A^1_{i_1}(x_1) A^2_{i_2}(x_2) \cdots A^n_{i_n}(x_n)}{\sum_{i_n=1}^{N_n} \cdots \sum_{i_1=1}^{N_1} A^1_{i_1}(x_1) A^2_{i_2}(x_2) \cdots A^n_{i_n}(x_n)}. \end{aligned}$$

Since the fuzzy sets $A^i_1, A^i_2, \ldots, A^i_{N_i}$ are complete GFScom on $U_i = [\alpha_i, \beta_i]$, at every point $x \in U$ there exists $i_1, i_2, \ldots, i_n$ such that $A^1_{i_1}(x_1) A^2_{i_2}(x_2) \cdots A^n_{i_n}(x_n) \neq 0$. Therefore, the fuzzy system (6) is well defined, that is, its denominator is always nonzero.

In the above procedure, we only consider the case of the membership function distribution over $[\alpha_i, \frac{\alpha_i+\beta_i}{2}]$. However, if the membership function distribution on $[\frac{\alpha_i+\beta_i}{2}, \beta_i]$ can be determined, then we may carry out the same work as in Steps 1 through 5 of the above design procedure in terms of Definition 2.

## 4.3 Design of Fuzzy System with Second-Order Accuracy

The design problem is the same as in Section 4.2. Next, on the basis of GFScom, we design the fuzzy system with second-order accuracy in a step-by-step manner.

**Step 1**. Define $\lceil \frac{N_j}{2} \rceil$ $(j = 1, 2, 3, \ldots, n)$ fuzzy sets $A^j_1, A^j_2, \ldots, A^j_{\lceil \frac{N_j}{2} \rceil}$ in $[\alpha_j, \frac{\alpha_j+\beta_j}{2}]$ which are normal, consistent and complete with the triangular membership functions

$$A^j_{i_j}(x_j) = \triangle^j_{i_j}(x_j;\ e^j_{i_j-1}, e^j_{i_j}, e^j_{i_j+1})$$

for $i_j = 1, 2, \ldots, \lceil \frac{N_j}{2} \rceil$, where $e^j_0 = e^j_1 = \alpha_j$, $e^j_1 < e^j_2 < \cdots < e^j_{\lceil \frac{N_j}{2} \rceil} \le e^j_{\lceil \frac{N_j}{2} \rceil + 1}$, and $e^j_{\lceil \frac{N_j}{2} \rceil} = e^j_{\lceil \frac{N_j}{2} \rceil + 1} = \frac{\alpha_j+\beta_j}{2}$ whenever $N_j$ is odd. Otherwise, $e^j_{\lceil \frac{N_j}{2} \rceil} < \frac{\alpha_j+\beta_j}{2}$, $e^j_{\lceil \frac{N_j}{2} \rceil} + e^j_{\lceil \frac{N_j}{2} \rceil + 1} = \alpha_j + \beta_j$ and $A^j_{\lceil \frac{N_j}{2} \rceil}(\frac{\alpha_j+\beta_j}{2}) \le 0.5$.

**Step 2**. When $N_j$ is even, calculate $(A^j_{i_j})^{\dashv}$ ($i_j = 1, 2, \ldots, \lceil \frac{N_j}{2} \rceil$) in $U_j = [\alpha_j, \beta_j]$. When $N_j$ is odd, calculate $(A^j_{i_j})^{\dashv}$ ($i_j = 1, 2, \ldots, \lceil \frac{N_j}{2} \rceil - 1$) in $U_j = [\alpha_j, \beta_j]$ and $A^{j'}_{\lceil \frac{N_j}{2} \rceil}(x) = A^j_{\lceil \frac{N_j}{2} \rceil}(\alpha_j + \beta_j - x)$ for all $x \in U_j$. Specially, for $j = 1, 2, \ldots, n$, we distinguish one case from the other as follows:

a) When $N_j$ is even, let $A^j_{N_j}(x_j) = \triangle^j_{N_j}(x_j;\ e^j_{\lceil \frac{N_j}{2} \rceil - 1}, e^j_{\lceil \frac{N_j}{2} \rceil}, e^j_{\lceil \frac{N_j}{2} \rceil + 1}) = (A^j_1)^{\dashv}, \ldots, A^j_{\lceil \frac{N_j}{2} \rceil + 1}(x_j) = \triangle^j_{\lceil \frac{N_j}{2} \rceil + 1}(x_j; e^j_{\lceil \frac{N_j}{2} \rceil}, e^j_{\lceil \frac{N_j}{2} \rceil + 1}, e^j_{\lceil \frac{N_j}{2} \rceil + 2}) = (A^j_{\lceil \frac{N_j}{2} \rceil})^{\dashv}$, where $e^j_{N_j} = e^j_{N_j+1} = \beta_j$, $e^j_{N_j-1} = \alpha_j + \beta_j - e^j_2$; $\ldots$; $e^j_{\lceil \frac{N_j}{2} \rceil} = e^j_{\lceil \frac{N_j}{2} \rceil}$, $e^j_{\lceil \frac{N_j}{2} \rceil + 1} = e^j_{\lceil \frac{N_j}{2} \rceil + 1}$, $e^j_{\lceil \frac{N_j}{2} \rceil + 2} = \alpha_j + \beta_j - e^j_{\lceil \frac{N_j}{2} \rceil - 1}$.

b) When $N_j$ is odd, $A^j_{N_j}, A^j_{N_j-1}, \ldots, A^j_{\lceil \frac{N_j}{2} \rceil + 1}$ are the same as the above a), and let $A^j_{\lceil \frac{N_j}{2} \rceil} = A^j_{\lceil \frac{N_j}{2} \rceil} \bigcup A^{j'}_{\lceil \frac{N_j}{2} \rceil}$ (Notice that $A^j_{\lceil \frac{N_j}{2} \rceil} \bigcup A^{j'}_{\lceil \frac{N_j}{2} \rceil}$ is a novel fuzzy set, still denoted as $A^j_{\lceil \frac{N_j}{2} \rceil}$ for the sake of simplicity), where $\bigcup$ denotes fuzzy union, i.e., max operator.

**Step 3 and Step 4**. The same as Steps 4 and 5 of the design procedure in Section 4.2. That is, the constructed fuzzy system is given by Eq.(6), where $\bar{y}_{i_1 i_2 \ldots i_n}$ is given by Eq.(5).

In the sequel, we make a few remarks on this above procedure of designing fuzzy systems.

*Remark 2* A fundamental difference between the designed fuzzy systems in Sections 4.2 and 4.3 is the former usually requires a large number of rules to approximate some simple functions. However, using the fuzzy system with second-order accuracy, we may use fewer rules to approximate the same function with the same accuracy. In summary, the difference between the constructed fuzzy systems in this paper is the same as for that between the traditional fuzzy system with first-order accuracy and fuzzy system with second-order accuracy (see Wang (1997) for more details).

