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COVERING A SYMMETRIC POSET BY SYMMETRIC CHAINS

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We prove a min-max result on special partially ordered sets, a conjecture of András Frank. As corollaries we deduce Dilworth's theorem and the well-known min-max formula for the minimum size edge cover of a graph.

1. Introduction

We prove the following result, conjectured by András Frank [4]:

Theorem 1.1. *Let $P=(V,\preceq,M)$ be a symmetric poset. The minimal number of symmetric chains needed to cover P is equal to the maximum value of a legal subpartition of P.*

Here $P=(V,\preceq,M)$ is a *symmetric poset* if $(V,\preceq)$ is a finite poset and M is a perfect matching on V such that $u\preceq v$ and $uu',vv'\in M$ implies $u'\succeq v'$. By $u\prec v$ we mean that $v\neq u\preceq v$. A subset $\{u_1v_1,u_2v_2,\dots,u_kv_k\}$ of M is a *symmetric chain* in the symmetric poset $P=(V,\preceq,M)$ if $u_i\prec u_{i+1}$ for $1\leq i<k$. Symmetric chains $S_1,S_2,\dots,S_t$ *cover* the symmetric poset P if $M=\bigcup_{i=1}^{t}S_i$.

$M_1,M_2,\dots,M_l\subset M$ is a *legal subpartition of* P if

(1) $u_1v_1\in M_i$, $u_2v_2\in M_j$ and $u_1\preceq u_2$ yields $i=j$ and

(2) there is no symmetric chain of length three contained in any M_i.

The *value* of the legal subpartition $\mathscr{L}$ is $\sum_{M_i\in\mathscr{L}}\left\lceil\frac{|M_i|}{2}\right\rceil$.

We use the following notation: $G-X+Y$ means the graph we obtain from G by deleting vertices (or edges) of X and adding edges of Y. For pairs (x,y) or two

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element sets $\{x,y\}$ we write xy, and remind the reader that $R=\{xy:xRy\}$ for any binary relation R.

We also recall some facts from matching theory. The Tutte–Berge formula [1] states that the size of a maximum matching of G is

$$\nu(G) = \min\left\{\frac{1}{2}\left[|V| + |X| - o(G-X)\right] : X \subset V\right\},$$

where $o(H)$ stands for the number of odd components of graph H. The Edmonds–Gallai decomposition [3, 5, 6] of a graph G is the partition of $V(G)$ into the following sets:

$D(G) = \{v \in V : \exists$ maximum matching of G not covering $v\}$,
$A(G) = \Gamma(D(G)) \setminus D(G)$, and
$C(G) = V \setminus (D(G) \cup A(G))$, where

$\Gamma(X) := \{v \in V : (\exists x \in X : vx \in E)\}$ denotes the set of neighbours of subset X of V. The main property of the decomposition is that

$$(*) \qquad \nu(G) = \frac{1}{2}\left[|V| + |A(G)| - o(G - A(G))\right].$$

Edmonds in [3] also gives a polynomial-time method to construct the above decomposition.

The proof of Theorem 1.1 is in Section 2. We also discuss the algorithmic aspects there. Section 3 contains two, in a sense extreme cases: Dilworth's theorem and the well-known result for the edge covering number of a graph are deduced as corollaries. A generalization of Theorem 1.1 is also obtained with the help of the well-known node splitting construction. Section 3 can be understood without knowing the proof of our main result.

2. The proof of the main result

In this section we prove our main result.

Proof of Theorem 1.1. Clearly any symmetric chain intersects at most one part M_i of a legal subpartition $\mathscr{L}$ and such an intersection contains at most two elements. So the value of a legal subpartition is a lower bound for the number of symmetric chains needed to cover M.

For the reverse inequality we prove that there is a special symmetric chain cover, namely a symmetric chain partition $\mathscr{S}$ of M and a legal subpartition $\mathscr{L}$ of P such that $|\mathscr{S}|$ is equal to the value of $\mathscr{L}$.

