On the Complexity of Computing the Diameter of a Polytope

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May 22, 2006

Abstract

In this paper, some results on the complexity of computing the combinatorial diameter of a polytope are presented. We show that it is D^P-hard to determine the diameter of a polytope specified by linear inequalities with integer data. Our result partially resolves a long-term open question.

1 Introduction

The basic idea of the simplex method for linear programming is to find a path from a vertex of the underlying polyhedron to an optimal one along edges. In graph-theoretic terms, the simplex method computes a path in graph $\Gamma(P)$, the *1–complex* formed by the vertices and edges of the input polytope P, from an initial vertex to an optimal one. The efficiency of the simplex method is determined by the length of the path it computes. Therefore, the diameter of the graph $\Gamma(P)$ provides a natural lower bound for the simplex method.

Although, the diameter of polyhedral graphs has been studied intensively (see Klee 1974 and Larman 1970), tight bounds on the diameter in terms of the number of facets are still not known. Kalai (1991) gave the first a subexponential upper bound on the maximum diameter of d-polytopes with

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n facets. Recently, Kalai and Kleitman (1992) further improved the upper bound to n^{logd+2}.

In this paper, we study the complexity of computing the diameter of a polytope. On one hand, it is easy to see that the diameter of a polytope can be computed in Δ_3, under the assumption that the diameter is bounded by a polynomial in the number of facets. On the other hand, we show that it is D^P-hard to determine the diameter of a polytope given by its facets, where D^P is the following class of languages defined by Papadimitriou and Yannakakis (1984).

$$D^P = \{L_1 \cap L_2 : L_1 \in NP \ \&\ L_2 \in co\text{-}NP\}.$$

Our result partially resolves a long-term open question.

2 Definitions and problems

Let $V(P)$ denote the set of all vertices of a polytope P, $E(P)$ the set of all edges of P, and $F(P)$ be the set of all facets of P.

Define $\Gamma(P)$ to be the *1–complex* formed by the vertices and edges of a polytope P, i.e., $\Gamma(P) = (V(P), E(P))$.

The *distance* of two vertices u and v in a graph G, denoted by $distance_G(u, v)$, is the length of the shortest path between u and v in G. The radius of a vertex v in G is

$$radius_G(v) = \max\{distance_G(v, u) : u \in G\}$$

and the diameter of a graph G is

$$diameter(G) = \max\{radius_G(v) : v \in G\}.$$

The *diameter of a polytope* P is defined to be the diameter of $\Gamma(P)$.

We study the following computational problem:

> **COMPUTING DIAMETER**: *given* $P = \{x : Ax \leq b\}$, a set of n half-spaces in m dimensions, *compute* the diameter of $\Gamma(P)$.

The following are two related decision problems,

- **DIAMETER**: *given* a polytope $P = \{x : Ax \leq b\}$ and an integer k, *is* k the diameter of $\Gamma(P)$?
- **RADIUS**: *given* a polytope $P = \{x : Ax \leq b\}$, a vertex v of P, and an integer k, *is* $radius_{\Gamma(P)}(v) = k$?

3 The NP-hardness of computing DIAMETER

When a graph has N vertices, using Dijkstra's shortest path algorithm, the diameter can be computed in $O(N^3)$ time. However, in general, the number of vertices of a polytope may be $\Omega(n^{m/2})$. Only when the dimension is fixed, can breadth-first search be used to compute the diameter of a polytope in polynomial time.

In this section, we give a proof that DIAMETER is NP-hard. The idea of the proof will be used in the next section to show that DIAMETER is in fact D^P-hard. The reduction is from the following NP-complete problem (see Karp 1972 and Garey & Johnson 1979).

> **EXACT PARTITION**: *given* a finite set $A = \{s_1, \ldots, s_{2m}\}$ of integers, *is* there a subset $A' \subset A$ with $|A'| = m$ and
>
> $$\sum_{s \in A'} s = \sum_{t \in A - A'} t?$$

3.1 The basic reduction

The basic idea is to show that for each instance A of EXACT PARTITION, we can, in polynomial time, construct a polytope P_A with a polynomial number of faces and an integer k such that

$$diameter(P_A) = \begin{cases} k & \text{if } A \text{ has an exact partition,} \\ k-1 & \text{if } A \text{ has no exact partition.} \end{cases} \tag{1}$$

