# Covering the edges of a random graph by cliques 

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## 1 Introduction

 The clique cover number $\theta_{1}(G)$ of a graph $G$ is the minimum number of cliques required to cover the edges of graph $G$. In this paper we consider $\theta_{1}\left(G_{n, p}\right)$, for $p$ constant. (Recall 'that in the random graph $G_{n, p}$, each of the $\binom{n}{2}$ edges occurs independently with probability $p$ ). Bollobás, Erdős, Spencer and West [1] proved that whp (i.e. with probability 1-o(1) as ' $n \rightarrow \infty$ )$$
\frac{(1-o(1)) n^{2}}{4\left(\log _{2} n\right)^{2}} \leq \theta_{1}\left(G_{n, 5}\right) \leq \frac{c n^{2} \ln \ln n}{(\ln n)^{2}}
$$

They implicitly conjecture that the $\ln \ln n$ factor in the upper bound is unnecessary and in this paper we prove

Theorem 1. There exist constants $c_{i}=c_{i}(p)>0, i=1,2$ such that whp

$$
\frac{c_{1} n^{2}}{(\ln n)^{2}} \leq \theta_{1}\left(G_{n, p}\right) \leq \frac{c_{2} n^{2}}{(\ln n)^{2}} .
$$

Remark 1: a simple use of a martingale tail inequality shows that $\theta_{1}$ is close to its mean with very high probability.

[^0]
## 2 Proof of Theorem 1

We write $a_{n} \approx b_{n}$ if $a_{n} / b_{n} \rightarrow 1$ as $n \rightarrow \infty$.

The lower bound is simple as the number of edges $m$ of $G_{n, p}$ whp satisfies

$$
m \approx \frac{n p^{2}}{2}
$$

and the size of the largest clique $\omega=\omega\left(G_{n, p}\right)$ whp satisfies

$$
\omega \approx 2 \log _{b} n
$$

where $b=1 / p$. We may thus choose $c_{1} \approx(\ln b)^{2} p / 2$.

The upper bound requires more work. Our method does not seem to yield the correct value for $c_{2}$ and so we will not work hard to keep $c_{2}$ small. Let $\alpha$ be some small constant and let

$$
k=\left\lfloor\alpha \log _{b} n\right\rfloor .
$$

We consider an algorithm for randomly selecting cliques to cover the edges of $G=G_{n, p}$. It bears some relation to part of the algorithm described in Pippenger and Spencer [2]. At iteration $i$ we randomly select cliques of size $k_{i}=\lfloor k / i\rfloor$ none of whose edges are covered by previously chosen cliques. Our idea is to choose these cliques so that at the start of iteration $i$ the graph $G_{i}$ formed by the set $E_{i}$ of edges which have not been covered behaves, for our purposes, similarly to $G_{n, p_{i}}, p_{i}=p e^{1-i}$. That is it will contain about $m_{i}=\binom{n}{2} p_{i}$ edges, it will have about $N_{i}=\binom{n}{k_{i}} p_{i}^{\binom{k_{i}}{2}}$ cliques of size $k_{i}$ and the intersection of these cliques will be similar to that for the $k_{i}$-cliques in $G_{n, p_{i}}$. In particular, in both $G_{n, p_{i}}$ and $G_{i}$ almost all of the edges are in about $\zeta_{i}=N_{i}\binom{k_{i}}{2} / m_{i} k_{i}$-cliques.

Now in iteration $i$ we choose a set $\mathcal{C}_{i}$ of $k_{i}$-cliques from $G_{i}$ to add to our cover. The available cliques are chosen independently with probability about $1 / \zeta_{i}$. By our assumptions on $G_{i}$, an edge is left uncovered with probability about $e^{-1}$. With a bit of care we can show that our assumptions continue to hold for $G_{i+1}$ as well.

