# Transport via Partial Galois Connections and Equivalences (Extended Version) 

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#### Abstract

Multiple types can represent the same concept. For example, lists and trees can both represent sets. Unfortunately, this easily leads to incomplete libraries: some set-operations may only be available on lists, others only on trees. Similarly, subtypes and quotients are commonly used to construct new type abstractions in formal verification. In such cases, one often wishes to reuse operations on the representation type for the new type abstraction, but to no avail: the types are not the same. To address these problems, we present a new framework that transports programs via equivalences. Existing transport frameworks are either designed for dependently typed, constructive proof assistants, use univalence, or are restricted to partial quotient types. Our framework (1) is designed for simple type theory, (2) generalises previous approaches working on partial quotient types, and (3) is based on standard mathematical concepts, particularly Galois connections and equivalences. We introduce the notions of partial Galois connection and equivalence and prove their closure properties under (dependent) function relators, (co)datatypes, and compositions. We formalised the framework in Isabelle/HOL and provide a prototype. ${ }^{1}$


Keywords: Galois connections • Equivalences • Relational parametricity

## 1 Introduction

Computer scientists often write programs and proofs in terms of representation types but provide their libraries in terms of different, though related, type abstractions. For example, the abstract type of finite sets may be represented by the type of lists: every finite set is related to every list containing the same elements and, conversely, every list is related to its set of elements. As such, every function on lists respecting this relation may be reused for a library on finite sets. To be more explicit, consider the following example in simple type theory:

[^0]A Simple Example Take the types of lists, $\alpha$ list, and finite sets, $\alpha$ fset. There is a function to fset : $\alpha$ list $\Rightarrow \alpha$ fset that turns a list into its set of elements. This allows us to define the relation LFS xs $s:=$ to fset $x s=s$ that identifies lists and finite sets, e.g. $\operatorname{LFS}[1,2,3]\{1,2,3\}$ and $\operatorname{LFS}[3,1,2]\{1,2,3\}$. Our goal is to use this identification to transport programs between these two types.

For instance, take the function max_list $x s:=$ foldr $\max x s 0$ of type $\mathbb{N}$ list $\Rightarrow \mathbb{N}$ that returns the maximum natural number contained in a list. After some thinking, one recognises that max_list respects the relation LFS in the following sense: if two lists correspond to the same set, then applying max_list to these lists returns equal results. Formally,

$$
\begin{equation*}
\forall x s y s . \text { to_fset } x s=\text { to_fset } y s \longrightarrow \text { max_list } x s=\text { max_list } y s \tag{1}
\end{equation*}
$$

Despite this insight, we still cannot directly compute the maximum of a finite set $s: \mathbb{N}$ fset using max_list; the term max_list $s$ does not even typecheck (for good reasons). But there is an indirect way if we are also given an "inverse" of to_fset, call it to_list ${ }^{\mathrm{fin}}: \alpha$ fset $\Rightarrow \alpha$ list, that returns an arbitrary list containing the same elements as the given set. The functions to _fset and to_list ${ }^{\text {fin }}$ form an equivalence between $\alpha$ list and $\alpha$ fset that respects the relation LFS:

$$
\begin{equation*}
\forall x s . \text { LFS } x s \text { (to_fset } x s) \quad \text { and } \quad \forall s . \text { LFS }\left(\text { to_list }^{\text {fin }} s\right) s . \tag{2}
\end{equation*}
$$

Thanks to this equivalence, we can compute the maximum of $s$ by simply transporting $s$ along the equivalence:

$$
\begin{equation*}
\text { max_fset } \left.s:=\text { max_list (to_list }{ }^{\text {fin }} s\right) . \tag{3}
\end{equation*}
$$

The correctness of this transport is guaranteed by (1)-(3):

$$
\begin{equation*}
\forall x s \text { s. LFS } x s s \longrightarrow \text { max_list } x s=\text { max_fset } s . \tag{4}
\end{equation*}
$$

We can now readily replace any occurrence of max_fset $s$ by max_list (to_list ${ }^{\text {fin }} s$ ) and, vice versa, any occurrence of max_list $x s$ by max_fset (to_fset $x s$ ). This process can be extended to many other functions, such as map, filter, intersect, by introducing new terms map_fset, filter_fset, intersect_fset and proving their respectfulness theorems. Indeed, it is a very repetitive task begging for automation.

State of the Art There are various frameworks to automate the transport of terms along equivalences. Most of them are designed for dependently typed, constructive proof assistants and are based on type equivalences [8, 9, 26, 28, 29], which play a central role in homotopy type theory. In a nutshell, type equivalences are pairs of functions $f, g$ that are mutually inverse (i.e. $g(f x)=x$ and $f(g y)=y)$ together with a compatibility condition. They cannot solve our problem since to fset and to_list ${ }^{\text {fin }}$ are not mutually inverse.

Angiuli et al. [1] note and address this issue in Cubical Agda [32]. Essentially, they first quotient both types and then obtain a type equivalence between the quotiented types. Their approach supports a restricted variant of quasi-partial
equivalence relations [16] but also uses univalence [33], which is unavailable in major proof assistants like Isabelle/HOL [24] and Lean 3 [22]/Lean 4 [23].

Another existing framework is Isabelle's Lifting package [13], which transports terms via partial quotient types:

Definition 1. A partial quotient type ( $T$, Abs, Rep) is given by a right-unique and right-total relation $T$ and two functions Abs, Rep respecting $T$, that is $T x y \longrightarrow A b s x=y$ and $T(R e p y) y$, for all $x, y$.

In fact, (LFS, to_fset, to_list ${ }^{\text {fin }}$ ) forms a partial quotient type. The Lifting package can thus transport our list library to finite sets ${ }^{2}$. However, the package also has its limitations:

Limitations of the Lifting Package Consider the previous example with one modification: rather than transporting max_list to finite sets, we want to transport it to the type of (potentially infinite) sets, $\alpha$ set. We cannot build a partial quotient type from $\alpha$ list to $\alpha$ set because the required relation $T: \alpha$ list $\Rightarrow$ $\alpha$ set $\Rightarrow$ bool is not right-total (we can only relate finite sets to lists). The Lifting package is stuck. But in theory, we can (almost) repeat the previous process: There is again a function to_set : $\alpha$ list $\Rightarrow \alpha$ set. We can define a relation LS xs $s:=$ to_set $x s=s$. We can again prove that max_list respects LS:

$$
\begin{equation*}
\forall x s y s . \text { to_set } x s=\text { to_set } y s \longrightarrow \text { max_list } x s=\text { max_list } y s . \tag{5}
\end{equation*}
$$

There is a function to_list : $\alpha$ set $\Rightarrow \alpha$ list, and we obtain a partial equivalence:

$$
\begin{equation*}
\forall x s \text {. LS } x s \text { (to_set } x s) \quad \text { and } \quad \forall s \text {. finite } s \longrightarrow \text { LS (to_list } s) s . \tag{6}
\end{equation*}
$$

We can define the function max_set $s:=$ max_list (to_list $s$ ). And we again obtain a correctness theorem: $\forall x s s$. LS $x s s \longrightarrow$ max_list $x s=$ max_set $s$. While this process looks rather similar, there is one subtle change: the second part of Eq. (6) only holds conditionally. As a contribution of this paper, we show that these conditions are not showstoppers, and that we can transport via such partial equivalences in general.

Now one may argue that we could still use partial quotient types to transport from lists to sets: First obtain a right-unique, right-total relation $T$ by building a subtype of the target type. Then transport to the new subtype and then inject to the original type. In spirit, this is close to the approach suggested by Angiuli et al. [1]. But the author finds this unsatisfactory from a practical and a conceptual perspective: From a practical perspective, it introduces unnecessary subtypes to our theory. And conceptually, the process for sets and lists was almost identical to the one for finite sets and lists - there was no detour via subtypes.

A second limitation of the Lifting package is that it does not support interargument dependencies. For example, take the types of natural numbers, $\mathbb{N}$,

[^1]and integers, $\mathbb{Z}$. We can construct a partial quotient type (ZN, to_nat, to_int), where to_int : $\mathbb{N} \Rightarrow \mathbb{Z}$ is the standard embedding, to_nat : $\mathbb{Z} \Rightarrow \mathbb{N}$ is its inverse (a partial function), and ZN $i n:=i=$ to int $n$. It then seems straightforward to transport subtraction $(-\mathbb{Z}): \mathbb{Z} \Rightarrow \mathbb{Z} \Rightarrow \overline{\mathbb{Z}}$ from integers to natural numbers in the following way:
\[

$$
\begin{equation*}
n_{1}-_{\mathbb{N}} n_{2}:=\text { to_nat }\left(\text { to_int } n_{1}-_{\mathbb{Z}} \text { to_int } n_{2}\right) . \tag{7}
\end{equation*}
$$

\]

And of course, we expect a correctness theorem:

$$
\begin{equation*}
\forall i_{1} n_{1} i_{2} n_{2} . \mathrm{ZN} i_{1} n_{1} \wedge \mathrm{ZN} i_{2} n_{2} \longrightarrow \mathrm{ZN}\left(i_{1}-_{\mathbb{Z}} i_{2}\right)\left(n_{1}-_{\mathbb{N}} n_{2}\right) \tag{8}
\end{equation*}
$$

But alas, the theorem does not hold: we need an extra dependency between the arguments of the respective subtractions, e.g. $i_{1} \geq i_{2}$ or $n_{1} \geq n_{2}$. Unfortunately, the Lifting package's theory [13] cannot account for such dependencies, and as such, the transport attempt for $\left(-_{\mathbb{Z}}\right)$ fails.

In a similar way, the list index operator (!!) : $\alpha$ list $\Rightarrow \mathbb{N} \Rightarrow \alpha$ can only be transported to the type of arrays for indices that are in bounds (cf. Section 5, Example 2). While solutions for dependently typed environments [1, $8,9,26,28,29]$ typically handle such examples by encoding the dependencies in a type, e.g. $(x s: \alpha$ list $) \Rightarrow\{0, \ldots$, length $x s-1\} \Rightarrow \alpha$, it is unclear how to support this in a simply typed environment. As a contribution of this paper, we show how to account for such dependencies with the help of dependent function relators.

Contributions and Outline We introduce a new transport framework - simply called Transport. Our framework (1) is applicable to simple type theory, (2) is richer than previous approaches working on partial quotient types, and (3) is based on standard mathematical notions, particularly Galois connections and equivalences. In Section 2, we distil the essence of what we expect when we transport terms via equivalences. The derived set of minimal expectations motivates us to base our framework on Galois connections.

To meet these expectations, we introduce the notion of partial Galois connections, which generalise (standard) Galois connections and partial quotient types, in Section 3.4. We also introduce a generalisation of the well-known function relator that allows for dependent relations in Section 3.2.

Section 4 builds the technical core of the paper. We derive closure conditions for partial Galois connections and equivalences as well as typical order properties (reflexivity, transitivity, etc.). Specifically, we show closure properties under (dependent) function relators, relators for (co)datatypes, and composition. All these results are novel and formalised in Isabelle/HOL.

Based on our theory, we implemented a prototype for automated transports in Isabelle/HOL and illustrate its usage in Section 5. We conclude with related work in Section 6 and future work in Section 7.

This article's supplementary material ${ }^{3}$ includes the formalisation and a guide linking all definitions, results, and examples to their formal counterpart in Isabelle/HOL.

[^2]
(a) Example of a type equivalence. Left and right-hand side relation are restricted to be equality.

(b) Example of a partial quotient type. The left relation can be an arbitrary partial equivalence relation. The right relation is restricted to be equality.

Fig. 1: Examples of equivalences used in prior work. Types are drawn solid, black. Transport functions are drawn dashed. Each equivalence gives rise to a number of equivalence classes on the left and right-hand side of the equivalence, which are drawn dotted. Arrows inside equivalence classes are omitted.

## 2 The Essence of Transport

Existing frameworks, although beneficial in practical contexts, are unapplicable to our introductory examples. We hence first want to find the essence of transport ${ }^{4}$. To find this essence, we have to answer the following question:

What are the minimum expectations when we transport terms via equivalences?
In this section, we argue that Galois connections are the right notion to cover this essence. Let us examine prior work to identify some guiding principles.

Type Equivalences Much recent work is based on type equivalences $[1,8,9,26$, 28, 29]. We denote a type equivalence between $\alpha$ and $\beta$ with mutual inverses $f: \alpha \Rightarrow \beta$ and $g: \beta \Rightarrow \alpha$ by $(\alpha \simeq \beta) f g$. Then, on a high level, given a set of equivalences $\left(\alpha_{i} \simeq \beta_{i}\right) f_{i} g_{i}$ for $1 \leq i \leq n$ and two target types $\alpha, \beta$ that may include $\alpha_{i}, \beta_{i}$, one tries to build an equivalence $(\alpha \simeq \beta) f g$. Given a term $t: \alpha$, we can then define $t^{\prime}:=f t$, satisfying $t=g t^{\prime}$. Symmetrically, for a term $t^{\prime}: \beta$, we can define $t:=g t^{\prime}$, satisfying $f t=t^{\prime}$. This situation is depicted in Fig. 1(a).

[^3]Partial Quotient Types The Lifting package [13] is based on partial quotient types $(T, A b s, R e p)$ (see Def. 1). Every partial quotient type induces a relation $(\approx): \alpha \Rightarrow \alpha \Rightarrow$ bool that identifies values in $\alpha$ that map to the same value in $\beta$ :

$$
\begin{equation*}
x_{1} \approx x_{2}:=\mathrm{in} \_\operatorname{dom} T x_{1} \wedge A b s x_{1}=A b s x_{2} . \tag{9}
\end{equation*}
$$

Given a set of partial quotient types $\left(T_{i}: \alpha_{i} \Rightarrow \beta_{i} \Rightarrow\right.$ bool, $\left.A b s_{i}, \operatorname{Rep}_{i}\right)$ for $1 \leq i \leq n$ and two target types $\alpha, \beta$ that may include $\alpha_{i}, \beta_{i}$, the Lifting package tries to build a partial quotient type $(T: \alpha \Rightarrow \beta \Rightarrow$ bool, $A b s, R e p)$. Given a term $t$ in the domain of $(\approx)$, we can then define $t^{\prime}:=A b s t$, satisfying $t \approx R e p t^{\prime}$. Symmetrically, for a term $t^{\prime}: \beta$, we can define $t:=R e p t^{\prime}$, satisfying $A b s t=t^{\prime}$. This situation is depicted in Fig. 1(b).