*Remark 3* Although the opposite negative operator $\dashv$ is only considered in the constructed fuzzy systems in Sections 4.2 and 4.3, the opposite negative operator $\dashv$ is not enough for a practical Mamdani fuzzy system, such as Mamdani fuzzy controller. Firstly, for an applied fuzzy system, the classical negative operator "not" (called the contradictory negative operator in this paper) is usually needed to be considered. This is the reason why we use the classical complement to define the proposed contradictory negative operator in this paper. Secondly, for each variable in the input space, if the designed fuzzy system consists of the evaluative trichotomy of the form "big-medium-small" (sometimes, the contradictory negative operator $\neg$, i.e., the classical negative operator "not", is needed), thus, we only need to obtain the membership function of each fuzzy set "big" ("small") over the domain $U \subset \mathbb{R}^n$, we can then design the desired fuzzy system by using the proposed methods of this paper (e.g., see Case 3 below). Consequently, considering the design of an actual Mamdani fuzzy system, it is necessary to introduce the other two negations $\neg$ and $\sim$.

# 5 Approximation Accuracy Analysis of the Fuzzy System Designed Based on GFScom

In this section, we build up the approximation bounds for the two classes of fuzzy systems constructed in Section 4.

We consider the case where the unknown function $g(x)$ is a continuous function on $U = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \cdots [\alpha_n, \beta_n] \subseteq \mathbb{R}^n$. Before giving the approximation bounds, we first introduce some formal notations and results as follows Zeng and Singh (1996):

Define the infinite norm for a bounded function $g$ in $U$ to be $\|g\|_\infty = \sup_{x \in U} |g(x)|$ and the modulus of continuity of $g$ in $U$ to be

$$\omega(g, h, U) = \sup\{|g(x) - g(y)| \,|\, |x_i - y_i| \leq h_i, i = 1, 2, \ldots, n\},$$

and let

$$U_{i_1 i_2 \cdots i_n} = [e_1^{i_1}, e_1^{i_1+1}] \times \cdots \times [e_n^{i_n}, e_n^{i_n+1}],$$

where $h = (h_1, h_2, \ldots, h_n)$ ($h_i \geq 0$ for all $1 \leq i \leq n$), $e_j^1 = \alpha_j$, $e_j^{N_j} = \beta_j$, and $e_j^{i_j} \in [b_j^{i_j}, c_j^{i_j}]$ ($i_j = 2, \ldots, N_j - 1$; $j = 1, 2, \ldots, n$).

**Theorem 6** *Zhang and Li (2016) Let $A_1, A_2, \ldots, A_N$ be any GFScom in $U = [\alpha, \beta]$. $A_1, A_2, \ldots, A_N$ are complete, consistent and normal fuzzy subsets with PTS functions on $U_1 = [\alpha, \frac{\alpha+\beta}{2}]$ if and only if $A_1^{\dashv}, A_2^{\dashv}, \cdots, A_N^{\dashv}$ are complete, consistent and normal fuzzy subsets with PTS functions on $U_2 = [\frac{\alpha+\beta}{2}, \beta]$. Further, if $A_1 < A_2 < \cdots < A_N$, then we have $A_1 < A_2 < \cdots < A_N < A_N^{\dashv} < \cdots < A_2^{\dashv} < A_1^{\dashv}$.*

**Theorem 7** *Let $f(x)$ be the fuzzy system in (6) and $g(x)$ be the unknown function in (5). Then*

$$max\{|g(x) - f(x)| \,|\, x \in U_{i_1 i_2 \cdots i_n}\} \leq \omega(g, h_{i_1 i_2 \cdots i_n}, U_{i_1 i_2 \cdots i_n}) \quad i_1 i_2 \cdots i_n \in \hat{I}, \tag{7}$$

$$\|g - f\|_\infty \leq \omega(g, h, U), \tag{8}$$

*where $\hat{I} = \{i_1 i_2 \cdots i_n | i_j = 1, \ldots, N_j - 1;\ j = 1, 2, \ldots, n\}$, $h_{i_1 i_2 \cdots i_n} = (h_{i_1}^1, h_{i_2}^2, \ldots, h_{i_n}^n)$, $h_{i_j}^j = e_j^{i_j+1} - e_j^{i_j}$, $h = (h_1, h_2, \ldots, h_n)$ and $h_j = max\{h_{i_j}^j | i_j = 1, 2, \ldots, N_j - 1\}$.*

*Further, if $g$ is continuously differentiable on $U$, then*

$$\|g - f\|_\infty \leq \sum_{j=1}^{n} \left\| \frac{\partial g}{\partial x_j} \right\|_\infty h_j \leq h \sum_{j=1}^{n} \left\| \frac{\partial g}{\partial x_j} \right\|_\infty \tag{9}$$

*where $h = max\{h_j | j = 1, 2, \ldots, n\}$.*

**Proof**. By assumptions, we can verify the following results:

(a) $U = \bigcup_{i_1 i_2 \cdots i_n \in \hat{I}} U_{i_1 i_2 \cdots i_n}$. In fact, since $[\alpha_i, \beta_i] = [e_i^1, e_i^2] \cup [e_i^2, e_i^3] \cup \cdots \cup [e_i^{N_i - 1}, e_i^{N_i}]$, $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} U &= [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \cdots [\alpha_n, \beta_n] \\ &= \bigcup_{i_1=1}^{N_1-1} \cdots \bigcup_{i_n=1}^{N_n-1} U_{i_1 i_2 \cdots i_n} = \bigcup_{i_1 i_2 \cdots i_n} U_{i_1 i_2 \cdots i_n} \end{aligned}$$

which means that for all $x \in U$, there exists $U_{i_1 i_2 \cdots i_n}$ such that $x \in U_{i_1 i_2 \cdots i_n}$.

(b) For any $x \in U_{i_1 i_2 \cdots i_n}$, we have

$$f(x) = \sum_{k_1 k_2 \cdots k_n \in I_{2^n}} B_{i_1+k_1\ i_2+k_2\ \cdots\ i_n+k_n}(x) \bar{y}_{i_1+k_1\ \cdots\ i_n+k_n} \tag{10}$$

where $I_{2^n} = \{k_1 k_2 \cdots k_n | k_j = 0, 1;\ \ j = 1, 2, \ldots, n\}$. In fact, suppose $x \in U_{i_1 i_2 \cdots i_n}$, that is $x_1 \in [e_1^{i_1}, e_1^{i_1+1}]$, $x_2 \in [e_2^{i_2}, e_2^{i_2+1}]$, …, $x_n \in [e_n^{i_n}, e_n^{i_n+1}]$. By Theorem 6 one can see the fuzzy sets $A_1^{(j)}, A_2^{(j)}, \ldots, A_{N_j}^{(j)}$ are normal, consistent and complete on $[\alpha_j, \beta_j]$ for $j = 1, 2, \ldots, n$, at least one and at most two $A_{i_j}^{(j)}(x_j)$ are nonzero for $i_j = 1, 2, \ldots, N_j$. By the definition of $e_j^{i_j}(i_j = 1, 2, \ldots, N_j - 1)$, these two possible nonzero are $A_{i_j}^{(j)}(x_j)$ and $A_{i_j+1}^{(j)}(x_j)$. Hence, the fuzzy system $f(x)$ in (6) is simplified to the equality (10).