We do this for $i_{0}=\lceil 4 \ln \ln n\rceil$ iterations. After this there are about $\binom{n}{2} p e(\ln n)^{-4}$ uncovered edges and we can add these as cliques of size two to the cover. In iteration $i$ we choose about
$m_{i} /\binom{k_{i}}{2} \approx n^{2} i^{2} p e^{1-i} k^{-2}$ cliques and so the total number of cliques used is $O\left(n^{2} /(\ln n)^{2}\right)$ as required.

We now need to describe our clique choosing process a little more formally: let $\mathcal{C}_{j, i}$ denote the set of $j$-cliques all of whose edges are in $E_{i}$. If

$$
c_{s, j, i}=\binom{n-s}{j-s}\left(b e^{i}\right)^{\binom{s}{2}-\binom{j}{2}},
$$

then $c_{s, j, i}$ is close to the expected number of cliques in $\mathcal{C}_{j, i}$ which contain a particular fixed clique in $\mathcal{C}_{s, i}$.

For a clique $S \in \mathcal{C}_{s, i}$ we let

$$
X_{S, j, i}=\left|\left\{C \in \mathcal{C}_{j, i}: C \supseteq S\right\}\right|
$$

and for integer $s \geq 0$,

$$
X_{s, j, i}^{*}=\max \left\{X_{S, j, i}: S \in \mathcal{C}_{s, i}\right\}
$$

## Algorithm COVER

## begin

$$
E_{1}:=E\left(G_{n, p}\right) ; \mathcal{C}_{\text {COVER }}:=\emptyset ;
$$

for $i=1$ to $i_{0}$ do
begin

A: independently place each $C \in \mathcal{C}_{\lfloor k / i\rfloor, i}$ into $\mathcal{C}_{\text {COVER }}$ with probability

$$
X_{2,\lfloor k / i\rfloor, i}^{*}{ }^{-1}
$$

B: for each $u \in E_{i}$ which is not covered by a clique in Step A, add $u$
(as a clique of size 2 ) to $\mathcal{C}_{\text {COVER }}$ with probability $\rho_{u}$ where

$$
e^{-1}-X_{2}^{*-1}=\left(1-\frac{1}{X_{2}^{*}}\right)^{X_{u}}\left(1-\rho_{u}\right)
$$

$$
X_{2}^{*}=X_{2,\lfloor k / i\rfloor, i}^{*} \text { and } X_{u}=X_{u,\lfloor k / i\rfloor, i} .
$$

end
$\mathcal{C}_{\text {COVER }}:=\mathcal{C}_{\text {COVER }} \cup E_{i_{0}+1}$.
end

Observe first that the definition of $\rho_{u}$ assumes that $X_{2}^{*}$ is large (which it is whp) and so

$$
\begin{aligned}
\left(1-\frac{1}{X_{2}^{*}}\right)^{X_{u}} & \geq\left(1-\frac{1}{X_{2}^{*}}\right)^{X_{2}^{*}} \\
& \geq e^{-1}-X_{2}^{*-1}
\end{aligned}
$$

and $\rho_{u}$ is properly defined.

The following lemma contains the main core of the proof:
Lemma 1. Let $\mathcal{E}_{i}$ refer to the following two conditions:
(a)

$$
X_{S, j, i} \leq\left(1+\beta_{i}\right) c_{s, j, i}, \quad 0 \leq s \leq j \leq k / i \text { and } S \in \mathcal{C}_{s, i}
$$

where $\beta_{i}=i n^{-1 / 4}$,
(b)

$$
X_{u, j, i} \geq\left(1-\gamma_{i}\right) c_{2, j, i}, \quad e \in E_{i} \text { and } 2 \leq j \leq k / i
$$

for all but at most $i n^{31 / 16}$ edges, where $\gamma_{i}=i n^{-1 / 16}$.

Then

$$
\begin{align*}
\operatorname{Pr}\left(\mathcal{E}_{1}\right) & =1-o\left(n^{-1}\right)  \tag{1}\\
\operatorname{Pr}\left(\mathcal{E}_{i+1} \mid \mathcal{E}_{i}\right) & \geq 1-O\left(n^{-1 / 16} \log n\right) \tag{2}
\end{align*}
$$

We defer the proof of the lemma to the next section and show how to use it to prove Thereom 1. Observe first that

$$
\begin{equation*}
\frac{c_{s+1, j, i}}{c_{s, j, i}}=\left(\frac{j-s}{n-s}\right)\left(b e^{i}\right)^{s} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
c_{s, j, i} \geq n^{7 / 8} \tag{4}
\end{equation*}
$$

when $\alpha$ is small and $0 \leq s<j \leq k / i$.