Define the undirected graph $G=(V,E)$ by

$$E := \{uv' : \exists v \text{ such that } vv' \in M \text{ and } u \prec v\}.$$

E is well defined, as the equivalence of $u \prec v$ and $u' \succ v'$ (where uu', $vv' \in M$) yields that $uv' \in E$ if and only if $v'u \in E$.

Observe that $\{m_1, m_2, \ldots, m_k\} \subset M$ is a symmetric chain if and only if there exist $e_1, e_2, \ldots, e_{k-1} \in E$ such that $m_1 e_1 m_2 e_2 \ldots m_{k-1} e_{k-1} m_k$ is a path (called an *ME-alternating path*). Observe moreover that transitivity of the partial order $\preceq$ means that if $vv' \in M$ and $u_1 v, u_2 v' \in E$ then $u_1 u_2 \in E$ (we refer to this property as the *transitivity* of E). Also, there is no ME-alternating cycle (i.e. a closed ME-alternating path) because of the acyclicity of the ordering.

So now we are looking for a decomposition of V into the minimum number of ME-alternating paths. A decomposition of V into k ME-alternating paths contains $|M| - k$ independent edges of E. On the other hand, if $I \subset E$ is a matching then $M \cup I$ contains exactly $|M| - |I|$ ME-alternating paths and some ME-alternating cycles that together cover V. As there is no ME-alternating cycle, we see that the minimum number of symmetric chains needed to cover is $|M| - \nu(G)$. Also, by contracting the edges of a maximum matching of G we can construct an optimal symmetric chain partition $\mathscr{S}$ of P.

Thus it remains to construct a legal subpartition $\mathscr{L}$ of P with value $|\mathscr{S}|$. From $(*)$ using the fact that $|M| = \frac{|V|}{2}$ we get that

$$|M| - \nu(G) = \frac{1}{2}\left[o(G - A(G)) - |A(G)|\right].$$

Observe that every node $x \in A(G)$ is adjacent to at least two components of $G - A(G)$, as otherwise this only component would be completely covered by every maximum matching, or there would be a maximum matching that does not cover x. This contradicts the definition of the Edmonds-Gallai decomposition. We claim that for every $x \in A(G)$ if $xx' \in M$ then $\{x'\}$ is a component of $G - A(G)$. Indeed: if not, x' is adjacent to some component of $G - A(G)$, and x must be adjacent to some other component of $G - A(G)$. From the transitivity of E we get the contradiction that two different components of $G - A(G)$ are adjacent.

Define $M^* := \{m \in M : m \text{ joins two different components of } G - A(G)\}$.
If $vv' \in M^*$ then again by the transitivity of E, v or v' is an isolated vertex of $G - A(G)$. Thus after contracting the edges of E in $G - A(G) + M^*$ each resulted component is a star of M^*-edges. Let $\mathscr{L}$ be the partition of M^* formed by these components. We claim that $\mathscr{L} = \{M_i : 1 \le i \le l\}$ is a legal subpartition of P with value $|\mathscr{S}|$.

We prove legality first. From the definition of $\mathscr{L}$ we see that there is no E-edge joining two different $M_i \in \mathscr{L}$, which proves (1). A symmetric chain of length three implies the existence of an ME-alternating path containing three M^*-edges. The middle edge of this path must connect two nonisolated vertices from different components of $G - A(G)$, which is impossible by the latest observation. Thus $\mathscr{L}$ is a legal subpartition of P.