Noting that $\sum_{k_1 k_2 \cdots k_n \in I_{2^n}} B_{i_1+k_1\ i_2+k_2\ \cdots\ i_n+k_n}(x) = 1$ and the equality (10), for any $x \in U_{i_1 i_2 \cdots i_n}$, we can obtain

$$\begin{aligned} & |g(x) - f(x)| \\ \leq & \sum_{k_1 k_2 \cdots k_n \in I_{2^n}} B_{i_1+k_1\ \cdots\ i_n+k_n} |g(x) - \bar{y}_{i_1+k_1\ \cdots\ i_n+k_n}| \\ \leq & \max\{|g(x) - \bar{y}_{i_1+k_1\ \cdots\ i_n+k_n}| | k_1 k_2 \cdots k_n \in I_{2^n}\}. \end{aligned} \tag{11}$$

Noting that $\bar{y}_{i_1+k_1\ i_2+k_2\ \cdots\ i_n+k_n} = g(e_1^{i_1+k_1}, e_2^{i_2+k_2}, \ldots, e_n^{i_n+k_n})$ and $(e_1^{i_1+k_1}, e_2^{i_2+k_2}, \ldots, e_n^{i_n+k_n}) \in U_{i_1 i_2 \ldots i_n}$ $(k_1 k_2 \cdots k_n \in I_{2^n})$, we have $|x_j - e_j^{i_j+k_j}| \leq e_j^{i_j+1} - e_j^{i_j}$ $(k_j = 0, 1;\ \ j = 1, 2, \ldots, n)$. Hence, for any $x \in U_{i_1 i_2 \ldots i_n}$, the following inequality

$$\begin{aligned} |g(x) - \bar{y}_{i_1+k_1\ i_2+k_2\ \cdots\ i_n+k_n}| \leq \omega(g, h_{i_1 i_2 \ldots i_n}, U_{i_1 i_2 \ldots i_n}) \\ (k_1 k_2 \cdots k_n \in I_{2^n}) \end{aligned} \tag{12}$$

holds. From (11) and (12), we can obtain the inequalities (7) and (8).

Further, if $g$ is continuously differentiable on $U$, using the Mean Value Theorem, we have

$$\begin{aligned}\omega(g,h,U) &= \sup\{|g(x)-g(y)|||x_j-y_j|\le h_j;\ j=1,\ldots,n\}\\ &\le \sum_{j=1}^{n}\left\|\tfrac{\partial g}{\partial x_j}\right\|_\infty h_j \le h\sum_{j=1}^{n}\left\|\tfrac{\partial g}{\partial x_j}\right\|_\infty\end{aligned} \tag{13}$$

which implies immediately the inequality (9). The proof is complete.

*Remark 4* From the proof of Theorem 7 we see that if we change $A^1_{i_1}(x_1)A^2_{i_2}(x_2)\cdots A^n_{i_n}(x_n)$ to $\min\{A^1_{i_1}(x_1),\ A^2_{i_2}(x_2),\ \cdots,A^n_{i_n}(x_n)\}$, the proof is still valid. Therefore, if we use minimum inference engine (i.e., take "and" as min operator, implication as Mamdani min implication operator) in the design procedure and keep the others unchange, the designed fuzzy system still has the approximation capability in Theorem 7.

**Theorem 8** *Let $f(x)$ be the fuzzy system designed through the above four Steps in Section 4.3, that is, the membership functions of fuzzy sets $A^j_{i_j}$ are the triangular-shaped functions $A^j_{i_j}(x_j) = \triangle^j_{i_j}(x_j;\ e^j_{i_j-1},e^j_{i_j},e^j_{i_j+1})$ $(i_j = 1,2,\ldots,N_j;\ j = 1,2,\ldots,n)$ with $e^j_0 = e^j_1 = \alpha_j$, $e^j_{N_j} = e^j_{N_j+1} = \beta_j$ and $e^j_1 < e^j_2 < \cdots < e^j_{N_j}$, and the fuzzy system constructed by using singleton fuzzifier, product inference engine and center average defuzzifier. Then*

*(1) $\forall x\in U$, we have*

$$f(x) = \sum_{i_1 i_2\cdots i_n\in I}[\prod_{j=1}^{n}A^j_{i_j}(x_j)]\bar{y}_{i_1 i_2\ldots i_n}, \tag{14}$$

*where $I = \{i_1 i_2\cdots i_n | i_j = 1,2,\ldots,N_j;\ j=1,2,\ldots,n\}$.*

*(2) If $g$ is a continuously differentiable function on $U$, then*

$$\|g-f\|_\infty \le \sum_{j=1}^{n}\frac{1}{2}h_j\left\|\tfrac{\partial g}{\partial x_j}\right\|_\infty \le \frac{1}{2}h\sum_{j=1}^{n}\left\|\tfrac{\partial g}{\partial x_j}\right\|_\infty, \tag{15}$$

*where $h^j_{i_j} = e^j_{i_j+1}-e^j_{i_j}$, $h_j = max\{h^j_{i_j}|i_j = 1,2,\ldots,N_j-1;\ j = 1,2,\ldots,n\}$, $h = max\{h_j|j=1,2,\ldots,n\}$.*

*(3) If $g$ is a twice continuously differentiable function on $U$, then*

$$\|g-f\|_\infty \le \sum_{j=1}^{n}\frac{1}{8}(h_j)^2\left\|\tfrac{\partial^2 g}{\partial (x_j)^2}\right\|_\infty \le \frac{1}{8}h^2\sum_{j=1}^{n}\left\|\tfrac{\partial^2 g}{\partial (x_j)^2}\right\|_\infty, \tag{16}$$

*where $h^j_{i_j} = e^j_{i_j+1}-e^j_{i_j}$, $h_j = max\{h^j_{i_j}|i_j = 1,2,\ldots,N_j-1;\ j = 1,2,\ldots,n\}$, $h = max\{h_j|j=1,2,\ldots,n\}$.*

**Proof**. (1) By Theorem 6, we see that $A^j_1$, $A^j_2$, ..., $A^j_{N_j}$ are normal, complete and consistent fuzzy sets on $U_j = [\alpha_j,\beta_j]$, and $A^j_1 < A^j_2 <$

$\cdots < A^{j}{}_{N_j}$. Moreover, the membership function of fuzzy set $A^{j}{}_{i_j}$ ($i_j$ $=$ $1, 2, \ldots, N_j;\ j = 1, 2, \ldots, n$) is a triangular-shaped function. Therefore, we can obtain $\sum_{i_j=1}^{N_j} A_{i_j}^j(x_j) = 1$ for any $x_j \in [\alpha_j, \beta_j]$. Hence, it is not difficult to see $\sum_{i_1 i_2 \cdots i_n \in I} \prod_{j=1}^{n} A_{i_j}^j(x_j) = 1$. As a result, it follows that $B_{i_1 i_2 \cdots i_n}(x) = \prod_{j=1}^{n} A_{i_j}^j(x_j)$ which implies that the equality (14) from (6).