Next let $Y_{i}$ and $Z_{i}$ denote the number of $\lfloor k / i\rfloor$-cliques and edges respectively added to $\mathcal{C}_{C O V E R}$ in iteration $i$.

$$
\begin{align*}
\mathbf{E}\left(Y_{i} \mid \mathcal{E}_{i}\right) & \leq \mathbf{E}\left(\left.\frac{X_{0,\lfloor k / i\rfloor, i}^{*}}{X_{2,\lfloor k / i\rfloor, i}^{*}} \right\rvert\, \mathcal{E}_{i}\right) \\
& \leq(1+o(1)) \frac{c_{0,\lfloor k / i\rfloor, i}}{c_{2,\lfloor k / i\rfloor, i}} \\
& \approx \frac{n^{2} i^{2}}{b k^{2} e^{i}} \tag{5}
\end{align*}
$$

on using (3)

Since $Y_{i}$ is binomially distributed, we see using standard bounds on the tails of the binomial, that

$$
\operatorname{Pr}\left(\left.Y_{i} \geq \frac{2 n^{2} i^{2}}{b k^{2} e^{i}} \right\rvert\, \mathcal{E}_{i}\right) \leq n^{-1}
$$

Thus

$$
\operatorname{Pr}\left(\left.\sum_{i=1}^{i_{0}} Y_{i} \geq \sum_{i=1}^{i_{0}} \frac{2 n^{2} i^{2}}{b k^{2} e^{i}} \right\rvert\, \mathcal{E}_{0}\right)=O\left(\frac{i_{0} \log n}{n^{1 / 16}}\right)
$$

and so

$$
\begin{equation*}
\operatorname{Pr}\left(\sum_{i=1}^{i_{0}} Y_{i} \geq \sum_{i=1}^{i_{0}} \frac{2 n^{2} i^{2}}{b k^{2} e^{i}}\right)=o(1) \tag{6}
\end{equation*}
$$

Now a simple calculation gives

$$
\begin{equation*}
\rho_{u}=O\left(\frac{X_{2}^{*}-X_{u}}{X_{2}^{*}}\right) \tag{7}
\end{equation*}
$$

and so

$$
\begin{aligned}
\mathbf{E}\left(Z_{i} \mid \mathcal{E}_{i}\right) & =O\left(i n^{31 / 16}+\beta_{i}\left|E_{i}\right|\right) \\
& =O\left(n^{31 / 16} \ln n\right)
\end{aligned}
$$

Thus

$$
\operatorname{Pr}\left(Z_{i} \geq n^{63 / 32} \mid \mathcal{E}_{i}\right)=O\left(n^{-1 / 32} \ln n\right)
$$

and so

$$
\operatorname{Pr}\left(\exists 1 \leq i \leq i_{0}: Z_{i} \geq n^{63 / 32} \mid \mathcal{E}_{0}\right)=O\left(n^{-1 / 32}(\ln n)^{2}\right)
$$

and

$$
\begin{equation*}
\operatorname{Pr}\left(\sum_{i=1}^{i_{0}} Z_{i} \geq i_{0} n^{63 / 32}\right)=o(1) \tag{8}
\end{equation*}
$$

Also

$$
\begin{aligned}
\operatorname{Pr}\left(u \in E_{i+1} \mid u \in E_{i}\right) & =\left(1-\frac{1}{X_{2}^{*}}\right)^{X_{u}}\left(1-\rho_{u}\right) \\
& <e^{-1}
\end{aligned}
$$