Note first that EXACT PARTITION with $A = \{s_1, \ldots, s_{2m}\}$ is equivalent to an integer linear program of the following simple form ILP1 (see Korte & Schrader 1981).

maximize $\sum_{i=1}^{2m} x_i$
subject to

$$\begin{aligned} \sum_{i=1}^{2m} s_i x_i &\le \frac{1}{2} S \\ \sum_{i=1}^{2m} d_i x_i &\le \frac{1}{2} \sum_{i=1}^{2m} d_i \\ x_i &\in \{0, 1\} \end{aligned}$$

where $S = \sum_{i=1}^{2m} s_i$, $s_{\max} = \max_i s_i$, and $d_i = s_{\max} - s_i$.

Lemma 1 (Korte and Schrader) *A has an exact partition iff ILP1 has an optimal solution of value m.* $\square$

Note also that all coefficients in ILP1 are non-negative. Let $M = (\sum_{i=1}^{2m} s_i) + 1$. We modify ILP1 to the following integer linear program: ILP2

maximize $\sum_{i=1}^{2m} x_i$
subject to

$$\begin{aligned}
\sum_{i=1}^{2m}(M+s_i)x_i &\leq \frac{1}{2}S + mM + \epsilon \\
\sum_{i=1}^{2m}(M+d_i)x_i &\leq \frac{1}{2}\sum_{i=1}^{2m} d_i + mM + \epsilon \\
x_i &\in \{0,1\}
\end{aligned}$$

Clearly, ILP1 has an optimal solution of value m iff ILP2 has an optimal solution of value m. Further we chose $\epsilon = \frac{1}{2M}$ in ILP2 so that the linear programming relaxation of ILP2 is *non-degenerate.*

Proposition 2 *When $\epsilon = \frac{1}{2M}$, the polytope defined by the linear programming relaxation of ILP2 is non-degenerate.*

Proof Suppose the polytope defined by the linear programming relaxation of ILP2 is degenerate, then there are $2m+1$ inequalities that are satisfied with equality by a common point x. This implies that

$$|\{i : x_i \in \{0,1\}\}| \geq 2m - 1.$$

Case 1: $x_i \in \{0,1\}$, $i = 1, 2, ..., 2m$. In this case, one of the first two inequalities is tight which implies that 2ϵ is an integer, a contradiction.

Case 2: $|\{i : x_i \in \{0,1\}\}| = 2m - 1$. Without loss of generality, assume $0 < x_1 < 1$. Then,

$$\begin{aligned}
\sum_{i=1}^{2m}(M+s_i)x_i &= \frac{1}{2}S + mM + \epsilon \\
\sum_{i=1}^{2m}(M+d_i)x_i &= \frac{1}{2}\sum_{i=1}^{2m} d_i + mM + \epsilon
\end{aligned}$$

After eliminating x_1, we see that there is an integer c, such that

$$\epsilon = \frac{c}{2d_1}$$

which contradicts with our assumption that $\epsilon = \frac{1}{2M}$, and therefore the proposition follows. $\square$

Proposition 3 *The value of the optimal solution of $ILP2$ is either m or $m-1$.*

Proof This is because all $\vec{x} = (x_1, \ldots, x_{2m})$ with $\sum_{i=1}^{2m} x_i = m-1$ are feasible solutions to ILP2 and no $\vec{x}$ with $\sum_{i=1}^{2m} x_i \geq m+1$ is a feasible solution to ILP2. $\square$

Consequently,

Lemma 4 *ILP2 has an optimal solution of value m if A has an exact partition, otherwise the optimal value is $m-1$.* $\square$

By relaxing the integrality constraints in ILP2, we obtain a polytope P'_A defined to be the set of $\vec{x}$ satisfying

$$\begin{aligned} \sum_{i=1}^{2m}(M+s_i)x_i &\leq \frac{1}{2}S + mM + \epsilon \\ \sum_{i=1}^{2m}(M+d_i)x_i &\leq \frac{1}{2}\sum_{i=1}^{2m} d_i + mM + \epsilon \\ 0 \leq x_i &\leq 1 \end{aligned} \tag{2}$$

Geometrically, P'_A is a polytope obtained from the unit $2m$-cube by cutting it with two half-spaces with non-negative coefficients. We denote the associated boundary hyperplanes by HS_1 and HS_2 in the following discussion. Now $\vec{0}$, the origin of $2m$–space, is a vertex of P'_A, and the diameter of P'_A is bounded by $2m+4$. This latter fact follows immediately from

Lemma 5

$$radius_{\Gamma(P'_A)}(\vec{0}) = \begin{cases} m+2 & \textit{if } A \textit{ has an exact partition,} \\ m+1 & \textit{if } A \textit{ has no exact partition.} \end{cases}$$

Proof We first prove that $radius_{\Gamma(P'_A)}(\vec{0}) \leq m+2$, if A has an exact partition, and $radius_{\Gamma(P'_A)}(\vec{0}) \leq m+1$ otherwise.