(2) By Theorem 6 and the constructed procedure of $f(x)$, i.e., Steps 1 through 4 in Section 4.3, we can see the fuzzy sets $A_1^j$, $A_2^j$, $\ldots$, $A_n^j$ ($j = 1, 2, \ldots, n$) are consistent, complete and normal on $U_j = [\alpha_j, \beta_j]$. For consistent and complete fuzzy sets, we have

$$f(x) = \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} \bar{y}_{i_1+k_1\ i_2+k_2\ \cdots\ i_n+k_n} \prod_{j=1}^{n} A_{i_j+k_j}^j(x_j)$$
$$= \sum_{j=1}^{n} \sum_{k_j=0}^{1} \bar{y}_{i_1+k_1\ \cdots\ i_n+k_n} A_{i_j+k_j}^j(x_j) \prod_{m=1, m\neq j}^{n} A_{i_m+k_m}^m(x_m)$$

where $\bar{y}_{i_1+k_1\ i_2+k_2\ \cdots\ i_n+k_n} = g(e_{i_1+k_1}^1, e_{i_2+k_2}^2, \ldots, e_{i_n+k_n}^n)$ for $i_j = 1, 2, \ldots, N_j - 1;\ j = 1, 2, \ldots, n$.

Hence,

$$|g(x) - f(x)| \leq \sum_{j=1}^{n} \sum_{k_j=0}^{1} \Big( |g(x) - \bar{y}_{i_1+k_1\ \cdots\ i_n+k_n}| A_{i_j+k_j}^j(x_j) \Big) \times \prod_{m=1, m\neq j}^{n} A_{i_m+k_m}^m(x_m).$$

For the normal fuzzy sets,

$$|g(x) - f(x)| \leq \sum_{j=1}^{n} \sum_{k_j=0}^{1} |g(x) - \bar{y}_{i_1+k_1\ i_2+k_2\ \cdots\ i_n+k_n}| A_{i_j+k_j}^j(x_j).$$

Using the Mean Value Theorem, then

$$|g(x) - f(x)| \leq \sum_{j=1}^{n} \sum_{k_j=0}^{1} \left| \frac{\partial g(x)}{\partial x_j} \right|_{x=\xi} |x_j - e_{i_j+k_j}^j| A_{i_j+k_j}^j(x_j).$$

Let $g'_{x_j}(x_j) = \bigvee_{x_i \in [\alpha_j, \beta_j], i \neq j} \left| \frac{\partial g}{\partial x_j} \right|$. Noting that $A_{i_j}^j(x_j) + A_{i_j+1}^j(x_j) = 1$ for any $x_j \in [e_{i_j}^j, e_{i_j+1}^j]$ ($j = 1, 2, \ldots, n$) and $A_{i_j}^j$ is a triangular-shaped function, we have

$$|g(x) - f(x)| \leq \sum_{j=1}^{n} \max_{e_{i_j}^j \leq x_j \leq e_{i_j+1}^j} |g'_{x_j}(x_j)| \frac{2(e_{i_j+1}^j - x_j)(x_j - e_{i_j}^j)}{h_{i_j}^j}, \tag{17}$$

where $h_{i_j}^j = e_{i_j+1}^j - e_{i_j}^j$.

Furthermore, it is easy to verify the identity

$$\max_{e^j_{i_j} \le x_j \le e^j_{i_j+1}} \{|x_j - e^j_{i_j}||x_j - e^j_{i_j+1}|\} = \frac{(h^j_{i_j})^2}{4} \tag{18}$$

follows. Substituting (18) into (17), we obtain

$$|g(x) - f(x)| \le \frac{1}{2}\sum_{j=1}^{n} h^j_{i_j}(\max_{e^j_{i_j} \le x_j \le e^j_{i_j+1}} |g^{'}_{x_j}(x_j)|).$$

Therefore, we have

$$\left\|g(x) - f(x)\right\|_\infty \le \frac{1}{2}\sum_{j=1}^{n} h_j \left\|\frac{\partial g}{\partial x_j}\right\|_\infty \le \frac{1}{2}h\sum_{j=1}^{n}\left\|\frac{\partial g}{\partial x_j}\right\|_\infty,$$

where $h_j = \max\{h^j_{i_j}|i_j = 1, 2, \dots, N_j - 1\}$, $h = \max\{h_j|j = 1, 2, \dots, n\}$. That is, the inequality (15) follows.

(3) Clearly, $f(x)$ is equal to $g(x)$ at points $x^* = (e^1_{i_1+k_1}, e^2_{i_2+k_2}, \dots, e^n_{i_n+k_n})$, where $i_j = 1, 2, \dots, N_j - 1$, $k_j = 0, 1$ for $j = 1, 2, \dots, n$, namely $g(x^*) - f(x^*) = 0$. Let $x \ne x^*$ be fixed, and without loss of generality, the approximation error can be expressed as

$$g(x) - f(x) = [(x_j - e^j_{i_j})(x_j - e^j_{i_j+1})]^T_j P(x). \tag{19}$$

Consider the following function of $s$, where $s = (s_1,\ s_2,\ \dots, s_n)$:

$$W(s) = g(s) - f(s) - [(x_j - e^j_{i_j})(x_j - e^j_{i_j+1})]^T_j P(x).$$

Obviously, the above constructed function $W(s) = 0$ at points $s = (e^1_{i_1+k_1}$, $e^2_{i_2+k_2}$, $\dots$, $e^n_{i_n+k_n})$ with $k_j = 0, 1$, and at the additional $s = x$. As a result, according to the well known generalized Rolle's Theorem Davis (1963), the function $\frac{\partial^2 W(s)}{\partial s_i^2}$ for $i = 1, 2, \dots, n$ must vanish at points including $\xi = (\xi_1, \xi_2, \dots, \xi_n)$. The vector of second-order derivatives of $W(s)$ is attained as

$$\begin{pmatrix} \frac{\partial^2 W(s)}{\partial s_1^2} \\ \vdots \\ \frac{\partial^2 W(s)}{\partial s_n^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 g(s)}{\partial s_1^2} \\ \vdots \\ \frac{\partial^2 g(s)}{\partial s_n^2} \end{pmatrix} - 2P(x).$$

Therefore, at arbitrary fixed point $x$, we have

$$P(x) = \frac{1}{2}\left[\frac{\partial^2 g(\xi)}{\partial x_1^2}, \cdots, \frac{\partial^2 g(\xi)}{\partial x_n^2}\right]^T. \tag{20}$$

Substituting (20) into (19), we can obtain

$$g(x) - f(x) = \frac{1}{2}\sum_{j=1}^{n}(x_j - e^j_{i_j})(x_j - e^j_{i_j+1})\frac{\partial^2 g(\xi)}{\partial x_j^2}.$$