Thus

$$
\mathbf{E}\left(\left|E_{i_{0}+1}\right|\right)=O\left(\frac{n^{2}}{(\ln n)^{4}}\right)
$$

and

$$
\begin{equation*}
\operatorname{Pr}\left(\left|E_{i_{0}+1}\right| \geq \frac{n^{2}}{(\ln n)^{3}}\right)=o(1) \tag{9}
\end{equation*}
$$

Theorem 1 follows from (6), (8) and (9) and

$$
\left|\mathcal{C}_{\text {COVER }}\right|=\sum_{i=1}^{i_{0}} Y_{i}+\sum_{i=1}^{i_{0}} Z_{i}+\left|E_{i_{0}+1}\right|
$$

As we only use estimates for $X_{0,\lfloor k / i\rfloor, i}^{*}$ and $X_{2,\lfloor k / i\rfloor, i}^{*}$ the reader may wonder why it is necessary to prove Lemma 1 (a) for $0 \leq s \leq j \leq k / i$. The reason is simply that the lemma is proved by induction and we use a stronger induction hypothesis than the needed outcome.

## 3 Proof of Lemma 1

If $s=j$ then $X_{S, j, i}=c_{s, j, i}=1$ and so we can assume $s<j$ from now on.

Let us first consider $\mathcal{E}_{1}$. Fix a set $S$ of size $s, 0 \leq s \leq k$. Assume it forms a clique in $G$. This does not condition any edges not contained in $S$. For a set $T$ let $N_{c}(T)$ denote the set of common neighbours of $T$ in $G$. We can enumerate the set of $j$-cliques containing $S$ as follows: choose $x_{1} \in N_{c}(S), x_{2} \in N_{c}\left(S \cup\left\{x_{1}\right\}\right), \ldots, x_{j-s} \in N_{c}\left(S \cup\left\{x_{1}, x_{2}, \ldots, x_{j-s-1}\right\}\right)$. The number
of choices $\nu_{t}$ for $x_{t}$ given $x_{1}, x_{2}, \ldots, x_{t-1}$ is distributed as $\operatorname{Bin}\left(n-(s-t+1), p^{s+t-1}\right)$. Thus for $0 \leq \epsilon \leq 1$

$$
\begin{aligned}
\operatorname{Pr}\left(\left|\frac{\nu_{t}}{(n-s-t+1) p^{s+t-1}}-1\right| \geq \epsilon\right) & \leq 2 \exp \left\{-\frac{\epsilon^{2}(n-s-t+1) p^{s+t-1}}{3}\right\} \\
& \leq 2 \exp \left\{-\epsilon^{2} n^{1-\alpha} / 4\right\}
\end{aligned}
$$

Putting $\epsilon=n^{-1 / 3}$ we see that since there are $n^{O(\ln n)}$ choices for $x_{1}, x_{2}, \ldots, x_{j-s}$,

$$
\operatorname{Pr}\left(\left|\frac{X_{S, j, 0}}{c_{s, j, 0}}-1\right| \geq n^{-1 / 3+o(1)}\right) \leq \exp \left\{-n^{1 / 4}\right\}
$$

There are $n^{O(\ln n)}$ choices for $S$ and (1) follows.

Assume now that $\mathcal{E}_{i}$ holds. We first prove
Lemma 2. Suppose $e_{1}, e_{2}, \ldots, e_{t} \in E_{i}$. Then

$$
\operatorname{Pr}\left(e_{t} \in E_{i+1} \mid e_{1}, e_{2}, \ldots, e_{t-1} \in E_{i+1}\right)=e^{-1}\left(1+O\left(\frac{t \ln n}{n}\right)\right)
$$

uniformly for $1 \leq t \leq n^{1 / 2}$.