The Essence Abstracting from these approaches, we note some commonalities:

- As input, they take base equivalences, which are then used to build more complex equivalences.
- The equivalences include a left transport function $l: \alpha \Rightarrow \beta$ and a right transport function $r: \beta \Rightarrow \alpha$. They can be used to move terms from one side of the equivalence to a "similar" term on the other side of the equivalence.
- Terms $t: \alpha$ and $t^{\prime}: \beta$ that are "similar" stand in particular relations: in the case of type equivalences, $t=r t^{\prime}$ and $l t=t^{\prime}$; in the case of Lifting, $t \approx r t^{\prime}$ and $l t=t^{\prime}$. More abstractly, $L t\left(r t^{\prime}\right)$ and $R(l t) t^{\prime}$ for some left relation $L: \alpha \Rightarrow \alpha \Rightarrow$ bool and right relation $R: \beta \Rightarrow \beta \Rightarrow$ bool $^{5}$
- More generally, $L$ and $R$ specify how terms ought to be related in $\alpha$ and $\beta$ and determine which terms can be meaningfully transported using $l$ and $r$.
- $L, R, l, r$ are compatible: if terms are related on one side (e.g. $L t_{1} t_{2}$ ), their transports are related on the other side (e.g. $\left.R\left(l t_{1}\right)\left(l t_{2}\right)\right)$.

Based on these commonalities, we can formulate six minimum expectations:
(1) We want to specify how terms in $\alpha$ and $\beta$ are related using relations $L, R$.
(2) Transports should be possible by means of functions $l: \alpha \Rightarrow \beta, r: \beta \Rightarrow \alpha$.
(3) The notion of equivalence should be closed under common relators, particularly those for functions and (co)datatypes.
(4) Terms related on one side have transports that are related on the other side.
(5) Transporting a term should result in a term that is "similar" to its input.
(6) "Similar" terms $t: \alpha$ and $t^{\prime}: \beta$ are related with each other's transports, i.e. $L t\left(r t^{\prime}\right)$ and $R(l t) t^{\prime}$.

Applying Expectation (6) to Expectation (5) then yields the requirements
(a) $L t(r(l t))$,
(b) $R\left(l\left(r t^{\prime}\right)\right) t^{\prime}$.

[^4]At this point, one may notice the similarity to Galois connections. A Galois connection between two preorders $\left(\leq_{L}\right)$ and $\left(\leq_{R}\right)$ consists of two functions $l$ and $r$ such that

- $l$ is monotone, that is $x_{1} \leq_{L} x_{2} \longrightarrow l x_{1} \leq_{R} l x_{2}$ for all $x_{1}, x_{2}$,
- $r$ is monotone, that is $y_{1} \leq_{R} y_{2} \longrightarrow r y_{1} \leq_{L} r y_{2}$ for all $y_{1}, y_{2}$, and
- $x \leq_{L} r(l x)$ and $l(r y) \leq_{R} y$ for all $x, y .{ }^{6}$

The final conditions correspond to Requirements (a) and (b) above, while the monotonicity conditions on $l$ and $r$ correspond to Expectation (4).

Other Motivations A second motivation to base our framework on Galois connections comes from category theory. There, an equivalence between two categories $L, R$ is given by two functors $l: L \rightarrow R$ and $r: R \rightarrow L$ and two natural isomorphisms $\eta: I d_{L} \rightarrow r \circ l$ and $\epsilon: l \circ r \rightarrow I d_{R}$. Applied to preorders $\left(\leq_{L}\right),\left(\leq_{R}\right)$ and monotone functions $l, r$, this translates to the four conditions
(a) $x \leq_{L} r(l x)$,
(b) $l(r y) \leq_{R} y$,
(c) $r(l x) \leq_{L} x$,
(d) $y \leq_{R} l(r y)$.

A related categorical concept is that of an adjunction. When applied to preorders and monotone functions, an adjunction is similar to an equivalence but is only required to satisfy Conditions (a) and (b). In fact, while Galois connections are not categorical equivalences, they are adjunctions. From this perspective, a Galois connection can be seen as a weak form of an (order) equivalence.

A final motivation is the applicability and wide-spread use of Galois connections. They are fundamental in the closely related field of abstract interpretation [5,7], where they are used to relate concrete to abstract domains. Moreover, they are pervasive throughout mathematics. In the words of Saunders Mac Lane:

The slogan is "Adjoint functors arise everywhere".
(Categories for the Working Mathematician)
We hope our exposition convinced the reader that Galois connections are a suitable notion to cover the essence of transport. The remaining challenges are

- to bring the notion of Galois connections to a partial world - the relations $L, R$ may only be defined on a subset of $\alpha, \beta$ - and
- to check the closure properties of our definitions under common relators.


## 3 Partial Galois Connections, Equivalences, and Relators

In the previous section, we singled out Galois connections as a promising candidate for Transport. Now we want to bring our ideas to the formal world of proof assistants. In this section, we introduce the required background theory for this endeavour. In the following, we fix two relations $L: \alpha \Rightarrow \alpha \Rightarrow$ bool, $R: \beta \Rightarrow \beta \Rightarrow$ bool and two functions $l: \alpha \Rightarrow \beta, r: \beta \Rightarrow \alpha$.

[^5]
## 3.1 (Order) Basics

We work in a polymorphic, simple type theory [3], as employed, for example, in Isabelle/HOL [24]. In particular, our formalisation uses function extensionality. We assume basic familiarity with Isabelle's syntax. Here, we only recap the most important concepts for our work. A complete list of definitions can be found in Appendix A.1.

A predicate on a type $\alpha$ is a function of type $\alpha \Rightarrow$ bool. A relation on $\alpha$ and $\beta$ is a function of type $\alpha \Rightarrow \beta \Rightarrow$ bool. Composition of two relations $R, S$ is defined as $(R \circ S) x y:=\exists z . R x z \wedge S z y$. A relation $R$ is finer than a relation $S$, written $R \leq S$, if $\forall x y \cdot R x y \longrightarrow S x y$. It will be convenient to interpret relations as infix operators. For every relation $R$, we hence introduce an infix operator $\left(\leq_{R}\right):=R$, that is $x \leq_{R} y \longleftrightarrow R x y$. We also write $\left(\geq_{R}\right):=\left(\leq_{R}\right)^{-1}$. The field predicate on a relation is defined as in_field $R x:=$ in_dom $R x \vee$ in_codom $R x$.

We use relativised versions of well-known order-theoretic concepts. For example, given a predicate $P$, we define reflexivity on $P$ and $R$ as reflexive_on $P R:=$ $\forall x . P x \longrightarrow R x x$. We proceed analogously for other standard order-theoretic concepts, such as transitivity, preorders, etc. (see Appendix A.1).

### 3.2 Function Relators and Monotonicity

We introduce a generalisation of the well-known function relator (see e.g. [25]). The slogan of the function relator is "related functions map related inputs to related outputs". Our generalisation - the dependent function relator - additionally allows its target relation to depend on both inputs:

$$
\begin{equation*}
([x y:: R] \Rightarrow S) f g:=\forall x y \cdot R x y \longrightarrow S(f x)(g y), \tag{10}
\end{equation*}
$$

where $x, y$ may occur freely in $S$. The well-known (non-dependent) function relator is given as a special case: $(R \Rightarrow S):=\left(\left[\__{~}:: R\right] \Rightarrow S\right)$. A function is monotone from $R$ to $S$ if it maps $R$-related inputs to $S$-related outputs:

$$
\begin{equation*}
\left([x y:: R] \Rightarrow_{\mathrm{m}} S\right) f:=([x y:: R] \Rightarrow S) f f, \tag{11}
\end{equation*}
$$

where $x, y$ may occur freely in $S$. A monotone function relator is like a function relator but additionally requires its members to be monotone:

$$
\begin{align*}
\left([x y:: R] \Rightarrow^{\oplus} S\right) f g:= & ([x y:: R] \Rightarrow S) f g \\
& \wedge\left([x y:: R] \Rightarrow_{\mathrm{m}} S\right) f \wedge\left([x y:: R] \Rightarrow_{\mathrm{m}} S\right) g, \tag{12}
\end{align*}
$$

where $x, y$ may occur freely in $S$. In some examples, we have to include conditionals in our relators. For this, we define the relational if conditional rel_if B S xy := $B \longrightarrow S x y$ and set the following notation:

$$
\begin{equation*}
([x y:: R \mid B] \Rightarrow S):=([x y:: R] \Rightarrow \text { rel_if } B S), \tag{13}
\end{equation*}
$$

where $x, y$ may occur freely in $B, S$.

### 3.3 Galois Relator

In Expectation (6) of Section 2, we noted that "similar" terms $t, t^{\prime}$ are related with each other's transports, i.e. $L t\left(r t^{\prime}\right)$ and $R(l t) t^{\prime}$. We now define this relation formally, calling it the Galois relator:

$$
\begin{equation*}
\text { Galois }\left(\leq_{L}\right)\left(\leq_{R}\right) r x y:=\text { in_codom }\left(\leq_{R}\right) y \wedge x \leq_{L} r y \tag{14}
\end{equation*}
$$

When the parameters are clear from the context, we will use the infix notation $\left(L_{L} \lesssim\right):=$ Galois $\left(\leq_{L}\right)\left(\leq_{R}\right) r$. It is easy to show that Galois relators generalise the transport relations of partial quotient types:

Lemma 1. For every partial quotient type ( $T, l, r$ ) with induced left relation $\left(\leq_{L}\right)$, we have $T=$ Galois $\left(\leq_{L}\right)(=) r$.

### 3.4 Partial Galois Connections and Equivalences

In their standard form, Galois connections are defined on preorders $\left(\leq_{L}\right),\left(\leq_{R}\right)$, where every $x: \alpha$ is in the domain of $\left(\leq_{L}\right)$ and every $y: \beta$ is in the domain of $\left(\leq_{R}\right)$. But as we have seen, this is not generally the case when transporting terms.

We hence lift the notion of Galois connections to a partial setting. We also do not assume any order axioms on $\left(\leq_{L}\right),\left(\leq_{R}\right)$ a priori but add them as needed. In our formalisation, we moreover break the concept of Galois connections down into smaller pieces that, to our knowledge, do not appear as such in the literature. This allows us to obtain very precise results when deriving the closure properties for our definitions (Section 4). But for reasons of brevity, we only state the main definitions and results here. Details can be found in Appendix A.4.

The (partial) Galois property is defined as:

$$
\begin{align*}
\left(\left(\leq_{L}\right) \unlhd\left(\leq_{R}\right)\right) l r:= & \forall x y . \text { in_dom }\left(\leq_{L}\right) x \wedge \text { in_codom }\left(\leq_{R}\right) y \longrightarrow \\
& \left(x \leq_{L} r y \longleftrightarrow l x \leq_{R} y\right) . \tag{15}
\end{align*}
$$

If $l$ and $r$ are also monotone, we obtain a (partial) Galois connection:

$$
\begin{align*}
\left(\left(\leq_{L}\right) \dashv\left(\leq_{R}\right)\right) l r:= & \left(\left(\leq_{L}\right) \unlhd\left(\leq_{R}\right)\right) l r  \tag{16}\\
& \wedge\left(\left(\leq_{L}\right) \Rightarrow_{\mathrm{m}}\left(\leq_{R}\right)\right) l \wedge\left(\left(\leq_{R}\right) \Rightarrow_{\mathrm{m}}\left(\leq_{L}\right)\right) r
\end{align*}
$$

We omit the qualifier "partial" when referring to these definitions, unless we want to avoid ambiguity. An example Galois connection can be found in Fig. 2(a).

As mentioned in Section 2, Galois connections can be seen as a weak form of an equivalence. Unfortunately, they are not in general closed under compositions (cf. Section 4.3), where we need a stronger form of an equivalence. We can obtain a suitable strengthening by requiring a two-sided Galois connection, which we call a (partial) Galois equivalence:

$$
\begin{equation*}
\left(\left(\leq_{L}\right) \equiv_{\mathrm{G}}\left(\leq_{R}\right)\right) l r:=\left(\left(\leq_{L}\right) \dashv\left(\leq_{R}\right)\right) l r \wedge\left(\left(\leq_{R}\right) \dashv\left(\leq_{L}\right)\right) r l \tag{17}
\end{equation*}
$$


(a) A partial Galois connection. Note that unlike in Fig. 1, the relations may not decompose into equivalence classes.

(b) A partial Galois equivalence. The relations decompose into "strongly connected components", drawn as dotted circles. Any two members in such a component are connected. These arrows are omitted.

Fig. 2: Examples of partial equivalences as defined in (16),(17). Types are drawn solid, black, transport functions dashed, and left and right relations dotted.

An example of a Galois equivalence can be found in Fig. 2(b). It can be shown that Galois equivalences are, under mild conditions, equivalent to the traditional notion of (partial) order equivalences (see Appendix A.4).

In practice, the relations $\left(\leq_{L}\right),\left(\leq_{R}\right)$ are often preorders or partial equivalence relations (PERs). Given some $\left(\left(\leq_{L}\right) \equiv_{\mathrm{G}}\left(\leq_{R}\right)\right) l r$, we hence introduce the notations $\left(\left(\leq_{L}\right) \equiv_{\text {pre }}\left(\leq_{R}\right)\right) l r$ and $\left(\left(\leq_{L}\right) \equiv_{\text {PER }}\left(\leq_{R}\right)\right) l r$ in case both relations $\left(\leq_{L}\right),\left(\leq_{R}\right)$ are preorders and PERs on their domain, respectively. It is easy to show that Galois equivalences generalise partial quotient types:

Lemma 2. ( $T, l, r$ ) is a partial quotient type with induced left relation $\left(\leq_{L}\right)$ if and only if $\left(\left(\leq_{L}\right) \equiv\right.$ PER $\left.(=)\right) l r$.

## 4 Closure Properties

We now explore the closure properties of partial Galois connections and equivalences, as well as standard order properties, such as reflexivity and transitivity. We will derive closure conditions for the dependent function relator, relators for (co)datatypes, and composition. In each case, we will also derive conditions under which the Galois relator aligns with the context-dependent notion of "similarity".

For reasons of brevity, we only show that our framework is robust under Galois equivalences on preorders and PERs here. The results for Galois connections (and proof sketches) can be found in Appendix B.1.

## 4.1 (Dependent) Function Relator

In the field of abstract interpretation, it is well-known that Galois connections, as usually defined in the literature, are closed under the non-dependent, monotone function relator (see for example [7]). We generalise this result to partial Galois connections and to dependent function relators.

Remark 1. The relations and functions we use are often non-dependent in practice. The following definitions and theorems are considerably simpler in this case. The reader hence might find instructive to first consult the results for this special case in Appendix B.1.