To calculate the value of $\mathscr{L}$ define $\mathscr{C}_i$ as the set of odd components of $G - A(G)$ that are incident with some edge in M_i. From the structure of $G - A(G)$ it is clear that either $\mathscr{C}_i$ consists of an even number of isolated vertices each joined by M^* to a certain even component in $C(G)$ or $\mathscr{C}_i$ is an odd number of isolated vertices together with an odd component of $D(G)$ joined by M^* to them. In both cases $\frac{|\mathscr{C}_i|}{2} = \left\lceil \frac{|M_i|}{2} \right\rceil$. Hence

$$|\mathscr{S}| = |M| - \nu(G) = \frac{1}{2}\left(o(G - A(G)) - |A(G)|\right) = \sum_{i=1}^{l} \frac{|\mathscr{C}_i|}{2} = \sum_{i=1}^{l} \left\lceil \frac{|M_i|}{2} \right\rceil,$$

the equality we need. ∎

From the proof it is clear that using any well-known maximum matching algorithm efficiently determining the Edmonds-Gallai decomposition we can construct an optimal symmetric chain cover and a legal subpartition with maximum value, both in polynomial-time. We remark that if X is an inclusionwise minimal subset of V attaining the maximum in the Tutte-Berge formula then the structure of $G-X+M$ is the similar to the described structure of $G-A(G)+M$. Also, it is easy to determine an optimal legal subpartition of P from any $X \subset V$ attaining the maximum in the Tutte-Berge formula.

3. Corollaries

As corollaries we deduce Dilworth's theorem and the well-known min-max formula for the minimum size edge cover of a graph. Note that this indicates the unifying nature of our result rather than provides a simple proof: in the first reduction instead of Tutte's theorem we rely only on Kőnig's (as the auxiliary graph is bipartite) and in the second case the edge cover formula itself is an immediate consequence of Tutte's theorem that has already been used in the proof of Theorem 1.1.

Corollary 3.1. (Dilworth's theorem [2]) *Let $P=(V,\preceq)$ be a finite poset. Then the minimal number of chains that cover V equals the maximum size of an antichain of P.*

Proof. Define $V' := \{v' : v \in V\}$, $M := \{vv' : v \in V\}$, and $\preceq' := \preceq \cup \{u'v' : v \preceq u\}$. Thus $P' := (V \cup V', \preceq', M)$ is a symmetric poset, and $S \subset M$ is a symmetric chain if and only if the elements of V covered by S form a chain in P. Observe that for an antichain A the system $\mathscr{L}_A := \{\{vv'\} : v \in A\}$ is a legal subpartition of P' with value $|A|$.

Thus it is enough to prove that for any legal subpartition $\mathscr{L}$ of P' there exists an antichain $A_{\mathscr{L}}$ of P with size not less than the value of $\mathscr{L}$. Let

$$\mathscr{L}^V = \{X \subset V : \{vv' : v \in X\} \in \mathscr{L}\} = \{V_i : 1 \le i \le l\}$$

be the subpartition of V corresponding to $\mathscr{L}$ in the natural way. From the definition of legality we see that different parts of $\mathscr{L}^V$ contain pairwise $\preceq$-incomparable elements, and from the lack of 3-chains we get that each part $V_i \in \mathscr{L}^V$ can be decomposed as $V_i = V_i^{max} \cup V_i^{min}$ the union of its $\preceq$-minimal and $\preceq$-maximal elements. Now define V_i^* as the set from V_i^{max} and V_i^{min} with greater cardinality. Clearly $|V_i^*| \geq \left\lceil \frac{|V_i|}{2} \right\rceil$, so $A_{\mathscr{L}} := \bigcup_{i=1}^{l} V_i^*$ is an antichain of P with size not less than the value of $\mathscr{L}$. ∎

Corollary 3.2. *Let $G = (V,E)$ be an undirected graph without isolated vertices. The minimum number of edges needed to cover V equals the maximum of $\frac{1}{2}[|V-X| + o(G-X)]$ for $X \subset V$.*

Proof. Define $V' := \{v' : v \in V\}$, $M := \{vv' : v \in V\}$, and $\prec := \{uv' : uv \in E\}$. Thus $P' := (V \cup V', \preceq, M)$ is a symmetric poset and $S \subset M$ is a maximal symmetric chain if and only if there is an edge $uv \in E$ such that $S = \{uu', vv'\}$. Thus a minimal symmetric chain cover of P' corresponds to an edge cover of G. Observe that for $X \subset V$ the system $\mathscr{L}_X := \{\{vv' : v \in C\} : C$ is a component of $G - X\}$ is a legal subpartition of P' with value $\frac{1}{2}[|V-X| + o(G-X)]$.