The set of vertices in P'_A can be partitioned into three subsets;

- V_0: the set of all vertices lying neither on HS_1 nor HS_2;
- V_1: the set of all vertices lying on one of HS_1 or HS_2, but not both, and
- V_2: the set of all vertices lying on both HS_1 and HS_2.

Note that all components of a vertex in V_0 are either 0 or 1. Since the linear system (2) is non-degenerate, all vertices in V_1 have exactly one non-integer co-ordinate and all vertices in V_2 have exactly two non-integer co-ordinates, and all other co-ordinates are either 0 or 1.

It follows from Proposition 3, that the number of components of value 1 of a vertex of P'_A is bounded from above by m if A has an exact partition, and by $m-1$ otherwise.

Note first that the distance from $\vec{0}$ to each vertex $\vec{v}$ in V_0 is equal to the number of components of value 1 in $\vec{v}$, which is bounded from above by m if A has an exact partition, and by $m-1$ otherwise.

Consider a vertex $\vec{v}$ in V_i for $i \in \{1,2\}$. Without loss of generality assume $\vec{v}$ is on HS_1 and $\vec{v} = (v_1, \ldots, v_{2m})$ with $v_j = 1$ for $1 \leq j \leq l$, v_{l+1} and v_{l+i} non-integral, $i \in \{1,2\}$, and $v_j = 0$ for $l+i < j \leq 2m$. Note that $l \leq m$ if A has an exact partition, and $l \leq m-1$ otherwise.

Let $\vec{v}' = (v'_1, \ldots, v'_{2m})$ with $v'_j = 1$ for $1 \leq j \leq l+i-1$ and $v'_j = 0$ for $l+1 \leq j \leq 2m$. Using non-degeneracy, we see that $\vec{v}' \in V(P'_A) \cap V_{i-1}$. Note that $(\vec{v}, \vec{v}') \in E(P'_A)$ because we can obtain the basic feasible solution associated with $\vec{v}$ from the one associated with $\vec{v}'$ by replacing $x_{l+i} = 0$ by HS_i. Consequently. the distance from $\vec{0}$ to a vertex $\vec{v}$ in V_i is bounded by $l+i$ (a two step induction.)

We now prove that $radius_{\Gamma(P'_A)}(\vec{0}) \geq m+2$, if A has an exact partition, and $radius_{\Gamma(P'_A)}(\vec{0}) \geq m+1$ otherwise.

By adding slack variables, the linear system (2) takes the form

$$
\begin{aligned}
\sum_{i=1}^{2m}(M+s_i)x_i + z_1 &= \frac{1}{2}S + mM + \epsilon \\
\sum_{i=1}^{2m}(M+d_i)x_i + z_2 &= \frac{1}{2}\sum_{i=1}^{2m} d_i + mM + \epsilon \qquad (3) \\
x_i + y_i &= 1 \\
x_i, y_i, z_k &\geq 0
\end{aligned}
$$

Note that, the set of *basic variables* associated with the vertex $\vec{0}$ is $BV_1 = \{y_1, \ldots, y_{2m}, z_1, z_2\}$.

First of all assume that A has an exact partition and without loss of generality, assume $\{s_1, ..., s_m\}$ and $\{s_{m+1}, ..., s_{2m}\}$ is one. Since the linear system (2), and hence (3) is non-degenerate, there is a basic feasible solution of the following form:

$$
\begin{aligned}
(x_i = 1) \ \& \ (y_i = 0) & \qquad 1 \leq i \leq m \\
(x_{m+1} = \zeta_1) \ \& \ (y_{m+1} = 1 - \zeta_1) \ \& \ (z_{m+1} = 0) & \qquad (4) \\
(x_{m+2} = \zeta_2) \ \& \ (y_{m+2} = 1 - \zeta_2) \ \& \ (z_{m+2} = 0) & \\
(x_j = 0) \ \& \ (y_j = 1) & \qquad m+3 \leq j \leq 2m
\end{aligned}
$$

where $0 < \zeta_1, \zeta_2 < 1$.