The Setup In Section 1, we highlighted the need of inter-argument dependencies when transporting functions. For example, we may only transport the index operator (!!) : $\alpha$ list $\Rightarrow \mathbb{N} \Rightarrow \alpha$ if a given index is not out of bounds for a given list. We can realise such dependencies with the help of the dependent function relator from Section 3.2. For this, we fix the following variables:

$$
\begin{array}{ll}
L_{1}: \alpha_{1} \Rightarrow \alpha_{1} \Rightarrow \text { bool }, & l_{1}: \alpha_{1} \Rightarrow \alpha_{2}, \\
R_{1}: \alpha_{2} \Rightarrow \alpha_{2} \Rightarrow \text { bool, } & r_{1}: \alpha_{2} \Rightarrow \alpha_{1}, \\
L_{2}: \alpha_{1} \Rightarrow \alpha_{1} \Rightarrow \beta_{1} \Rightarrow \beta_{1} \Rightarrow \text { bool, } & l_{2}: \alpha_{2} \Rightarrow \alpha_{1} \Rightarrow \beta_{1} \Rightarrow \beta_{2}, \\
R_{2}: \alpha_{2} \Rightarrow \alpha_{2} \Rightarrow \beta_{2} \Rightarrow \beta_{2} \Rightarrow \text { bool, } & r_{2}: \alpha_{1} \Rightarrow \alpha_{2} \Rightarrow \beta_{2} \Rightarrow \beta_{1} .
\end{array}
$$

Each variable $L_{2}, R_{2}, l_{2}, r_{2}$ takes parameters from $\alpha_{1}, \alpha_{2}$. These parameters enable the expression of inter-argument dependencies (cf. Section 5, Example 2). We hence call $L_{2}, R_{2}, l_{2}, r_{2}$ the dependent variables. Intuitively, we are in a situation where
(1) we are given an equivalence between $\left(\leq_{L_{1}}\right)$ and $\left(\leq_{R_{1}}\right)$, using $l_{1}$ and $r_{1}$,
(2) whenever $x_{L_{1}} \lesssim x^{\prime}$, we are given an equivalence between $\left(\leq_{L_{2} x\left(r_{1} x^{\prime}\right)}\right)$ and $\left(\leq_{R_{1}\left(l_{1} x\right) x^{\prime}}\right)$, using the transport functions $l_{2} x^{\prime} x$ and $r_{2} x x^{\prime}$, and
(3) we want to construct an equivalence for functions between

$$
\left(\left[x_{1} x_{2}::\left(\leq_{L_{1}}\right)\right] \Rightarrow \Rightarrow^{\oplus}\left(\leq_{L_{2} x_{1} x_{2}}\right)\right) \text { and }\left(\left[x_{1}^{\prime} x_{2}^{\prime}::\left(\leq_{R_{1}}\right)\right] \Rightarrow^{\oplus}\left(\leq_{R_{2} x_{1}^{\prime} x_{2}^{\prime}}\right)\right)
$$

To define suitable transport functions, we use the dependent function mapper:

$$
\begin{equation*}
([x:: f] \rightarrow g) h x:=g(f x)(h(f x)), \tag{18}
\end{equation*}
$$

where $x$ may occur freely in $g$. We can now define the target relations and transport functions:

$$
\begin{array}{ll}
L:=\left(\left[x_{1} x_{2}::\left(\leq_{L_{1}}\right)\right] \Rightarrow^{\oplus}\left(\leq_{L_{2} x_{1} x_{2}}\right)\right), \quad l:=\left(\left[x^{\prime}:: r_{1}\right] \rightarrow l_{2} x^{\prime}\right), \\
R:=\left(\left[x_{1}^{\prime} x_{2}^{\prime}::\left(\leq_{R_{1}}\right)\right] \Rightarrow^{\oplus}\left(\leq_{R_{2} x_{1}^{\prime} x_{2}^{\prime}}\right)\right), \quad r:=\left(\left[x:: l_{1}\right] \rightarrow r_{2} x\right) . \tag{19}
\end{array}
$$

In particular, $l f x^{\prime}=l_{2} x^{\prime}\left(r_{1} x^{\prime}\right)\left(f\left(r_{1} x^{\prime}\right)\right)$ and $r g x=r_{2} x\left(l_{1} x\right)\left(g\left(l_{1} x\right)\right)$.

Closure Theorems Checking the closure of order-theoretic concepts, such as reflexivity, transitivity, and symmetry, is fairly straightforward. Verifying the closure of Galois connections and equivalences, however, is nuanced, requiring careful alignment of the dependent variables' parameters. These alignments require the following monotonicity conditions, which, broadly speaking, say that (1) $L_{2}, R_{2}$ are antimonotone in their first and monotone in their second parameter, and (2) $l_{2}, r_{2}$ are monotone in both parameters:
(i) If $x_{1} \leq_{L_{1}} x_{2} \leq_{L_{1}} x_{3} \leq_{L_{1}} x_{4}$ then $\left(\leq_{L_{2} x_{2} x_{3}}\right) \leq\left(\leq_{L_{2} x_{1} x_{4}}\right)$.
(ii) If $x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime} \leq_{R_{1}} x_{3}^{\prime} \leq_{R_{1}} x_{4}^{\prime}$ then $\left(\leq_{R_{2} x_{2}^{\prime} x_{3}^{\prime}}\right) \leq\left(\leq_{R_{2} x_{1}^{\prime} x_{4}^{\prime}}\right)$.
(iii) If $x_{1} \leq_{L_{1}} x_{2 L_{1}} \lesssim x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime}$ and in_field $\left(\leq_{L_{2} x_{1}\left(r_{1} x_{2}^{\prime}\right)}\right) y$ then $\left(l_{2} x_{1}^{\prime} x_{1} y\right) \leq_{R_{2}\left(l_{1} x_{1}\right) x_{2}^{\prime}}\left(l_{2} x_{2}^{\prime} x_{2} y\right)$.
(iv) If $x_{1} \leq_{L_{1}} x_{2 L_{1}} \lesssim x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime}$ and in_field $\left(\leq_{R_{2}\left(l_{1} x_{1}\right)} x_{2}^{\prime}\right) y^{\prime}$ then $\left(r_{2} x_{1} x_{1}^{\prime} y^{\prime}\right) \leq_{L_{2} x_{1}\left(r_{1} x_{2}^{\prime}\right)}\left(r_{2} x_{2} x_{2}^{\prime} y^{\prime}\right)$.
We are now ready to state our main result for Galois equivalences on preorders and PERs. The result for Galois connections (and a proof sketch) can be found in Appendix B.1. All other results can be found in our formalisation.
Theorem 1. Let $\star \in\left\{\equiv_{\text {pre }}, \equiv_{\text {PER }}\right\}$ and assume
(1) $\left(\left(\leq_{L_{1}}\right) \star\left(\leq_{R_{1}}\right)\right) l r$,
(2) if $x_{L_{1}} \lesssim x^{\prime}$ then $\left(\left(\leq_{L_{2} x\left(r_{1} x^{\prime}\right)}\right) \star\left(\leq_{R_{2}\left(l_{1} x\right) x^{\prime}}\right)\right)\left(l_{2} x^{\prime} x\right)\left(r_{2} x x^{\prime}\right)$,
(3) Monotonicity Conditions (i)-(iv).

Then $\left(\left(\leq_{L}\right) \star\left(\leq_{R}\right)\right)$ lr.
"Similarity" Given the closure theorem, we can readily transport a function $f$ from $\left(\leq_{L}\right)$ to a function $g$ in $\left(\leq_{R}\right)$. Due to Expectations (4) and (6), we also know that $f L \lesssim g$, that is $\left(\left[x_{1} x_{2}::\left(\leq_{L_{1}}\right)\right] \Rightarrow^{\oplus}\left(\leq_{L_{2} x_{1} x_{2}}\right)\right) f(r g)$ and $\left(\left[x_{1}^{\prime} x_{2}^{\prime}::\left(\leq_{R_{1}}\right)\right] \Rightarrow{ }^{\oplus}\left(\leq_{R_{2} x_{1}^{\prime} x_{2}^{\prime}}\right)\right)(l f) r$. But arguably, this is not quite enough:

Remember the slogan of the function relator: "related functions map related inputs to related outputs". We know how to relate terms between $\left(\leq_{L_{1}}\right)$ and $\left(\leq_{R_{1}}\right)$ : we can use $\left(L_{1} \lesssim\right)$. Whenever $x_{L_{1}} \lesssim x^{\prime}$, we also know how to relate terms between $\left(\leq_{L_{2} x\left(r_{1} x^{\prime}\right)}\right)$ and $\left(\leq_{R_{2}\left(l_{1} x\right) x^{\prime}}\right)$ : we can use

$$
\begin{equation*}
\left(L_{2} x x^{\prime} \lesssim\right):=\text { Galois }\left(\leq_{L_{2} x\left(r_{1} x^{\prime}\right)}\right)\left(\leq_{R_{2}\left(l_{1} x\right) x^{\prime}}\right)\left(r_{2} x x^{\prime}\right) . \tag{20}
\end{equation*}
$$

So when we say that " $f$ and $g$ are similar", we may actually desire that

$$
\begin{equation*}
\left(\left[x x^{\prime}::\left(L_{1} \lesssim\right)\right] \Rightarrow\left(L_{2} x x^{\prime} \lesssim\right)\right) f g . \tag{21}
\end{equation*}
$$

The following theorem answers when $(L \lesssim)$ aligns with this definition of similarity for preordered Galois equivalences. Preciser results can be found in Appendix B. 1 and the formalisation.
Theorem 2. Assume
(1) $\left(\left(\leq_{L_{1}}\right) \equiv_{\text {pre }}\left(\leq_{R_{1}}\right)\right) l_{1} r_{1}$,
(2) if $x_{L_{1}} \lesssim x^{\prime}$ then $\left(\left(\leq_{L_{2} x\left(r_{1} x^{\prime}\right)}\right) \equiv_{\text {pre }}\left(\leq_{R_{2}\left(l_{1} x\right) x^{\prime}}\right)\right)\left(l_{2} x^{\prime} x\right)\left(r_{2} x x^{\prime}\right)$,
(3) Monotonicity Conditions (i) and (iv),
(4) in_dom $\left(\leq_{L}\right) f$, and in_codom $\left(\leq_{R}\right) g$.

Then $f_{L} \lesssim g \longleftrightarrow\left(\left[x x^{\prime}::\left(L_{1} \lesssim\right)\right] \Rightarrow\left(L_{2} x x^{\prime} \lesssim\right)\right) f g$.

## 4.2 (Co)datatypes

Different proof assistants ground (co)datatypes in different ways. For instance, Coq and Lean introduce them axiomatically, whereas Isabelle/HOL proves their existence using the theory of bounded natural functors [30]. As our formalisation takes place in Isabelle/HOL, we use the latter theory. Nonetheless, the results presented in this section are relatively straightforward and can likely be adapted to other "reasonable" definitions of (co)datatypes.

In this section, we derive closure properties for arbitrary natural functors. A natural functor is a bounded natural functor without cardinality constraints. The exact axioms can be found elsewhere [30]. For our purposes, it suffices to say that natural functors are equipped with a mapper and a relator. More precisely, for every $n$-ary natural functor $\left(\alpha_{1}, \ldots, \alpha_{n}\right) F$, there are two functions:

$$
\begin{aligned}
\operatorname{map}_{F} & :\left(\alpha_{1} \Rightarrow \beta_{1}\right) \Rightarrow \cdots \Rightarrow\left(\alpha_{n} \Rightarrow \beta_{n}\right) \Rightarrow\left(\alpha_{1}, \ldots, \alpha_{n}\right) F \Rightarrow\left(\beta_{1}, \ldots, \beta_{n}\right) F \\
\operatorname{rel}_{F}: & \left(\alpha_{1} \Rightarrow \beta_{1} \Rightarrow \text { bool }\right) \Rightarrow \cdots \Rightarrow\left(\alpha_{n} \Rightarrow \beta_{n} \Rightarrow \text { bool }\right) \Rightarrow \\
& \left(\alpha_{1}, \ldots, \alpha_{n}\right) F \Rightarrow\left(\beta_{1}, \ldots, \beta_{n}\right) F \Rightarrow \text { bool }
\end{aligned}
$$

The former lifts functions on the functor's type arguments to the functorial structure, the latter lifts relations on the functor's type arguments to the functorial structure. Using the mapper and relator, it is straightforward to define appropriate target relations and transport functions. First we fix the following variables for $1 \leq i \leq n$ :

$$
L_{i}: \alpha_{i} \Rightarrow \alpha_{i} \Rightarrow \text { bool, } \quad l_{i}: \alpha_{i} \Rightarrow \beta_{i}, \quad R_{i}: \beta_{i} \Rightarrow \beta_{i} \Rightarrow \text { bool, } \quad r_{i}: \beta_{i} \Rightarrow \alpha_{i}
$$

Then we define the new target relations and transport functions as follows:

$$
\begin{array}{ll}
L:=\operatorname{rel}_{F}\left(\leq_{L_{1}}\right) \ldots\left(\leq_{L_{n}}\right), & l:=\operatorname{map}_{F} l_{1} \ldots l_{n} \\
R:=\operatorname{rel}_{F}\left(\leq_{R_{1}}\right) \ldots\left(\leq_{R_{n}}\right), & r:=\operatorname{map}_{F} r_{1} \ldots r_{n} \tag{22}
\end{array}
$$

The closure properties follow without any difficulty:
Theorem 3. Let $\star \in\left\{\dashv_{,} \equiv_{\mathrm{G}}, \equiv_{\mathrm{pre}}, \equiv \mathrm{PER}\right\}$ and assume $\left(\left(\leq_{L_{i}}\right) \star\left(\leq_{R_{i}}\right)\right) l_{i} r_{i}$ for $1 \leq i \leq n$. Then $\left(\left(\leq_{L}\right) \star\left(\leq_{R}\right)\right) l r$.

As in the previous section, we can ponder whether the relation ( $L_{\lesssim}$ ) adequately captures our desired notion of "similarity". Again, we already know how to relate terms between $\left(\leq_{L_{i}}\right)$ and $\left(\leq_{R_{i}}\right)$ for $1 \leq i \leq n$ : we can use $\left(L_{i} \lesssim\right)$. We also know how to relate two functors: we can use rel ${ }_{F}$. We thus may desire that " $t$ and $t^{\prime}$ are similar" when $\operatorname{rel}_{F}\left(L_{1} \lesssim\right) \ldots\left(L_{n} \gtrsim\right) t t^{\prime}$. It is easy to show that $\left(L_{\gtrsim} \lesssim\right)$ aligns with this desire:

Theorem 4. $\left(L^{\lesssim} \lesssim\right)=\operatorname{rel}_{F}\left(L_{1} \gtrsim\right) \ldots\left(L_{n} \lesssim\right)$.
Proof details for this section can be found in our formalisation. The formalisation includes tactic scripts that are applicable to functors of arbitrary arity. Integrating them into Isabelle/HOL's datatype package is left as future work.

### 4.3 Compositions

It is well-known that Galois connections, as defined in the literature, are closed under composition in the following sense: given Galois connections between $\left(\leq_{L_{1}}\right),\left(\leq_{R_{1}}\right)$ and $\left(\leq_{L_{2}}\right),\left(\leq_{R_{2}}\right)$ with $\left(\leq_{R_{1}}\right)=\left(\leq_{L_{2}}\right)$, we can build a Galois connection between $\left(\leq_{L_{1}}\right),\left(\leq_{R_{2}}\right)$. This result readily generalises to our partial setting (see Appendix B.2). However, $\left(\leq_{R_{1}}\right)$ and $\left(\leq_{L_{2}}\right)$ usually do not coincide in our context. We need a more general result.