## Proof

$$
\begin{align*}
\operatorname{Pr}\left(e_{t} \in E_{i+1} \mid e_{1}, e_{2}, \ldots, e_{t-1} \in E_{i+1}\right) & \geq \operatorname{Pr}\left(e_{t} \in E_{i+1}\right)  \tag{10}\\
& =\left(1-\frac{1}{X_{2}^{*}}\right)^{X_{u}}\left(1-\rho_{u}\right) \\
& =e^{-1}-X_{2}^{*-1}
\end{align*}
$$

Here $u=e_{t}, X_{u}=X_{u,\lfloor k / i\rfloor, i}$ and $X_{2}^{*}=X_{2,\lfloor k / i\rfloor, i}^{*}$ and inequality (10) follows from the fact that knowing $e_{1}, e_{2}, \ldots e_{t-1} \in E_{i+1}$ tells us that certain cliques (and edges) were not chosen for $\mathcal{C}_{\text {COVER }}$. On the other hand

$$
\begin{align*}
\operatorname{Pr}\left(e_{t} \in E_{i+1} \mid e_{1}, e_{2}, \ldots, e_{t-1} \in E_{i+1}\right) & \leq\left(1-\frac{1}{X_{2}^{*}}\right)^{X_{u}-t X_{3}^{*}}\left(1-\rho_{u}\right)  \tag{11}\\
& =\left(e^{-1}-X_{2}^{*-1}\right)\left(1-\frac{1}{X_{2}^{*}}\right)^{t X_{3}^{*}} \\
& =e^{-1}\left(1+O\left(\frac{t X_{3}^{*}}{X_{2}^{*}}\right)\right)
\end{align*}
$$

where $X_{3}^{*}=X_{3,\lfloor k / i\rfloor, i}^{*}$. If $\mathcal{E}_{i}$ holds then $X_{3}^{*} / X_{2}^{*}=O(\ln n / n)$.
Inequality (11) follows from the fact that $e_{t}=u$ lies in at least $X_{u}-(t-1) X_{3}^{*}$ cliques which contain none of $e_{1}, e_{2}, \ldots, e_{t-1}$. This in turn arises from a two term inclusion-exclusion inequality and the fact that $e_{t}$ and $e_{i}$ together lie in at most $X_{3}^{*}$ cliques, for $1 \leq i \leq t-1$.

Now fix a set $S \in \mathcal{C}_{s, i}$ and let $X=X_{S, j, i+1}$ for some $j \leq k /(i+1)$. Condition on $S \in \mathcal{C}_{s, i+1}$. Let $\mathcal{C}_{S, j, i}=\left\{C \in \mathcal{C}_{j, i}: C \supseteq S\right\}$. Then on using Lemma 2, we have

$$
\begin{align*}
\mathbf{E}(X) & =\sum_{C \in \mathcal{C}_{S, j, i}} \operatorname{Pr}\left(C \in \mathcal{C}_{j, i+1} \mid S \in \mathcal{C}_{s, i+1}\right) \\
& =X_{S, j, i} \exp \left\{\binom{s}{2}-\binom{j}{2}\right\}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right)  \tag{12}\\
& =\mathbf{E}\left(X_{S, j, 0}\right) \exp \left\{(i+1)\left(\binom{s}{2}-\binom{j}{2}\right)\right\}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right)
\end{align*}
$$

by induction on $i$

$$
\begin{align*}
& =c_{s, j, 0} \exp \left\{(i+1)\left(\binom{s}{2}-\binom{j}{2}\right)\right\}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right) \\
& =c_{s, j, i+1}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right) \tag{13}
\end{align*}
$$

We are going to use the Markov inequality

$$
\begin{equation*}
\operatorname{Pr}(X \geq x) \leq \frac{\mathbf{E}\left((X)_{r}\right)}{(x)_{r}} \tag{14}
\end{equation*}
$$

where $(x)_{r}=x(x-1)(x-2) \ldots(x-r+1)$ and $r=\left\lfloor n^{3 / 8}\right\rfloor$.