So the following inequality

$$\begin{aligned} &\left\|g(x) - f(x)\right\|_\infty \\ \leq &\frac{1}{2}\sum_{j=1}^{n}\vee_{e^j_{i_j}\leq x_j \leq e^j_{i_j+1}}|x_j - e^j_{i_j}||x_j - e^j_{i_j+1}|\left\|\tfrac{\partial^2 g(x)}{\partial x_j^2}\right\|_\infty \end{aligned}$$

holds. Therefore, we have

$$\left\|g(x) - f(x)\right\|_\infty \leq \frac{1}{8}\sum_{j=1}^{n}h_j^2\left\|\tfrac{\partial^2 g(x)}{\partial x_j^2}\right\|_\infty \leq \frac{1}{8}h^2\sum_{j=1}^{n}\left\|\tfrac{\partial^2 g(x)}{\partial x_j^2}\right\|_\infty$$

by using (18), where $h_j = \max\{h^j_{i_j}|i_j = 1, 2, \ldots, N_j - 1\}$, $h = \max\{h_j|j = 1, 2, \ldots, n\}$. That is, the inequality (16) holds. The proof is completed.

Theorem 8 implies immediately the following corollary which shows the fuzzy systems constructed by Section 4.3 can duplicate any linear (or affine) function and multilinear (or mutliaffine) function.

**Corollary 1** *Suppose that the following two conditions holds:*

*1) $g_1(x)$ is any affine function on $U = [\alpha_1, \beta_1]\times[\alpha_2, \beta_2]\times\cdots\times[\alpha_n, \beta_n]$ given by*

$$g_1(x_1, x_2, \ldots, x_n) = a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

*and $g_2(x)$ is any multiaffine function on $U$ defined by*

$$\begin{aligned} &g_2(x_1, x_2, \ldots, x_n) \\ &= a_0 + \sum_{(k_1,k_2,\ldots,k_n)\in K} a_{i_1i_2\cdots i_n}(x_1)^{k_1}(x_2)^{k_2}\cdots(x_n)^{k_n}, \end{aligned}$$

*where $K = \{(k_1, k_2, \ldots, k_n)|k_j = 0, 1;\ \ j = 1, 2, \ldots, n$ and $\sum_{j=1}^n k_j > 0\}$, $i_1i_2\cdots i_n \in I = \{i_1i_2\cdots i_n|i_j = 1, 2, \ldots, N_j;\ j = 1, 2, \ldots, n\}$;*

*2) fuzzy system $f(x)$ is constructed by Section 4.3.*

*For a given $k \in K$, thus, we have $f(x) = g_k(x)$ for any $x \in U$, i.e., $f \equiv g_k$ on $U$.*

*Remark 5* If we consider the error bound of Theorem 2 in Zeng and Singh (1996) for a second-order approximator, we can see that the error bounds of Theorem 8 uses the identical number of membership functions with those of Zeng and Singh (1996) for the same approximation accuracy degree. However, providing only fewer membership functions than those of Zeng and Singh (1996), we can construct the desired fuzzy system by using the developed approach in this paper.

In Section 4.3, if $\lceil \frac{N_j}{2} \rceil$ $(j = 1, 2, 3, \ldots, n)$ fuzzy sets $A_1^j, A_2^j, \ldots, A^j_{\lceil \frac{N_j}{2} \rceil}$ in the interval $[\alpha_j, \frac{\alpha_j+\beta_j}{2}]$ are determined by the proposed approach in Sonbol et al (2012), we can obtain the new better claim as follows.

Firstly, we start with some notations. If a function is continuous, with all of its partial derivatives up to the $l$th-order continuity, we say that the function is $C^l$. Let the function $g(x)$ be $C^{l-1}$ on $U = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \cdots [\alpha_n, \beta_n] \subseteq \mathbb{R}^n$ and define

$$g^{(l)}_{x_j}(x_j) = \max_{\substack{x_i \in [\alpha_i, \beta_i] \\ i=1,2,\ldots,n \\ i \neq j}} \left| \frac{\partial^l g(x)}{\partial x_j^l} \right|,$$

$$\alpha_j = e_1^j < e_2^j < \cdots < e^j_{N_j} = \beta_j,$$

and

$$\varepsilon^j_{i_j,l} = \min_{e^j_{i_j} \le a^j_{i_j} \le e^j_{i_j+1}} \max_{e^j_{i_j} \le x_j \le e^j_{i_j+1}} \left| g^{(l)}_{x_j}(x_j) - g^{(l)}_{x_j}(a^j_{i_j}) \right|,$$

where $N_j$ is odd, $e^j_{\lceil \frac{N_j}{2} \rceil} = \frac{\alpha_j+\beta_j}{2}$, $j = 1, 2, \ldots, n$, $i_j = 1, 2, \ldots, N_j - 1$, and $a^j_{i_j} \in [e^j_{i_j}, e^j_{i_j+1}]$.

In addition, we refer to $U' = [u_1, v_1] \times \cdots \times [u_n, v_n]$, where $u_i \in \{\alpha_i, \frac{\alpha_i+\beta_i}{2}\}$, $v_i \in \{\frac{\alpha_i+\beta_i}{2}, \beta_i\}$, $u_i \neq v_i$ for all $i = 1, 2, \ldots, n$, as the half-input space of $U$.

Now, we give the another claim as follows.

**Theorem 9** *Let $U'$ be a half-input space of $U$, $N_j$ odd for every $j = 1, 2, \ldots, n$. If the $\lceil \frac{N_j}{2} \rceil$ fuzzy sets over $U'$ are obtained by means of the proposed approach in Sonbol et al (2012), then the constructed fuzzy system $f(x)$ built from Sec.4.3 has the following:*

*(1) If $g$ is a continuously differentiable function on $U$, then*

$$\|g - f\|_\infty \le \frac{1}{2} \sum_{j=1}^{n} h_j^{i_j} \left( \left| g^{(1)}_{x_j}(a^j_{i_j}) \right| + \varepsilon^j_{i_j,1} \right), \tag{21}$$

*where $h^j_{i_j} = e^j_{i_j+1} - e^j_{i_j}$ and $a^j_{i_j} \in [e^j_{i_j}, e^j_{i_j+1}]$.*

*(2) If $g$ is a twice continuously differentiable function on $U$, then*

$$\|g - f\|_\infty \le \frac{1}{8} \sum_{j=1}^{n} \left( h_j^{i_j} \right)^2 \left( \left| g^{(2)}_{x_j}(a^j_{i_j}) \right| + \varepsilon^j_{i_j,2} \right), \tag{22}$$

*where $h^j_{i_j} = e^j_{i_j+1} - e^j_{i_j}$ and $a^j_{i_j} \in [e^j_{i_j}, e^j_{i_j+1}]$.*

**Proof**. It follows from Lemma 1 of Sonbol et al (2012) and Theorem 6. Also, we refer the reader to the proof of the above Theorem 8, since the proving methods of both these claims are similar. The proof is completed.

Furthermore, compared with the Theorems 1 and 2 in Sonbol et al (2012), we have the following corollary.