The Setup Our goal is to define a notion of composition that works even if $\left(\leq_{R_{1}}\right)$ and ( $\leq_{L_{2}}$ ) do not coincide. For this, we fix the variables

$$
\begin{aligned}
& L_{1}: \alpha \Rightarrow \alpha \Rightarrow \text { bool, } \quad l_{1}: \alpha \Rightarrow \beta, \quad R_{1}: \beta \Rightarrow \beta \Rightarrow \text { bool }, \quad r_{1}: \beta \Rightarrow \alpha, \\
& L_{2}: \beta \Rightarrow \beta \Rightarrow \text { bool, } \quad l_{2}: \beta \Rightarrow \gamma, \quad R_{2}: \gamma \Rightarrow \gamma \Rightarrow \text { bool, } \quad r_{2}: \gamma \Rightarrow \beta .
\end{aligned}
$$

Intuitively, we are in a situation where
(1) we are given an equivalence between $\left(\leq_{L_{1}}\right)$ and $\left(\leq_{R_{1}}\right)$, using $l_{1}$ and $r_{1}$,
(2) we are given an equivalence between $\left(\leq_{L_{2}}\right)$ and $\left(\leq_{R_{2}}\right)$, using $l_{2}$ and $r_{2}$, and
(3) we want to construct an equivalence with transport functions $l_{2} \circ l_{1}$ and $r_{1} \circ r_{2}$ between those parts of $\left(\leq_{L_{1}}\right)$ and $\left(\leq_{R_{2}}\right)$ that can be made "compatible" with respect to these functions. This particularly means that we can apply the transport functions on these parts without leaving the domains of the input equivalences.

The question is: how do we find those parts and how can we make them compatible? The solution we propose is inspired by and generalises the approach of Huffman and Kunčar [13]. We provide details and intuitions for the constructions in Appendix B.2. The resulting target relations and transport functions are defined as follows (where $\left(R_{i} \lesssim\right):=$ Galois $\left.\left(\leq_{R_{i}}\right)\left(\leq_{L_{i}}\right) l_{i}\right)$ :

$$
\begin{array}{ll}
L:=\left(L_{1} \lesssim\right) \circ\left(\leq_{L_{2}}\right) \circ\left(R_{1} \gtrsim\right), & l:=l_{2} \circ l_{1}  \tag{23}\\
R:=\left(R_{2} \gtrsim\right) \circ\left(\leq_{R_{1}}\right) \circ\left(L_{2} \gtrsim\right), & r:=r_{1} \circ r_{2}
\end{array}
$$

Closure Theorems Again, we only state our main result for Galois equivalences on preorders and PERs. Preciser results can be found in Appendix B. 2 (including a proof sketch) and in our formalisation.
Theorem 5. Let $\star \in\left\{\equiv_{\text {pre }}, \equiv_{\operatorname{PER}}\right\}$ and assume
(1) $\forall i \in\{1,2\} .\left(\left(\leq_{L_{i}}\right) \star\left(\leq_{R_{i}}\right)\right) l_{i} r_{i}, \quad$ (2) $\left(\left(\leq_{R_{1}}\right) \circ\left(\leq_{L_{2}}\right)\right)=\left(\left(\leq_{L_{2}}\right) \circ\left(\leq_{R_{1}}\right)\right)$. Then $\left(\left(\leq_{L}\right) \star\left(\leq_{R}\right)\right) l r$.
"Similarity" For a final time, we can ponder whether the relation $\left(L_{\lesssim}\right)$ is sufficient to capture our desired notion of "similarity": Again, we already know how to relate terms between $\left(\leq_{L_{i}}\right)$ and $\left(\leq_{R_{i}}\right)$ for $i \in\{1,2\}$ : we can use $\left(L_{i} \lesssim\right)$. We also have a natural way to combine these relations, namely composition. We thus may desire that " $t$ and $t^{\prime}$ are similar" when $\left(\left(L_{1} \lesssim\right) \circ\left(L_{2} \lesssim\right)\right) t t^{\prime}$. The next theorem answers when $(L \lesssim)$ aligns with this desire for Galois equivalences. Preciser results can be found in Appendix B. 2 and the formalisation.

Theorem 6. Assume
(1) $\forall i \in\{1,2\} .\left(\left(\leq_{L_{i}}\right) \equiv_{\text {pre }}\left(\leq_{R_{i}}\right)\right) l_{i} r_{i}$, (2) $\left(\left(\leq_{R_{1}}\right) \circ\left(\leq_{L_{2}}\right)\right)=\left(\left(\leq_{L_{2}}\right) \circ\left(\leq_{R_{1}}\right)\right)$. Then $\left(L_{\gtrsim}\right)=\left(\left(L_{1} \lesssim\right) \circ\left(L_{2} \gtrsim\right)\right)$.

## 5 Application Examples

As all our results are formalised in Isabelle/HOL, we can directly use them to manually transport terms in said environment. But that would be rather tiresome. We thus implemented a prototype in Isabelle/ML to automate transports.

The Prototype The method trprover uses registered base equivalences, along with the closure theorems from Section 4, to construct more complex equivalences. The prototype is currently restricted to equivalences on partial equivalence relations (PERs) for pragmatic reasons: their closure theorems have fewer assumptions and are hence simpler to apply. Providing automation for weaker equivalences is future work. The current prototype also does not build composition closures (Section 4.3) and automates only a fragment of dependent function relators for simplicity reasons. Again, these extensions are future work.

The prototype provides a command trp. As input, it takes a term $t: \alpha$ (the term to be transported) and two optional target relations $L: \alpha \Rightarrow \alpha \Rightarrow$ bool, $R: \beta \Rightarrow \beta \Rightarrow$ bool. This is unlike other transport frameworks $[9,13,26,29]$, which only take the term $t: \alpha$ and a target type $\beta$. This design decision is crucial since we can neither assume a unique correspondence between types and target relations in practice (cf. Example 3), nor can we express dependencies in types, but we express them using dependent relators (cf. Example 2). The command then opens two goals. The first one asks for an equivalence $\left(\left(\leq_{L}\right) \equiv_{\text {PER }}\left(\leq_{R}\right)\right) l r$, the second one for a proof that in_dom $\left(\leq_{L}\right) t$. On success, it registers a new term $t^{\prime}$ and a theorem that $t_{L} \lesssim t^{\prime}$. It also registers a second theorem where the relator $(L \lesssim)$ has been rewritten to its desired form as described in Theorems 2,4 , and 6 .

The following examples are best explored interactively in our formalisation. We define the restricted equality relation on predicates as $x=_{P} y:=P x \wedge x=y$ and the restricted equality relation on sets as $x=_{S} y:=x \in S \wedge x=y$.

Example 1. It is easy to transport the list and set examples from Section 1. We just have to prove the equivalence between $\operatorname{LFS}_{\mathrm{L}} x s x s^{\prime}:=\mathrm{LFS} x s\left(\right.$ to_fset $\left.x s^{\prime}\right)$ and $(=): \mathbb{N}$ fset $\Rightarrow \mathbb{N}$ fset $\Rightarrow$ bool and invoke our prototype on max_list:

$$
\begin{aligned}
& \text { lemma [per_intro]: }\left(\mathrm{LFS}_{L} \equiv \text { } \text { PER }(=)\right) \text { to_fset to_list fin } \\
& \text { trp max_fset : } \mathbb{N} \text { fset } \Rightarrow \mathbb{N} \text { where } t=\text { max_list by trprover }
\end{aligned}
$$

The [per_intro] tag is used by trprover to discharge the closure theorems' side conditions. $\operatorname{trp}$ registers the theorem (LFS $\Rightarrow(=)$ ) max_list max_fset and the definition max_fset $s:=$ max_list (to_listin $s$ ) as a result. We can also readily transport in the opposite direction or use sets rather than fsets if we define

LS $\left\llcorner x s x s^{\prime}:=\right.$ LS $x s$ (to_set $x s^{\prime}$ ):

$$
\begin{aligned}
& \operatorname{trp} \max \text { list }: \mathbb{N} \text { list } \Rightarrow \mathbb{N} \text { where } t=\text { max_fset by trprover } \\
& \text { lemma }\left[\text { per_intro }:\left(\text { LS }_{\mathrm{L}} \equiv \text { PER }(=\text { finite })\right)\right. \text { to set to_list } \\
& \text { trp max_set }: \mathbb{N} \text { set } \Rightarrow \mathbb{N} \text { where }=\text { max_list by trprover }
\end{aligned}
$$

Example 2. As noted in Section 1, transporting subtractions $i_{1}-\mathbb{Z} i_{2}$ from $\mathbb{Z}$ to $\mathbb{N}$ requires a dependency $i_{1} \geq i_{2}$. We model this dependency using dependent function relators. We first define $\mathrm{Zpos}:=\left(=_{(\leq) 0}\right)$ and then proceed as usual:

```
lemma [per_intro]: \((Z \operatorname{pos} \equiv\) PER ( \(=)\) ) to_nat to_int
\(\operatorname{trp}(-\mathbb{N}): \mathbb{N} \Rightarrow \mathbb{N} \Rightarrow \mathbb{N}\) where \(\mathrm{t}=(-\mathbb{Z})\)
    and \(\mathrm{L}=\left(\left[i_{1} \_::\right.\right.\)Zpos \(] \Rightarrow\left[i_{2} \_: Z \mathrm{Zpos} \mid i_{1} \geq i_{2}\right] \Rightarrow\) Zpos \()\)
    and \(\mathrm{R}=\left(\left[n_{1} \_::(=)\right] \Rightarrow\left[n_{2} \quad::(=) \mid n_{1} \geq n_{2}\right] \Rightarrow(=)\right)\) by trprover
```

Similarly, operations on datatypes may only conditionally be transportable. For example, we may only transport the index operator (!!) : $\alpha$ list $\Rightarrow \mathbb{N} \Rightarrow \alpha$ to the type of immutable arrays ( $\alpha$ iarray) if the index is not out of bounds. In the following, let $S$ be an arbitrary partial equivalence relation:

```
lemma [per_intro]: \((\operatorname{ListRel} S \equiv\) PER IArrRel \(S)\) to_iarr to_list
trp iarr_ind : \(\alpha\) iarray \(\Rightarrow \mathbb{N} \Rightarrow \alpha\) where \(\mathrm{t}=(!!)\)
    and \(\overline{\mathrm{L}}=\left(\left[x s{ }_{-}:: \operatorname{ListRel} S\right] \Rightarrow\left[i_{-}::(=) \mid i<\right.\right.\) length \(\left.\left.x s\right] \Rightarrow S\right)\)
    and \(\mathrm{R}=\left([\right.\) arr_ \(\quad::\) IArrRel \(S] \Rightarrow\left[\bar{i}_{-}::(=) \mid i<\right.\) iarr_length \(\left.\left.\operatorname{arr}\right] \Rightarrow S\right)\)
    by trprover
```

Example 3. Isabelle/Set [14] is a set-theoretic environment in Isabelle/HOL. Its type of sets is called set. Isabelle/Set provides a set-extension mechanism: As input, it takes two sets $A$ : set and $B$ : set and an injection from $A$ to $B$. It then creates a new set $B^{\prime} \supseteq A$ together with a bijection between $B$ and $B^{\prime}$ with mutual inverses $l, r$ : set $\Rightarrow$ set. This mechanism is used to enforce subset relationships. For instance, it first uses a construction of the integers $\mathbb{Z}$ : set where $\mathbb{N} \nsubseteq \mathbb{Z}$. It then uses the set-extension mechanism to create a copy $\mathbb{Z}^{\prime} \supseteq \mathbb{N}$ with inverses $l, r$. Doing so necessitates a manual transport of all definitions from $\mathbb{Z}$ to $\mathbb{Z}^{\prime}$. Using Transport, it is possible to automate this process:

$$
\begin{aligned}
& \text { lemma [per_intro]: }\left(\left(=_{\mathbb{Z}}\right) \equiv \text { PER }\left(=_{\mathbb{Z}^{\prime}}\right)\right) l r \\
& \operatorname{trp}\left(+_{\mathbb{Z}^{\prime}}\right) \text { wheret }=\left(+_{\mathbb{Z}}\right) \text { and } L=\left(\left(=_{\mathbb{Z}}\right) \Rightarrow(=\mathbb{Z}) \Rightarrow(=\mathbb{Z})\right) \\
& \text { and } \mathrm{R}=\left(\left(=_{\mathbb{Z}^{\prime}}\right) \Rightarrow\left(=_{\mathbb{Z}^{\prime}}\right) \Rightarrow\left(=_{\mathbb{Z}^{\prime}}\right)\right) \text { by trprover } \\
& \operatorname{trp}\left(-\mathbb{Z}^{\prime}\right) \text { wheret }=(-\mathbb{Z}) \text { and } \mathrm{L}=\left(\left(=_{\mathbb{Z}}\right) \Rightarrow\left(=_{\mathbb{Z}}\right) \Rightarrow\left(=_{\mathbb{Z}}\right)\right) \\
& \text { and } R=\left(\left(=_{\mathbb{Z}^{\prime}}\right) \Rightarrow\left(=_{\mathbb{Z}^{\prime}}\right) \Rightarrow\left(=_{\mathbb{Z}^{\prime}}\right)\right) \text { by trprover }
\end{aligned}
$$

Note that all constants $\left(+_{\mathbb{Z}}\right),\left(+_{\mathbb{Z}^{\prime}}\right),(-\mathbb{Z}),\left(-_{\mathbb{Z}^{\prime}}\right)$ are of the same type set $\Rightarrow$ set $\Rightarrow$ set. This stresses the point that users must be able to specify target relations and not just target types.

## 6 Related Work

Transport in Proof Assistants Our work was chiefly inspired by Isabelle's Lifting package [13, 17], which transports terms via partial quotient types. All closure theorems in this work generalise the ones in [13]. Besides this source of inspiration, the theory of automated transports has seen prolific work in recent years:

Tabareau et al. [28] proved a strengthened relational parametricity result, called univalent parametricity, for the Calculus of Inductive Constructions. Their approach ensures that all relations are compatible with type equivalences. One can then use univalence [33] to seamlessly transport terms between related types. The framework is implemented using Coq's typeclass mechanism [27].

Tabareau et al. [29] extended their work to integrate what they call "whitebox transports". White-box transports structurally rewrite a term $t$ to $t^{\prime}$ using user-specified correspondences. In contrast, "black-box transports" transport $t$ without looking at its syntactic structure. For instance, given an equivalence between unary and binary numbers $(\mathbb{N} \simeq \operatorname{Bin}) l r$, black-box transporting the term $0+_{\mathbb{N}} 0$ results in $l\left(0+_{\mathbb{N}} 0\right)$. In contrast, given correspondences between the functions $(+)_{\mathbb{N}},(+)_{\text {Bin }}$ and constants $0,0_{\text {Bin }}$, white-box transporting the term results in $0_{\text {Bin }}+$ Bin $0_{\text {Bin }}$. These modes can also be mixed: given just the equivalence $(\mathbb{N} \simeq \operatorname{Bin}) l r$ and correspondence between $(+)_{\mathbb{N}},(+)_{\text {Bin }}$, we obtain $(l 0)+{ }_{\operatorname{Bin}}(l 0)$. Isabelle's Lifting package also supports white-box transports via the transfer method [17]. While our work is concerned with black-box transports, our prototype also contains experimental support for white-box transports. This integration will be further polished in future work.