Let $\mathcal{B}\left(\ell_{2}, \ell_{3}, \ldots, \ell_{r}\right)=\left\{\left(C_{1}, C_{2}, \ldots, C_{r}\right):(i) C_{t} \neq C_{t^{\prime}}\right.$ for $t \neq t^{\prime}$, (ii) $C_{t} \in \mathcal{C}_{S, j, i}$, (iii) $\mid \mathcal{C}_{t} \cap\left(C_{1} \cup\right.$ $\left.C_{2} \cup \cdots C_{t-1}\right) \mid=s+\ell_{t}$, for $\left.t, t^{\prime}=2,3, \ldots, r\right\}$. Then

$$
\mathbf{E}\left((X)_{r}\right)=\sum_{\ell_{2}, \ell_{3}, \ldots, \ell_{r}} \sum_{\mathcal{B}\left(\ell_{2}, \ell_{3}, \ldots, \ell_{r}\right)} \operatorname{Pr}\left(C_{1}, C_{2}, \ldots, C_{r} \in \mathcal{C}_{j, i+1} \mid S \in \mathcal{C}_{s, i+1}\right) .
$$

From (12)

$$
\operatorname{Pr}\left(C_{1} \in \mathcal{C}_{j, i+1} \mid S \in \mathcal{C}_{s, i+1}\right)=\exp \left\{\binom{s}{2}-\binom{j}{2}\right\}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right)
$$

and

$$
\begin{aligned}
\operatorname{Pr}\left(C_{t} \in \mathcal{C}_{j, i+1} \mid C_{1}, C_{2}, \ldots, C_{t-1} \in \mathcal{C}_{j, i+1}\right) & =\exp \left\{\binom{s+\ell_{t}}{2}-\binom{j}{2}\right\}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right) \\
& =\exp \left\{\binom{s+\ell_{t}}{2}-\binom{s}{2}\right\} \frac{c_{s, j, i+1}}{c_{s, j, i}}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\left|\mathcal{B}\left(\ell_{2}, \ell_{3}, \ldots, \ell_{r}\right)\right| & \leq \prod_{t=1}^{r}\left(\binom{(t-1) j-s}{\ell_{t}} X_{s+\ell_{t}, j, i}^{*}\right) \\
& \leq \prod_{t=1}^{r}(r j)^{\ell_{t}}\left(1+\beta_{i}\right)\left(\frac{b^{s+\ell_{t}} j e^{i\left(s+\ell_{t}\right)}}{n}\right)^{\ell_{t}} c_{s, j, i} .
\end{aligned}
$$

Hence,

$$
\begin{align*}
\frac{\mathbf{E}\left((X)_{r}\right)}{c_{s, j, i+1}^{r}} & \leq\left(1+O\left(\frac{(\ln n)^{4} r}{n}\right)\right) \sum_{\ell_{2}, \ell_{3}, \ldots, \ell_{r}} \prod_{t=1}^{r}\left(1+\beta_{i}\right)\left(\frac{e^{\left(\ell_{t}+2 s-1\right) / 2} r j^{2}\left(b e^{i}\right)^{s+\ell_{t}}}{n}\right)^{\ell_{t}} \\
& \leq\left(1+O\left(\frac{(\ln n)^{4} r}{n}\right)\right)\left(1+\beta_{i}\right)^{r} \sum_{\ell_{2}, \ell_{3}, \ldots, \ell_{r}}\left(\frac{r k^{2} e^{3 k} b^{2 k}}{n}\right)^{\ell_{2}+\cdots+\ell_{t}}  \tag{15}\\
& \leq\left(1+r n^{-3 / 4}\right)\left(1+\beta_{i}\right)^{r}, \tag{16}
\end{align*}
$$

for $\alpha$ sufficiently small.

Hence, using (14),

$$
\begin{array}{rlr}
\operatorname{Pr}\left(X \geq\left(1+\beta_{i+1}\right) c_{s, j, i+1}\right) & \leq \frac{2\left(1+\beta_{i}\right)^{r} c_{s, j, i+1}^{r}}{\left(\left(1+\beta_{i+1}\right) c_{s, j, i+1}\right)_{r}}, \quad \text { by }(16) \\
& \leq 3\left(\frac{1+\beta_{i}}{1+\beta_{i+1}}\right)^{r}, & \text { using }(4) \\
& \leq 3 \exp \left\{-\frac{r\left(\beta_{i+1}-\beta_{i}\right)}{1+\beta_{i+1}}\right\} & \\
& =\exp \left\{-n^{1 / 8-o(1)}\right\}
\end{array}
$$