**Corollary 2** *The fuzzy system $f(x)$ constructed from Theorem 9 requires a smaller number of known membership functions than the counterparts of Sonbol et al (2012) for the same guaranteed error bound.*

**Proof**. From Theorem 9, we only verify that it follows whenever $N_j$ is even for some $j \in \{1, 2, \dots, n\}$. Indeed, when $N_j$ is even for some $j \in \{1, 2, \dots, n\}$, it is trivial that $N_j \geq \left\lceil \frac{N_j+1}{2} \right\rceil$.

*Remark 6* From Sonbol et al (2012), the constructed Mamdani fuzzy system requires a smaller number of membership functions than all previously published fuzzy systems. By Corollary 2, we can get that the sufficient conditions in Theorem 9 require a smaller number of known membership functions than all previously published conditions. Furthermore, the computation cost of the developed approach in this paper is smaller than that of the Mamdani fuzzy systems developed in Sonbol et al (2012) for the same guaranteed error bound.

# 6 Illustrative Cases

*Case 1* Zeng and Singh (1996); Sonbol et al (2012). Based on GFScom, design a fuzzy system $f(x)$ to approximate the continuous function $g(x) = \sin(x)/x$ defined on $U = [-3, 3]$ with a given accuracy of $\epsilon = 0.2$.

First, we use the error bound of Eq.(16) and obtain 9 fuzzy sets to approximate the function $g(x) = \sin(x)/x$ with the desired degree. So it is sufficient that the membership functions of only $\lceil \frac{9}{2} \rceil = 5$ fuzzy sets are defined appropriately in the interval $[-3, 0]$ (or $[0, 3]$).

Next, we use the error bound of Eq.(21) and obtain the results in the interval $[-3, 0]$ listed in Table 1.

**Table 1** Approximation results over the interval $[-3, 0]$ for Case 1 using the error bound of Eq.(21)

| $i_1$ | 1 | 2 | 3 |
|---|---|---|---|
| $h_1^{i_1}$ | 0.918 | 0.918 | 1.164 |

Finally, since $g(x) = \sin(x)/x$ is a second-order differential function, we can use the error bound of Eq.(22) and obtain the results in the interval $[-3, 0]$ listed in Table 2. Although in this case fewer known fuzzy sets are required when using the bound of Eq.(22) than the bound of Eq.(21), in general, this is not true.

**Table 2** Approximation results over the interval $[-3, 0]$ for Case 1 using the error bound of Eq.(22)

| $i_1$ | 1 | 2 |
|---|---|---|
| $h_1^{i_1}$ | 2.305 | 0.695 |

Table 3 compares our results with the results in Sonbol et al (2012), Zeng and Singh (1996) and Ying (1994) for approximation accuracy $\epsilon = 0.2$.

**Table 3** Comparison of the number of known fuzzy sets needed to achieve the error bound $\epsilon = 0.2$

| Method | No. of Known Fuzzy Sets for Input Variable |
|---|---|
| (3) of Theorem 8 | 5 |
| (1) of Theorem 9 | 4 |
| (2) of Theorem 9 | 3 |
| Theorem 1 of Sonbol et al (2012) | 7 |
| Theorem 2 of Sonbol et al (2012)<br>(Second order approximation) | 4 |
| Theorem 2 of Zeng and Singh (1996)<br>(Second order approximation) | 9 |
| Theorem 3.4 of Ying (1994) | 207 |

*Case 2* Assume $g(x_1, x_2) = e^{x_1+x_2}$ (unknown) is defined on $U = [-0.5, 0.5]^2$. If two of membership functions of fuzzy sets in the designed fuzzy system $f(x_1, x_2)$ with a desired degree of accuracy $\epsilon = 0.2$ to approximate $g(x_1, x_2)$ is, respectively, given by

$$A_1^1(x_1) = \begin{cases} -2x_1, & x_1 \in [-0.5, 0] \\ 0, & x_1 \in [0, 0.5] \end{cases} \tag{23}$$

and

$$A_1^2(x_2) = \begin{cases} -2x_2, & x_2 \in [-0.5, 0] \\ 0, & x_2 \in [0, 0.5] \end{cases} \tag{24}$$

thus, how can we design a fuzzy system to roughly approximate $g(x_1, x_2)$?

Obviously, in terms of the traditional design methods of fuzzy systems(see Wang (1997) and therein references for more details), it is not feasible to construct the fuzzy system under the above conditions. However, using the method developed in this paper, i.e., the design procedure in Section 4.3, we can do it.

First, from the Eq.(23) and Definition 2, we can compute the opposite negative set $A_1^{1\lrcorner}$ and medium negative set $A_1^{1\sim}$ of $A_1^1$ as follows:

$$A_3^1(x_1) = A_1^{1\lrcorner}(x_1) = \begin{cases} 2x_1, & x_1 \in [0, 0.5] \\ 0, & x_1 \in [-0.5, 0] \end{cases} \text{ and}$$

$$A_2^1(x_1) = A_1^{1\sim}(x_1) = \begin{cases} 2x_1 + 1, & x_1 \in [-0.5, 0) \\ 1, & x_1 = 0 \\ 1 - 2x_1, & x_1 \in (0, 0.5] \end{cases}$$

By symmetry and the Eq.(24), the same applies to the opposite negative set $A_3^2 = A_1^{2\lrcorner}$ and medium negative set $A_2^2 = A_1^{2\sim}$ of $A_1^2$.

If we choose $t_1^1 = t_1^2 = -0.5$, $t_2^1 = t_2^2 = 0$, $t_3^1 = t_3^2 = 0.5$, then we can construct the following fuzzy system:

$$f(x_1, x_2) = \sum_{i_1=1}^{3} \sum_{i_2=1}^{3} A_{i_1}^1(x_1) A_{i_2}^2(x_2) e^{t_{i_1}^1 + t_{i_2}^2}$$

which approximates $g(x_1, x_2)$ with the accuracy degree $\epsilon = 0.2$. The figure of system function $f(x_1, x_2)$ is depicted as follows (Fig.1). A comparison of the figure of $g(x_1, x_2)$ and the figure of the errors of the system function (or approximation function) and the origin function $g(x_1, x_2)$ are also described in Figs. 2 and 3.

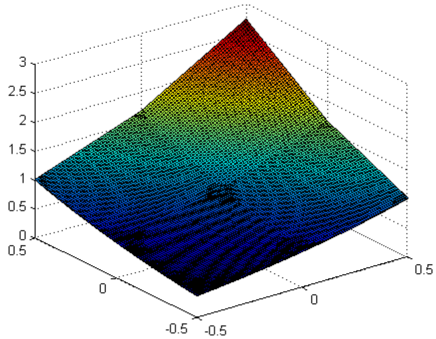


**Fig. 1** The system function $y = f(x_1, x_2)$ of the fuzzy system in Case 2

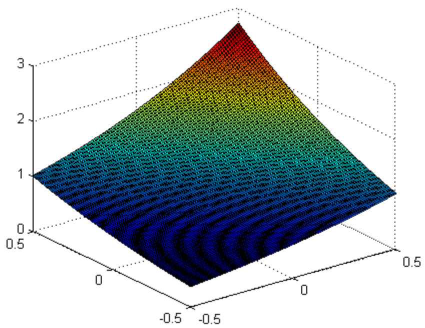


**Fig. 2** The origin function $y = g(x_1, x_2)$ in Case 2

For Case 2, Table 4 compares our result with the results in Sonbol et al (2012), Zeng et al (2000) and Ying (1994) for approximation accuracy $\epsilon = 0.2$.