Angiuli et al. [1] establish representation independence results in Cubical Agda [32]. Their approach applies to a restricted variant of quasi-partial equivalence relations [16]. Essentially, they quotient two types by a given correspondence to obtain a type equivalence between the quotiented types.

Dagand et al. [8,9] introduce what they call "type-theoretic partial Galois connections", which are essentially partial type equivalences on an enriched $\alpha$ option type. They allow for partiality on one side of the equivalence but not the other. Their framework is designed for effective program extraction and implemented using Coq's typeclass mechanism.

Ringer et al. [26] developed a Coq plugin to transport proof terms via type equivalences for inductive types. Their theory shares similarities with [28, 29], but it directly transforms proof terms. This way, one can remove all references to the old datatype once the proof terms have been transported to the new target type. This is not readily achievable using other mentioned frameworks, including ours.

Type equivalences enjoy the property of having total and mutually inverse transport functions. This is not the case for partial Galois connections, which makes the transport of proofs harder. For example, the parametricity law for equality $(T \Rightarrow T \Rightarrow(\longleftrightarrow))(=)(=)$ holds only if $T$ is left-unique and injective. This is the case if $T$ is described by a type equivalence but not in general by a Galois connection. Kunčar [17] provides parametricity rules for all prominent
logical connectives. These rules also apply to our setting and will be crucial when we polish the integration of white-box transports in our prototype.

The works mentioned above all transport terms via certain notions of equivalences. But there are also other approaches, particularly in the field of data refinement. An example is the CoqEAL framework [4], which automatically derives parametricity results using typeclass search. Another one is Isabelle's Autoref framework [18], which derives relational parametricity results using white-box transports. The core inspiration in both cases goes back to [21, 25, 34]. A comprehensive comparison of these frameworks can be found in [19].

Galois Connections in Computer Science Galois connections are fundamental in the field of abstract interpretation. Cousot and Cousot's recent book [5] provides an overview of their applications. The closure of Galois connections under nondependent function relators goes back to at least [6]. We generalised this result to partial Galois connections and dependent function relators in Section 4.1. Most work in abstract interpretation does not consider partially defined Galois connections and assumes partial orderings on relations. The work of Miné [20] is an exception, allowing for partiality on one side of the connection but not the other. Darais and Van Horn [10] formalise Galois connections constructively and apply it to tasks in abstract interpretation. An early application of Galois connections was by Hartmanis and Stearns [12]. Though they did not use Galois connections, they introduced an equivalent notion of pair algebras [11]. Our Galois relator indeed describes the pair algebra induced by a Galois connection.

## 7 Conclusion and Future Work

We explored existing notions of equivalences used for automatic transport. Based on this exploration, we identified a set of minimal expectations when transporting terms via equivalences. This essence led us to introduce a new class of equivalences, namely partial Galois connections. Partial Galois connections generalise (standard) Galois connections and apply to relations that are only defined on subsets of their types. We derived closure conditions for partial Galois connections and equivalences, and typical order properties under (dependent) function relators, relators for (co)datatypes, and composition. Our framework applies to simple type theory and - unlike prior solutions for simple type theory - can handle inter-argument dependencies. We implemented a prototype in Isabelle/HOL based on our results. The prototype needs to be further polished, but it can already handle relevant examples that are out of scope for existing tools.

Future work As our theory subsumes the one of Isabelle's Lifting package, one goal is to replace the package by a more general tool. To this end, we have to integrate our results into Isabelle's (co)datatypes package [2], extend our prototype to automate the construction of compositions, and polish the support of white-box transports (cf. Section 6).

Finally, based on our formalisation insights, we conjecture that one can adopt our theory to constructive logics, but only a formalisation in a constructive prover will give a definite answer.

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## References

1. Angiuli, C., Cavallo, E., Mörtberg, A., Zeuner, M.: Internalizing Representation Independence with Univalence. Proc. ACM Program. Lang. 5(POPL) (jan 2021). https://doi.org/10.1145/3434293
2. Blanchette, J.C., Hölzl, J., Lochbihler, A., Panny, L., Popescu, A., Traytel, D.: Truly Modular (Co)datatypes for Isabelle/HOL. In: Klein, G., Gamboa, R. (eds.) Interactive Theorem Proving. pp. 93-110. Springer International Publishing, Cham (2014). https://doi.org/10.1007/978-3-319-08970-6_7
3. Church, A.: A Formulation of the Simple Theory of Types. The Journal of Symbolic Logic 5(2), 56-68 (1940). https://doi.org/10.2307/2266170
4. Cohen, C., Dénès, M., Mörtberg, A.: Refinements for Free! In: Gonthier, G., Norrish, M. (eds.) Certified Programs and Proofs. pp. 147-162. Springer International Publishing, Cham (2013). https://doi.org/10.1007/978-3-319-03545-1_10
5. Cousot, P.: Principles of Abstract Interpretation. MIT Press (2021)
6. Cousot, P., Cousot, R.: Static Determination of Dynamic Properties of Recursive Procedures. In: Neuhold, E. (ed.) IFIP Conf. on Formal Description of Programming Concepts, St-Andrews, N.B., CA. pp. 237-277. North-Holland (1977)
7. Cousot, P., Cousot, R.: Abstract Interpretation Frameworks. Journal of Logic and Computation 2(4), 511-547 (08 1992). https://doi.org/10.1093/logcom/2.4.511
8. Dagand, P.E., Tabareau, N., Tanter, E.: Partial Type Equivalences for Verified Dependent Interoperability. SIGPLAN Not. 51(9), 298-310 (sep 2016). https://doi.org/10.1145/3022670.2951933
9. Dagand, P.E., Tabareau, N., Tanter, E.: Foundations of Dependent Interoperability. Journal of Functional Programming 28 (2018). https://doi.org/10.1017/S0956796818000011
10. Darais, D., Van Horn, D.: Constructive Galois Connections. Journal of Functional Programming 29 (2019). https://doi.org/10.1017/S0956796819000066
11. Derderian, J.C.: Galois Connections and Pair Algebras. Canadian Journal of Mathematics 21, 498-501 (1969). https://doi.org/10.4153/CJM-1969-056-x
12. Hartmanis, J., Stearns, R.: Pair Algebra and Its Application to Automata Theory. Information and Control 7(4), 485-507 (1964). https://doi.org/https://doi.org/10.1016/S0019-9958(64)90181-0
13. Huffman, B., Kunčar, O.: Lifting and Transfer: A Modular Design for Quotients in Isabelle/HOL. In: Gonthier, G., Norrish, M. (eds.) Certified Programs and Proofs Third International Conference, CPP 2013, Melbourne, VIC, Australia, December 11-13, 2013, Proceedings. Lecture Notes in Computer Science, vol. 8307, pp. 131146. Springer (2013). https://doi.org/10.1007/978-3-319-03545-1_9
14. Kappelmann, K., Josh, C., Krauss, A.: Isabelle/Set (2023), https://github.com/kappelmann/Isabelle-Set
15. Kappelmann, Kevin: Transport via Partial Galois Connections and Equivalences. In: Hur, Chung-Kil (ed.) Asian Symposium on Programming Languages and Systems. pp. 225-245. Springer, Singapore (2023). https://doi.org/\{10.1007/978-981-99-8311-7_11\}
16. Krishnaswami, N.R., Dreyer, D.: Internalizing Relational Parametricity in the Extensional Calculus of Constructions. In: Rocca, S.R.D. (ed.) Computer Science Logic 2013 (CSL 2013). Leibniz International Proceedings in Informatics (LIPIcs), vol. 23, pp. 432-451. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany (2013). https://doi.org/10.4230/LIPIcs.CSL. 2013.432
17. Kunčar, O.: Types, Abstraction and Parametric Polymorphism in Higher-Order Logic. Ph.D. thesis, Technische Universität München (2016)
18. Lammich, P.: Automatic Data Refinement. In: Blazy, S., PaulinMohring, C., Pichardie, D. (eds.) Interactive Theorem Proving. pp. 84-99. Springer Berlin Heidelberg, Berlin, Heidelberg (2013). https://doi.org/10.1007/978-3-642-39634-2_9
19. Lammich, P., Lochbihler, A.: Automatic Refinement to Efficient Data Structures: A Comparison of Two Approaches. Journal of Automated Reasoning 63(1), 53-94 (Jun 2019). https://doi.org/10.1007/s10817-018-9461-9
20. Miné, A.: Weakly Relational Numerical Abstract Domains. Theses, Ecole Polytechnique X (Dec 2004), https://pastel.archives-ouvertes.fr/tel-00136630
21. Mitchell, J.C.: Representation Independence and Data Abstraction. In: Proceedings of the 13th ACM SIGACT-SIGPLAN Symposium on Principles of Programming Languages. p. 263-276. POPL '86, Association for Computing Machinery, New York, NY, USA (1986). https://doi.org/10.1145/512644.512669
22. Moura, L.d., Kong, S., Avigad, J., van Doorn, F., von Raumer, J.: The Lean Theorem Prover (System Description). In: Felty, A.P., Middeldorp, A. (eds.) Automated Deduction - CADE-25. pp. 378-388. Springer International Publishing, Cham (2015). https://doi.org/10.1007/978-3-319-21401-6_26
23. Moura, L.d., Ullrich, S.: The Lean 4 Theorem Prover and Programming Language. In: Platzer, A., Sutcliffe, G. (eds.) Automated Deduction CADE 28. pp. 625-635. Springer International Publishing, Cham (2021). https://doi.org/10.1007/978-3-030-79876-5_37
24. Nipkow, T., Wenzel, M., Paulson, L.C.: Isabelle/HOL: A Proof Assistant for Higher-Order Logic. Springer-Verlag, Berlin, Heidelberg (2002). https://doi.org/10.1007/3-540-45949-9
25. Reynolds, J.C.: Types, Abstraction and Parametric Polymorphism. In: Mason, R.E.A. (ed.) Information Processing 83, Proceedings of the IFIP 9th World Computer Congress, Paris, France, September 19-23, 1983. pp. 513-523. NorthHolland/IFIP (1983)
26. Ringer, T., Porter, R., Yazdani, N., Leo, J., Grossman, D.: Proof Repair across Type Equivalences. In: Proceedings of the 42nd ACM SIGPLAN International Conference on Programming Language Design and Implementation. p. 112-127. PLDI 2021, Association for Computing Machinery, New York, NY, USA (2021). https://doi.org/10.1145/3453483.3454033
27. Sozeau, M., Oury, N.: First-Class Type Classes. In: Mohamed, O.A., Muñoz, C., Tahar, S. (eds.) Theorem Proving in Higher Order Logics. pp. 278-293. Springer Berlin Heidelberg, Berlin, Heidelberg (2008). https://doi.org/10.1007/978-3-540-71067-7_23
28. Tabareau, N., Tanter, E., Sozeau, M.: Equivalences for Free: Univalent Parametricity for Effective Transport. Proc. ACM Program. Lang. 2(ICFP) (jul 2018). https://doi.org/10.1145/3236787
29. Tabareau, N., Tanter, E., Sozeau, M.: The Marriage of Univalence and Parametricity. J. ACM 68(1) (jan 2021). https://doi.org/10.1145/3429979
30. Traytel, D., Popescu, A., Blanchette, J.C.: Foundational, Compositional (Co)datatypes for Higher-Order Logic: Category Theory Applied to Theorem Proving. In: 2012 27th Annual IEEE Symposium on Logic in Computer Science. pp. 596-605 (2012). https://doi.org/10.1109/LICS. 2012.75
31. Univalent Foundations Program, T.: Homotopy Type Theory: Univalent Foundations of Mathematics. https://homotopytypetheory.org/book, Institute for Advanced Study (2013)
32. Vezzosi, A., Mörtberg, A., Abel, A.: Cubical Agda: A Dependently Typed Programming Language with Univalence and Higher Inductive Types. Proc. ACM Program. Lang. 3(ICFP) (jul 2019). https://doi.org/10.1145/3341691
33. Voevodsky, V.: The equivalence axiom and univalent models of type theory (2010). https://doi.org/10.48550/ARXIV.1402.5556
34. Wadler, P.: Theorems for free! In: Proceedings of the Fourth International Conference on Functional Programming Languages and Computer Architecture. p. 347-359. FPCA '89, Association for Computing Machinery, New York, NY, USA (1989). https://doi.org/10.1145/99370.99404

## A Partial Galois Connections, Equivalences, and Relators

## A. 1 (Order) Basics

Given types $\alpha$ and $\beta$, the type of functions from $\alpha$ to $\beta$ is written $\alpha \Rightarrow \beta$.
The composition of functions is defined as $(f \circ g) x:=f(g x)$.
A predicate on a type $\alpha$ is a function of type $\alpha \Rightarrow$ bool.
The predicate mapping all inputs to True is denoted by $\top:=\lambda_{\ldots}$. True.
A relation on $\alpha$ and $\beta$ is a function of type $\alpha \Rightarrow \beta \Rightarrow$ bool.
For every relation $R$, we introduce an infix operator $\left(\leq_{R}\right):=R$, that is $x \leq_{R} y \longleftrightarrow R x y$.

The inverse of a relation is defined as $R^{-1} x y:=R y x$.
The composition of two relations $R, S$ is defined as $(R \circ S) x y:=\exists z . R x z \wedge$ $S z y$.

A relation $R$ is finer than another relation $S$, written $R \leq S$, if $\forall x y . R x y \longrightarrow$ $S x y$.

The domain, codomain, and field predicates on a relation $R$ are defined as

$$
\begin{aligned}
\text { in_dom } R x & :=\exists y \cdot R x y \\
\text { in_codom } R y & :=\exists x \cdot R x y \\
\text { in_field } R x & :=\text { in_dom } R x \vee \text { in_codom } R x
\end{aligned}
$$

A relation $R$ is right-total if $\forall y . \exists x . R x y$ and right-unique if $\forall x, y, y^{\prime} . R x y \wedge$ $R x y^{\prime} \rightarrow y=y^{\prime}$.