The vertex associated with the above basic feasible solution is

$$\vec{v} = (1, ..., 1, \zeta_1, \zeta_2, 0, ...0),$$

and the associated basic variables are

$$BV_2 = \{x_1, \ldots, x_m, x_{m+1}, x_{m+2}, y_{m+1}, ..., y_{2m}\}.$$

¿From $|BV_1 - BV_2| \geq m+2$ it follows that $distance(\vec{0}, \vec{v}) \geq m+2$ and hence $radius_{\Gamma(P'_A)}(\vec{0}) \geq m+2$.

Similarly, if A does not have an exact partition then we can show $radius_{\Gamma(P'_A)}(\vec{0})$ $\geq m+1$, by considering the basic feasible solution $(1, ..., 1, \zeta_1, \zeta_2, 0, ...0)$ where there are $m-1$ 1's. □

Consequently,

Theorem 6 *RADIUS is NP–hard.* □

We now show how to construct the polytope P_A from P'_A, which satisfies (1).

¿From the definition of P'_A, we have for all $i : 1 \leq i \leq 2m$, $\vec{w}_i = (\mu_1, \ldots, \mu_{2m})$, with $\mu_i = 1$ and $\mu_j = 0$, for all $j \neq i : 1 \leq j \leq 2m$, are vertices of P'_A. Moreover, they are exactly the set of all neighbors of $\vec{0}$ in $\Gamma(P'_A)$.

By adding the constraint $H_0 = \{x : \sum_{i=1}^{2m} x_i \geq 1\}$ to P'_A, we obtain a new polytope P''_A in which $F_0 = H_0 \cap P'_A$ is a face. F_0 is a $(2m-1)$–simplex with the set of vertices $\{\vec{w}_1, \ldots, \vec{w}_{2m}\}$.

Now the idea is to construct a polytope $P_{2m}(F_0, m+6)$, a stack of simplices with *base* F_0, which has the following properties

1. there is a vertex $\vec{o}$ in $P_{2m}(F_0, m+6)$ such that the distance between $\vec{o}$ and any vertex in F_0 is $m+6$.

2. $P_A = P''_A \cup P_{2m}(F_0, m+6)$ forms a polytope.

First, we give the construction of $P_{2m}(F_0, m+6)$. Then, we shall show that

$$diameter(P_A) = radius_{\Gamma(P_A)}(\vec{o}) = radius_{\Gamma(P'_A)}(\vec{0}) + m + 5.$$

In the following procedure, let $\Delta = \sqrt{1/2m}$, the distance from $\vec{0}$ to the hyperplane $\sum_i x_i = 1$ and the symbol $\approx$ denotes the rational approximation with a predefined precision.

Procedure to Define $P_{2m}(F_0, k)$:

1. $P_{2m}(F_0, 0)$ is defined to be the simplex with the set of vertices $\{\vec{w}_1, \ldots, \vec{w}_{2m}\} \cup \{\vec{o}_0\}$, where $\vec{o}_0 = \vec{0}$;

2. for all $k > 0$, $P_{2m}(F_0, k)$ can be constructed from $P_{2m}(F_0, k-1)$ as follows:

   (a) $d_k = \frac{1}{2}\Delta + \frac{1}{2^{k+1}}\Delta$;

   (b) $c_k \approx \sqrt{2m} d_k$;

   (c) $d'_k = \frac{1}{2}\Delta - \frac{1}{2^{k+1}}\Delta$;

   (d) Let $H_k = \{x : \sum_{i=1}^{2m} x_i \geq c_k\}$;

   (e) Let $\vec{o}_k = (\alpha_k, ..., \alpha_k)$, where $\alpha_k \approx \frac{d'_k}{\sqrt{2m}}$;

(f) Let polytope $Q_k = P_{2m}(F_0, k-1) \cap H_k$.

(g) Let polytope $P_{2m}(F_0, k)$ be the convex hull of $\{\vec{o}_k\} \cup V(Q_k)$.

Note that $\vec{o}_k$ in the above procedure is the point on the ray $\{x_i = x_j : 1 \leq i, j \leq 2m\}$, whose distance from $\vec{0}$ is d'_k, and H_k is the hyperplane parallel to H_0, whose distance from $\vec{0}$ is d_k.