*Case 3* Consider a simple two-input-single-output liquid level control problem: in some liquid level control system, the fluid mass in the container often changes randomly. By adjusting the opening degree of the valve, we can control the liquid level in the container such that the level keeps the steady state error small. Both input variables, named *level* (denoted $l$) and *rate* (denoted $r$), represents the liquid level and the flow input rate, respectively. The output variable, named *valve* (denoted

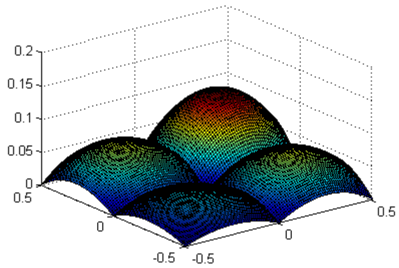


**Fig. 3** The errors of the system function $y = f(x_1, x_2)$ and the origin function $y = g(x_1, x_2)$ in Case 2

**Table 4** Comparison of the number of known fuzzy sets needed to achieve the error bound $\epsilon = 0.2$

| Method | No. of Known Fuzzy Sets for Each Input Variable |
|---|---|
| GFScom | 1 |
| Theorem 2 of Sonbol et al (2012) | 4 |
| Theorem 2 of Zeng et al (2000) | 6 |
| Theorem 3.4 of Ying (1994) | 309 |

$v$), denotes the opening degree of the valve. Suppose that the range of the *level* be $[-1, 1]$, *rate* be $[-0.1, 0.1]$ and *valve* be $[-1, 1]$.

By inquiring skilled experts in the field, we get that the range of the input variable $l$ is covered by 3 fuzzy sets, named *High*, *Okay* and *Low*, and the membership function of *High* is expressed as follows:

$$High(l) = \exp\left(-\frac{(l+1)^2}{2 \cdot (0.3)^2}\right)$$

where $l \in [-1, 1]$.

Also, the range of the other input $r$ is covered by 3 fuzzy sets, termed *Negative*, *Zero* and *Positive*, and the membership function of *Negative* is defined as follows

$$Negative(r) = \exp\left(-\frac{(r+0.1)^2}{2 \cdot (0.03)^2}\right)$$

where $r \in [-0.1, 0.1]$.

In the sequel, we simulate the above liquid level control procedure by using the well-known *sltank* in Matlab 7.10.0(R2010a).

First, for simplicity, we use the square wave with amplitude 0.5 and frequency 0.1 rad/s to imitate change of the liquid level in the container. Moreover, the mathematical model of the controlled object in *sltank* is directly used as that in our system.

Considering the definition of GFScom and the above simulation conditions, we design the following five fuzzy sets (named $close\_fast$, $close\_slow$, $no\_change$, $open\_slow$ and $open\_fast$) over the output interval $[-1, 1]$, shown in Fig.4

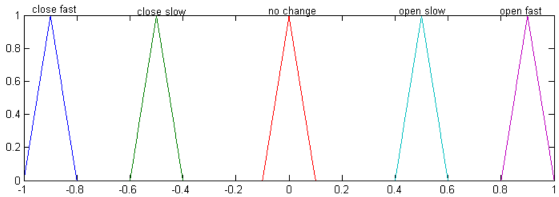


**Fig. 4** The distribution of membership functions over the output space in Case 3

Next, from previous operating experiences, we conclude the following control rules listed in Table 5. Note that the fuzzy rule base is small and incomplete, but these fuzzy rules are sufficient for our simulation.

**Table 5** Fuzzy rule base for the liquid level control problem in Case 3

| $l$ | $r$ | | | |
|---|---|---|---|---|
| | $None$ | $Negative$ | $Zero$ | $Positive$ |
| $None$ | | | | |
| $High$ | $close\_fast$ | | | |
| $Okay$ | $no\_change$ | $open\_slow$ | | $close\_slow$ |
| $Low$ | $open\_fast$ | | | |

Finally, by using the proposed method in this paper, we can obtain the results shown in Figs.5 and 6. From the resulting simulation, one can see that the constructed fuzzy controller in this paper successfully resolves the liquid level control problem in Example 4. Note that the model of the controlled plant in *sltank* is nonlinear, so this demonstration illustrates that the proposed fuzzy controller in this paper can implement the control of nonlinear systems effectively.

# 7 Conclusion and Future Work

In this paper, we have continually developed the theory of GFScom introduced in Zhang and Li (2017). By considering triangular norm operations, we

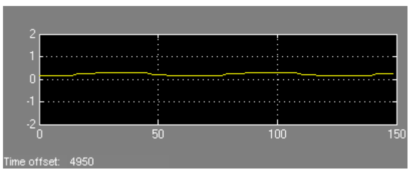


**Fig. 5** The change of liquid level in Case 3

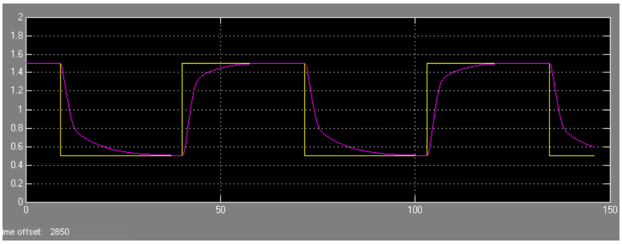


**Fig. 6** Comparison of the square wave (yellow) and its response curve (red) in Case 3

exploited the generalized properties of GFScom. On the basis of GFScom, the constructive approaches of the Mamdani-type fuzzy system that can approximate any continuous function on a compact set to a given degree of accuracy have been presented. We established the approximation bounds for the classes of the constructed fuzzy systems. Illustrative cases and numerical comparisons are provided to show the effectiveness and advantage of the developed approaches. At last, we need further to point out that the developed model in this paper can not give the one-to-one relationship of fuzzy sets between the input and output variables.

This is the first step of GFScom. There are many issues that can be done in the future. We list several suggestions as follows.

- What is the detailed structure of GFScom? For example, what are the relations of opposite negative operator, medium negative operator and other connectives such as conjunction, disjunction, implication? How should they be depicted in fuzzy logic? Although our work is based upon the medium logic ML system, we think ML is not very suitable for acting as the structure of GFScom. Therefore, how to construct the logic structure of GFScom should be studied.
- How can we design other types of fuzzy systems, such as T-S type fuzzy systems and Boolean type fuzzy systems by using GFScom.
- In Fernandez-Peralta et al (2022), when $N$ is a continuous function the characterization of $(S, N)$-implications is explored, and also a first characterization of this family of implications is illustrated. Analogously, when the contradictory, opposite and medium negation in fuzzy logic and reasoning are distinguished, what is the characterization of $(S, N)$-implications? This is interesting and next work.

