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A REMARK ON SETS HAVING THE STEINHAUS PROPERTY

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A point set satisfies the Steinhaus property if no matter how it is placed on a plane, it covers exactly one integer lattice point. Whether or not such a set exists, is an open problem. Beck has proved [1] that any bounded set satisfying the Steinhaus property is not Lebesgue measurable. We show that any such set (bounded or not) must have empty interior. As a corollary, we deduce that closed sets do not have the Steinhaus property, fact noted by Sierpinski [3] under the additional assumption of boundedness.

The purpose of this paper is to prove the following

Theorem. *Any set S having the Steinhaus property has empty interior.*

Our proof requires a number of preliminary lemmas. For m, $n \in \mathbb{Z}$, denote by $A_{m,n}$ the unit square of the lattice $\mathbb{Z}^2$ having its upper-left corner at (m,n). For any subset $M \subset A_{m,n}$ we denote by $M(\bmod\ 1)$ the set $M+(-m,-n)$. For $\theta \in [0,2\pi)$, $S(\theta)$ is the set obtained from S by a rotation of angle θ around the origin. The unit squares $A_{m,n}$ are considered to contain their north and west sides, and not their south and east sides, so that they form a partition of $\mathbb{R}^2$. We use matrix-style coordinates for the points of $\mathbb{Z}^2$, i.e., coordinates increase downward and to the right.

Lemma 1. *A set S satisfies the Steinhaus property if and only if $\{S(\theta) \cap A_{m,n} \ (\bmod\ 1)\}_{m,n}$ is a partition of $A_{0,0}$, for all $\theta \in [0,2\pi)$.*

Proof. We prove that for any set $E \subset \mathbb{R}^2$, $\{E \cap A_{m,n} \ (\bmod\ 1)\}_{m,n}$ is a partition of $A_{0,0}$ iff every translation of E contains exactly one lattice point. This implies the statement of the lemma.

Indeed, suppose E satisfies the latter, and let $E_{m,n} = E \cap A_{m,n}(\bmod\ 1)$. To see that the $E_{m,n}$'s are disjoint, suppose $x \in E_{m,n} \cap E_{m',n'}$. Then $x+(m,n), x+$

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$(m',n') \in E$. Therefore, the translation of E by vector $(-x)$ contains the lattice points (m,n) and (m',n'), so they must coincide. To show that the $E_{m,n}$'s cover $A_{0,0}$, suppose towards a contradiction that there exists $y \in A_{0,0} \setminus \bigcup_{m,n} E_{m,n}$. Let (k,l) be the lattice point that $E' = -y + E$ contains. Then $y + (k,l)$ belongs to both E and $A_{k,l}$, so $y \in E \cap A_{k,l} \pmod 1 = E_{k,l}$, a contradiction. Conversely, if E has the above partition property, then the translates of E contain at most one lattice point because the $E_{m,n}$'s are disjoint, and at least one since the $E_{m,n}$'s cover $A_{0,0}$. ∎

Lemma 2. *If* $\operatorname{int} S \neq \emptyset$, *then* S *must be bounded.*

Proof. Assume that S contains a closed disc of radius $r > 0$. Without loss of generality we may consider it centered at the origin. Then for any integer n representable in the form $x^2 + y^2$, where x and y are integers, the annulus $A(0; \sqrt{n} - r, \sqrt{n} + r)$ cannot contain points of S. For if it contained $u \in S$, then since the point v on the ray Ou such that $|v| = |u| - \sqrt{n}$ lies inside the closed disc of radius r centered at the origin, we would obtain two points of S $\sqrt{x^2+y^2}$ units apart, and we could map these onto two lattice points by a suitable isometry. Therefore,

$$S \subseteq \mathbb{R}^2 \setminus \bigcup_{n=x^2+y^2} A(0; \sqrt{n} - r, \sqrt{n} + r). \tag{1}$$

But it is easy to see that if $a_1, a_2, \ldots$ is the increasing sequence of all integers that are sums of two squares, then $\lim_{n\to\infty}(\sqrt{a_{n+1}} - \sqrt{a_n}) = 0$. This shows that the union in (1) contains the exterior of some circle centered at the origin. So S is bounded. ∎

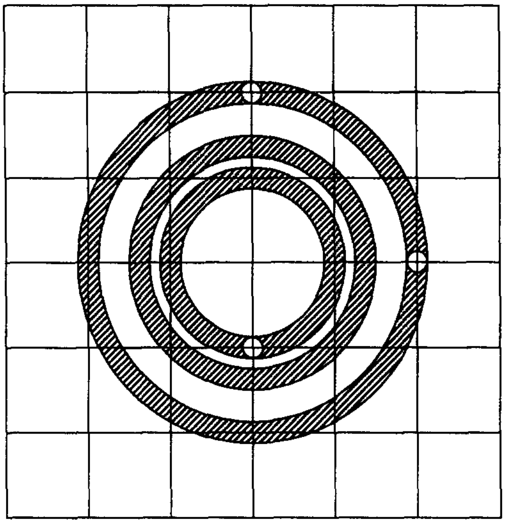
Assume that S has the Steinhaus property and contains a closed disc of radius $r > 0$, centered at the origin. By Lemma 2, S is bounded. Let $S \subseteq [-N, N]^2$, for some positive integer N. For each lattice point $(x,y) \neq (0,0)$ with $-N \leq x, y \leq N$, consider the annulus $A(0; \sqrt{x^2+y^2} - r, \sqrt{x^2+y^2} + r)$ (see Figure 1). Let D be the union of these annuli. Then, by the Steinhaus property, $S \cap D = \emptyset$.