Lemma 7 *for all $k \geq 0$,*

1. *The polytope $P_{2m}(F_0, k)$ has $2(k+1)m+1$ vertices and $2(k+1)m+1$ faces;*
2. *$P''_A \cup P_{2m}(F_0, k)$ forms a polytope;*
3. *All coefficients in the new faces have size bounded polynomially in m and k.*

Proof We prove the theorem by induction on k.
Clearly, the lemma and the following statements are true when $k = 0$.

1. $\vec{o}_k$ is a vertex of $P_{2m}(F_0, k)$ and is the intersection of exactly $2m$ faces;
2. All neighbors of $\vec{o}_k$ in $P_{2m}(F_0, k)$ are on the hyperplane defined by H_k.

Assume the Lemma and the above statements are true for $k-1$. We now prove that they are true for k.

Note that the hyperplane defined by H_k is parallel to the hyperplane defined by H_{k-1} and H_k separates $\vec{o}_{k-1}$ and $\vec{o}_k$ from Q_k. Moreover, $V(P_{2m}(F_0, k-1)) - \{\vec{o}_{k-1}\} \subset V(Q_k)$.

Applying the induction hypotheses, we see that $\vec{o}_{k-1}$ is a vertex of $P_{2m}(F_0, k-1)$ and is the intersection of exactly $2m$ faces and all neighbors of $\vec{o}_k$ in $P_{2m}(F_0, k-1)$ are on the hyperplane defined by H_{k-1}. Therefore, $H_k \cap Q_k$ contains exactly $2m$ vertices of Q_k, and those $2m$ vertices are all the new vertices introduced in Q_k, which do not belong to $V(P_{2m}(F_0, k-1))$, and hence,

$$|F(Q_k)| = |F(P_{2m}(F_0, k-1))| + 1.$$

Since, $\vec{o}_k \in \text{int } P_{2m}(F_0, k-1)$ and the hyperplane defined by H_k, which contains $2m$ vertices of Q_k, separates $\vec{o}_k$ from Q_k, $P''_A \cup P_{2m}(F_0, k)$ forms a polytope, and $\vec{o}_k$ is a vertex of $P_{2m}(F_0, k)$ and is the intersection of $2m$ faces of $P_{2m}(F_0, k)$ and all neighbors of $\vec{o}_k$ in $P_{2m}(F_0, k)$ are on the hyperplane H_k. Moreover,

$$V(P_{2m}(F_0, k)) = V(Q_k) \cup \{\vec{o}_k\}.$$

Therefore,

$$|V(P_{2m}(F_0, k))| = |V(P_{2m}(F_0, k-1))| + 2m = 2(k+1)m + 1.$$

Similarly,

$$|F(P_{2m}(F_0, k))| = |F(P_{2m}(F_0, k-1))| + 2m = 2(k+1)m + 1.$$

Note also, all coefficients in the new face have size bounded polynomially in m and k. $\square$

Lemma 8

$$diameter(P_A) = radius_{\Gamma(P'_A)}(\vec{0}) + m + 5.$$

Proof ¿From the construction of P_A, we have for all vertices $\vec{v} \in V(P_A) - V(P'_A)$, $distance(\vec{o}, \vec{v}) \leq m+5$, and $distance(\vec{o}, \vec{w}_i) = m+6$ $(1 \leq i \leq 2m)$. Thus, for all $\vec{u} \in V(P'_A)$, $distance_{P_A}(\vec{o}, \vec{u}) = distance_{P'_A}(\vec{0}, \vec{u}) + m + 5$. Therefore,

$$radius_{\Gamma(P_A)}(\vec{o}) = radius_{\Gamma(P'_A)}(\vec{0}) + m + 5 \geq 2m + 6.$$

We now prove that $diameter(P_A) = radius_{\Gamma(P_A)}(\vec{o})$.

This is true because for all pairs of vertices in P'_A, their distance in P_A is no more than their distance in P'_A, which is bounded by $2m+4$; and for all vertices in $V(P_A) - V(P'_A)$, $\vec{o}$ is the vertex with the largest radius. $\square$

Theorem 9 *DIAMETER is NP–hard.*

Proof The theorem is a simple consequence of Lemmas 5 and 8. $\square$

4 Polytope products and D^P-hardness

Let P_1 and P_2 be two polytopes in, respectively, m_1 and m_2 space. $P_1 \odot P_2$, the *product of polytopes* P_1 and P_2, is a polytope in $m_1 + m_2$ space, such that

$$P_1 \odot P_2 = \{(x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}) | (x_1, \ldots, x_{m_1}) \in P_1 \ \& \ (y_1, \ldots, y_{m_2}) \in P_2\}.$$

Note that $\odot$ is associative.