Given a predicate $P$ and relation $R$, we define reflexivity, transitivity, and symmetry on $P$ and $R$ as follows:

$$
\begin{aligned}
\text { reflexive_on } P R & :=\forall x . P x \longrightarrow R x x \\
\text { transitive_on } P R & :=\forall x y z . P x \wedge P y \wedge P z \wedge R x y \wedge R y z \longrightarrow R x z \\
\text { symmetric_on } P R & :=\forall x y . P x \wedge P y \wedge R x y \longrightarrow R y x
\end{aligned}
$$

Preorders and partial equivalence relations (PERs) are then defined in the expected way:

$$
\begin{aligned}
& \text { preorder_on } P R:=\operatorname{transitive\_ on~} P R \wedge \text { reflexive_on } P R \\
& \text { partial_equivalence_rel_on } P R:=\operatorname{transitive\_ on~} P R \wedge \text { symmetric_on } P R
\end{aligned}
$$

For all relativised concepts, we introduce their unrelativised analogue:

$$
\begin{aligned}
\text { reflexive } R & :=\text { reflexive_on } \top R \\
\text { transitive } R & :=\text { transitive_on } \top R \\
& \vdots \\
\text { partial_equivalence_rel } R & :=\text { partial_equivalence_rel_on } \top R
\end{aligned}
$$

Given a predicate $P$ and relation $R$, we say that $f$ is inflationary (sometimes also called extensive) on $P$ and $R$, written inflationary_on $P R f$, if $\forall x . P x \longrightarrow$ $x \leq_{R} f x$. Similarly, we say that $f$ is deflationary on $P$ and $R$, written as deflationary_on $P R f$, if $\forall x . P x \longrightarrow f x \leq_{R} x$. If $f$ is inflationary and deflationary on $P$ and $R$, it is a relational equivalence on $P$ and $R$ :

$$
\text { rel_equivalence_on } P R f:=\text { inflationary_on } P R f \wedge \text { deflationary_on } P R f \text {. }
$$

## A. 2 Function Relators and Monotonicity

The dependent function relator is defined as

$$
([x y:: R] \Rightarrow S) f g:=\forall x y . R x y \longrightarrow S(f x)(g y)
$$

where $x, y$ may occur freely in $S$. The (non-dependent) function relator is given as a special case: $(R \Rightarrow S):=\left(\left[{ }_{-}:: R\right] \Rightarrow S\right)$. A function is monotone from $R$ to $S$ if it maps $R$-related inputs to $S$-related outputs:

$$
\left([x y:: R] \Rightarrow_{\mathrm{m}} S\right) f:=([x y:: R] \Rightarrow S) f f
$$

where $x, y$ may occur freely in $S$. The non-dependent variant is given as a special case: $\left(R \Rightarrow_{\mathrm{m}} S\right):=\left(\left[\__{-}:: R\right] \Rightarrow_{\mathrm{m}} S\right)$. A monotone function relator is like a function relator but additionally requires its members to be monotone:

$$
\begin{aligned}
\left([x y:: R] \Rightarrow{ }^{\oplus} S\right) f g:= & ([x y:: R] \Rightarrow S) f g \\
& \wedge\left([x y:: R] \Rightarrow_{\mathrm{m}} S\right) f \wedge\left([x y:: R] \Rightarrow_{\mathrm{m}} S\right) g
\end{aligned}
$$

where $x, y$ may occur freely in $S$. The non-dependent variant is given as a special case: $\left(R \nRightarrow^{\oplus} S\right):=\left([\ldots \quad:: R] \Rightarrow^{\oplus} S\right)$. We define the relational if conditional and the following notation:

$$
\begin{aligned}
\text { rel_if } B S x y & :=B \longrightarrow S x y, \\
([x y:: R \mid B] \Rightarrow S) & :=([x y:: R] \Rightarrow \text { rel_if } B S),
\end{aligned}
$$

where in the latter two cases, $x, y$ may occur freely in $B, S$.

## A. 3 Galois Relator

We define the dual of $\left(L_{\gtrsim}\right)$ as $\left(\gtrsim_{\approx}\right):=$ Galois $\left(\geq_{R}\right)\left(\geq_{L}\right) l$, that is $x \lesssim_{R} y \longleftrightarrow$ in_dom $\left(\leq_{L}\right) x \wedge l x \leq_{R} y$.
Lemma 3. Assume $\left(\left(\leq_{L}\right) \unlhd\left(\leq_{R}\right)\right) l r$. Then $x_{L} \lesssim y \longleftrightarrow x \lesssim_{\approx} y$.

## A. 4 Partial Galois Connections and Equivalences

Typically, Galois connections are defined on preorders, distinguished by the characteristic property $x \leq_{L} r y \longleftrightarrow l x \leq_{R} y$ for all $x, y$. We break the concept down into smaller pieces and lift it to a partial setting. The (partial) half Galois property on the left is defined as

$$
\left(\left(\leq_{L}\right)_{\mathrm{h}} \unlhd\left(\leq_{R}\right)\right) l r:=\forall x y \cdot x{ }_{L} \lesssim y \longrightarrow l x \leq_{R} y
$$

and dually, the (partial) half Galois property on the right as

$$
\left(\left(\leq_{L}\right) \unlhd_{\mathrm{h}}\left(\leq_{R}\right)\right) l r:=\forall x y \cdot x \lesssim_{R} y \longrightarrow x \leq_{L} r y,
$$

Both halves combined constitute the (partial) Galois property:

$$
\left(\left(\leq_{L}\right) \unlhd\left(\leq_{R}\right)\right) l r:=\left(\left(\leq_{L}\right)_{\mathrm{h}} \unlhd\left(\leq_{R}\right)\right) \operatorname{lr} \wedge\left(\left(\leq_{L}\right) \unlhd_{\mathrm{h}}\left(\leq_{R}\right)\right) \operatorname{lr} .
$$

If $l$ and $r$ are also monotone, we obtain a (partial) Galois connection:
$\left(\left(\leq_{L}\right) \dashv\left(\leq_{R}\right)\right) l r:=\left(\left(\leq_{L}\right) \Rightarrow_{\mathrm{m}}\left(\leq_{R}\right)\right) l \wedge\left(\left(\leq_{R}\right) \Rightarrow_{\mathrm{m}}\left(\leq_{L}\right)\right) r \wedge\left(\left(\leq_{L}\right) \unlhd\left(\leq_{R}\right)\right) l r$.
Note that we neither require $\left(\leq_{L}\right),\left(\leq_{R}\right)$ to be transitive nor reflexive. An example Galois connection can be found in Fig. 2(a).

By requiring a two-sided Galois connection, we obtain a (partial) Galois equivalence:

$$
\left(\left(\leq_{L}\right) \equiv_{\mathrm{G}}\left(\leq_{R}\right)\right) l r:=\left(\left(\leq_{L}\right) \dashv\left(\leq_{R}\right)\right) l r \wedge\left(\left(\leq_{R}\right) \dashv\left(\leq_{L}\right)\right) r l
$$

An example of a Galois equivalence can be found in Fig. 2(b). It can be shown that Galois equivalences are, in many circumstances, equivalent to the traditional notion of (partial) order equivalences (see Appendix A.4).

Since the relations $\left(\leq_{L}\right),\left(\leq_{R}\right)$ are preorders or partial equivalence relations in many practical cases, we introduce two more definitions for convenience:

$$
\begin{aligned}
\left(\left(\leq_{L}\right) \equiv_{\text {pre }}\left(\leq_{R}\right)\right) l r:= & \left(\left(\leq_{L}\right) \equiv_{\mathrm{G}}\left(\leq_{R}\right)\right) l r \\
& \wedge \text { preorder_on }\left(\text { in_field }\left(\leq_{L}\right)\right)\left(\leq_{L}\right) \\
& \wedge \text { preorder_on }\left(\text { in_field }\left(\leq_{R}\right)\right)\left(\leq_{R}\right) \\
\left(\left(\leq_{L}\right) \equiv_{\text {PER }}\left(\leq_{R}\right)\right) l r:= & \left(\left(\leq_{L}\right) \equiv_{\mathrm{G}}\left(\leq_{R}\right)\right) l r \\
& \left.\wedge \text { partial_equivalence_rel_on (in_field }\left(\leq_{L}\right)\right)\left(\leq_{L}\right) \\
& \left.\wedge \text { partial_equivalence_rel_on (in_field }\left(\leq_{R}\right)\right)\left(\leq_{R}\right)
\end{aligned}
$$

Order Equivalences To define the concept of an order equivalence, we first define the unit and counit functions:

$$
\text { unit } l r:=r \circ l \quad \text { counit } l r:=l \circ r
$$

When $l$ and $r$ are clear from the context, we will write $\eta:=$ unit $l r$ and $\epsilon:=$ counit $l r$. A (partial) order equivalence is then defined as

$$
\begin{aligned}
\left(\left(\leq_{L}\right) \equiv_{\mathrm{o}}\left(\leq_{R}\right)\right) l r:= & \left(\left(\leq_{L}\right) \Rightarrow_{\mathrm{m}}\left(\leq_{R}\right)\right) l \wedge\left(\left(\leq_{R}\right) \Rightarrow_{\mathrm{m}}\left(\leq_{L}\right)\right) r \\
& \left.\wedge \text { rel_equivalence_on (in_field }\left(\leq_{L}\right)\right)\left(\leq_{L}\right) \eta \\
& \left.\wedge \text { rel_equivalence_on (in_field }\left(\leq_{R}\right)\right)\left(\leq_{R}\right) \epsilon .
\end{aligned}
$$

In practice, we will commonly work with preorders, where the notions of Galois equivalences and order equivalences coincide:

Lemma 4. Assume

$$
\begin{aligned}
\text { (1) }\left(\left(\leq_{L}\right)\right. & \left.\equiv_{\mathrm{o}}\left(\leq_{R}\right)\right) l r, \quad(\mathbf{2}) \text { transitive }\left(\leq_{L}\right), \quad(\mathbf{3}) \text { transitive }\left(\leq_{R}\right) . \\
\text { Then }\left(\left(\leq_{L}\right)\right. & \left.\equiv_{\mathrm{G}}\left(\leq_{R}\right)\right) l r .
\end{aligned}
$$

Lemma 5. Assume
(1) $\left(\left(\leq_{L}\right) \equiv_{\mathrm{G}}\left(\leq_{R}\right)\right) l r$ (2) reflexive_on (in_field $\left.\left(\leq_{L}\right)\right)\left(\leq_{L}\right)$,
(3) reflexive_on (in_field $\left.\left(\leq_{R}\right)\right)\left(\leq_{R}\right)$.

Then $\left(\left(\leq_{L}\right) \equiv_{\circ}\left(\leq_{R}\right)\right) l r$.

## B Closure Properties

## B. 1 (Dependent) Function Relator

In Section 4.1, we only stated our results for Galois equivalences on preorders and partial equivalence relations and the dependent function relator. In this section, we show the more general results for Galois connections for both the (non-dependent) and dependent function relator. We also clarify the need of the monotone function relator.

Function Relator In practice, the relations and functions we use are often nondependent. The definitions in (19) then simplify to the standard, non-dependent function relator and mapper. Moreover, the closure theorems will have considerably simpler assumptions. We hence find it instructive to present the results for this special case. Let us fix the following variables:

$$
\begin{array}{ll}
L_{1}: \alpha_{1} \Rightarrow \alpha_{1} \Rightarrow \text { bool, } & l_{1}: \alpha_{1} \Rightarrow \alpha_{2}, \\
R_{1}: \alpha_{2} \Rightarrow \alpha_{2} \Rightarrow \text { bool, } & r_{1}: \alpha_{2} \Rightarrow \alpha_{1}, \\
L_{2}: \beta_{1} \Rightarrow \beta_{1} \Rightarrow \text { bool, } & l_{2}: \beta_{1} \Rightarrow \beta_{2}, \\
R_{2}: \beta_{2} \Rightarrow \beta_{2} \Rightarrow \text { bool, } & r_{2}: \beta_{2} \Rightarrow \beta_{1} .
\end{array}
$$

Compared to Eq. (19), the target relations and transport functions then simplify to

$$
\begin{aligned}
L & :=\left(\left(\leq_{L_{1}}\right) \Rightarrow^{\oplus}\left(\leq_{L_{2}}\right)\right), & R & :=\left(\left(\leq_{R_{1}}\right) \Rightarrow^{\oplus}\left(\leq_{R_{2}}\right)\right), \\
l & :=\left(r_{1} \rightarrow l_{2}\right), & r & :=\left(l_{1} \rightarrow r_{2}\right),
\end{aligned}
$$

where $(f \rightarrow g):=\left(\left[\_:: f\right] \rightarrow g\right)$ is the (non-dependent) function mapper. In other words: $(f \rightarrow g) \bar{h}=g \circ h \circ f$.

Lemma 6. Assume
(1) $\left(\left(\leq_{L_{1}}\right) \dashv\left(\leq_{R_{1}}\right)\right) l_{1} r_{1}$,
(2) reflexive_on (in_field $\left.\left(\leq_{L_{1}}\right)\right)\left(\leq_{L_{1}}\right)$,
(3) reflexive_on (in_field $\left.\left(\leq_{R_{1}}\right)\right)\left(\leq_{R_{1}}\right)$,
(4) $\left(\left(\leq_{L_{2}}\right) \dashv\left(\leq_{R_{2}}\right)\right) l_{2} r_{2}$,
(5) transitive $\left(\leq_{L_{2}}\right)$,
(6) transitive $\left(\leq_{R_{2}}\right)$.

Then $\left(\left(\leq_{L}\right) \dashv\left(\leq_{R}\right)\right) l r$.
Proof. The theorem is a direct consequence of Theorem 7, but it is instructive to consider the proof of this simpler theorem first. We only show the case for $\left(\left(\leq_{L}\right)_{\mathrm{h}} \unlhd\left(\leq_{R}\right)\right) l r$. The other cases are similar. Assume
(a) in_codom $\left(\leq_{R}\right) g$,
(b) $f \leq_{L} r g$,
(c) $x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime}$.

Our goal is $l f x_{1}^{\prime} \leq_{R_{2}} g x_{2}^{\prime}$. Due to monotonicity of $r_{1}$ (Assumption (1)), we get $r_{1} x_{1}^{\prime} \leq_{L_{1}} r_{1} x_{2}^{\prime}$. Due to Assumption (b), we get

$$
f\left(r_{1} x_{1}^{\prime}\right) \leq_{L_{2}} r g\left(r_{1} x_{2}^{\prime}\right)=r_{2}\left(g\left(\epsilon_{1} x_{2}^{\prime}\right)\right) .
$$

Since $\left(\left(\leq_{L_{2}}\right)_{\mathrm{h}} \unlhd\left(\leq_{R_{2}}\right)\right) l_{2} r_{2}$ (Assumption (4)), we get

$$
l_{2}\left(f\left(r_{1} x_{1}^{\prime}\right)\right)=l f x_{1}^{\prime} \leq_{R_{2}} g\left(\epsilon_{1} x_{2}^{\prime}\right) . .^{7}
$$

Due to transitivity (Assumption (6)), it remains to show that $g\left(\epsilon_{1} x_{2}^{\prime}\right) \leq_{R_{2}} g x_{2}^{\prime}$. This follows from the first Galois connection, reflexivity of $\left(\leq_{R_{1}}\right)$, monotonicity of $g$, and in_codom ( $\left.\leq_{R_{1}}\right) x_{2}^{\prime}$ (Assumptions (1), (2), (a), and (c)).

Specialising Theorem 8 to the non-dependent function relator yields:

[^6]Lemma 7. Assume
(1) $\left(\left(\leq_{L_{1}}\right) \dashv\left(\leq_{R_{1}}\right)\right) l_{1} r_{1}$,
(2) reflexive_on (in_field $\left.\left(\leq_{L_{1}}\right)\right)\left(\leq_{L_{1}}\right)$,
(3) $\left(\left(\leq_{R_{2}}\right) \Rightarrow_{\mathrm{m}}\left(\leq_{L_{2}}\right)\right) r_{2}$,
(4) transitive $\left(\leq_{L_{2}}\right)$,
(5) in_ $_{-} \operatorname{dom}\left(\leq_{L}\right) f$,
(6) in_codom $\left(\leq_{R}\right) g$.
Then $f L_{\gtrsim} g \longleftrightarrow\left(\left(L_{1} \gtrsim\right) \Rightarrow\left(L_{2} \gtrsim\right)\right) f g$.