There are $n^{O(\ln n)}$ choices for $S$ and $j$ and so part (a) of the lemma is proven.
It remains only to deal with $X_{u, j, i+1}$ for an edge $u \in E_{i}$. It follows from (13) that if $X=X_{u, j, i+1}$ then

$$
\begin{equation*}
\mathbf{E}(X)=c_{2, j, i}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right) \tag{17}
\end{equation*}
$$

and from (16) that

$$
\begin{equation*}
\mathbf{E}(X(X-1)) \leq\left(1+\frac{3 i}{n^{1 / 4}}\right) c_{2, j, i+1}^{2} . \tag{18}
\end{equation*}
$$

Suppose now that $X_{u, j, i} \geq\left(1-\gamma_{i}\right) c_{2, j, i}$. Then (17) and (18) imply that

$$
\begin{align*}
& \operatorname{Pr}\left(X \leq\left(1-\gamma_{i+1}\right) c_{2, j, i+1}\right)= \\
& \operatorname{Pr}\left(\mathbf{E}(X)-X \geq \mathbf{E}(X)-\left(1-\gamma_{i+1}\right) c_{2, j, i+1}\right) \leq \\
& \operatorname{Pr}\left(\mathbf{E}(X)-X \geq\left(1-\gamma_{i}\right) c_{2, j, i} \exp \left\{1-\binom{j}{2}\right\}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right)-\left(1-\gamma_{i+1}\right) c_{2, j, i+1}\right)= \\
& \operatorname{Pr}\left(\mathbf{E}(X)-X \geq\left(1-\gamma_{i}\right) c_{2, j, i+1}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right)-\left(1-\gamma_{i+1}\right) c_{2, j, i+1}\right)= \\
& \operatorname{Pr}\left(\mathbf{E}(X)-X \geq(1-o(1)) n^{-1 / 16} c_{2, j, i+1}\right) \leq \\
& \frac{(\mathbf{E}(X)-X)^{2}}{(1-o(1)) n^{-1 / 8} c_{2, j, i+1}^{2}}= \\
& \frac{\mathbf{E}(X(X-1))+\mathbf{E}(X)-\mathbf{E}(X)^{2}}{(1-o(1)) n^{-1 / 8} c_{2, j, i+1}^{2}} \leq \\
& \frac{\left(1+\frac{3 i}{n^{1 / 4}}\right) c_{2, j, i+1}^{2}+\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right) c_{2, j, i+1}-c_{2, j, i+1}^{2}\left(1+O\left(\frac{j^{4} \ln n}{n}\right)\right)}{(1-o(1)) n^{-1 / 8} c_{2, j, i+1}^{2}} \leq \\
& \leq 4 i n^{-1 / 8} . \tag{19}
\end{align*}
$$

Now let $Z_{i+1}$ denote the number of edges $u \in E_{i+1}$ for which $X_{u, j, i+1} \leq\left(1-\gamma_{i+1}\right) c_{2, j, i+1}$ and $\hat{Z}_{i+1}$ those $u$ counted in $Z_{i+1}$ for which $X_{u, j, i} \geq\left(1-\gamma_{i}\right) c_{2, j, i}$. Then

$$
Z_{i+1} \leq Z_{i}+\hat{Z}_{i+1}
$$

and from (19)

$$
\mathbf{E}\left(\hat{Z}_{i+1} \mid \mathcal{E}_{i}\right) \leq 4 i\left|E_{i}\right| n^{-1 / 8} .
$$

So

$$
\begin{aligned}
\operatorname{Pr}\left(Z_{i+1} \geq(i+1) n^{31 / 16} \mid \mathcal{E}_{i}\right) & \leq \operatorname{Pr}\left(\hat{Z}_{i+1} \geq n^{31 / 16} \mid \mathcal{E}_{i}\right) \\
& =O\left(n^{-1 / 16} \log n\right)
\end{aligned}
$$

this completes the proof of Lemma 1.

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## References

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