Algebraicly, if $P_1 = \{x : A_1 x \leq b_1\}$ and $P_2 = \{y : A_2 y \leq b_2\}$ then

$$P_1 \odot P_2 = \{(x, y) : A_1 x \leq b_1, A_2 x \leq b_2\}.$$

Therefore, we have $f(P_1 \odot P_2) = f(P_1) + f(P_2)$, where $f(P)$ denotes the number of faces of polytope P.

We now show how $\Gamma(P_1 \odot P_2)$ is defined in term of $\Gamma(P_1)$ and $\Gamma(P_2)$.

The *product* of two graphs $G(V_1, E_2)$ and $G(V_2, E_2)$, denoted by $G_1 \odot G_2$, is a new graph $G(V, E)$ with $V = V_1 \times V_2$ and

$$E = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \in E_1 \ \& \ v_1 = v_2 \text{ or } \mathrm{u_1 = u_2 \ \& \ (v_1, v_2) \in E_2}\}.$$

Proposition 10

$$\Gamma(P_1 \odot P_2) = \Gamma(P_1) \odot \Gamma(P_2).$$

□

Proposition 11

$$Diameter(G_1 \odot G_2) = Diameter(G_1) + Diameter(G_2).$$

□

Let PARTITON–UNPARTITON be the problem of "given (A_1, A_2), where $A_1 = \{s_1, \ldots, s_{2n_1}\}$ and $A_2 = \{s'_1, \ldots, s'_{2n_2}\}$, does A_1 have an exact partition, while A_2 does not?".

Lemma 12 *PARTITION–UNPARTITION is complete for D^P.*

Proof Clearly, PARTITION–UNPARTITION is in D^P. To prove it is complete, we see that from any instance x of a problem in D^P, we can construct two sets A_1 and A_2, one for the NP-predicate of A and one for the co–NP one. □

Theorem 13 *DIAMETER is hard for* D^P.

Proof We reduce PARTITION–UNPARTITION to DIAMETER. Given (A_1, A_2), we construct two polytopes P_1 and P_2, respectively, for A_1 and A_2, such that A_i has an exact partition iff $DIAMETER(\Gamma(P_i)) = k_i$, and has no exact partition iff $DIAMETER(\Gamma(P_i)) = k_i - 1$, where $k_1 \neq k_2$. Let $P = P_1 \odot P_1 \odot P_2$. It is easy to see that $(A_1, A_2) \in PARTITION–UNPARTITION$ iff $DIAMETER(P) = 2k_1 + k_2 - 1$. □

Similarly we can prove,

Theorem 14 *RADIUS is hard for* D^P.

5 Upper bound for DIAMETER

Consider the following decision problem:

> **DIAMETER-DECISION**: *given* $P = \{Ax \leq b\}$ and $k \in \mathcal{N}$, *is* diameter $G(P) \leq k$?

Lemma 15 *The problem DIAMETER-DECISION is in* Π_2 *under the assumption that the diameters of polytopes are polynomially bounded by the number of faces.*

Proof The lemma follows from the facts that in polynomial time, we can decide whether a point is a vertex of a polytope and whether a pair of vertices is an edge of a polytope. □

Using binary search and the Π_2 oracle for DIAMETER-DECISION, we can show that both DIAMETER and RADIUS are in Δ_3 (under the polynomial assumption).

6 Open questions

1. Is DIAMETER in the polynomial time hierarchy (see Stockmeyer 1977) (without any assumption)?

2. Is DIAMETER complete for Δ_3 and the DIAMETER-DECISION complete for Π_2?

3. Can we approximate the diameters of polytopes in random polynomial time?

4. Can we improve the straightforward method for COMPUTING DIAMETER (especially for fixed dimension)?

Acknowledgement The research of Alan Frieze was supported in part by National Science Foundation Grant CCR-089-00112. The research of Shang-Hua Teng was supported in part by NSF Grant CCR-9016641. His current address is: Department of Mathematics and Laboratory for Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139.

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