We will address these issues in subsequent papers.

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# Compliance with Ethical Standards

**Conflicts of interest.** Shengli Zhang and Jing Chen declare that they have no conflict of interest.

**Human and animal rights.** This article does not contain any studies with human participants or animals performed by any of the authors.

# References


Analyti A, Antoniou G, Damásio CV, et al (2008) Extended RDF as a semantic foundation of rule markup languages. J of Artif Intell Res 32(1):37–94

Bustince H, Campin M, De Miguel L, et al (2022) Strong negations and restricted equivalence functions revisited: An analytical and topological approach. Fuzzy Sets and Systems 441:110–129. https://doi.org/10.1016/j.fss.2021.10.013

Cintula P, Klement EP, Mesiar R, et al (2010) Fuzzy logics with an additional involutive negation. Fuzzy Sets and Syst 161(3):390–411

Davis PJ (1963) Interpolation and Approximation. Blaisdell, New York

Esteva F, Godo L, Hájek P, et al (2000) Residuated fuzzy logics with an involutive negation. Archive Mathematical Logic 39(2):103–124

Fernandez-Peralta R, Massanet S, Mesiarová-Zemánková A, et al (2022) A general framework for the characterization of (S,N)-implications with a non-continuous negation based on completions of t-conorms. Fuzzy Sets and Systems 441:1–32. https://doi.org/10.1016/j.fss.2021.06.009

Ferré S (2006) Negation, opposition, and possibility in logical concept analysis. In: Ganter B, Kwuida L (eds) Proc. of the fourth International Conference on Formal Concept Analysis. Springer Verlag, Berlin Heidelberg, no. 3874 in Lecture Notes in Artificial Intelligence, pp 130–145

Hájek P (1998) Metamathematics of Fuzzy Logic. Kluwer Academic Publishers, Dordrecht

Jiang L, Bosselut A, Bhagavatula C, et al (2021) “I’m not mad”: Commonsense implications of negation and contradiction. Preprint at https://arxiv.org/abs/2104.06511

Kaneiwa K (2007) Description logics with contraries, contradictories, and subcontraries. New Generation Computing 25(4):443–468

Kassner N, Krojer B, Schútze H (2020) Are pretrained language models symbolic reasoners over knowledge? In: Proceedings of the 24th Conference on Computational Natural Language Learning. Association for Computational Linguistics, pp 552–564, https://doi.org/10.18653/v1/P17, online

Klement EP, Mesiar R, Pap E (2000) Triangular Norms. Kluwer, Dordrrecht, The Netherlands

Klir GJ, Yuan B (1995) Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prantice-Hall, Upper Saddle River, NJ

Murinová P, Novák V (2014) Analysis of generalized square of opposition with intermediate quantifiers. Fuzzy Sets and Syst 242:89–113

Nourouzi K, Aghajani A (2008) Convexity in triangular norm of fuzzy sets. Chaos, Solitons and Fractals 36:883–889

Novák V (2001) Antonyms and linguistic quantifers in fuzzy logic. Fuzzy Sets and Syst 124:335–351

Novák V (2005) On fuzzy type theory. Fuzzy Sets and Syst 149:235–273

Novák V (2008) A comprehensive theory of trichotomous evaluative linguistic expressions. Fuzzy Sets and Syst 159(22):2939–2969

Pan Y, Wu WM (1990) Medium algebras. J of Math Res & Exposition 10(2):265–270

Pan ZH (2010) Fuzzy set with three kinds of negations in fuzzy knowledge processing. In: Proceedings of The Ninth International Conference on Machine Learning and Cybernatics, vol 5. IEEE Computer Society Press, Piscataway,USA, pp 2730–2735

Pan ZH (2012) Three kinds of fuzzy knowledge and their base of set. Chinese J of Comput 35(7):1421–1428. (in Chinese)

Pan ZH (2013) Three kinds of negation of fuzzy knowledge and their base of logic. In: D.S.Huang, K.H.Jo, Y.Q.Zhou, et al (eds) Pro. Of 9th international conference on intelligent computing. Springer Verlag, Berlin Heidelberg, no. 7996 in Lecture Notes in Artificial Intelligence, pp 83–93

Sonbol AH, Fadali MS, Jafarzadeh S (2012) TSK fuzzy approximators: design and accuracy analysis 42(3):702–712

Torres-Blanc C, Cubillo S, Hernndez-Varela P (2019) New negations on the membership functions of type-2 fuzzy sets. IEEE Transactions on Fuzzy Systems 27(7):1397–1406

Wagner G (2003) Web rules need two kinds of negation. In: F.Bry, N.Henze, J.Maluszynski (eds) Proc.of the 1st international workshop on Principles and Practice of Semantic Web Reasoning. Springer Verlag, Berlin Heidelberg, no. 2901 in Lecture Notes in Computer Science, pp 33–50

Wang LX (1997) A Course in Fuzzy Systems and Control. Prentice Hall PTR, Englewood Cliff, USA

Xiao XA, Zhu WJ (1988) Propositional calculus system of medium logic(I). J of Math Res & Exposition 8(2):327–331

Ying H (1994) Sufficient conditions on general fuzzy systems as function approximators. Automatica 30(3):521–525

Zeng K, Zhang NY, Xu WL (2000) A comparative study on sufficient conditions for Takagi-Sugeno fuzzy systems as universal approximators 8(6):773–780

Zeng XJ, Singh MG (1996) Approximation accuracy analysis of fuzzy systems as function approximators 4(1):44–63

Zhang SL (2014) Formal deductive system of fuzzy propositional logic with different negations. J of Frontiers of Comput Sci and Technol 8(4):494–505. (in Chinese)

Zhang SL, Li YM (2016) Algebraic representation of negative knowledge and its application to design of fuzzy systems. Chinese Journal of Computer 39(12):2527–2546

Zhang SL, Li YM (2017) A novel table look-up scheme based on GFScom and its application. Soft Comput 21(22):6767–6781

Zhu WJ, Xiao XA (1988a) On the naive mathematical models of medium mathematical system MM. J of Math Res & Exposition 8(1):139–151

Zhu WJ, Xiao XA (1988b) Predicate calculus system of medium logic(I). J of Nanjing University 24(4):583–596

Zhu WJ, Xiao XA (1988c) Propositional calculus system of medium logic(II). J of Math Res & Exposition 8(3):457–466

Zou J (1988) Semantic interpretation of propositional calculus system $MP^*$ of medium logic and its soundness and completeness. J of Math Res & Exposition 8(3):467–468. (in Chinese)

Zou J (1989) Semantic interpretation of predicate calculus system of medium logic $ME^*$ and its soundness and completeness. Chinese Sci Bulletin 34(6):448–451