Dependent Function Relator As in Theorem 1, the closure theorem requires monotonicity conditions for each of the dependent variables (Assumptions (7)(10) below). Morally speaking, these assumptions say that (1) $L_{2}$ is antimonotone in its first and restricted antimonotone in its second parameter, (2) $R_{2}$ is restricted monotone in its first and monotone in its second parameter, and (3) $l_{2}, r_{2}$ are monotone in both parameters.

Theorem 7. Define $\eta_{1}:=$ unit $l_{1} r_{1}$ and $\epsilon_{1}:=\operatorname{counit} l_{1} r_{1}$. Assume
(1) $\left(\left(\leq_{L_{1}}\right) \dashv\left(\leq_{R_{1}}\right)\right) l_{1} r_{1}$,
(2) reflexive_on (in_field $\left.\left(\leq_{L_{1}}\right)\right)\left(\leq_{L_{1}}\right)$,
(3) reflexive_on (in_field $\left.\left(\leq_{R_{1}}\right)\right)\left(\leq_{R_{1}}\right)$,
(4) if $x_{L_{1}} \lesssim x^{\prime}$ then $\left(\left(\leq_{L_{2} x\left(r_{1} x^{\prime}\right)}\right) \dashv\left(\leq_{R_{2}\left(l_{1} x\right) x^{\prime}}\right)\right)\left(l_{2} x^{\prime} x\right)\left(r_{2} x x^{\prime}\right)$,
(5) if $x_{1} \leq_{L_{1}} x_{2}$ then transitive $\left(\leq_{L_{2} x_{1} x_{2}}\right)$,
(6) if $x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime}$ then transitive $\left(\leq_{R_{2}} x_{1}^{\prime} x_{2}^{\prime}\right)$,
(7) if $x_{1} \leq_{L_{1}} x_{2} \leq_{L_{1}} x_{3} \leq_{L_{1}} x_{4} \leq_{L_{1}} \eta_{1} x_{3}$ then $\left(\leq_{L_{2} x_{2} x_{4}}\right) \leq\left(\leq_{L_{2} x_{1} x_{3}}\right)$,
(8) if $\epsilon_{1} x_{2}^{\prime} \leq_{R_{1}} x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime} \leq_{R_{1}} x_{3}^{\prime} \leq_{R_{1}} x_{4}^{\prime}$ then $\left(\leq_{R_{2} x_{1}^{\prime} x_{3}^{\prime}}\right) \leq\left(\leq_{R_{2} x_{2}^{\prime} x_{4}^{\prime}}\right)$,
(9) if $x_{1} \leq_{L_{1}} x_{2} L_{1} \lesssim x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime}$ and in_field $\left(\leq_{L_{2} x_{1}\left(r_{1} x_{2}^{\prime}\right)}\right)$ y then $\left(l_{2} x_{1}^{\prime} x_{1} y\right) \leq_{R_{2}\left(l_{1} x_{1}\right) x_{2}^{\prime}}\left(l_{2} x_{2}^{\prime} x_{2} y\right)$,
(10) if $x_{1} \leq_{L_{1}} x_{2} L_{1} \gtrsim x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime}$ and in_field $\left(\leq_{R_{2}\left(l_{1} x_{1}\right) x_{2}^{\prime}}\right) y^{\prime}$ then $\left(r_{2} x_{1} x_{1}^{\prime} y^{\prime}\right) \leq_{L_{2} x_{1}\left(r_{1} x_{2}^{\prime}\right)}\left(r_{2} x_{2} x_{2}^{\prime} y^{\prime}\right)$.
Then $\left(\left(\leq_{L}\right) \dashv\left(\leq_{R}\right)\right) l r$.
Proof. We will only prove that $\left(\left(\leq_{L}\right)_{\mathrm{h}} \unlhd\left(\leq_{R}\right)\right) l r$. This should primarily illustrate how the monotonicity requirements arise as part of the proof. The rest of the proof can be found in our formalisation. It is also instructive to first consider the proof for the non-dependent function relator as it uses the same core ideas (see Lemma 6).

A visualisation of the following proof can be found in Fig. 3. Assume
(a) in_codom $\left(\leq_{R}\right) g$,
(b) $f \leq_{L} r g$,
(c) $x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime}$.

We have to show that ( $l f x_{1}^{\prime}$ ) $\leq_{R_{2} x_{1}^{\prime} x_{2}^{\prime}}\left(g x_{2}^{\prime}\right)$, which unfolds to

$$
\left(l_{2} x_{1}^{\prime}\left(r_{1} x_{1}^{\prime}\right)\left(f\left(r_{1} x_{1}^{\prime}\right)\right)\right) \leq_{R_{2} x_{1}^{\prime} x_{2}^{\prime}}\left(g x_{2}^{\prime}\right) .
$$

First we apply reflexivity of $\left(\leq_{R_{1}}\right)$ to obtain $x_{1}^{\prime} \leq_{R_{1}} x_{1}^{\prime}$. With monotonicity of $r_{1}$ (Assumption (1)), we get $r_{1} x_{1}^{\prime} \leq_{L_{1}} r_{1} x_{1}^{\prime}$. Due to Assumption (b), we get

$$
\left(f\left(r_{1} x_{1}^{\prime}\right)\right) \leq_{L_{2}\left(r_{1} x_{1}^{\prime}\right)\left(r_{1} x_{1}^{\prime}\right)}\left(r g\left(r_{1} x_{1}^{\prime}\right)\right)=r_{2}\left(r_{1} x_{1}^{\prime}\right)\left(\epsilon_{1} x_{1}^{\prime}\right)\left(g\left(\epsilon_{1} x_{1}^{\prime}\right)\right) .
$$



Fig. 3: Proof of $\left(\left(\leq_{L}\right)_{\mathrm{h}} \unlhd\left(\leq_{R}\right)\right) l r$ as explained in Theorem 7. Types are drawn solid, black, transport functions dashed, relations dotted and dashed-dotted.

Now unlike in Lemma 6, we cannot directly apply Assumption (4): the parameters of $\left(\leq_{L_{2}\left(r_{1} x_{1}^{\prime}\right)\left(r_{1} x_{1}^{\prime}\right)}\right)$ and $r_{2}\left(r_{1} x_{1}^{\prime}\right)\left(\epsilon_{1} x_{1}^{\prime}\right)$ do not match up. We first have to use monotonicity of $r_{2}$ (Assumption (10)) to obtain

$$
\left(r_{2}\left(r_{1} x_{1}^{\prime}\right)\left(\epsilon_{1} x_{1}^{\prime}\right)\left(g\left(\epsilon_{1} x_{1}^{\prime}\right)\right)\right) \leq_{L_{2}\left(r_{1} x_{1}^{\prime}\right)\left(r_{1} x_{1}^{\prime}\right)}\left(r_{2}\left(r_{1} x_{1}^{\prime}\right) x_{1}^{\prime}\left(g\left(\epsilon_{1} x_{1}^{\prime}\right)\right)\right)
$$

With transitivity (Assumption (5)), we then get

$$
\left(f\left(r_{1} x_{1}^{\prime}\right)\right) \leq_{L_{2}\left(r_{1} x_{1}^{\prime}\right)\left(r_{1} x_{1}^{\prime}\right)}\left(r_{2}\left(r_{1} x_{1}^{\prime}\right) x_{1}^{\prime}\left(g\left(\epsilon_{1} x_{1}^{\prime}\right)\right)\right) .
$$

Now we apply Assumption (4) to obtain

$$
l_{2} x_{1}^{\prime}\left(r_{1} x_{1}^{\prime}\right)\left(f\left(r_{1} x_{1}^{\prime}\right)\right)=\left(l f x_{1}^{\prime}\right) \leq_{R_{2}\left(\epsilon_{1} x_{1}^{\prime}\right) x_{1}^{\prime}}\left(g\left(\epsilon_{1} x_{1}^{\prime}\right)\right) .8
$$

With monotonicity of $g$ and Assumption (1), one can show that

$$
\left(g\left(\epsilon_{1} x_{1}^{\prime}\right)\right) \leq_{R_{2}\left(\epsilon_{1} x_{1}^{\prime}\right) x_{1}^{\prime}}\left(g x_{1}^{\prime}\right)
$$

[^7]Thus with transitivity (Assumption (6)), ( $\left.l f x_{1}^{\prime}\right) \leq_{R_{2}\left(\epsilon_{1} x_{1}^{\prime}\right) x_{1}^{\prime}}\left(g x_{1}^{\prime}\right)$. Using monotonicity of $R_{2}$ (Assumption (8)), we can adapt the parameters of $R_{2}$ and obtain $\left(l f x_{1}^{\prime}\right) \leq_{R_{2} x_{1}^{\prime} x_{2}^{\prime}}\left(g x_{1}^{\prime}\right)$. Finally, we obtain $\left(g x_{1}^{\prime}\right) \leq_{R_{2} x_{1}^{\prime} x_{2}^{\prime}}\left(g x_{2}^{\prime}\right)$ from $x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime}$ and monotonicity of $g$. We can conclude using transitivity.

We can also prove a generalisation of Theorem 2:

## Theorem 8. Assume

(1) $\left(\left(\leq_{L_{1}}\right) \dashv\left(\leq_{R_{1}}\right)\right) l_{1} r_{1}$,
(2) reflexive_on (in_field $\left.\left(\leq_{L_{1}}\right)\right)\left(\leq_{L_{1}}\right)$,
(3) if $x_{L_{1}} \lesssim x^{\prime}$ then $\left(\left(\leq_{R_{2}\left(l_{1} x\right) x^{\prime}}\right) \Rightarrow_{\mathrm{m}}\left(\leq_{L_{2} x\left(r_{1} x^{\prime}\right)}\right)\right)\left(r_{2} x x^{\prime}\right)$,
(4) if $x_{1} \leq_{L_{1}} x_{2}$ then transitive $\left(\leq_{L_{2} x_{1} x_{2}}\right)$,
(5) if $x_{1} \leq_{L_{1}} x_{2} \leq_{L_{1}} x_{3}$ then $\left(\leq_{L_{2} x_{1} x_{2}}\right) \leq\left(\leq_{L_{2} x_{1} x_{3}}\right)$,
(6) if $x_{1} \leq_{L_{1}} x_{2} \leq_{L_{1}} x_{3} \leq_{L_{1}} \eta_{1} x_{2}$ then $\left(\leq_{L_{2} x_{1} x_{3}}\right) \leq\left(\leq_{L_{2} x_{1} x_{2}}\right)$,
(7) if $x_{1} \leq_{L_{1}} x_{2 L_{1}} \lesssim x_{1}^{\prime} \leq_{R_{1}} x_{2}^{\prime}$ and in_field $\left(\leq_{R_{2}\left(l_{1} x_{1}\right) x_{2}^{\prime}}\right) y^{\prime}$ then $\left(r_{2} x_{1} x_{1}^{\prime} y^{\prime}\right) \leq_{L_{2} x_{1}\left(r_{1} x_{2}^{\prime}\right)}\left(r_{2} x_{2} x_{2}^{\prime} y^{\prime}\right)$,
(8) in_dom $\left(\leq_{L}\right) f$, and in_codom $\left(\leq_{R}\right) g$.

Then $f L_{L} g \longleftrightarrow\left(\left[x x^{\prime}::\left(L_{1} \lesssim\right)\right] \Rightarrow\left(L_{2} x x^{\prime} \lesssim\right)\right) f g$.

Regarding Monotonicity Finally, we want to mention a subtlety: while work in abstract interpretation points out the necessity to use monotone function relators, for example [7], related work dealing with the concept of transports in proof assistants does not talk about any such monotonicity restriction [1, 8, 9, $13,26,28,29]$. The reason is not that the monotonicity restriction is unnecessary, but rather that the function relators in latter works are monotone by default. This can be made precise with the following lemma:

Lemma 8. Assume
(1) reflexive_on (in_field $\left.\left(\leq_{L_{1}}\right)\right)\left(\leq_{L_{1}}\right)$,
(2) if $x_{1} \leq_{L_{1}} x_{2}$ then $\left(\leq_{L_{2} x_{2} x_{2}}\right) \leq\left(\leq_{L_{2} x_{1} x_{2}}\right)$,
(3) if $x_{1} \leq_{L_{1}} x_{2}$ then $\left(\leq_{L_{2} x_{1} x_{1}}\right) \leq\left(\leq_{L_{2} x_{1} x_{2}}\right)$,
(4) if $x_{1} \leq_{L_{1}} x_{2}$ then partial_equivalence_rel $\left(\leq_{L_{2} x_{1} x_{2}}\right)$.

Then $\left(\left[x_{1} x_{2}::\left(\leq_{L_{1}}\right)\right] \Rightarrow^{\oplus}\left(\leq_{L_{2} x_{1} x_{2}}\right)\right)=\left(\left[x_{1} x_{2}::\left(\leq_{L_{1}}\right)\right] \Rightarrow\left(\leq_{L_{2} x_{1} x_{2}}\right)\right)$.
Again, we can specialise this to the non-dependent function relator:

## Lemma 9. Assume

(1) reflexive_on (in_field $\left.\left(\leq_{L_{1}}\right)\right)\left(\leq_{L_{1}}\right)$,
(2) partial_equivalence_rel $\left(\leq_{L_{2}}\right)$.

Then $\left(\left(\leq_{L_{1}}\right) \Rightarrow{ }^{\oplus}\left(\leq_{L_{2}}\right)\right)=\left(\left(\leq_{L_{1}}\right) \Rightarrow\left(\leq_{L_{2}}\right)\right)$.
It is easy to check that these assumptions are met by type equivalences and partial quotient types.

## B. 2 Compositions

In this section, we provide some intuition for the constructions from Section 4.3, provide preciser results, and compare the construction with Isabelle's Lifting package.

## Closure for Coinciding Relations

Theorem 9. Let $\star \in\left\{\dashv, \equiv_{\mathrm{G}}, \equiv_{\mathrm{o}}, \equiv_{\mathrm{pre}}, \equiv\right.$ PER $\}$ and assume
(1) $\left(\left(\leq_{L_{1}}\right) \star\left(\leq_{R_{1}}\right)\right) l_{1} r_{1}, \quad(\mathbf{2})\left(\left(\leq_{R_{1}}\right) \star\left(\leq_{R_{2}}\right)\right) l_{2} r_{2}, \quad(\mathbf{3})\left(\leq_{R_{1}}\right)=\left(\leq_{L_{2}}\right)$.

Then $\left(\left(\leq_{L_{1}}\right) \star\left(\leq_{R_{2}}\right)\right)\left(l_{2} \circ l_{1}\right)\left(r_{1} \circ r_{2}\right)$.
Proof. The proof can be found in the formalisation ${ }^{9}$.