Denote by $C_{m,n}$ the connected component of $A_{m,n} \cap D \pmod 1$ containing the origin. Consider

$$I(r) = \bigcap_{m,n=-N,(m,n)\neq(0,0)}^{N} C_{m,n}. \tag{2}$$

Lemma 3. $I(r) \subseteq S$.

Proof. Let $x \in I(r) \subseteq A_{0,0}$. By Lemma 1, there exist m, n such that $x \in S \cap A_{m,n} \pmod 1$, i.e., $x + (m,n) \in S$. We claim that $(m,n) = (0,0)$. Indeed, suppose this is not the case. Then, since $x \in I(r)$, we have $x \in C_{m,n} \subseteq A_{m,n} \cap D \pmod 1$,



and hence $x+(m,n)\in D$. So $x+(m,n)$ would be in the intersection of S and D, a contradiction. Therefore, $x=x+(0,0)\in S$. ∎

Lemma 4. $I(r)=\bigcap_{m,n=0,(m,n)\neq(0,0)}^{N} C_{m,n}$.

Proof. Let us denote the expression on the right by $I_0(r)$. Suppose first that exactly one of m and n is negative. Then $C_{m,n}$ contains the lower-right quarter P of the square that circumscribes the disc of radius r centered at the origin and has sides parallel to the coordinate axes. Since $C_{1,0}\cap C_{0,1}$ is contained in P, we obtain that $I_0(r)\subseteq C_{m,n}$ in this case. Second, if both m and n are negative, then we have $C_{-m,-n}\subseteq C_{m,n}$, so again $I_0(r)\subseteq C_{m,n}$. ∎

Proof of Theorem. Suppose that $\operatorname{int} S\neq\emptyset$. Define

(3) $$R=\sup\{r>0 : S \text{ contains some closed disc of radius } r\}.$$

Observe that for any $(m,n)\neq(0,0)$, $0\leq m,n\leq N$, $C_{m,n}$ contains the lower-right quarter Q of the closed disc of radius r centered at the origin. Moreover, for each such m and n, there is a unique ray of Q that ends on the boundary of $C_{m,n}$ (since the disc of radius r centered at (m,n) touches the boundary of the annulus that contains it at two diametrically opposite points). Therefore, $I(r)$ contains Q, and there are only a finite number of rays of Q that end on the boundary of $I(r)$. This implies that, if $M(r)=\max\{|x| : x\in I(r)\}$, then $M(r)/r>1$. Note that as long as $S\subseteq[-N,N]^2$, $I(r)$ and $M(r)$ are independent of S.

Let N be such that $S(\theta)\subseteq[-N,N]^2$ for all $\theta\in[0,2\pi)$. Choose $x\in I(r)$ with $|x|=M(r)$. By Lemma 3 applied to $S(\theta)$, we obtain that the line segment Ox is contained in $S(\theta)$, for all $\theta\in[0,2\pi)$. This implies that the closed disc of radius

$M(r)$ centered at the origin is contained in S. Lemma 5 shows that r can be chosen such that $M(r)$ is larger than R, a contradiction. ∎

Lemma 5. *If $r_1 < r_2$, then $M(r_1)/r_1 \geq M(r_2)/r_2$. In particular, $M(r)/r \geq M(R)/R > 1$ for any r such that S contains a closed disc of radius r centered at the origin.*

Proof. For any positive number a and plane set X let $aX = \{ax : x \in X\}$. Since dilations preserve proportions, we have that $M(r)/r = \max\{|x| : x \in (1/r)I(r)\}$. Therefore it suffices to prove that $(1/r_2)I(r_2) \subseteq (1/r_1)I(r_1)$.

Let $m,n \geq 0, (m,n) \neq (0,0)$ be fixed and let $\alpha = \sqrt{m^2+n^2}$. For simplicity, write C_1 for the set $C_{m,n}$ corresponding to r_1 and C_2 for the one corresponding to r_2. By Lemma 4, it is enough to show that

$$\frac{1}{r_2}C_2 \subseteq \frac{1}{r_1}C_1. \tag{4}$$

Both $(1/r_1)C_1$ and $(1/r_2)C_2$ are convex sets bounded by the coordinate axes and a circle. The two circles contain the unit circle and touch it at the same point. Thus, the set bounded by the circle with larger curvature is contained in the other one. Since the two curvatures are $r_1/(\alpha+r_1)$ and $r_2/(\alpha+r_2)$, respectively, we obtain relation (4). ∎

Corollary. *If S has the Steinhaus property, then it cannot be closed.*

Proof. Suppose there exists a closed set S satisfying the Steinhaus property. Consider the closed square $P = [0,1/2] \times [-1/2,0] \subset A_{0,0}$. Let $P_{m,n} = P + (m,n)$. By Lemma 1, $\{P_{m,n} \cap S \pmod 1\}_{m,n}$ is a partition of P into closed sets. Therefore, by Baire's theorem (e.g., [2], p. 387), there exist m,n such that $\operatorname{int}\{P_{m,n} \cap S \pmod 1\} \neq \emptyset$. This implies $\operatorname{int} S \neq \emptyset$, a contradiction. ∎

References

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