Thus it is enough to find for any legal subpartition $\mathscr{L}$ of P' a subset $X_{\mathscr{L}}$ of V such that $\frac{1}{2}[|V - X_{\mathscr{L}}| + o(G - X_{\mathscr{L}})]$ is not less than the value of $\mathscr{L}$. Let

$$\mathscr{L}^V = \{Y \subset V : \{vv' : v \in Y\} \in \mathscr{L}\} = \{V_i : 1 \leq i \leq l\}$$

be the subpartition of V corresponding to $\mathscr{L}$ in the natural way. Let $X_{\mathscr{L}} := V - \bigcup \mathscr{L}^V$. From the definition of legality we see that each part $V_i \in \mathscr{L}^V$ is the union of components $C_i^1, \ldots, C_i^{k_i}$ of $G - X_{\mathscr{L}}$. This means that

$$\frac{1}{2}[|V - X_{\mathscr{L}}| + o(G - X_{\mathscr{L}})] = \frac{1}{2}\left[o(G - X_{\mathscr{L}}) + \sum_{i=1}^{l}\sum_{j=1}^{k_i} |C_i^j|\right] =$$

$$= \sum_{i=1}^{l}\sum_{j=1}^{k_i} \left\lceil \frac{|C_i^j|}{2} \right\rceil \geq \sum_{i=1}^{l} \left\lceil \sum_{j=1}^{k_i} \frac{|C_i^j|}{2} \right\rceil = \sum_{i=1}^{l} \left\lceil \frac{|V_i|}{2} \right\rceil, \qquad \text{the value of } \mathscr{L}.$$ ∎

At last we make use of the plain node splitting construction and prove a weighted generalization of the main result corresponding to the weighted version of Dilworth's theorem and to the so called b-matching problem.

Corollary 3.3. *Let $P = (V, \preceq, M)$ be a symmetric poset and $w : M \longrightarrow \mathbb{N}$ a multiplicity function. The minimal number of symmetric chains needed to cover P with multiplicity w is equal to the maximal w-value of a legal subpartition of P.*

We say that symmetric chains $S_1, S_2, \ldots, S_t$ *cover* the symmetric poset P *with multiplicity* w if $\sum_{i=1}^{t} \chi^{S_i} \geq w$. The *w-value* of a legal subpartition $\mathscr{L}$ is

$$\sum_{M_i \in \mathscr{L}} w^*(M_i), \text{ where } w^*(M_i) := \begin{cases} \left\lceil \frac{1}{2} \sum_{m \in M_i} w(m) \right\rceil, & \text{if } |M_i| > 1, \\ w(m), & \text{if } M_i = \{m\}. \end{cases}$$

Proof. Define

$V' := \{v_i : vv' \in M, 1 \leq i \leq w(vv')\}, \quad M' := \{v_i v_i' : vv' \in M, 1 \leq i \leq w(vv')\}$
and $\prec' := \{u_i v_j : uu', vv' \in M, 1 \leq i \leq w(vv'), 1 \leq j \leq w(uu'), u \prec v\}$.

Apply Theorem 1.1 to the symmetric poset $P' = (V', \preceq', M')$. Observe that if two copies $v_i v_i', v_j v_j' \in M'$ of a certain edge $vv' \in M$ lie in different parts of the optimal legal subpartition $\mathscr{L}'$ of P' then we can assume that each copy of the edge vv' forms a separate part of $\mathscr{L}'$ by itself. Thus the maximum value of a legal subpartition of P' equals the maximum w-value of a legal subpartition of P. ∎

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