Construction Idea As mentioned in Section 4.3, our construction is inspired by Huffman and and Kunčar's construction in [13]. Unfortunately, they do not provide any intuition about their constructions, nor does Kunčar [17] in his thesis. We try our best to fill this gap: In the following, we call $\left(\leq_{L_{1}}\right)$ the leftmost relation, $\left(\leq_{R_{1}}\right),\left(\leq_{L_{2}}\right)$ the middle relations, and $\left(\leq_{R_{2}}\right)$ the rightmost relation. We will explain the definition of $\left(\leq_{L}\right)$. The case for $\left(\leq_{R}\right)$ is symmetric.

Fix some $x: \alpha$ of the leftmost type. We want to (a) make sure that applying $l=l_{2} \circ l_{1}$ on $x$ does not leave the domain/codomain of our equivalences, and (b) find all elements $x^{\prime}: \alpha$ that are greater or equal than $x$ while doing so. We make a first approximation to satisfy these conditions using three "chase" steps:
(1) check whether in_dom $\left(\leq_{L_{1}}\right) x$ and find some $y$ such that $l_{1} x \leq_{R_{1}} y$,
(2) find some $y^{\prime}$ such that $y \leq_{L_{2}} y^{\prime}$, and
(3) check whether in_dom $\left(\leq_{R_{1}}\right) y^{\prime}$ and find some $x^{\prime}$ such that $r_{1} y^{\prime} \leq_{L_{1}} x^{\prime}$.

These steps are not enough: we may have $l_{1} x \leq_{R_{1}} y \leq_{L_{2}} y^{\prime}$ but not necessarily $l_{1} x \leq_{L_{2}} y \leq_{L_{2}} y^{\prime}$, as required for Property (a) and Step (3). But if we further require that $\left(\leq_{R_{1}}\right)$ and $\left(\leq_{L_{2}}\right)$ commute, that is $\left(\left(\leq_{R_{1}}\right) \circ\left(\leq_{L_{2}}\right)\right)=\left(\left(\leq_{L_{2}}\right) \circ\right.$ $\left.\left(\leq_{R_{1}}\right)\right)$, the steps become sufficient. Finally note that

- $x{ }_{L_{1}} \lesssim y \longleftrightarrow$ in_dom $\left(\leq_{L_{1}}\right) x \wedge l_{1} x \leq_{R_{1}} y$ whenever $\left(\left(\leq_{L_{1}}\right) \unlhd\left(\leq_{R_{1}}\right)\right) l_{1} r_{1}$, and
- $y^{\prime} R_{1} \lesssim x^{\prime} \longleftrightarrow$ in_dom $\left(\leq_{R_{1}}\right) y^{\prime} \wedge r_{1} y^{\prime} \leq_{L_{1}} x^{\prime}$ whenever $\left(\left(\leq_{R_{1}}\right) \unlhd\left(\leq_{L_{1}}\right)\right) r_{1} l_{1}$
due to Lemma 3. For Galois equivalences $\left(\left(\leq_{L_{1}}\right) \equiv_{\mathrm{G}}\left(\leq_{R_{1}}\right)\right) l_{1} r_{1}$, it is thus sufficient to search for a chain $x L_{1} \lesssim y \leq_{L_{2}} y^{\prime} R_{1} \lesssim x^{\prime}$, which is equivalent to $\left(\left(L_{1} \lesssim\right) \circ\left(\leq_{L_{2}}\right) \circ\left(R_{1} \lesssim\right)\right) x x^{\prime}$. Hence the definition of $\left(\leq_{L}\right)$.

[^8]Remark 2. A Galois connection $\left(\left(\leq_{L_{1}}\right) \dashv\left(\leq_{R_{1}}\right)\right) l_{1} r_{1}$ would not be sufficient due to Step (3): We are given some $y^{\prime}: \beta$ and $x^{\prime}: \alpha$ and need to check whether $y^{\prime}$ is "smaller" than $x^{\prime}$. We may check this by either transporting $y^{\prime}$ to the left (i.e. $r_{1} y^{\prime} \leq_{L_{1}} x^{\prime}$ ) or $x^{\prime}$ to the right (i.e. $y^{\prime} \leq_{R_{1}} l_{1} x^{\prime}$ ). However, right adjoints only preserve infima while left adjoints only preserve suprema. Hence the need for $\left(\left(\leq_{R_{1}}\right) \dashv\left(\leq_{L_{1}}\right)\right) r_{1} l_{1}$.

Now it is not to be excluded that there is an alternative way that avoids the need of a Galois equivalence. But at least thus far, it has eluded the author.

Remark 3. As noted, the relations $\left(\leq_{L}\right)$ and $\left(\leq_{R}\right)$ may not be equal to $\left(\leq_{L_{1}}\right)$ and ( $\leq_{R_{2}}$ ), but, in some sense, describe those parts that were made "compatible" with respect to $l$ and $r$. While our formalisation includes conditions under which we can obtain an equality, they do not apply to all practical examples. It is indeed a challenge on its own to find particular conditions under which the relations $\left(\leq_{L}\right)$ and $\left(\leq_{R}\right)$ may be rewritten to a simpler form. In this direction, the thesis of Kunčar [17] includes ideas applicable to total quotients and partial subtypes.

Closure and Similarity Theorems The next result generalises Theorem 5.
Theorem 10. Assume
(1) $\left(\left(\leq_{L_{i}}\right) \equiv_{\mathrm{G}}\left(\leq_{R_{i}}\right)\right) l_{i} r_{i}$ for $i \in\{1,2\}$, (2) preorder_on (in_field $\left.\left(\leq_{R_{1}}\right)\right)\left(\leq_{R_{1}}\right)$,
(3) preorder_on (in_field $\left.\left(\leq_{L_{2}}\right)\right)\left(\leq_{L_{2}}\right)$, (4) $\left(\left(\leq_{R_{1}}\right) \circ\left(\leq_{L_{2}}\right)\right)=\left(\left(\leq_{L_{2}}\right) \circ\left(\leq_{R_{1}}\right)\right)$.

Then $\left(\left(\leq_{L}\right) \dashv\left(\leq_{R}\right)\right)$ l $r$.
Proof. We will only show that $\left(\left(\leq_{L}\right)_{\mathrm{h}} \unlhd\left(\leq_{R}\right)\right) l r$ to illustrate the usage of the compatibility condition (Assumption (4)). The rest of the proof can be found in our formalisation. A visualisation of the following proof can be found in Fig. 4.

Assume that
(a) in_codom $\left(\leq_{R}\right) z$,
(b) $x \leq_{L} r z$.

We have to show that $l x \leq_{R} z$, which unfolds to $\left(\left(R_{2} \lesssim\right) \circ\left(\leq_{R_{1}}\right) \circ\left(L_{2} \lesssim\right)\right)\left(l_{2}\left(l_{1} x\right)\right) z$. From Assumption (b), we obtain $y, y^{\prime}$ such that

$$
l_{1} x \leq_{R_{1}} y \leq_{L_{2}} y^{\prime} \leq_{R_{1}} l_{1}(r z)=\epsilon_{1}\left(r_{2} z\right)
$$

where $\epsilon_{1}:=$ counit $l_{1} r_{1}$. We wish to obtain $\epsilon_{1}\left(r_{2} z\right) \leq_{R_{1}} r_{2} z$; this only holds if in_codom $\left(\leq_{R_{1}}\right)\left(r_{2} z\right)$, however. For this purpose, take Assumptions (1) and (a). We obtain $w, w^{\prime}$ such that $w \leq_{R_{1}} w^{\prime} \leq_{L_{2}} r_{2} z$. Thus, by Assumption (4), there is $w^{\prime \prime}$ such that $w \leq_{L_{2}} w^{\prime \prime} \leq_{R_{1}} r_{2} z$. Hence, in_codom $\left(\leq_{R_{1}}\right)\left(r_{2} z\right)$.

Then by transitivity, we get $y \leq_{L_{2}} y^{\prime} \leq_{R_{1}} r_{2} z$. Thus, by Assumption (4), there is $y^{\prime \prime}$ such that $y \leq_{R_{1}} y^{\prime \prime} \leq_{L_{2}} r_{2} z$. From $y^{\prime \prime} \leq_{L_{2}} r_{2} z$ and Assumption (a), we get $y^{\prime \prime} L_{2} \lesssim z$. From $l_{1} x \leq_{R_{1}} y \leq_{R_{1}} y^{\prime \prime}$ and transitivity, we get $l_{1} x \leq_{R_{1}} y^{\prime \prime}$. It remains to show that $l x{ }_{R_{2}} \lesssim l_{1} x$, that is $l x \leq_{R_{2}} l x$ and in_codom $\left(\leq_{L_{2}}\right)\left(l_{1} x\right)$.

From $l_{1} x \leq_{R_{1}} y \leq_{L_{2}} y^{\prime}$ and Assumption (4), we obtain $u$ such that $l_{1} x \leq_{L_{2}}$ $u \leq_{R_{1}} y^{\prime}$. Thus, in_dom $\left(\leq_{L_{2}}\right)\left(l_{1} x\right)$. Then by reflexivity (Assumption (3)), $l_{1} x \leq_{L_{2}} l_{1} x$. Finally, $l x \leq_{R_{2}} l x$ by monotonicity of $l_{2}$ (Assumption (1)).


Fig. 4: Proof of $\left(\left(\leq_{L}\right)_{\mathrm{h}} \unlhd\left(\leq_{R}\right)\right) l r$ as explained in Theorem 10. Types are drawn solid, black, transport functions dashed, relations dotted and dashed-dotted.

We can also prove a generalisation of Theorem 6:
Theorem 11. Assume
(1) $\left(\left(\leq_{R_{1}}\right) \Rightarrow_{\mathrm{m}}\left(\leq_{L_{1}}\right)\right) r_{1}$,
(2) $\left(\left(\leq_{L_{1}}\right) \unlhd\left(\leq_{R_{1}}\right)\right) l_{1} r_{1}$,
(3) $\left(\left(\leq_{R_{1}}\right)_{\mathrm{h}} \unlhd\left(\leq_{L_{1}}\right)\right) r_{1} l_{1}$,
(4) preorder_on (in_field $\left.\left(\leq_{R_{1}}\right)\right)\left(\leq_{R_{1}}\right)$,
(5) $\left(\left(\leq_{L_{2}}\right) \Rightarrow_{\mathrm{m}}\left(\leq_{R_{2}}\right)\right) l_{2}$,
(6) $\left(\left(\leq_{R_{2}}\right)_{\mathrm{h}} \unlhd\left(\leq_{L_{2}}\right)\right) r_{2} l_{2}$,
(7) reflexive_on $\left(\right.$ in_dom $\left.\left(\leq_{L_{2}}\right)\right)\left(\leq_{L_{2}}\right)$,
(8) $\left(\left(\leq_{R_{1}}\right) \circ\left(\leq_{L_{2}}\right)\right)=\left(\left(\leq_{L_{2}}\right) \circ\left(\leq_{R_{1}}\right)\right)$.

Then $\left(L^{\lesssim}\right)=\left(\left(L_{1} \gtrsim\right) \circ\left(L_{2} \lesssim\right)\right)$.

Comparison To Isabelle's Lifting Package As mentioned, our definitions are inspired by [13]: Let $\left(T_{1}, l_{1}, r_{1}\right)$ and $\left(T_{2}, l_{2}, r_{2}\right)$ be two partial quotient types with induced left relations $\left(\leq_{L_{1}}\right)$ and $\left(\leq_{L_{2}}\right)$. Huffman and and Kunčar then
construct the composition $\left(T_{1} \circ T_{2}, l_{2} \circ l_{1}, r_{1} \circ r_{2}\right)$. Moreover, they prove that the induced left relation $\left(\leq_{L}\right)$ of this composed partial quotient type satisfies $\left(\leq_{L}\right)=T_{1} \circ\left(\leq_{L_{2}}\right) \circ T_{1}^{-1}$. This insight sparked the idea of our definitions.

Indeed, we can show that our definitions faithfully generalise their work. Just as Lemma 1 shows that $T_{1}=$ Galois $\left(\leq_{L_{1}}\right)(=) r_{1}$, we can show that $T_{1}^{-1}=$ Galois $(=)\left(\leq_{L_{1}}\right) l_{1}$. It then follows that

$$
\left(T_{1} \circ\left(\leq_{L_{2}}\right) \circ T_{1}^{-1}\right)=\left(\text { Galois }\left(\leq_{L_{1}}\right)(=) r_{1} \circ\left(\leq_{L_{2}}\right) \circ \text { Galois }(=)\left(\leq_{L_{1}}\right) l_{1}\right)
$$

Moreover, it is easy to show that the compatibility condition is vacuously true for partial quotient types.


[^0]:    ${ }^{1}$ Non-peer reviewed, extended version of "Transport via Partial Galois Connections and Equivalences", 21st Asian Symposium on Programming Languages and Systems (APLAS), 2023 [15]

[^1]:    ${ }^{2}$ The Lifting package is indeed used pervasively for such purposes. At the time of writing, Isabelle/HOL and the Archive of Formal Proofs (www.isa-afp.org) contain more than 2800 invocations of the package.

[^2]:    ${ }^{3}$ https://www.isa-afp.org/entries/Transport.html

[^3]:    ${ }^{4}$ To avoid confusion, our work is not about the transport map from homotopy type theory [31, Chapter 2]. We focus on the general task of transporting a term $t$ to another term $t^{\prime}$ along some notion of equivalence (not necessarily a type equivalence).

[^4]:    ${ }^{5}$ The choice of $L t\left(r t^{\prime}\right), R(l t) t^{\prime}$ may seem arbitrary - why not pick $L t\left(r t^{\prime}\right), R t^{\prime}(l t)$ instead? In the end, the choice does not matter: While the former leads us to (monotone) Galois connections, the latter leads us to antitone Galois connections. Using that $L, R$ form a Galois connection if and only if $L, R^{-1}$ form an antitone Galois connection, every result in this paper can be transformed to its corresponding result on antitone Galois connections by an appropriate instantiation of the framework.

[^5]:    ${ }^{6}$ These two conditions are equivalent to requiring $x \leq_{L} r y \longleftrightarrow l x \leq_{R} y$ for all $x, y$.

[^6]:    ${ }^{7}$ Strictly speaking, we first need to show that in_codom $\left(\leq_{R_{2}}\right)\left(g\left(\epsilon_{1} x_{2}^{\prime}\right)\right)$, but we omit that technical step here.

[^7]:    ${ }^{8}$ Again, we omit the step showing that in_codom $\left(\leq_{R_{2}\left(\epsilon_{1} x_{1}^{\prime}\right) x_{1}^{\prime}}\right)\left(g\left(\epsilon_{1} x_{2}^{\prime}\right)\right)$.

[^8]:    ${ }^{9}$ We actually prove a more general result where the right and left relations of the input Galois connections need not be equal but only need to "agree whenever required". But we suspect that such an agreement rarely holds in practice and hence omit it.

